Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 4, pp. 527–535
doi: https://doi.org/10.15407/mag20.04.08

On a Uniqueness Property of n-th Convolutions and Extensions of Titchmarsh Convolution Theorem

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This note is an attempt to present a brief story of how a question of Kolmogorov on Gaussian measures had led to an number of function-theoretic results obtained mostly by Ostrovskii and some of his students. It is intended to outline the general development, rather than give a full account of the results.

Key words: probability measure, infinitely divisible measure, characteristic function, analytic function, Titchmarsh convolution theorem, value-distribution theory

Mathematical Subject Classification 2020: 30D, 60E

1. Introduction

1.1. Notations. This note is an attempt to present a brief story of how a question of Kolmogorov on Gaussian measures had led to a number of function-theoretic results obtained mostly by Ostrovskii and some of his students. It is intended to outline the general development, rather than give a full account of the (large number of) results. The reference list is therefore far from being complete. For a more detailed account of the results and a fuller list of references, see [18]. This story also illustrates the efficiency of complex-analytic methods in some problems of probability theory.

Given a real-valued random variable ξ, its probability measure (distribution) is defined by

$$\nu((-\infty, t)) := \mathrm{Prob}(\{\xi < t\}).$$

The Fourier–Stieltjes transform

$$\widehat{\nu}(x) = \int_{\mathbb{R}} e^{itx} \nu(dt)$$

is often called the characteristic function of ξ.

Denote by $\mathcal{P}$ the set of all probability measures on the real line and by $\mathcal{I}$ the set of all infinitely divisible probability measures. Recall that a probability measure μ is infinitely divisible if every n-th "convolution root" of μ is again a probability measure. This means that for every natural number $n \geq 2$ there exists $\mu_n \in \mathcal{P}$ such that $\mu = \mu_n^{n*}$ (convolution of μ_n with itself n times). Infinitely divisible measures play an important role in Probability Theory. The famous Gaussian (or normal) measures ϕ are infinitely divisible. The corresponding characteristic functions are given by

$$\widehat{\phi}(x) = e^{-\sigma x^2 + i\alpha x}, \quad \sigma > 0,\ \alpha \in \mathbb{R}.$$

1.2. Kolmogorov's question. The following problem was raised by Kolmogorov in the 50-ies at a Moscow university seminar: Show that the normal measure is uniquely determined in the class of all infinitely divisible measures by its values (restriction) on a half-line.

In what follows we consider the "left" half-lines $(-\infty, a)$, $a \in \mathbb{R}$, though similar results are true for the "right" half-lines (a, ∞).

The fist proof was given by Rossberg [21] in 1974. He proved the implication:

$$\mu \in \mathcal{I},\ \mu|_{(-\infty,a)} = \phi|_{(-\infty,a)} \quad \text{for some } a \in \mathbb{R} \Rightarrow \mu = \phi. \tag{1.1}$$

It was proved in [23] that condition $\mu \in \mathcal{I}$ in (1.1) can be replaced by the weaker condition $\mu = \mu_1^{n*}, n \geq 2$, where $\mu_1 \in \mathcal{P}$. Also, implication (1.1) remains true under the weaker conditions $\mu \in \mathcal{I}$ and

$$\lim_{x \to -\infty} \mu(-\infty, x)/\phi(-\infty, x) = 1,$$

see [20]. For different extensions of these results (including extensions to finite complex-valued measures and extensions to LCA groups), see [1, 2, 10–12, 26, 27]. Note that these results have applications in the theory of addition of random variables (see e.g. [22, 25, 30]). Some natural generalizations of Kolmogorov's problem were suggested by Zolotarev [30].

1.3. Ibragimov's result. Denote by $\mathbb{C}$ the complex plane and by $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ the upper half-plane.

A key development was due to Ibragimov [9] who proved a far reaching generalization of (1.1):

Theorem 1.1. *Let $\mu, \nu \in \mathcal{I}$. Assume that the characteristic function $\widehat{\nu}$ admits analytic continuation to $\mathbb{C}_+$. If*

$$\mu|_{(-\infty,a)} = \nu|_{(-\infty,a)} \quad \textit{for some } a \in \mathbb{R},$$

then either $\nu|_{(-\infty,b)} = 0$ for some $b \in \mathbb{R}$, or $\mu = \nu$.

The analyticity of $\widehat{\nu}$ in $\mathbb{C}_+$ is essential in this result. Moreover, Ostrovskii in [14] and [15] presented some conditions on a positive measure ν defined in a ball $|x| < r$ in $\mathbb{R}^n$ $(r > 0,\ n \geq 1)$, to have infinitely many infinitely divisible extensions, i.e. there are infinitely many different measures $\mu \in \mathcal{I}$ such that $\mu = \nu$ outside the ball.

The proof of Theorem 1.1 used the value distribution theory of Nevanlinna. Both Ibragimov's theorem and the method of proof indicated that the phenomenon may have purely analytic nature.

2. A simple lemma

Denote by $\mathcal{M}$ the set of all finite complex-valued Borel measures on $\mathbb{R}$. Given $\nu \in \mathcal{M}$, we denote by $l(\mu) := \inf(\operatorname{supp} \mu)$ the left boundary of support of ν.

In what follows we consider measures $\nu \in \mathcal{M}$ satisfying one of the following decay conditions at $-\infty$:

$$\lim_{x \to -\infty} \frac{\log |\nu|(-\infty, x)}{|x|} = -\infty, \tag{2.1}$$

$$\lim_{x \to -\infty} \frac{\log |\nu|(-\infty, x)}{|x| \log |x|} = -\infty. \tag{2.2}$$

Let us now "transfer" certain conditions on measures to conditions on analytical functions.

Lemma 2.1.

(i) *If $\nu \in \mathcal{M}$ satisfies (2.1) then its Fourier–Stieltjes transform $\widehat{\nu}$ admits analytic extension to $\mathbb{C}_+$ and satisfies*

$$\sup_{x \in \mathbb{R}, 0<y<R} |\widehat{\nu}(x+iy)| < \infty, \quad \textit{for every } R > 0. \tag{2.3}$$

If ν is a non-negative finite measure, condition (2.1) is necessary and sufficient for $\widehat{\nu}$ to have the above property.

(ii) *If $\nu \in \mathcal{M}$ satisfies (2.2) then $\widehat{\nu}$ admits analytic extension to $\mathbb{C}_+$ and satisfies*

$$\lim_{y \to +\infty} \frac{\log \log \sup_{x \in \mathbb{R}} |\widehat{\nu}(x+iy)|}{y} = 0. \tag{2.4}$$

(iii) *If $\nu \in \mathcal{I}$ satisfies (2.1), then $\widehat{\nu}$ does not vanish in $\overline{\mathbb{C}_+}$.*

(iv) *If $\nu \in \mathcal{M}$ and $\nu|_{(-\infty,b)} = 0$, for some $b \in \mathbb{R}$, then*

$$|\widehat{\nu}(x+iy)| \le Ce^{-by}, \quad \textit{for some } C = C(\nu) > 0 \textit{ and all } x \in \mathbb{R}, y \ge 0. \tag{2.5}$$

3. Uniqueness property of n-fold convolutions

Using Lemma 2.1 one may check that Theorem 1.1 is an immediate consequence of the following result on analytic functions: Assume that two functions f, g are analytic in $\mathbb{C}_+$, do not vanish there and satisfy (2.3). If the difference $f - g$ satisfies (2.5) with some b, then either $f = g$ or both f and g satisfy (2.5) with some number b.

Ostrovskii proved in [13] that the condition $f, g \ne 0$ in $\mathbb{C}_+$ can be replaced with a weaker condition $f = f_1^n, g = g_1^n$, where $n \ge 3$, f_1, g_1 are analytic in $\mathbb{C}_+$ and satisfy (2.3). Therefore, Theorem 1.1 can be generalized in two directions:

(a) The condition of infinite divisibility of μ and ν can be weakened by replacing it with the condition $\mu = \mu_1^{*n}$, $\nu = \nu_1^{*n}$, for some $n \ge 3$;

(b) The result remains true for complex-valued measures from $\mathcal{M}$.

Theorem 3.1. *Given measures $\mu=\mu_1^{n*}$ and $\nu=\nu_1^{n*}$, where $n\geq 3$, $\mu_1,\nu_1\in\mathcal{M}$ and ν_1 satisfies (2.1). If*

$$\mu|_{(-\infty,a)}=\nu|_{(-\infty,a)}\quad \textit{for some } a\in\mathbb{R},$$

then either $l(\nu_1)>-\infty$, or $\mu=\nu$. In the latter case we have $\mu_1=e^{2\pi ij/n}\nu_1$, for some integer $j, 0\leq j\leq n-1$.

Corollary 3.2 (Uniqueness property of n-th convolutions)**.** *Let $n\geq 3$ and let $\nu_1\in\mathcal{M}$ satisfy (2.1) and $l(\nu_1)=-\infty$. Then every half-line $(-\infty,a)$ is a uniqueness set for the n-th convolution ν_1^{n*} in the sense that the implication holds:*

$$\mu_1\in\mathcal{M},\ \mu_1^{*n}|_{(-\infty,a)}=\nu_1^{*n}|_{(-\infty,a)}\quad \textit{for some } a\in\mathbb{R}\Rightarrow \mu_1^{*n}=\nu_1^{*n}.$$

The property that $\hat{\nu}_1$ is analytic in $\mathbb{C}_+$ cannot be replaced with the property that $\hat{\nu}_1$ is only analytic in some strip $\{0<\Im z<R\}$. However, a variant of Theorem 3.1 for the strip is true provided $\hat{\nu}_1$ has "sufficiently fast" growth when approaching the line $z=x+iR$, see [4].

Corollary 3.2 ceases to be true for $n=2$.

Example 3.3. Define μ_1,ν_1 by

$$\hat{\mu}_1(x):=e^{e^{-aix}}+e^{-e^{-aix}},\quad \hat{\nu}_1(x):=e^{e^{-aix}}-e^{-e^{-aix}}\quad (a>0).$$

One may check that $\mu_1,\nu_1\in\mathcal{M}$, satisfy (2.1), $l(\nu_1)=l(\mu_1)=-\infty$ and $\mu_1^{2*}-\nu_1^{2*}=4\delta_0$, so that $\mu_1^{2*}|_{(-\infty,0)}=\nu_1^{2*}|_{(-\infty,0)}=0$.

However, it was proved in [5] that there cannot be more than two different second convolutions which agree on a half-line:

Theorem 3.4. *Assume $\mu_1\in\mathcal{M}$ satisfies (2.1) and $l(\mu_1)=-\infty$. If there exist measures $\mu_2,\mu_3\in\mathcal{M}$ such that*

$$\mu_1^{2*}|_{(-\infty,a)}=\mu_2^{2*}|_{(-\infty,a)}=\mu_3^{2*}|_{(-\infty,a)}\quad \textit{for some } a\in\mathbb{R},$$

then either $\mu_1^{2*}=\mu_2^{2*}$, or $\mu_1^{2*}=\mu_3^{2*}$.

It follows that for every measure $\mu_1\in\mathcal{M}$ satisfying (2.1) there is a number $a\in\mathbb{R}$ such that the second convolution μ_1^{2*} is uniquely determined by its values on $(-\infty,a)$.

An analogue of Theorem 3.1 for $n=2$ is true under a sharper condition on the decay of μ (see [17]):

Theorem 3.5. *Assume $\nu_1\in\mathcal{M}$ satisfies (2.2). If a measure $\mu_1\in\mathcal{M}$ satisfies*

$$\mu_1^{2*}|_{(-\infty,a)}=\nu_1^{2*}|_{(-\infty,a)}\quad \textit{for some } a\in\mathbb{R},$$

then either $l(\nu_1)>-\infty$ or $\mu_1^{2*}=\nu_1^{*2}$.

Example 3.3 shows that condition (2.2) in Theorem 3.5 cannot be essentially weakened.

4. Extensions of Titchmarsh convolution theorem

Given measures $\mu_1, \mu_2 \in \mathcal{M}$. Assume that

$$l(\mu_j) > -\infty, \quad j = 1, 2. \tag{4.1}$$

Then the classical Titchmarsh convolution theorem states that

$$l(\mu_1 * \mu_2) = l(\mu_1) + l(\mu_2). \tag{4.2}$$

Let μ_1, ν_1 be measure defined in Example 3.3. Then

$$l(\mu_1^{2*} - \nu_1^{2*}) = l((\mu_1 + \nu_1) * (\mu_1 - \nu_1)) = 0, \text{ while } l(\mu_1 \pm \nu_1) = -\infty,$$

which proves that condition (4.1) in Titchmarsh's theorem is essential. One may ask if condition (4.1) can be weakened by replacing it with sufficiently rapid decay of $|\mu_j|((-\infty, x))$ as $x \to -\infty$. A positive answer to this question was given by Domar [7] for a somewhat wider class than $\mathcal{M}$. More general results are mentioned below.

The best condition on the decay of measures from $\mathcal{M}$ was found by Ostrovskii in [17]:

Theorem 4.1. *If $\mu_1, \mu_2 \in \mathcal{M}$ satisfy (2.2) and $l(\mu_1 * \mu_2) > -\infty$ then $l(\mu_j) > -\infty$, $j = 1, 2$, so that (4.2) holds true.*

Since $\mu_1^{2*} - \nu_1^{2*} = (\mu_1 + \nu_1) * (\mu_1 - \nu_1)$, one may check that Theorems 3.5 and 4.1 are in fact equivalent. Example 3.3 shows that restriction (2.2) cannot be essentially improved.

When $n \geq 3$, we have

$$\mu_1^{n*} - \nu_1^{n*} = (\mu_1 - \nu_1) * (\mu_1 - \epsilon\nu_1) * \cdots * (\mu_1 - \epsilon^{n-1}\nu_1), \quad \epsilon = \exp(2\pi i/n).$$

Therefore, $\mu_1^{n*} - \nu_1^{n*}$ is a convolution of linearly dependent measures. One may ask if Theorem 3.1 can be deduced from an extension of Titchmarsh convolution theorem for convolution of $n \geq 3$ linearly dependent measures. This was confirmed in [8]. We will state one result from [8] showing that condition (2.2) can be relaxed:

Theorem 4.2. *If the measures $\mu_1, \ldots, \mu_{n-1} \in \mathcal{M}, n \geq 3$, are linearly independent over $\mathbb{C}$, satisfy (2.1) and $\mu_n = \mu_1 + \cdots + \mu_n$, then*

$$l(\mu_1 * \cdots * \mu_n) = \sum_{j=1}^{n} l(\mu_j).$$

5. Methods of proof

Some of the methods used in the proofs of Theorems 1–6 above are described in the survey [18]. These include the theory of distribution of values, theory of divisibility of quasi-polynomials, factorization in Hardy classes, and results on the connection between the growth and decrease of analytic functions.

For the convenience of the reader, we recall very briefly the main ideas of the proof of Theorem 4.2 in [8].

Assume that $n = 3$. Then it suffices to verify the implication

$$l(\mu_1 * \mu_2 * (\mu_1 + \mu_2)) > -\infty \Rightarrow l(\mu_j) > -\infty, \ j = 1, 2.$$

One may assume that $\mu_1 * \mu_2 * (\mu_1 + \mu_2) = 0$ on $(-\infty, 0)$. Then the product of the Fourier–Stieltjes transforms $\hat{\mu}_1 \hat{\mu}_2 (\hat{\mu}_1 + \hat{\mu}_2)$ belongs to the Hardy space $H_\infty(\mathbb{C}_+)$. Hence, the product (and so each factor) has "few" zeros in the sense that the zeros satisfy the Blaschke condition. Now one can use the following argument: If functions $f_j, j = 1, \dots, n, \ n \geq 2$, are analytic in the unit disk, linearly independent and such that the zeros of each f_j and the sum $f_1 + \dots + f_n$ satisfy the Blaschke condition in the disk, then each f_j must have "slow" growth in the disk. A sharp statement follows from Cartan's second main theorem for analytic curves. This argument proves that the growth of each $\hat{\mu}_j$ in the upper half-plane satisfies a certain restriction. Next, we have additional information (see condition (2.3) in Lemma 2.1 above) that each function $\hat{\mu}_j$ is bounded in every horizontal strip in $\mathbb{C}_+$. This allows one to improve the previous estimate to show that numbers b_j exist such that

$$\hat{\mu}_j(z) \exp(i b_j z) \in H_\infty(\mathbb{C}_+).$$

This implies $l(\mu_j) > -\infty, j = 1, 2$.

6. Extensions and related results

6.1. Zaidenberg's results. The following result follows from Theorem 3.1 as a corollary for the measures concentrated on $\mathbb{Z}$: Given an integer $n \geq 3$ and polynomial $p(x, y) := x^n - y^n$. Assume functions f and g are analytic in $D := \{0 < |z| < 1\}$ and f has an essential singularity at the origin. If the function $p(f, g)$ is meromorphic in $\{z : |z| < 1\}$, then $p(f, g) = 0$, i.e. $f^n = g^n$. One may ask if and how this result can be extended to other polynomials $p \in \mathbb{C}[x, y]$. This was one of the questions considered by Zaidenberg in [28]. In particular, it was proved that that every "sufficiently general" polynomial p in $\mathbb{C}^2$ has a somewhat analogous property, see details in [28]. See also Theorem 3.4 in [29].

6.2. Borichev's results. Observe that the convolution $\mu_1 * \mu_2$ is well defined for *unbounded* complex-valued measures μ_1, μ_2 having "fast" decay at $-\infty$ and "moderate" growth at $+\infty$. One may wonder if Theorem 4.1 admits extension to such measures. The paper [6] presents in quite general situations exact relations (provided some regularity conditions are satisfied) between growth on $(0, \infty)$ and decay on $(-\infty, 0)$ which imply condition (4.2). Also, the problem is considered under what conditions the relation $\mu_1 * \mu_2 = 0$ implies $\mu_1 = 0$ or $\mu_2 = 0$.

Acknowledgments. I would like to thank Misha Sodin for the useful comments and suggestions.

References

[1] N. Blank and I.V. Ostrovskii, *Functions of bounded variation that are close to a normal distribution function*, Operator Theory in Functional Spaces and its Applications, Naukova Dumka, Kiev, 1981, 3–10 (Russian).

[2] N. Blank, *Unique determination of functions of bounded variation and distribution functions by asymptotic behavior as $x \to -\infty$.* Problems Stochastic Models (Panevezhis, 1980) Vsesoyuz. Nauchno-Issled. Inst. Sistem. Issled., Moscow, 1981, 10–15 (Russian).

[3] N. Blank, *Distributions whose convolutions are identical on the half line.* Dokl. Akad. Nauk Ukrain. SSR Ser. A (1983), No. 6, 3–4 (Russian).

[4] N. Blank, *On uniqueness conditions for functions of bounded variation and distribution functions with values on the half-line*, Theory Probab. Appl. **27** (1983), No. 4, 844–848.

[5] N. Blank, *On a uniqueness property of second convolutions*, Math. Scand. **104** (2009), No. 2, 182–187.

[6] A. Borichev, *The generalized Fourier transform, the Titchmarsh theorem and almost analytic functions*, Algebra i Analiz **1** (1989), No. 4, 17–53 (Russian). *Errata: The generalized Fourier transform, the Titchmarsh theorem and almost analytic functions.* Algebra i Analiz, **2** (1990), No. 5, 236–237 (Russian).

[7] Y. Domar, *Extension of the Titchmarsh convolution theorem with applications in the theory of invariant subspaces*, Proc. London Math. Soc. **46** (1983), 288–300.

[8] S. Gergün, I.V. Ostrovskii, and A. Ulanovskii, *On the Titchmarsh convolution theorem*, Ark. Mat. **40** (2002), No. 1, 55–71.

[9] I.A. Ibragimov, *On determining an infinitely divisible distribution function by its values on the half-line*, Theory Prob. Appl. **22** (1977), 384–390.

[10] A.I. Il'inskii, *Normality of a multidimensional infinitely divisible distribution which coincides with a normal distribution in a cone.* Lect. Notes Math., **1155**, Springer–Verlag, Berlin, 1985, 47–59.

[11] A.I. Il'inskii, *Some remarks on the uniqueness of the extension of measures from subsets of the group $Z_2 \times \mathbb{R}$.* Stability Probl. Stochastic Models (Sukhumi, 1987), 55–59 (Russian).

[12] V.M. Kruglov and A. Ulanovskii, *Mixtures of probability distribution, which are determined uniquely by their behavior on a half-line*, Teor. Veroyatn. Primen. **32** (1987), No. 4, 670–678 (Russian).

[13] I.V. Ostrovskii, *A class of functions of bounded variation on the line defined by their values on a half line*, Investigations on linear operators and the theory of functions, **IX**, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) (1979), No. 92, 220–229, 323 (Russian, English summary).

[14] I.V. Ostrovskii, *Measures that have infinitely many infinitely divisible extensions*, Dokl. Akad. Nauk Ukrain. Ser. A (1982), No. 6, 18–20.

[15] I.V. Ostrovskii, *Measures with infinitely many infinitely divisible extensions*, Sankhya A, **45** (1983), No. 3, 257–270.

[16] I.V. Ostrovskii, *Support of a convolution of finite measures, and measures determined uniquely by the restriction to a half-line*, Dokl. Akad. Nauk Ukrain. Ser. A (1984), No. 3, 8–12 (Russian).

[17] I.V. Ostrovskii, *Generalization of the Titchmarsh convolution theorem and the complex–valued measures uniquely determined by their restrictions to a half–line*, Stability Problems for Stochastic Models (Uzhgorod, 1984), Lect. Notes Math.. **1155**, Springer, Berlin, 1985, 256–283.

[18] I.V. Ostrovskii and A. Ulanovskii, *Classes of complex-valued Borel measures that can be uniquely determined by restrictions*, Issled. Linein. Oper. Teorii Funktsii. **17**, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **170**, 1989, 233–253, 325 (Russian).

[19] I.V. Ostrovskii and A. Ulanovskii, *Probability distributions and Borel measures uniquely determined by their restrictions to a half-space*. Probab. Theory Math. Stat., **II** (Vilnius, 1989), 278–287.

[20] M. Riedel, *On the one-sided tails of infinitely divisible distributions*. Collection of articles dedicated to the memory of Wolfgang Richter, Math. Nachr. **70** (1975), 155–163.

[21] H.-J. Rossberg, *On a problem of Kolmogorov concerning the normal distribution*, Theory Prob. Appl. **19** (1974), 795–798.

[22] H.-J. Rossberg and G. Siegel, *Continuation of convergence in the central limit theorem*, Theory Prob. Appl., **20** (1975), 866–868.

[23] Titov, A. N., *On the determination of a convolution of identical distribution functions from its values on a half line*. Teor. Veroyatnost. Primenen. **26** (1981), no. 3, 610–611.

[24] A. Ulanovskii, *On functions of bounded variation that are determined by restriction to a semi-axis*. Math. USSR Sb., **60** (1988), 427–436.

[25] A. Ulanovskii, *When is half-line convergence to an infinitely divisible law ν equivalent to weak convergence to ν?* J. London Math. Soc. (2) **57** (1998), No. 1, 245–256.

[26] A. Ulanovskii, *On the unique determination of convolutions of one-type distribution functions by their restriction to a half-line*, Teor. Veroyatnost. Primenen. **32** (1987), No. 1, 135–137 (Russian).

[27] A. Ulanovskii, *Measures on LCA groups whose n-fold convolutions are determined by the restriction to a massive set*, Issled. Mat. Statist., **9**, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 184, (1990), 264–270, 326.

[28] M.G. Zaidenberg, *Transcendental solutions of Diophantine equations*, Funktsional. Anal. Prilozhen. **22** (1988), No. 4, 78–79 (Russian).

[29] M.G. Zaidenberg, *Tests for hyperbolicity and families of curves*. Teor. Funktsii, Funktsional. Anal. Prilozhen. (1989), No. 52, 40–54 (Russian).

[30] V.M. Zolotarev, *Some remarks on the paper "On the Unique Determination of Stable Distribution Functions" by H.-J. Rossberg and B. Jesiak*. Math. Nachr. **82** (1978), 301–304.

Received January 30, 2024.

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Про властивість єдиності n-ї згортки та посилення теореми Тітчмарша про згортку

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Ця замітка є спробою представити коротку історію про те, як питання Колмогорова про міри Гауса привело до низки теоретико-функціональних результатів, одержаних переважно Островським та деякими з його учнів. Замітка має на меті окреслити загальний розвиток, а не подати повний звіт про результати.

Ключові слова: ймовірнісна міра, нескінченно подільна міра, характеристична функція, аналітична функція, теорема Тітчмарша про згортку, теорія розподілу Неванлінни