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On Gaussian Divisors of Characteristic Functions

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Dedicated to the memory of I.V. Ostrovskii and L.S. Kudina

We prove the following facts: 1) For every natural number $n \geqslant 3$ there are n characteristic functions each of which does not have a Gaussian divisor, and the products of all proper subsets of the set of these characteristic functions also does not have a Gaussian divisor, but the product of all of these characteristic functions has a Gaussian divisor; 2) Every non-degenerate distribution with bounded spectrum has rudiments of a Gaussian component in the following sense: for each such distribution there is a distribution without Gaussian component, whose convolution with the original one has a Gaussian component. We also indicate a wide class of functions on the real axis, which are the ratio of two characteristic functions.

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1. On Gaussian components of sums of independent random variables

In one of the letters to the author, A.M. Kagan put the following question:

Are there three independent random variables X, Y, Z such that none of them, and also none of the quantities $X+Y$, $Y+Z$, $Z+X$ do not has a Gaussian component, but $X+Y+Z$ has a Gaussian component? (*Similar question for ν* ($\nu > 3$) *random variables.*)

We recall that one says that the random variable U has a Gaussian component if its distribution law coincides with the distribution law of the sum $V + W$ of independent random variables V and W, one of which has a Gaussian distribution. In terms of characteristic functions (ch.f.s), this means that the ch.f. $\varphi_U(t)$ of the random variable U can be represented as a product $\psi(t)\exp(-\gamma t^2)$, where $\gamma > 0$, and $\psi(t)$ is a ch.f.

The purpose of this paper is to provide an example that gives a positive answer to the A.M. Kagan's question.

Theorem 1.1. *For every natural number $\nu \geqslant 3$ there are ν characteristic functions $\varphi_1, \varphi_2, \ldots, \varphi_\nu$ each of which does not have a Gaussian divisor, and the* *products of all proper subsets of the set of these characteristic functions also does not have a Gaussian divisor, but the product $\prod_{i=1}^{\nu} \varphi_i$ has a Gaussian divisor.*

We will give the proof of this theorem for $\nu = 3$. In what follows we give a construction of the *three ch.f.s φ_1, φ_2, φ_3 each of which does not have a Gaussian divisor, and each of the products $\varphi_1\varphi_2$, $\varphi_2\varphi_3$, $\varphi_3\varphi_1$ also does not have a Gaussian divisor, but the product $\varphi_1\varphi_2\varphi_3$ has a Gaussian divisor.*

In the cases $\nu > 3$, the proof is analogues to this one.

Note that for the case $\nu = 2$ a corresponding example was given by R. Fischer and D. Dugué [1] in 1948. (See also [7], pp. 118–119.) Our method is different from method of R. Fischer and D. Dugué.

Proof of Theorem 1.1. **1.** Let us denote

$$\varkappa(t) = \begin{cases} 4((1/2) - t)^2, & 0 \leqslant t \leqslant 1/2, \\ 0, & t \geqslant 1/2, \\ \varkappa(-t), & t \leqslant 0. \end{cases}$$

The function $\varkappa(t)$ is a ch.f. of absolutely continuous law with the density

$$k(x) = \frac{4}{\pi x^2}\left(1 - \frac{\sin(x/2)}{x/2}\right).$$

Its explicit form is not used further. The only important thing for us is that the density $k(x)$ satisfies the condition

$$\exists \varepsilon > 0 \quad \forall x \in \mathbb{R} \quad k(x) \geqslant \varepsilon \min\left(1, x^{-2}\right). \tag{1.1}$$

Let $q(x)$ be the density function of the law whose ch.f. is equal to $\varkappa(t)\exp(-t^2/2)$. It is positive everywhere and satisfies the condition similar to condition (1.1)

$$\exists \delta > 0 \quad \forall x \in \mathbb{R} \quad q(x) \geqslant \delta \min\left(1, x^{-2}\right). \tag{1.2}$$

Indeed, when $x > 2$ we have

$$q(x) = \int_{-\infty}^{\infty} k(x-v)\frac{1}{\sqrt{2\pi}}e^{-v^2/2}\,dv > \int_{-1}^{1} k(x-v)\frac{1}{\sqrt{2\pi}}e^{-1/2}\,dv$$
$$> c_1(x+1)^{-2} > c_2 x^{-2}.$$

A similar estimate exists for $x < -2$, since function $q(x)$ is even. It follows from the continuity and positivity of the function $q(x)$ that the condition (1.2) is fulfilled.

2. Let us denote $\psi(t)$ any even real function of the class $C^2(\mathbb{R})$ which is equal to zero for $|t| \geqslant 1/2$ and is equal to 1 at $t = 0$. Let

$$p(x) = \frac{1}{2\pi}\int_{-1/2}^{1/2} \psi(t)e^{-itx}\,dt$$

be the Fourier transform of the function ψ. Integrating by parts, we see that the function $p(x)$ satisfies the condition

$$\exists B>0 \quad \forall x\in\mathbb{R} \quad |p(x)|\leqslant B\min\bigl(1,x^{-2}\bigr). \tag{1.3}$$

Let us denote for each natural number n

$$\psi_n(t)=\psi(t-n)+\psi(t+n).$$

The Fourier transform of the function $\psi_n(t)$ equals

$$p_n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_n(t)e^{-itx}\,dt=2p(x)\cos(nx)$$

and therefore satisfies the condition

$$|p_n(x)|\leqslant 2B\min(1,x^{-2}) \quad (x\in\mathbb{R}). \tag{1.4}$$

We note that

$$\varkappa(t)\cdot\psi_n(t)\equiv 0 \quad (\forall\ n=1,2,\dots), \tag{1.5}$$

$$\psi_m(t)\psi_n(t)\equiv 0 \quad (\forall\ m,n=1,2,\dots,m\neq n). \tag{1.6}$$

3. Let $\beta>0$. This number will then be taken small. For $j=1,2,3$ and $n=1,2,\dots$, let us put

$$\Delta_{jn}=\begin{cases}0, & n\equiv j \pmod 3,\\ 1, & n\not\equiv j \pmod 3.\end{cases}$$

For $j=1,2,3$ we put

$$\varphi_j(t)=\varkappa(t)e^{-t^2/2}+\sum_{n=1}^{\infty}\beta^n\psi_n(t)\Delta_{jn}. \tag{1.7}$$

Let us show that for every sufficiently small $\beta>0$ the functions φ_j will be what we are looking for.

The modulus of the Fourier transform of the function $\varphi_j(t)-\varkappa(t)e^{-t^2/2}$ ($j=1,2,3$) can be estimated in the same way for every j (see (1.4))

$$\left|\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}(\varphi_j(t)-\varkappa(t)e^{-t^2/2})e^{-itx}\,dt\right|\leqslant\sum_{n=1}^{\infty}\beta^n|p_n(x)|\leqslant\frac{\beta}{1-\beta}2B\min\bigl(1,x^{-2}\bigr)$$
$$\leqslant\varepsilon\min\bigl(1,x^{-2}\bigr).$$

The last inequality holds if β is sufficiently small.

Therefore (see (1.1)) the functions φ_j are ch.f.s provided β is sufficiently small (because $\varphi_j(0)=1$).

Ch.f. $\varphi_j(t)$ has no Gaussian divisor because on the arithmetic progression $t=a+bm$ ($m=1,2,3,\dots$) it has a slower decay than $\exp(-cm^2)$. For example,

$$\varphi_1(2+3m)=\beta^{2+3m}\psi_{2+3m}(2+3m)=\beta^{2+3m}.$$

For the same reason, the ch.f.s $\varphi_j(t)\varphi_k(t)$ $(j \neq k)$ do not have a Gaussian divisor. For example, (see (1.5), (1.6))

$$\varphi_1(t)\varphi_2(t) = \varkappa^2(t)e^{-t^2} + \beta^6\psi_3^2(t) + \beta^{12}\psi_6^2(t) + \cdots .$$

Therefore

$$\varphi_1(3+3m)\varphi_2(3+3m) = \beta^{(3+3m)2}, \quad m = 1, 2, \ldots .$$

However, we have

$$\varphi_1(t)\varphi_2(t)\varphi_3(t) = \varkappa^3(t)e^{-(3/2)t^2},$$

where $\varkappa^3(t)$ is a ch.f. □

Obviously, similar constructions can be implemented for any number $\nu > 3$ of random variables.

2. On strongly non-Gaussian random variables

The second A.M. Kagan's question is the following one:

Let us call a random variable X strongly non-Gaussian if the sum $X + Y$ where Y is independent of X, has a Gaussian component if and only if Y has it. How to describe such X?

It is evident that the constant random variables are strongly non-Gaussian. We will show in this section that any bounded non-constant random variable is not strongly non-Gaussian.

We will use the language of ch.f.s. Let $\mathcal{CF}$ be the set of all ch.f.s on $\mathbb{R}$. Let us denote by $\mathcal{G}$ the class of ch.f.s that have a Gaussian divisor:

$$\mathcal{G} = \{\varphi \in \mathcal{CF} : \exists \varepsilon > 0 \;\; \varphi(t)e^{\varepsilon t^2} \in \mathcal{CF}\}.$$

Let $\overline{\mathcal{G}} = \mathcal{CF} \setminus \mathcal{G}$ be the set of ch.f.s that do not have a Gaussian divisor. Let us say that ch.f. $\varphi \in \overline{\mathcal{G}}$ *has rudiments of a Gaussian divisor* if there is a ch.f. $\psi \in \overline{\mathcal{G}}$ such that $\varphi \cdot \psi \in \mathcal{G}$. Let us denote by $\mathcal{K}$ the set of ch.f.s that do not have rudiments of a Gaussian divisor:

$$\mathcal{K} = \{\varphi \in \overline{\mathcal{G}} : \forall \psi \in \mathcal{CF} \;\; \varphi\psi \in \mathcal{G} \Rightarrow \psi \in \mathcal{G}\}. \tag{2.1}$$

This is the set of ch.f.s of the strong non-Gaussian random variables. It is required to describe class $\mathcal{K}$. It is clear that the ch.f.s $\exp(i\beta t)$ of the constant variables belong to the class $\mathcal{K}$. It turns out that these are the only laws with bounded spectrum that belong to $\mathcal{K}$. Recall that the spectrum $S(F)$ of a law F is the set

$$S(F) = \{x \in \mathbb{R} : \forall \varepsilon > 0 \;\; F((x - \varepsilon, x + \varepsilon)) > 0\}.$$

Theorem 2.1. *Let φ be the characteristic function of a law with a bounded spectrum containing at least two points. Then $\varphi \notin \mathcal{K}$.*

In particular, $\cos t$ and $\sin t/t$ have rudiments of Gaussian divisor.

Proof. It is easy to check (see, for example, [7], Ch. 3, § 3) that the function

$$\psi_1(t) = (1-t^2)e^{-t^2/2} \tag{2.2}$$

is a ch.f. of the law with the density

$$s_1(x) = \frac{1}{\sqrt{2\pi}} x^2 e^{-x^2/2}. \tag{2.3}$$

Let $0 < d < 1$. We denote

$$\psi_d(t) = (1-t^2)e^{-d^2t^2/2}. \tag{2.4}$$

It is easy to show that

$$\psi_d(t) = \int_{-\infty}^{\infty} e^{itx} s_d(x)\, dx, \tag{2.5}$$

where

$$s_d(x) = \frac{1}{\sqrt{2\pi} d^5} \left(x^2 - d^2(1-d^2)\right) e^{-x^2/2d^2}. \tag{2.6}$$

Let

$$\xi_d = d\sqrt{1-d^2}$$

be the positive root of the function $s_d(x)$ ($s_d(x) < 0$ for $|x| < \xi_d$, $s_d(x) > 0$ for $|x| > \xi_d$). Let us put

$$\eta_d = \frac{1}{\sqrt{2\pi} d^3}(1-d^2).$$

Therefore $s_d(0) = -\eta_d$ is a minimum of the function $s_d(x)$. We have $\xi_d \downarrow 0$, $\eta_d \downarrow 0$, and $s_d(x) \to s_1(x)$ for $d \uparrow 1$ uniformly on $\mathbb{R}$.

Let F be a distribution law satisfying the conditions: 1) F is non-degenerate ($S(F)$has at least two points), 2) the spectrum $S(F)$ of law F is a bounded set. Let

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} F(dx). \tag{2.7}$$

It is evident, that $\varphi \in \overline{\mathcal{G}}$ and, since $s_1(0) = 0$, it is easy to see that $\psi_1(t) \in \overline{\mathcal{G}}$. Let us show that $\varphi \cdot \psi_1 \in \mathcal{G}$. To do this it is necessary to show that the function

$$\varphi(t)\psi_d(t) = \varphi(t)(1-t^2)e^{-d^2t^2/2} \tag{2.8}$$

is a ch.f. for some $d < 1$ sufficiently closed to 1. We have

$$\varphi(t)\psi_d(t) = \int_{-\infty}^{\infty} e^{itx} q_d(x)\, dx. \tag{2.9}$$

where

$$q_d(x) = \int_{-\infty}^{\infty} s_d(x-v)F(dx) = \int_{S(F)} s_d(x-v)F(dx). \tag{2.10}$$

Our aim is to show that $q_d(x) > 0$ for all $x \in \mathbb{R}$ if $d < 1$ is sufficiently close to 1.

Let $a = \inf S(F)$ and $b = \sup S(F)$ be the extreme points of the spectrum of the law F. Since F is a non degenerate law, we have $b > a$. We fix the number $0 < \varrho < (b-a)/8$. Let us denote

$$\alpha_\varrho = F([a, a+\varrho]), \quad \beta_\varrho = F([b-\varrho, b]).$$

Numbers α_ϱ and β_ϱ are positive. They do not depend on d. We will choose parameter d so close to 1 that $\xi_d < \varrho$. Let us consider two cases: 1) $x > b+\varrho$, 2) $(a+b)/2 \leqslant x \leqslant b+\varrho$.

1) Let $x > b+\varrho$. Then $x - v > \varrho$ for all $v \in S(F) \subset [a, b]$. Therefore $x - v > \xi_d$. This means that for all $v \in [a, b]$ we have $s_d(x-v) > 0$. So, $q_d(x) > 0$.

2) Let $(a+b)/2 \leqslant x \leqslant b+\varrho$. We divide the integral in (2.10) into the sum of three integrals as follows:

$$q_d(x) = \left(\int_{A_1} + \int_{A_2} + \int_{A_3}\right) s_d(x-v)F(dv) = J_1 + J_2 + J_3, \tag{2.11}$$

where

$$A_1 = [a, a+\varrho], \quad A_2 = (x-\xi_d, x+\xi_d), \quad A_3 = (a+\varrho, b] \setminus (x-\xi_d, x+\xi_d). \tag{2.12}$$

We estimate the integral J_1 from below. We have

$$x - v > \frac{b-a}{8} \cdot 2 = \frac{b-a}{4}.$$

We have also that $x - v \leqslant b - a$. Therefore, for some constant $c > 0$ with the indicated x and v, inequality $s_d(x-v) \geqslant c$ is satisfied. Therefore

$$J_1 \geqslant cF([a, a+\varrho]) = c\alpha_\varrho. \tag{2.13}$$

The constant on the right side of (2.13) does not depend on d.

The integral J_3 is non-negative, since if $v \in (a+\varrho, b] \setminus (x-\xi_d, x+\xi_d)$ is satisfied, then $|x-v| > \xi_d$ and, therefore, the integrand function is non-negative. So,

$$J_3 \geqslant 0. \tag{2.14}$$

Let us estimate the integral J_2 from above. We have

$$|J_2| \leqslant \int_{A_2} |s_d(x-v)|F(dv) \leqslant \eta_d = \frac{1-d^2}{\sqrt{2\pi}d^3} \leqslant \frac{2^3}{\sqrt{2\pi}}(1-d^2), \tag{2.15}$$

since we can assume that $d > 1/2$.

Comparison of formulas (2.13), (2.14), (2.15) shows that for $d < 1$ sufficiently close to 1, inequality $q_d(x) > 0$ is valid.

Cases $x < a - \varrho$ and $a - \varrho \leqslant x \leqslant (a+b)/2$ are similar to those considered. □

Remark 2.2. There exist laws with unbounded spectrum that do not belong to class $\mathcal{K}$. For example, if φ is a ch.f. of some law with a bounded spectrum that has at least two points, $c_k > 0$, $\sum_{k=1}^{\infty} c_k = 1$, and ψ is a ch.f. such that $\psi \in \overline{\mathcal{G}}$, $\varphi\psi \in \mathcal{G}$, then the ch.f.

$$\Phi(t) = \varphi(t) \sum_{k=1}^{\infty} c_k e^{ikt}$$

satisfies the conditions: $\Phi \in \overline{\mathcal{G}}$, $\Phi\psi = \varphi\psi \sum_{k=1}^{\infty} c_k e^{ikt} \in \mathcal{G}$. Therefore $\Phi \notin \mathcal{K}$.

Remark 2.3. The author is not aware of examples of ch.f.s $\Phi \in \mathcal{K}$ of laws with an unbounded spectrum.

3. On representations of functions in the form of a ratio of characteristic functions

Equation of the form

$$\psi(t) = \varphi(t) e^{P(t)}, \tag{3.1}$$

where $\varphi(t)$, $\psi(t)$ are ch.f.s that do not equal to zero on the real axis, $P(t)$ ($P(0) = 0$) is a polynomial, play important role in characterization problems of probability distributions by properties of linear statistics ([5], [8]). (See also [4], [3].) A.A. Goldberg [2] showed that for every real even polynomial P there are infinitely differentiable characteristic functions φ, ψ satisfying the equality (3.1). (From the method of constructing of these characteristic functions it is not clear whether they have zeros or not). In this section we consider the equation (3.1) in the case when the function $e^{P(t)}$ on the right side of this equation is replaced by a more general one.

Our purpose is to prove the following theorem.

Theorem 3.1. *Let $F(t)$ ($F\colon \mathbb{R} \to \mathbb{C}$) be a twice continuously differentiable function on $\mathbb{R}$ and four times continuously differentiable on some neighborhood of the point $t = 0$, such that $F(-t) = \overline{F(t)}$, $F(0) = 1$.*

Then there exists a characteristic function $\varphi(t)$ that is positive on the whole of real axis, such that the function $\psi(t) = \varphi(t) \cdot F(t)$ is also a characteristic function.

In contrast to work [2], the function $\varphi(t)$ from Theorem 3.1 is not differentiable at point $t = 0$, so the corresponding distribution does not have a finite moment of order 1.

Proof of Theorem 3.1. **1**. Let us first prove the theorem with the additional assumption that $F(t) > 0$ in some neighborhood of the point $t = 0$. In this situation it suffices to assume that $F \in C^2(\mathbb{R})$. Let $a_0 > 0$ be such, that $F(t) > 0$ for $-a_0 \leqslant t \leqslant a_0$. We can assume that $a_0 \leqslant 1$. Let $0 < a < a_0$. We denote by $\varkappa_a(t)$ the ch.f.

$$\varkappa_a(t) = \begin{cases} \big(1 - (t/a)\big)^2, & 0 \leqslant t \leqslant a, \\ 0, & t > a, \\ \varkappa_a(-t), & t < 0. \end{cases} \tag{3.2}$$

The density of this ch.f. is given by

$$\widehat{\varkappa_a}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx}\varkappa_a(t)\,dt = \frac{2}{\pi a x^2}\left(1 - \frac{\sin ax}{ax}\right) \tag{3.3}$$

for $x \neq 0$ ($\widehat{\varkappa_a}(0) = a/(3\pi)$). Since the function $\widehat{\varkappa_a}(x)$ is continuous and does not vanish, it follows from (3.3) that

$$\exists c_1 > 0 \quad \forall x \in \mathbb{R} \quad \widehat{\varkappa_a}(x) \geqslant c_1 \min\left(1, x^{-2}\right). \tag{3.4}$$

2. Let us consider the ch.f. $\varkappa_a^2(t)$. The following estimate from below for its density follows from (3.4)

$$\exists c_2 > 0 \quad \forall x \in \mathbb{R} \quad \widehat{\varkappa_a^2}(x) = \widehat{\varkappa_a} * \widehat{\varkappa_a}(x) \geqslant c_2 \min\left(1, x^{-2}\right). \tag{3.5}$$

3. Let $F(t)$ satisfies the assumptions of Theorem 3.1. Let us show that for sufficiently small $a > 0$ the function $\varkappa_a(t)F(t)$ is a ch.f. By Pólya's theorem (see, for example, [6, Ch. 2, § 1]), it suffices to show that the derivative of order 2 of this function is non-negative on the interval $(0, a)$. We have for $0 < t < a$

$$\left(\varkappa_a(t)F(t)\right)'' = \frac{2}{a^2}F(t) - \frac{4}{a}\left(1 - \frac{t}{a}\right)F'(t) + \left(1 - \frac{t}{a}\right)^2 F''(t). \tag{3.6}$$

Let us denote

$$\begin{aligned} b &= \min\{F(t) : 0 \leqslant t \leqslant a_0\},\\ B &= \max\{B_1, B_2\}, \quad B_j = \max\{|F^{(j)}(t)| : 0 \leqslant t \leqslant a_0\}. \end{aligned}$$

Then it follows from (3.6) for $0 < t < a$

$$\left(\varkappa_a(t)F(t)\right)'' \geqslant \frac{2b}{a^2} - \frac{4B}{a} - B > 0, \tag{3.7}$$

for sufficiently small a. For such an a the function $\varkappa_a(t)F(t)$ is the ch.f. of a probability distribution with the density $\widehat{\varkappa_a F}(x)$.

4. Similar to item **2**, we can state that the following estimate is valid

$$\exists c_3 > 0 \quad \forall x \in \mathbb{R} \quad \widehat{\varkappa_a^2 F}(x) = \widehat{\varkappa_a} * \widehat{\varkappa_a F}(x) \geqslant c_3 \min\left(1, x^{-2}\right). \tag{3.8}$$

We put $c = \min\{c_1, c_2, c_3\}$. Then estimates (3.4), (3.5), and (3.8) will be carried out with this constant c, the same in these inequalities.

5. Let $a \in (0, a_0)$ be number fixed in item **3**. Let us denote for $n \in \mathbb{N}$

$$I_n = (\alpha_n, \beta_n) = \left(\frac{(n+1)a}{2}, \frac{(n+3)a}{2}\right), \quad -I_n = (-\beta_n, -\alpha_n), \quad J_n = I_n \cup (-I_n).$$

We denote by $\chi_n(t)$ a function on $\mathbb{R}$ that satisfies the conditions: $\chi_n(t) \in C^2(\mathbb{R})$, $\chi_n(t)$ is even, $\chi_n(t) > 0$ for $t \in I_n$, $\chi_n(t) = 0$ for $t \notin J_n$, $\chi_n(\alpha_n) = \chi_n(\beta_n) = 0$, $\chi_n'(\alpha_n) = \chi_n'(\beta_n) = 0$. We put

$$\mu_n = \max\{|\chi_n''(t)| : \alpha_n \leqslant t \leqslant \beta_n\}.$$

We will show that with a suitable choice of functions $\chi_n(t)$, determined by the values of μ_n, the required function will be

$$\varphi(t) = \varkappa_a^2(t) + \sum_{n=1}^{\infty} \chi_n(t). \tag{3.9}$$

6. We obtain an upper bound for $\widehat{\chi_n}(x)$. A simple calculation shows that

$$\widehat{\chi_n}(x) = \frac{1}{2\pi}\int_{J_n} e^{-itx}\chi_n(t)\,dt = -\frac{1}{\pi x^2}\int_{\alpha_n}^{\beta_n} \cos tx\,\chi_n''(t)\,dt \tag{3.10}$$

for $x \neq 0$. Therefore, we have for $|x| > 1$ $(0 < a < 1)$

$$|\widehat{\chi_n}(x)| \leqslant \frac{a\mu_n}{\pi x^2} \leqslant \frac{\mu_n}{\pi x^2} < \frac{c}{4^n x^2}. \tag{3.11}$$

where c is a constant defined in item **4** and μ_n satisfy the following condition

$$\mu_n < c\pi 4^{-n}. \tag{3.12}$$

Since also

$$|\widehat{\chi_n}(x)| = \left|\frac{1}{2\pi}\int_{J_n} \cos tx\,\chi_n(t)\,dt\right| \leqslant \frac{a}{\pi}\max_{[\alpha_n,\beta_n]}|\chi_n(t)| \leqslant \frac{a^3}{\pi}\mu_n < \frac{1}{\pi}\mu_n, \tag{3.13}$$

then for $|x| \leqslant 1$ the inequality

$$|\widehat{\chi_n}(x)| \leqslant c\,4^{-n} \tag{3.14}$$

will take place if (3.12) is valid. So,

$$|\widehat{\chi_n}(x)| \leqslant 4^{-n}c\,\min(1, x^{-2}). \tag{3.15}$$

Since

$$\begin{aligned}\widehat{\varphi}(x) &\geqslant \widehat{\varkappa_a^2}(x) - \sum_{n=1}^{\infty}|\widehat{\chi_n}(x)| \\ &\geqslant c\min(1, x^{-2}) - \sum_{n=1}^{\infty} 4^{-n}c\min(1, x^{-2}) = (c/3)\min(1, x^{-2}) > 0,\end{aligned}$$

the function $\varphi(t)$ is a ch.f. if the condition (3.12) is satisfied for all $n \geqslant 1$.

7. We will find the conditions on numbers μ_n for which the function $\psi(t) = \varphi(t)F(t)$ is a ch.f. Let us put

$$M_{nj} = \max\{|F^{(j)}(t)| : \alpha_n \leqslant t \leqslant \beta_n\}\ (j = 0, 1, 2), \quad M_n = \max\{M_{n0}, M_{n1}, M_{n2}\}.$$

Let us estimate from above the modulus of the Fourier transform of the function $\chi_n(t)F(t)$. We have for $|x| > 1$

$$\left|\widehat{\chi_n F}(x)\right| = \left|\frac{1}{2\pi}\int_{J_n} \cos tx\,\chi_n(t)F(t)\,dt\right|$$

$$= \left| -\frac{1}{\pi x^2} \int_{\alpha_n}^{\beta_n} \cos tx \big(\chi_n''(t)F(t) + 2\chi_n'(t)F'(t) + \chi_n(t)F''(t)\big)\, dt \right|$$
$$\leqslant \frac{4}{\pi x^2}\mu_n M_n < \frac{c}{x^2 4^n}, \tag{3.16}$$

if μ_n is taken so small that

$$\mu_n \leqslant c\pi 4^{-(n+1)} M_n^{-1}. \tag{3.17}$$

In the previous calculation we took advantage of the fact that

$$\max\{|\chi_n^{(j)}| : \alpha_n \leqslant t \leqslant \beta_n\} \leqslant \mu_n \quad (j = 0, 1, 2)$$

with $j = 0, 1$, since $0 < a < 1$.

We have for $|x| < 1$

$$\left|\widehat{\chi_n F}(x)\right| = \left| \frac{1}{\pi} \int_{\alpha_n}^{\beta_n} \cos tx\, \chi_n(t) F(t)\, dt \right| = \frac{1}{\pi}\mu_n M_n < c4^{-n} \tag{3.18}$$

if

$$\mu_n < c\pi 4^{-n} M_n^{-1}. \tag{3.19}$$

Therefore, we have for all $x \in \mathbb{R}$

$$\left|\widehat{\chi_n F}(x)\right| \leqslant \frac{c}{4^n} \min(1, x^{-2}). \tag{3.20}$$

This means that if μ_n satisfies condition (3.17), then the function $\psi(t) = \varphi(t)F(t)$ is a ch.f., since

$$\widehat{\psi}(x) = \widehat{\varkappa_a^2}(x) + \sum_{n=1}^{\infty} \widehat{\chi_n}(x)$$
$$\geqslant c\min(1, x^{-2}) - \sum_{n=1}^{\infty} 4^{-n} c \min(1, x^{-2}) = (c/3)\min(1, x^{-2}) > 0.$$

So, the functions φ and ψ are ch.f.s if conditions (3.12)and (3.17) are valid.

8. Let us now show how the general case reduces to considered one.

Let $F(t) \neq 0$ for $|t| \leqslant a_0$ and four times continuously differentiable on $[-a_0, a_0]$. Let us denote $\omega(t) = \arg F(t)$. This is an odd real function of class $C^4(-a_0, a_0)$, such that $F(t) = |F(t)|e^{i\omega(t)}$. Let $\Delta(t)$ be arbitrary real function on real axis that satisfies the conditions $\Delta(t) \in C^\infty(\mathbb{R})$, $\Delta(t) = 1$ at $|t| \leqslant a_0/2$, $\Delta(t) = 0$ at $|t| \geqslant a_0$.

Let us put

$$\omega_1(t) = \begin{cases} \omega(t)\Delta(t), & |t| \leqslant a_0, \\ 0, & |t| \geqslant a_0. \end{cases} \tag{3.21}$$

We denote ω_2 the $2a_0$-periodic continuation of the function ω_1 from the segment $[-a_0, a_0]$ to $\mathbb{R}$. Obviously, $\omega_2 \in C^4(\mathbb{R})$. Therefore, it can be represented in the form of an absolutely convergent trigonometric series

$$\omega_2(t) = \sum_{k=1}^{\infty} c_k \sin(k\pi t/a_0), \tag{3.22}$$

where $c_k \in \mathbb{R}$.

Let $j \in \mathbb{Z}$. We define the numbers d_j as follows: $d_0 = 0$, if $j > 0$ and $c_j \geqslant 0$, then $d_j = c_j$, $d_{-j} = 0$; if $j > 0$ and $c_j < 0$, then $d_j = 0$, $d_{-j} = -c_j = |c_j|$. Then the representation (3.22) can be written in the form

$$\omega_2(t) = \sum_{j\in\mathbb{Z}} d_j \sin(j\pi t/a_0), \tag{3.23}$$

where $d_j \geqslant 0$ for all j. We consider the infinitely divisible ch.f. of the class $C^2(\mathbb{R})$

$$\gamma(t) = \exp\left(\sum_{j\in\mathbb{Z}} d_j (e^{ij\pi t/a_0} - 1)\right). \tag{3.24}$$

We have

$$\arg\gamma(t) = \sum_{j\in\mathbb{Z}} d_j \sin(j\pi t/a_0) = \sum_{k=1}^{\infty} c_k \sin(k\pi t/a_0) = \omega_2(t) = \omega(t). \tag{3.25}$$

(The last equality in (3.25) holds at $|t| \leqslant a_0$.)

We will look for the function $\psi(t)$ in the equation $\psi(t) = F(t)\varphi(t)$ in the form $\psi(t) = \psi_1(t)\gamma(t)$, where $\psi_1(t)$ is a ch.f. We get equation $\psi_1(t) = G(t)\varphi(t)$, where $G(t) = F(t)/\gamma(t)$. Moreover $G(t) \in C^2(\mathbb{R})$ and $G(t) > 0$ for $|t| \leqslant a_0$. Under these conditions, the required statement has already been proven. □

It is easy to see that Theorem 3.1 is valid in the following slightly more general form.

Theorem 3.2. *Let A be a closed subset of $\mathbb{R}$ such that $0 \notin A$, $-A = A$. Let $F(t)$ be a twice continuously differentiable function on $\mathbb{R} \setminus A$ and four times continuously differentiable on some neighborhood of the point $t = 0$ ($F\colon \mathbb{R} \setminus A \to \mathbb{C}$), such that $F(-t) = \overline{F(t)}$, $F(0) = 1$.*

Then there exists characteristic function $\varphi(t)$, positive on $\mathbb{R} \setminus A$ and equal to zero on A such that the function $\psi(t) = \varphi(t) \cdot F(t)$ is also a characteristic function.

Remark 3.3. Is it possible to replace in this theorem the differentiability condition with the local Wiener property, which is obviously necessary?

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Про гауссівські дільники характеристичних функцій

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Доведено такі факти: 1) Для кожного натурального числа n існують n характеристичних функцій, кожна з яких не має гауссівських дільників, також добуток кожної власної підмножини цієї множини характеристичних функцій не має гауссівських дільників, але добуток їх всіх має гауссівські дільники; 2) Кожен невироджений ймовірнісний розподіл з обмеженим спектром має рудименти гауссівських компонент в такому розумінні: для кожного такого розподілу існує розподіл, який не має гауссівських компонент, але їх згортка має гауссівську компоненту. Ми також вказуємо широкий клас функцій на дійсній осі, які можуть бути представлені як відношення двох характеристичних функцій.

Ключові слова: характеристична функція, гаусівський розподіл, випадкова величина, згортка