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On Kinds of Weak Solutions to an Initial Boundary Value Problem for 1D Linear Degenerate Wave Equation

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In this article, we discuss the existence and uniqueness of mild, variational, and the so-called non-variational solutions to an initial-boundary value problem (IBVP) for linear wave equation with strong interior degeneracy of the coefficient in the principal part of the elliptic operator. The objective is to provide a well-posedness analysis of the IBVP and find out how the density property of smooth functions in the corresponding weighted Sobolev space affects the uniqueness of its solutions. We show that, in general, the uniqueness of solutions may be violated if the 'degree of degeneracy' corresponds to the strong degeneracy case.

Key words: strongly degenerate wave equation, existence and uniqueness of solutions, weighted Sobolev spaces, mild solutions, variational solutions, non-variational solutions, weak solutions

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1. Introduction and some preliminaries

We consider the following initial-boundary value problem (IBVP) with respect to $y(t,x)$:

$$y_{tt} - (a(x)\, y_x)_x = 0 \quad \text{in } Q := (0,T) \times \Omega, \tag{1.1}$$

$$y(t,-1) = 0, \quad y(t,1) = 0 \quad \text{on } (0,T), \tag{1.2}$$

$$y(0,\cdot) = y_0, \quad y_t(0,\cdot) = y_1 \quad \text{in } \Omega, \tag{1.3}$$

and discuss some issues related to the notion of weak solution to the above problem. Here, $[0,T]$ is a time interval; $\Omega = (-1,+1)$; $y_0(x)$ and $y_1(x)$ are given functions; and the weight function $a\colon \overline{\Omega} \to \mathbb{R}_+$ has the following properties:

(i) $a \in C^1(\overline{\Omega})$;

(ii) $a(0) = 0$, $a(x) > 0$ for all $x \in \overline{\Omega}_0$, $\overline{\Omega}_0 := \overline{\Omega} \setminus \{0\}$, and there exist two positive values $x_1^*, x_2^* \in (0,1)$ such that $a(\cdot)$ is monotonically decreasing on $(-x_1^*, 0)$, monotonically increasing on $(0, x_2^*)$, and

$$2 > \mu_{a,1} := \sup_{x\in[-x_1^*,0)} \frac{(-x)|a'(x)|}{a(x)} = \lim_{x\nearrow 0} \frac{(-x)|a'(x)|}{a(x)} \geqslant 1, \tag{1.4}$$

$$2 > \mu_{a,2} := \sup_{x\in(0,x_2^*]} \frac{x|a'(x)|}{a(x)} = \lim_{x\searrow 0} \frac{x|a'(x)|}{a(x)} \geqslant 1. \tag{1.5}$$

Since the weight function a is positive in $\Omega \setminus \{0\}$ but vanishes at zero, the wave equation (1.1) is degenerate. We recall that the degeneracy of (1.1) at $x = 0$ is measured by the parameter μ_a defined by $\mu_a = \max\{\mu_{a,1}, \mu_{a,2}\}$, where $\mu_{a,i}$, $i = 1, 2$, are defined in (1.4) and (1.5), and one says that (1.1) degenerates weakly if $\mu_a \in [0, 1)$, and strongly if $\mu_a \geq 1$ (for more details, we refer to [1, 5, 6]).

So, we deal with an imaginary degenerate 'string' vibrating on the segment $[-1, 1]$ that has a defect at the midpoint $x = 0$ (the degeneracy, or the damage point). Moreover, since $\min\{\mu_{a,1}, \mu_{a,2}\} \geqslant 1$, it means that we will focus on the case of the strong degeneracy.

We make use of the following result (see Proposition 2.1 in [7]).

Proposition 1.1. *Let $a\colon \overline{\Omega} \to \mathbb{R}_+$ be a weight function satisfying properties (i) and (ii). Then*

$$a(x) \geqslant a(x_1^*) \frac{|x|^{\mu_{1,a}}}{|x_1^*|^{\mu_{1,a}}}, \quad x \in [-x_1^*, 0] \subset [-1, 0], \tag{1.6}$$

$$a(x) \geqslant a(x_2^*) \frac{|x|^{\mu_{2,a}}}{|x_2^*|^{\mu_{2,a}}}, \quad x \in [0, x_2^*] \subset [0, 1]. \tag{1.7}$$

Moreover, as immediately follows from (1.4) and (1.5), there exists a positive value $\varepsilon > 0$ and a neighborhood $\mathcal{U}(0) \subset (-x_1^*, x_2^*)$ of $x = 0$ such that

$$2 - \varepsilon > \max\{\mu_{1,a}, \mu_{2,a}\} \quad \text{and} \quad \frac{|x||a'(x)|}{a(x)} \leqslant 2 - \varepsilon, \quad x \in \mathcal{U}(0).$$

From this we deduce the existence of a constant $C > 0$ depending only on the values of $a(x)$ on $\partial\mathcal{U}(0)$ and such that

$$a(x) \leqslant C|x|^{2-\varepsilon}, \quad x \in \mathcal{U}(0). \tag{1.8}$$

Before moving on to the analysis of the different notions of weak solutions to the IBVP (1.1)–(1.3), we remind the standard definition of weighted Sobolev spaces naturally related to the above problem. We set

$$W_a^{1,2}(\Omega) = \left\{ u \in L^2(\Omega)\colon\ u \in AC_{\mathrm{loc}}(\overline{\Omega}_0),\ \int_{-1}^{1} a(x)|u_x(x)|^2\, dx < +\infty \right\}. \tag{1.9}$$

It is easy to check that $W_a^{1,2}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{W_a^{1,2}(\Omega)} = \left(\int_{-1}^{1} \left[|u(x)|^2 + a(x)|u_x(x)|^2 \right] dx \right)^{1/2}.$$

We also introduce its proper closed subspace

$$W_{a,0}^{1,2}(\Omega) = \left\{ u \in W_a^{1,2}(\Omega)\colon\ u(-1) = u(1) = 0 \right\}. \tag{1.10}$$

Taking into account the following version of Poincaré inequality (see [7])

$$\|u\|_{L^2(\Omega)} \leqslant \min\{D_a, C_a\} \left(\int_{-1}^{1} a(x)|u_x(x)|^2 \, dx \right)^{1/2}, \quad u \in W^{1,2}_{a,0}(\Omega), \tag{1.11}$$

where

$$C_a^2 = 4 \max \left\{ \frac{1}{\min\limits_{x \in [-1,-x_1^*]} a(x)}, \frac{(-x_1^*)^{\mu_{1,a}}}{a(-x_1^*)}, \frac{1}{\min\limits_{x \in [x_2^*,1]} a(x)}, \frac{(x_2^*)^{\mu_{2,a}}}{a(x_2^*)} \right\},$$

$$D_a^2 = \max\{D_{1,a}, D_{2,a}\},$$

$$D_{1,a} = \frac{1-(x_1^*)^2}{2 \min\limits_{x \in [-1,-x_1^*]} a(x)} + \frac{(x_1^*)^2}{a(-x_1^*)\,(2-\mu_{1,a})},$$

$$D_{2,a} = \frac{1-(x_2^*)^2}{2 \min\limits_{x \in [x_2^*,1]} a(x)} + \frac{(x_2^*)^2}{a(x_2^*)\,(2-\mu_{2,a})},$$

we see that an equivalent norm on $W^{1,2}_{a,0}(\Omega)$ can be defined as

$$\|u\|_{W^{1,2}_{a,0}(\Omega)} = \left(\int_{-1}^{1} a(x)|u_x(x)|^2 \, dx \right)^{1/2}.$$

For our further analysis, we also make use of the weight spaces $H^1_a(\Omega)$ and $H^1_{a,0}(\Omega)$ which we define as follows:

- $H^1_a(\Omega)$ is the closure of the set $C^\infty(\overline{\Omega})$ with respect to the $\|\cdot\|_{W^{1,2}_a(\Omega)}$-norm;
- $H^1_{a,0}(\Omega)$ is the closure of the set $C^\infty_c(\Omega)$ with respect to the $\|\cdot\|_{W^{1,2}_a(\Omega)}$-norm.

First note that since $C^\infty_c(\Omega) \subset C^\infty(\overline{\Omega})$, we have that $H^1_{a,0}(\Omega) \subset H^1_a(\Omega)$. Moreover, due to compactness of the embedding

$$H^1_a((-1,\varepsilon) \cup (\varepsilon,1)) \hookrightarrow C^{0,1}([-1,-\varepsilon] \cup [\varepsilon,1]), \qquad \text{for all } \varepsilon > 0 \text{ small enough,}$$

we see that, if $y \in H^1_a(\Omega)$, then $y(\cdot)$ is an absolutely continuous function in $\overline{\Omega} \setminus \{0\}$. So, the conditions $y(-1) = 0$ and $y(1) = 0$ are consistent for all $y \in H^1_{a,0}(\Omega)$. Therefore, $H^1_{a,0}(\Omega)$ can be equivalently defined as the closed subspace of $H^1_a(\Omega)$ such that

$$H^1_{a,0}(\Omega) := \left\{ y \in H^1_a(\Omega) \ : \ y(-1) = y(1) = 0 \right\}.$$

It is worth to notice that, unlike classical Sobolev space, the subspace of smooth functions are not necessarily dense in $W^{1,2}_a(\Omega)$. So, for 'typical' weight functions $a : \overline{\Omega} \to \mathbb{R}$ with properties (i) and (ii), it is unknown whether the identity $H^1_a(\Omega) = W^{1,2}_a(\Omega)$ is valid. Therefore, $H^1_a(\Omega) \subseteq W^{1,2}_a(\Omega)$, in general.

Before we move on to the analysis of solvability issues of problem (1.1)–(1.3) in the above mentioned function spaces, we make use of the following observation.

Let us define the weight coefficient as follows $a(x) = |x|^{7/4}$. It is easy to see that in this case conditions (i) and (ii) holds true with $\mu_{a,1} = \mu_{a,2} = 7/4$. Then setting

$$y(x) = \begin{cases} |x|^{-\frac{1}{4}} - 1 & \text{if } x \in [-1, 0), \\ |x|^{+\frac{1}{2}} - 1 & \text{if } x \in [0, 1], \end{cases}$$

we see that the function $y\colon \overline{\Omega} \to \mathbb{R}$ has a discontinuity of the second kind at $x = 0$. Furthermore, direct calculations show that $y(-1) = y(1) = 0$, and

$$\|y\|^2_{W^{1,2}_{a,0}(\Omega)} = \int_{-1}^{1} a(x)\, y_x^2(x)\, dx < +\infty,$$

whereas

$$a(x)\, y_x(x) = \begin{cases} -\frac{1}{4}|x|^{\frac{1}{2}} & \text{if } x \in [-1, 0), \\ +\frac{1}{2}|x|^{\frac{5}{4}} & \text{if } x \in [0, 1]. \end{cases}$$

So, at the degeneracy point $x = 0$, for the given function y with finite $W^{1,2}_{a,0}(\Omega)$-norm, we have the following transmission conditions

$$\lim_{x \nearrow 0} a(x)\, y_x(x) = \lim_{x \searrow 0} a(x)\, y_x(x) = 0. \tag{1.12}$$

Remark 1.2. Apparently, in the case of strong degeneration (see (1.8)), the indicated property is a crucial characteristics of all elements of the weighted space $W^{1,2}_a(\Omega)$ for which the one-sided limits in (1.12) are well defined. In fact, we have the following result (see also Theorem 2.7 in [7]): If $a\colon \overline{\Omega} \to \mathbb{R}_+$ is a weight function with properties (i) and (ii), then (1.12) holds true for all $y \in W^{1,2}_a(\Omega)$ such that $a(x)\, y_x(x)$ is continuous in some neighborhood of $x = 0$. Indeed, if $\lim_{x\to 0} a(x) y_x(x) = L$, then $a(x)|y_x(x)|^2 \sim L^2/a(x)$ and, therefore, $L = 0$ otherwise $y \notin W^{1,2}_a(\Omega)$.

2. Mild solutions

Let $a\colon \overline{\Omega} \to \mathbb{R}_+$ be a given function with properties (i) and (ii). We begin with the following concept.

Definition 2.1. A function $y = y(t, x)$ is called a mild solution to the IBVP (1.1)–(1.3) if the following three conditions hold true:

1. it has a finite energy, i.e., $E_y(t) < \infty$ for all $t > 0$, where

$$E_y(t) = \frac{1}{2} \int_{-1}^{1} \left[y_t^2(t, x) + a(x)\, y_x^2(t, x) \right] dx; \tag{2.1}$$

2. $y \in C([0, T]; W^{1,2}_{a,0}(\Omega))$;
3. it satisfies

$$\int_0^t \int_0^s y(r, x)\, dr\, ds = \int_0^t (t - s)\, y(s, x)\, ds \in D(A_0), \tag{2.2}$$

$$y(t,x) = y_0(x) + t\,y_1(x) + A_0\left(\int_0^t (t-s)\,y(s,x)\,ds\right), \quad t\in[0,T], \tag{2.3}$$

where $A_0\colon D(A_0)\subset W^{1,2}_{a,0}(\Omega)\to W^{1,2}_{a,0}(\Omega)$ is an unbounded operator $A_0u = -(au_x)_x$ with domain

$$D(A_0) = \left\{u\in W^{1,2}_{a,0}(\Omega)\colon au_x\in W^{1,2}(\Omega)\right\}.$$

In this case one can prove the following existence result (see Section 3 in [7]).

Theorem 2.2. *Given $y_0\in W^{1,2}_{a,0}(\Omega)$ and $y_1\in L^2(\Omega)$, there exists a unique mild solution y to the problem (1.1)–(1.3) such that*

$$y\in C^1([0,T];L^2(\Omega))\cap C([0,T];V^1_{a,0}(\Omega)),$$

$V^1_{a,0}(\Omega)$ being an intermediate space: $H^1_{a,0}(\Omega)\subseteq V^1_{a,0}(\Omega)\subseteq W^1_{a,0}(\Omega)$.

In order to prove this result, it is enough to introduce $\mathcal{H}_a := W^{1,2}_{a,0}(\Omega)\times L^2(\Omega)$, the Hilbert space endowed with the scalar product (see [1,7,8])

$$\left\langle \begin{bmatrix}u\\v\end{bmatrix}, \begin{bmatrix}\widetilde{u}\\\widetilde{v}\end{bmatrix}\right\rangle_{\mathcal{H}_a} = \int_{-1}^1 v(x)\,\widetilde{v}(x)\,dx + \int_{-1}^1 a(x)\,u_x(x)\,\widetilde{u}_x(x)\,dx,$$

and show that the unbounded operator $\mathcal{A}\colon D(\mathcal{A})\subset\mathcal{H}_a\to\mathcal{H}_a$, associated with problem (1.1)–(1.3) and defined as

$$\mathcal{A}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}v\\(au_x)_x\end{bmatrix}, \tag{2.4}$$

where

$$\begin{aligned} D(\mathcal{A}) &= W^{2,2}_a(-1,1)\times W^{1,2}_{a,0}(\Omega),\\ W^{2,2}_a(\Omega) &= \left\{u\in W^{1,2}_{a,0}(\Omega)\colon\ au_x\in W^{1,2}(\Omega)\right\}, \end{aligned} \tag{2.5}$$

being maximal dissipative on $\mathcal{H}_a$ and, therefore, being the infinitesimal generator of a strongly continuous semigroup of contractions $e^{\mathcal{A}t}$ in $\mathcal{H}_a$. Moreover, $e^{\mathcal{A}t}$ is analytic. For the details, we refer to [1] (see also [8] and Lemma 3.1 in [7]).

On the other hand, utilizing property (1.12), which holds true for all elements $y\in D(A_0)$, and Remark 1.2, we immediately arrive at the following distinctive conclusion:

Proposition 2.3. *Given $y_0\in W^{1,2}_{a,0}(\Omega)$ and $y_1\in L^2(\Omega)$, the mild solution $y(t,x)$ to problem (1.1)–(1.3) is such that $y\in C([0,T];W^{1,2}_{a,0}(\Omega))$ and it can be represented as follows*

$$y(t,x) = \begin{cases}\underline{u}(t,x), & x\in[-1,0),\ t\in[0,T],\\ \overline{u}(t,x), & x\in(0,1],\ t\in[0,T].\end{cases} \tag{2.6}$$

Here, $\underline{u}$ and $\overline{u}$ are the mild solutions to the IBVPs

$$\underline{u}_{tt} - (a(x)\, \underline{u}_x)_x = 0 \quad in \quad (0,T)\times(-1,0), \tag{2.7}$$

$$\underline{u}(t,-1) = 0, \quad \lim_{x\nearrow 0} a(x)\, \underline{u}_x(x) = 0 \quad on \quad (0,T), \tag{2.8}$$

$$\underline{u}(0,\cdot) = y_0, \quad \underline{u}_t(0,\cdot) = y_1 \quad in \quad (-1,0), \tag{2.9}$$

and

$$\overline{u}_{tt} - (a(x)\, \overline{u}_x)_x = 0 \quad in \ (0,T)\times(0,1), \tag{2.10}$$

$$\lim_{x\searrow 0} a(x)\, \overline{u}_x(x) = 0, \quad \overline{u}(t,1) = 0 \quad on \ (0,T), \tag{2.11}$$

$$\overline{u}(0,\cdot) = y_0, \quad \overline{u}_t(0,\cdot) = y_1 \quad in \ (0,1), \tag{2.12}$$

respectively.

It is worth to notice that the existence and uniqueness of the mild solutions to the problems (2.7)–(2.9) and (2.10)–(2.12), respectively, have been proven in [1].

3. Variational solutions

Our main intention in this section is to discuss the existence of the so-called variational solutions to the IBVP (1.1)–(1.3). It is well-known that, unlike classical Sobolev spaces, smooth functions $C_0^\infty(\Omega)$ are not necessarily dense in each $V_{a,0}^1(\Omega)$. Furthermore, in the next section it will be shown that this property is definitely wrong for a special choice of the weight function $a : \overline{\Omega} \to \mathbb{R}_+$. So, it is plausible to suppose that $W_{a,0}^1(\Omega) \setminus H_{a,0}^1(\Omega) \neq \varnothing$.

Let $V_{a,0}^1(\Omega)$ be an intermediate space, i.e., $H_{a,0}^1(\Omega) \subseteq V_{a,0}^1(\Omega) \subseteq W_{a,0}^1(\Omega)$. In particular, if a function ψ is chosen such that $\psi \in W_{a,0}^1(\Omega) \setminus H_{a,0}^1(\Omega)$, then we can associate with it the following intermediate space

$$V_{a,0}^1(\Omega) = \text{closure}_{\|\cdot\|_{W_{a,0}^1(\Omega)}} \left\{ c_1\varphi + c_2\psi \ : \ \varphi \in C_0^\infty(\Omega),\ c_1, c_2 \in \mathrm{R} \right\}.$$

We define $V_a^{-1}(\Omega)$ as the dual space of $V_{a,0}^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$. Then, due to inequality (1.11), one can prove that $A_0 = -\frac{d}{dx}\left(a(x)\frac{d}{dx}\right)$ is an isomorphism from $V_{a,0}^1(\Omega)$ onto $V_a^{-1}(\Omega)$, and, in particular, $V_a^{-1}(\Omega) = A_0\left[V_{a,0}^1(\Omega)\right]$. We make use of the notation $\langle\cdot,\cdot\rangle_* = \langle\cdot,\cdot\rangle_{V_a^{-1}(\Omega);V_{a,0}^1(\Omega)}$ for the duality between $V_a^{-1}(\Omega)$ and $V_{a,0}^1(\Omega)$.

Definition 3.1. Let $y_0 \in V_{a,0}^1(\Omega)$ and $y_1 \in L^2(\Omega)$ be fixed arbitrary. We say that a function y is a $V_{a,0}^1(\Omega)$-variational solution to the problem (1.1)–(1.3) (in the sense of distributions) if:

1. it has finite energy (2.1), i.e., $E_y(t) < \infty$ for all $t > 0$;
2. $y \in L^2(0,T;V_{a,0}^1(\Omega))$, $y_t \in L^2(0,T;L^2(\Omega))$, $y_{tt} \in L^2(0,T;V_a^{-1}(\Omega))$;
3. for a.a. $t \in [0,T]$ and for all test functions $\varphi \in V_{a,0}^1(\Omega)$, y satisfies the integral identity

$$\langle y_{tt}, \varphi\rangle_* + \int_{-1}^{1} a\, y_x\, \varphi_x\, dx = 0; \tag{3.1}$$

4. for a.a. $t \in [0,T]$, the following energy equality holds

$$\langle y_{tt}, y\rangle_* + \int_{-1}^{1} a\,(y_x)^2\,dx = 0; \tag{3.2}$$

5. $y(0,\cdot) = y_0$ and $y_t(0,\cdot) = y_1$.

As follows from this definition, each intermediate space $V^1_{a,0}(\Omega)$ can be associated with a $V^1_{a,0}(\Omega)$-variational solution provided its existence. Let us show uniqueness of variational solutions for each $V^1_{a,0}(\Omega)$.

Assume that $y_0 \equiv 0$ and $y_1 \equiv 0$. Let y be a $V^1_{a,0}(\Omega)$-variational solution to the original problem. We want to show that $y \equiv 0$. To do so, for fixed $s \in [0,T]$, we set

$$v(t) = \begin{cases} \displaystyle\int_t^s y(\tau)\,d\tau & \text{if } 0 \le t \le s,\\ 0 & \text{if } s \le t \le T. \end{cases}$$

Since $v(t) \in V^1_{a,0}(\Omega)$ for all $t \in [0,T]$, it follows that we can utilize v as a test function in (3.1). Then, after an integration over $(0,T)$, we deduce

$$\int_0^s \left[\langle y_{tt}, v\rangle_* + \int_{-1}^{1} a\,y_x\,v_x\,dx\right] dt = 0. \tag{3.3}$$

An integration by parts yields

$$\begin{aligned}\int_0^s \langle y_{tt}(t), v(t)\rangle_*\,dt &= -\int_0^s\int_{-1}^{1} y_t(t)v_t(t)\,dx\,dt = \int_0^s\int_{-1}^{1} y_t(t)y(t)\,dx\,dt\\ &= \frac{1}{2}\int_0^s \frac{d}{dt}\|y(t)\|^2_{L^2(\Omega)}\,dt\end{aligned}$$

since $v(s) = 0$ and $v_t(t) = -y(t)$ if $0 < t < s$.

On the other hand,

$$\begin{aligned}\int_0^s\int_{-1}^{1} a\,y_x(t)\,v_x(t)\,dx\,dt &= -\int_0^s\int_{-1}^{1} a\,v_{tx}(t)\,v_x(t)\,dx\,dt\\ &= -\frac{1}{2}\int_0^s \frac{d}{dt}\|a^{1/2}v_x(t)\|^2_{L^2(\Omega)}\,dt.\end{aligned}$$

Hence, from (3.3), we obtain

$$\int_0^s \frac{d}{dt}\left[\|y(t)\|^2_{L^2(\Omega)} - \|a^{1/2}v_x(t)\|^2_{L^2(\Omega)}\right] dt = 0,$$

or, in other form,

$$\|y(s)\|^2_{L^2(\Omega)} + \|a^{1/2}v_x(0)\|^2_{L^2(\Omega)} = 0 \quad s \in [0,T],$$

with entails $y(s) \equiv 0$.

Thus, $V^1_{a,0}(\Omega)$-variational solution to the problem (1.1)–(1.3), if it exists, is unique for each intermediate space $V^1_{a,0}(\Omega)$.

Keeping in mind the fact that the original problem (1.1)–(1.3) can be decoupled into two sub-problems (2.7)–(2.9) and (2.10)–(2.12), we can give the following remark concerning existence issues for variational solutions. If for a chosen intermediate space $V^1_{a,0}(\Omega)$ can be selected a sequence of functions $\{w_k\}_{k=1}^{\infty}$ constituting an orthogonal basis in $V^1_{a,0}(\Omega)$ and an orthonormal basis in $L^2(\Omega)$, then the existence of a $V^1_{a,0}(\Omega)$-variational solution can be established following the standard Faedo–Galerking method. On the other hand, this issue can be highlighted by the following observation, which is crucial for this section.

Proposition 3.2. *Let $y \in C([0,T]; W^{1,2}_{a,0}(\Omega))$ be a mild solution of the problem (1.1)–(1.3). Then y is a unique V-variational solution for each intermediate space V such that $V_y \subseteq V \subseteq W^{1,2}_{a,0}(\Omega)$, where*

$$V_y = \text{closure}_{\|\cdot\|_{W^1_{a,0}(\Omega)}} \{c_1\varphi + c_2 y : \varphi \in C_0^\infty(\Omega),\ c_1, c_2 \in \mathrm{R}\}. \tag{3.4}$$

Proof. Since y is a mild solution, it follows from condition (2.2) that

$$\frac{\partial}{\partial x}\left(a(x)\int_0^t (t-s)\, y_x(s,x)\, ds\right) \in L^2(\Omega), \quad \text{for a.a. } t \in [0,T].$$

Hence, multiplying the equality

$$-\left(a(x)\int_0^t (t-s)\, y_x(s,x)\, ds\right)_x = y(t,x) - y_0(x) - t y_1(x)$$

by $v_{tt}(t,x)$ with properties

$$v \in L^2(0,T; W^{1,2}_{a,0}(\Omega)), \quad v_t \in W^{1,2}(0,T; L^2(\Omega)), \tag{3.5}$$

$$v(T,x) = 0, \quad v_t(T,x) = 0 \ \text{ a.e. in } (\Omega), \tag{3.6}$$

and then integrating over $(0,T)\times\Omega$, we arrive at the relation

$$\begin{aligned} &-\int_0^T \int_{-1}^1 \left(a(x)\int_0^t (t-s)\, y_x(s,x)\, ds\right)_x v_{tt}(t,x)\, dx\, dt \\ &= \lim_{s\searrow 0} \int_s^T \int_{-1}^1 [y(t,x) - y_0(x) - t y_1(x)]\ v_{tt}(t,x)\, dx\, dt. \end{aligned} \tag{3.7}$$

Taking into account conditions (3.6), we get

$$\int_s^T \int_{-1}^1 y\, v_{tt}\, dx\, dt = -\int_{-1}^1 y(s,x)\, v_t(s,x)\, dx - \int_s^T \int_{-1}^1 y_t\, v_t\, dx\, dt \tag{3.8}$$

$$\int_s^T \int_{-1}^1 y_0\, v_{tt}\, dx\, dt = -\int_{-1}^1 y_0\, v_t(s,x)\, dx, \tag{3.9}$$

$$\begin{aligned} \int_s^T \int_{-1}^1 y_1\, t\, v_{tt}\, dx\, dt &= -\int_{-1}^1 y_1\, s\, v_t(s,x)\, dx - \int_s^T \int_{-1}^1 y_1\, v_t\, dx\, dt \\ &= -\int_{-1}^1 y_1\, s\, v_t(s,x)\, dx + \int_{-1}^1 y_1\, v(s,x)\, dx. \end{aligned} \tag{3.10}$$

Since $v_t \in W^{1,2}(0,T;L^2(\Omega))$, it follows that $v_t\colon [0,T] \to L^2(\Omega)$ is a continuous function, by Sobolev embedding theorem. Hence, $v \in C([0,T], L^2(\Omega))$, and this allows us to pass to the limit in the right hand side of (3.7) as $s \searrow 0$ using the representations (3.8)–(3.10). As a result, we obtain

$$
\begin{aligned}
&\lim_{s\searrow 0}\int_s^T \int_{-1}^1 [y(t,x) - y_0(x) - t\, y_1(x)]\; v_{tt}(t,x)\,dx\,dt\\
&\quad = -\int_0^T\int_{-1}^1 y_t\, v_t\,dx\,dt - \int_{-1}^1 y_1\, v(0,x)\,dx\\
&\qquad + \int_{-1}^1 y_0\, v_t(0,x)\,dx - \lim_{s\searrow 0}\int_{-1}^1 y(s,x)\, v_t(s,x)\,dx\\
&\quad \underset{(2.3)}{=} -\int_0^T\int_{-1}^1 y_t\, v_t\,dx\,dt - \int_{-1}^1 y_1\, v(0)\,dx = \int_0^T \langle y_{tt}(t), v\rangle_*\,dt.
\end{aligned}\tag{3.11}
$$

As for the left hand side of (3.7), it can be rewritten as follows

$$
\begin{aligned}
&-\int_0^T\int_{-1}^1 \left(a(x)\int_0^t (t-s)\, y_x(s,x)\,ds\right)_x v_{tt}(t,x)\,dx\,dt\\
&= -\int_0^T\int_{-1}^1 (a(x)\, y_x(s,x))_x \int_s^T (t-s)\, v_{tt}(t,x)\,dt\,dx\,ds\\
&= \int_0^T\int_{-1}^1 (a(x)\, y_x(s,x))_x \int_s^T v_t(t,x)\,dt\,dx\,ds\\
&= -\int_0^T\int_{-1}^1 (a(x)\, y_x(t,x))_x\; v(t,x)\,dx\,dt = \int_0^T\int_{-1}^1 a\, y_x\, v_x\,dx\,dt.
\end{aligned}\tag{3.12}
$$

Combining in (3.7) the obtained equalities (3.11) and (3.12), we see that the mild solution of the problem (1.1)–(1.3) satisfies the integral identity

$$
\int_0^T \langle y_{tt}, v\rangle_*\,dt + \int_0^T\int_{-1}^1 a\, y_x\, v_x\,dx\,dt = 0 \tag{3.13}
$$

for all test functions v with properties (3.5)–(3.6). Let V be an arbitrary intermediate space satisfying condition $V_y \subseteq V \subseteq W^{1,2}_{a,0}(\Omega)$, where V_y is defined in (3.4). Then integral identity (3.1) immediately follows from (3.13). As for the energy relation (3.2), it is enough to notice that equality (3.13) implies its validity for all test functions $v = v(t,x)$ stepwise with respect to variable t, hence for all $v \in L^2(0,T;V)$. Taking into account the fact that V_y is the 'minimal' space containing element y, the mild solution y is in $C([0,T];V)$ for any V such that $V_y \subseteq V \subseteq W^{1,2}_{a,0}(\Omega)$. Thus, the energy relation (3.2) is a direct consequence of (3.13).

As for fulfilment of the initial conditions $y(0,\cdot) = y_0$ and $y_t(0,\cdot) = y_1$, they follow from equality (2.3). Thus, the mild solution y is a V-variational one to the same problem. □

As a direct consequence of Proposition 3.2 and Theorem 2.2, we can give the following conclusion: since the mild solution is unique and this solution belong to the space $C([0,T];W^{1,2}_{a,0}(\Omega))$, it follows that for given $y_0 \in W^{1,2}_{a,0}(\Omega)$ and $y_1 \in L^2(\Omega)$, the IBVP (1.1)–(1.3) has a unique V_y-variational solution in the sense of Definition 3.1. Thus, this solution is the same for all intermediate spaces V such that $V_y \subseteq V \subseteq W^{1,2}_{a,0}(\Omega)$. In addition, this solution admits the decomposition (2.6), where $\underline{u}$ and $\overline{u}$ are variational solutions to the one-side problems (2.7)–(2.9) and (2.10)–(2.12), respectively.

4. On Non-variational solutions

Let $a\colon \overline{\Omega} \to \mathbb{R}$ be a given function with properties (i) and (ii). Since $1/a(\cdot) \notin L^1(\Omega)$, it is unknown whether each element $u \in W^{1,2}_{a,0}(\Omega)$ can be successfully approximated by smooth functions, i.e., for any $\varepsilon > 0$ small enough it can be found a function $\varphi \in C^1_0(\Omega)$ such that $\|u - \varphi\|_{W^{1,2}_{a,0}(\Omega)} < \varepsilon$. This circumstance motivates us to introduce the following concept.

Definition 4.1. We say that a function $y = y(t,x)$ is a weak solution to the IBVP (1.1)–(1.3) if:

(a) $y \in L^2(0,T;W^{1,1}_{a,0}(\Omega))$ and $y_t \in L^2(0,T;L^1(\Omega))$;

(b) for all $\varphi \in C^1([0,T];C^\infty_0(\Omega))$: $\varphi(T,x) = 0$, $x \in \Omega$, the integral identity holds

$$-\int_0^T\int_{-1}^1 y_t\,\varphi_t\,dx\,dt + \int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt + \int_{-1}^1 y_1(x)\,\varphi(0,x)\,dx = 0; \quad (4.1)$$

(c) for all $\psi \in C^\infty_0(\Omega)$ the integral identity holds

$$\lim_{t\searrow 0}\int_{-1}^1 y(t,x)\,\psi(x)\,dx = \int_{-1}^1 y_0(x)\,\psi(x)\,dx. \quad (4.2)$$

Before proceeding further, we notice that the integral identity (4.1) is well defined for all $\varphi \in C^1([0,T];C^\infty_0(\Omega))$ and $y \in L^2(0,T;W^{1,1}_{a,0}(\Omega))$ such that $y_t \in L^2(0,T;L^1(\Omega))$. Indeed,

$$\left|\int_0^T\int_{-1}^1 y_t\,\varphi_t\,dx\,dt\right| \leqslant \sqrt{T}\,\|\varphi\|_{C^1([0,T];C(\overline{\Omega}))}\,\|y_t\|_{L^2(0,T;L^1(\Omega))},$$

$$\left|\int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt\right| \leqslant \sqrt{T}\,\|\varphi\|_{C([0,T];C^1(\overline{\Omega}))}\,\|ay_x\|_{L^2(0,T;L^1(\Omega))},$$

$$\left|\int_{-1}^1 y_1(x)\,\varphi(0,x)\,dx\right| \leqslant \sqrt{2}\,\|\varphi\|_{C([0,T];C(\overline{\Omega}))}\,\|y_1\|_{L^2(\Omega)}.$$

However, the continuity of the mapping

$$L^2(0,T;W^{1,1}_{a,0}(\Omega)) \ni \varphi \mapsto \int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt$$

is not evident. It makes the problem to extend the identity (4.1) onto a wider class of the test functions rather questionable.

Let us enumerate the principal properties of the weak solutions that can be deduced from Definition 4.1:

(1) if $y(t,x)$ is a variational solution to the IBVP (1.1)–(1.3) then it is a weak solution in the sense of Definition 4.1. Moreover, if y^1 and y^2 are two different variational solutions (corresponding to two different intermediate spaces), then $y^* = (y^1 + y^2)/2$ is not a variational solution, but it remains a weak solution;

(2) for given $y_0(\cdot) \in W^{1,2}_{a,0}(\Omega)$ and $y_1 \in L^2(\Omega)$, the set $\Xi_{(y_0,y_1)}$ of all weak solutions to the IBVP (1.1)–(1.3) is nonempty, convex and closed;

(3) if $y \in \Xi_{(y_0,y_1)}$ then $y \in C([0,T]; L^1(\Omega))$, so relation (4.2) makes sense;

(4) the weak solutions to the IBVP (1.1)–(1.3) may have unbounded energy, i.e.,

$$\lim_{s\nearrow 0}\int_{-1}^{s}\left[y_t^2(t,x) + a(x)\,y_x^2(t,x)\right]\,dx + \lim_{s\searrow 0}\int_{s}^{1}\left[y_t^2(t,x) + a(x)\,y_x^2(t,x)\right]\,dx = +\infty.$$

Hereinafter, we call the weak solutions with unbounded energy just non-variational solutions;

(5) if a weak solution $y(t,x)$ possesses a finite energy, i.e., $E_y(t) < +\infty$ for all $t \in [0,T]$, and the set $C_0^\infty(\Omega)$ is dense in $W^{1,2}_{a,0}(\Omega)$, then $y(t,x)$ is a variational solution to the IBVP (1.1)–(1.3);

(6) if $y(t,x)$ is a weak solution such that $a(\cdot)\,y(t,\cdot)_x \in L^2(0,T; W^{1,1}(\Omega))$, then there exists a constant K (in general, $K \neq 0$) such that for a.a. $t \in [0,T]$

$$\lim_{x\nearrow 0} a(x)\,y_x(t,x) = \lim_{x\searrow 0} a(x)\,y_x(t,x) = K; \tag{4.3}$$

(7) if $y(t,x)$ is a weak solution to the IBVP (1.1)–(1.3), then for a.a. $t \in [0,T]$

$$\lim_{x\nearrow 0} |x|^2\,y(t,x) = \lim_{x\searrow 0} |x|^2\,y(t,x) = 0. \tag{4.4}$$

Since the properties (1)–(4) are direct consequences of Definition 4.1 and the Sobolev embedding theorem, we give some details of the proof of the remaining properties (5)–(7). Starting from the property (5), we notice that a weak solution with finite energy belongs to the space $L^2(0,T; W^{1,2}_{a,0}(\Omega))$. Hence, the second term in (4.1) can be estimated as follows

$$\left|\int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt\right| \leqslant \|a^{1/2}\,y_x\|_{L^2(0,T;L^2(\Omega))}\|a^{1/2}\,\varphi_x\|_{L^2(0,T;L^2(\Omega))}. \tag{4.5}$$

In this case the mapping $\varphi \mapsto \int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt$ can be defined for all $\varphi \in L^2(0,T; W^{1,2}_{a,0}(\Omega))$ using (4.5), the density of $C_0^\infty(\Omega)$ in $W^{1,2}_{a,0}(\Omega)$, and the standard rule

$$\int_0^T\int_{-1}^1 a\,y_x\,\varphi_x\,dx\,dt = \lim_{\varepsilon\to 0}\int_0^T\int_{-1}^1 a\,y_x\,(\varphi_\varepsilon)_x\,dx\,dt,$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \subset C^1([0,T]; C_0^\infty(\Omega))$ such that $\varphi_\varepsilon(T,x) = 0$, $x \in \Omega$ and $\varphi_\varepsilon \to \varphi$ strongly in $L^2(0,T; W^{1,2}_{a,0}(\Omega))$. A similar reasoning can be addressed to the first term in (4.1). So, utilizing this way for the entire integral identity (4.1) and taking into account that

$$\int_0^T \langle y_{tt}, \varphi\rangle_* \, dt = -\int_0^T \int_{-1}^1 y_t\, \varphi_t \, dx\, dt + \int_{-1}^1 y_1(x)\, \varphi(0,x)\, dx,$$

after localization on the step-wise test functions φ with respect to t, we arrive at the relation (3.1) with an arbitrary function $\varphi \in W^{1,2}_{a,0}(\Omega)$. From this we deduce the fulfilment of (3.2). Thus, $y(t,x)$ is a $W^{1,2}_{a,0}(\Omega)$-variational solution of problem (1.1)–(1.3).

Property (6) immediately follows from the fact that the function $x \mapsto a(x)\, y_x(t,x)$ is absolutely continuous. It remains to check equality (4.4). With that in mind, let us show that the function

$$v(x) = \begin{cases} x^2\, y(x), & -1 \leqslant x < 0 \ \text{ or } \ 0 < x \leqslant 1, \\ 0, & x = 1, \end{cases}$$

is absolutely continuous on $\overline{\Omega}$ for a.a. $t \in (0,T)$. Indeed, for all $\varepsilon > 0$ small enough, due to compactness of the embedding

$$W^{1,2}_{a,0}((-1,-\varepsilon)\cup(\varepsilon,1)) \hookrightarrow C^{0,1}([-1,-\varepsilon]\cup[\varepsilon,1]),$$

$v \in AC_{loc}(\overline{\Omega}_0)$ and

$$v_x = 2x\, y(t,x) + x^2 y_x(t,x), \quad \text{a.e. in } \ (0,T)\times\overline{\Omega}.$$

Without loss of generality, we assume (for the simplicity) that $x_1^* = -1$ and $x_2^* = 1$. Since $y(t,\cdot) \in L^2(\Omega)$ and

$$\begin{aligned} \int_{-1}^1 x^2 |y_x(t,x)|\, dx &\leqslant \int_{-1}^0 |x|^{\mu_{1,a}} |y_x(t,x)|\, dx + \int_0^1 |x|^{\mu_{2,a}} |y_x(t,x)|\, dx \\ &\underset{(1.6)-(1.7)}{\leqslant} \frac{1}{a(-1)} \int_{-1}^0 a(x)\, |y_x(t,x)|\, dx + \frac{1}{a(1)} \int_0^1 a(x)\, |y_x(t,x)|\, dx \\ &\leqslant \max\left\{\frac{1}{a(-1)}, \frac{1}{a(1)}\right\} \|y(t,\cdot)\|_{W^{1,1}_{a,0}(\Omega)}, \end{aligned} \tag{4.6}$$

$$\int_{-1}^1 x\, |y(t,x)|\, dx \leqslant \|y(t,\cdot)\|_{L^1(-1,1)}\,, \tag{4.7}$$

it follows that $v_x \in L^1(\Omega)$. Hence, $v \in AC(\overline{\Omega})$ and, as a consequence, the one-sided limits $\lim_{x\nearrow 0} x^2\, y(t,x) = \lim_{x\searrow 0} x^2\, y(t,x) = N$ do exist and must vanish. Indeed, if $N \neq 0$ then $y(t,x) \sim N/|x|^2$ in a neighborhood of the degeneracy point and it would not be integrable. However, this is in contradiction with the fact that $y(t,\cdot) \in L^2(\Omega)$.

To conclude this section, we prove the following result (for comparison one should refer to [1]).

Theorem 4.2. *Let $y_0 \in W^{1,2}_{a,0}(\Omega)$, $y_1 \in L^2(\Omega)$ be given distributions, and let $y \in L^2(0,T; W^{1,1}_{a,0}(\Omega))$ be a weak solution to the problem (1.1)–(1.3) in the sense of Definition 4.1. Then for a.a. $t \in [0,T]$*

$$\mathcal{E}_y(t) := \int_{-1}^{1} \Big[|(a(x)\,y(t,x))_t| + |(a(x)\,y(t,x))_x| \Big]\,dx < +\infty. \tag{4.8}$$

Proof. To prove (4.8), it is enough to show that the terms $(a(x)\,y(t,x))_t$ and $(a(x)\,y(t,x))_x$ are absolutely integrable on Ω for a.a. $t \in [0,T]$. At this stage, we observe that

$$|(a(x)\,y(t,x))_x| \le |a'(x)||y(t,x)| + a(x)|y_x(t,x)|,$$

where $a(\cdot)|y_x(t,\cdot)| \in L^1(\Omega)$ for all $t \in [0,T]$ by the inclusions $y \in L^2(0,T; W^{1,1}_{a,0}(\Omega))$ and $y \in C([0,T]; L^1(\Omega))$. It remains to establish the same inclusion for the term $|a'(x)||y(t,x)|$. To this end, we make use of the neighborhood $\mathcal{U}(0)$ of $x = 0$ that was given by inequality (1.8). Then, we see that

$$\begin{aligned}
|a'(x)||y(t,x)| &\leqslant \max\left\{ \sup_{x\in[-x_1^*,0)} \frac{(-x)|a'(x)|}{a(x)},\ \sup_{x\in(0,x_2^*]} \frac{x|a'(x)|}{a(x)} \right\} \frac{a(x)}{|x|}|y(t,x)| \\
&= \max\{\mu_{1,a}, \mu_{2,a}\}\, \frac{a(x)}{|x|}|y(t,x)| \\
&\underset{(1.8)}{\leqslant} \max\{\mu_{1,a}, \mu_{2,a}\}\, C\frac{|x|^{2-\varepsilon}}{|x|}|y(t,x)|, \quad \text{a.e. in } (0,T)\times\mathcal{U}(0),
\end{aligned} \tag{4.9}$$

where $\varepsilon > 0$ is some threshold: $2 - \varepsilon > \max\{\mu_{1,a}, \mu_{2,a}\}$. Since $y \in C([0,T]; L^1(\Omega))$, we can suppose that $|y(t,x)| < 1/|x|$ for almost all $(t,x) \in (0,T)\times\mathcal{U}(0)$. Hence, as follows from estimate (4.9), there is a constant $K > 0$ such that $|a'(x)||y(t,x)| \leqslant K|x|^{-\varepsilon}$ a.e. in $(0,T)\times\mathcal{U}(0)$. Since $0 < \varepsilon < 2 - \max\{\mu_{1,a}, \mu_{2,a}\}$ and $|a'(x)||y(t,x)| \in C((0,T)\times\overline{\Omega}\setminus\mathcal{U}(0))$, it follows that $a'(\cdot)\,y(t,\cdot) \in L^1(\Omega)$ for all $t \in [0,T]$. Thus, combining the inferences given above, we can claim that the function $(a(x)\,y(t,x))_x$ is absolutely integrable on Ω.

Arguing in a similar manner, the same conclusion can be made with respect to the function $(a(x)\,y(t,x))_t$. Thus, the energy functional (4.8) for any weak solution to the problem (1.1)–(1.3) is finite for a.a. $t \in [0,T]$. □

As follows from the established property (4.8), we see that the weak solutions to the problem (1.1)–(1.3) possess a finite energy if the energy functional is defined as in (4.8). At the same time, the standard energy functional $E_y(t)$ (2.1) is always unbounded for non-variational solutions.

Remark 4.3. The natual question that may arise here is about the transmission conditions for the weak solutions at the degeneracy point $x = 0$. One of them takes the form (4.3) and can be considered as a good option saying that: even in the case of the strong degeneracy ($1 \leqslant \mu_{1,a}, \mu_{2,a} < 2$) a finite part of the energy of the weak solutions can be transfered through the damage point from one side of 'the string' to the opposite one. As for another kind of transmission conditions, it can be deduced from Theorem 4.2. Indeed, if $y(t,x)$ is a weak solution to the problem (1.1)–(1.3), then $(a(\cdot)\,y(t,\cdot))_x \in L^1(\Omega)$, due to property (4.8). Since $a(\cdot)\,y(t,\cdot) \in L^1(\Omega)$, it follows that $a(\cdot)\,y(t,\cdot) \in AC(\overline{\Omega})$ for all $t \in [0,T]$ (because $y \in C([0,T];L^1(\Omega))$). Hence, there is a constant M such that

$$\lim_{x\nearrow 0} a(x)\,y(t,x) = \lim_{x\searrow 0} a(x)\,y(t,x) = M, \quad t \in [0,T]. \tag{4.10}$$

Utilizing the above mentioned properties, we can give the following conclusion. In spite of the fact that the system

$$y_{tt} - (a(x)\,y_x)_x = 0 \quad \text{in } Q := (0,T)\times\Omega, \tag{4.11}$$

$$y(t,-1) = v(t), \quad y(t,1) = 0 \quad \text{on } (0,T), \tag{4.12}$$

$$y(0,\cdot) = y_0, \quad y_t(0,\cdot) = y_1 \quad \text{in } \Omega, \tag{4.13}$$

where $v(t)$ is a control, is not exactly controllable on the class of mild solutions [2, Theorem 1.4] (because in decomposition (2.6) the mild solution $\overline{u}(t,x)$ of the IBVP (2.10)–(2.12) is the same for any control action $v(t)$), the property of exact controllability can be realized on the class of non-variational solutions. However, the rigorous proof of this affirmation is a rather challenging problem. For more details, how the non-variational solutions transfer the energy through damage point $x = 0$, we refer to the next section.

5. Example of IBVP, admitting infinitely many weak solutions

Consider a special case of the problem (1.1)–(1.3) with the weight function $a(x) = |x|^\alpha$, $\alpha \in [1,2)$. Then, it is easy to see that $a(x)$ satisfies properties (i) and (ii) with $\mu_{1,a} = \mu_{2,a} = \alpha$. Let $y_0(x) \in W^{1,2}_{a,0}(\Omega)$ and $y_1(x) \in L^2(\Omega)$ be fixed arbitrary and meet the boundary conditions (1.2): $y_0(\mp 1) = 0$, $y_1(\mp 1) = 0$. Then Proposition 2.3 implies the existence of a unique function $y^I(t,x) \in C([0,T];W^{1,2}_{a,0}(\Omega))$ being a mild solution to the problem (1.1)–(1.3) with the given weight $a(x)$. Hence, $y^I(t,x)$ is a weak solution to the same problem in the sense of Definition 4.1.

Further, we make use of the solution to the above problem, that has been recently proposed in [3] (see also [4]) in the following series form

$$y^{II}(t,x) = \sum_{\mu=1}^{\infty} O_{1,\mu}(t) X_{1,\mu}(x) + \sum_{\mu=1}^{\infty} O_{2,\mu}(t) X_{2,\mu}(x), \tag{5.1}$$

where

$$\begin{cases} X_{1,\mu}(x) = |x|^{\frac{1-\alpha}{2}}\, J_{-\rho}\left(s_{1,\mu}|x|^{\frac{2-\alpha}{2}}\right), \\ X_{2,\mu}(x) = \operatorname{sgn} x\,|x|^{\frac{1-\alpha}{2}}\, J_{+\rho}\left(s_{2,\mu}|x|^{\frac{2-\alpha}{2}}\right), \end{cases} \tag{5.2}$$

with $\rho = (1-\alpha)/(2-\alpha) \notin \mathbb{Z}$, $\{s_{k,\mu}\}_{\mu=1}^{\infty}$ being the monotonically increasing sequences of the zeros of the Bessel functions of the first kind of orders $\mp\rho$ [9]

(respectively, for $k = 1, 2$)

$$J_{\mp\rho}(s) = \left(\frac{s}{2}\right)^{\mp\rho} \sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma}{\gamma!\,\Gamma(1 \mp \rho + \gamma)} \left(\frac{s}{2}\right)^{2\gamma}, \tag{5.3}$$

and $\Gamma(s)$ being the Euler gamma function. In the case $\rho = 0$, the functions (5.2) are replaced with the following

$$\begin{cases} X_{1,\mu}(x) = \ J_0\left(s_{1,\mu}|x|^{\frac{1}{2}}\right), \\ X_{2,\mu}(x) = \operatorname{sgn} x \ Y_0\left(s_{2,\mu}|x|^{\frac{1}{2}}\right), \end{cases} \tag{5.4}$$

where $\{s_{2,\mu}\}_{\mu=1}^{\infty}$ is now the monotonically increasing sequence of the zeros of the Bessel function $Y_0(s)$ of the second kind of order zero:

$$Y_0(s) = \frac{2}{\pi}\left(C + \ln\frac{s}{2}\right) J_0(s) - \frac{2}{\pi}\sum_{\gamma=0}^{\infty} \frac{(-1)^\gamma\,\Phi(\gamma)}{(\gamma!)^2}\left(\frac{s}{2}\right)^{2\gamma}, \quad \Phi(\gamma) = \sum_{\beta=1}^{\gamma}\frac{1}{\beta}, \tag{5.5}$$

where $C = 0.5772156649\ldots$ is the Euler constant and $\Phi(0) = 0$ [9].

The functions $X_{k,\mu}(x)$ were shown [3, (2.8), p. 96] to be biorthogonal [10, Sect. VIII, pp. 261–286] in $L^2(\Omega)$ and meet the boundary conditions $X_{k,\mu}(\mp 1) = 0$. The limit values of the functions (5.2) (see Fig. 5.1)

$$\begin{cases} X_{1,\mu}(0) = \left(\frac{s_{1,\mu}}{2}\right)^{-\varrho} \dfrac{1}{\Gamma(1-\varrho)}, \\ \operatorname{sgn} x \lim\limits_{|x|\to 0} |x|^{\alpha-1} X_{2,\mu}(x) = \left(\frac{s_{2,\mu}}{2}\right)^{+\varrho} \dfrac{1}{\Gamma(1+\varrho)}, \end{cases} \tag{5.6}$$

and their fluxes $D_{k,\mu}(x) = a(x)X'_{k,\mu}(x)$ (see Fig. 5.2)

$$\begin{cases} D_{1,\mu}(0) = 0, \\ D_{2,\mu}(0) = \left(\frac{s_{2,\mu}}{2}\right)^{\rho} \dfrac{1-\alpha}{\Gamma(1+\varrho)} \neq 0, \end{cases} \tag{5.7}$$

and the limit values of the functions (5.4): $X_{1,\mu}(0) = 0$, $X_{2,\mu}(\mp 0) = \pm\infty$, and their fluxes: $D_{1,\mu}(0) = 0$, $\pi\,D_{2,\mu}(0) = 1$, can be calculated directly, due to the series (5.3) and (5.5), respectively.

Thus, the functions $X_{2,\mu}(x)$ have discontinuity of the second kind at the degeneracy point, hence the continuity transmission condition does not hold

$$\lim_{x\nearrow 0} y^{II}(t,x) \neq \lim_{x\searrow 0} y^{II}(t,x), \quad t \in [0,T].$$

Then, the second sum in the representation (5.1) of $y^{II}(t,x)$ may contribute to the transmission condition (4.3) with nonzero constant K, whereas the first sum satisfies (4.3) with $K = 0$. Moreover, the additional transmission condition (4.10) holds true for $y^{II}(t,x)$ (5.1), provided that $M = 0$.

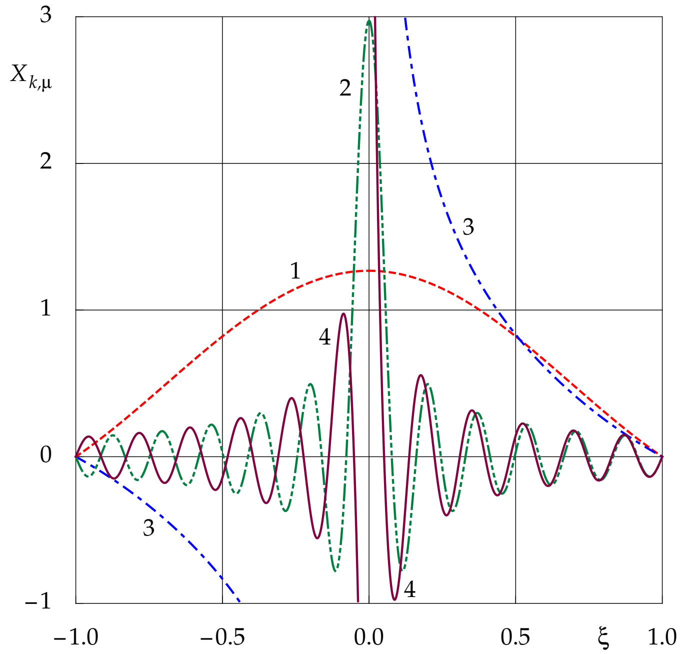

Fig. 5.1: The functions $X_{k,\mu}(x)$ (5.2) (curve 1: $k = 1$, $\mu = 1$; curve 2: $k = 1$, $\mu = 12$; curve 3: $k = 2$, $\mu = 1$; curve 4: $k = 2$, $\mu = 12$; both branches of curves 3, 4 are anti-symmetric). The transformation $|\xi| = |x|^{\frac{2-\alpha}{2}}$ stretches the neighborhood of the degeneracy point, acting like a magnifying glass.

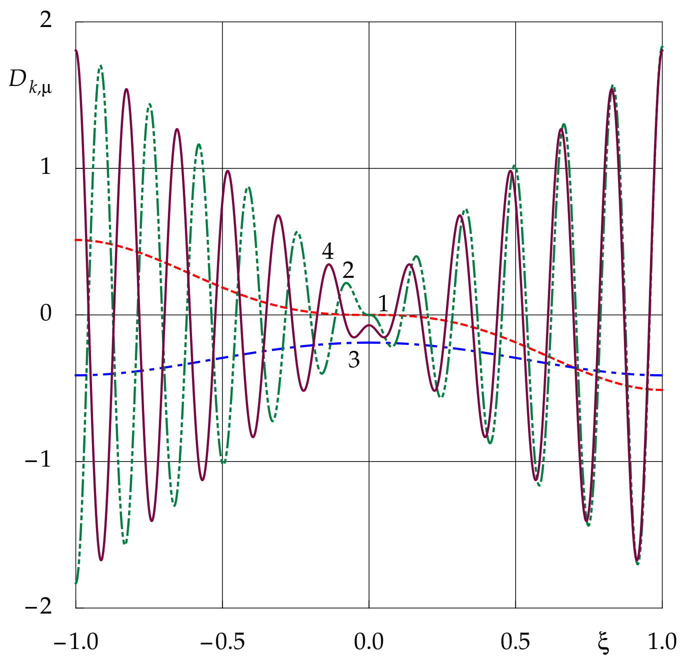

Fig. 5.2: The fluxes $D_{k,\mu}(x)$ of the functions $X_{k,\mu}(x)$ (5.2). Nomenclature of the curves is the same as in Fig. 5.1.

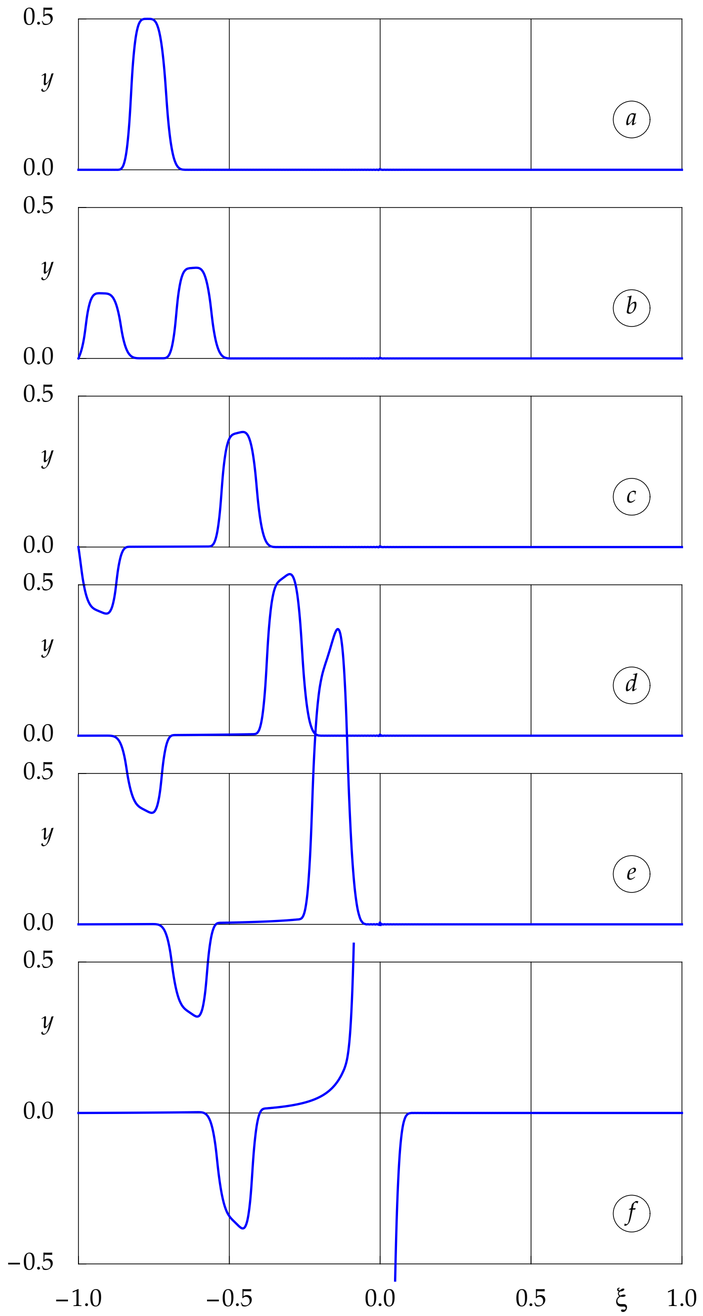

Fig. 5.3: The shape of 'the string' at time instants t: (a) 0.0, (b) 0.4, (c) 0.8, (d) 1.2, (e) 1.6, (f) 2.0

As immediately follows from representation (5.1), (5.2), (5.4), the boundary conditions (1.2) hold: $y^{II}(t,-1) = y^{II}(t,1) = 0$, whereas the initial conditions (1.3) can be met, due to the proper choice of the functions $O_{k,\mu}(t)$. In addition, the following inclusions

$$y^{II} \in L^2(0,T;L^1(\Omega)), \quad y_t^{II} \in L^2(0,T;L^1(\Omega)), \quad |x|^\alpha y_x^{II} \in L^2(0,T;L^1(\Omega))$$

hold true. Hence, $y^{II} \in L^2(0,T;W^{1,1}_{a,0}(\Omega))$, whereas $y^{II} \notin L^2(0,T;W^{1,2}_{a,0}(\Omega))$ and

$$\lim_{s\nearrow 0}\int_{-1}^{s}\left[y_t^2+|x|^\alpha y_x^2\right]\Big|_{y=y_t^{II}(t,x)}dx+\lim_{s\searrow 0}\int_{s}^{1}\left[y_t^2+|x|^\alpha y_x^2\right]\Big|_{y=y_t^{II}(t,x)}dx=+\infty.$$

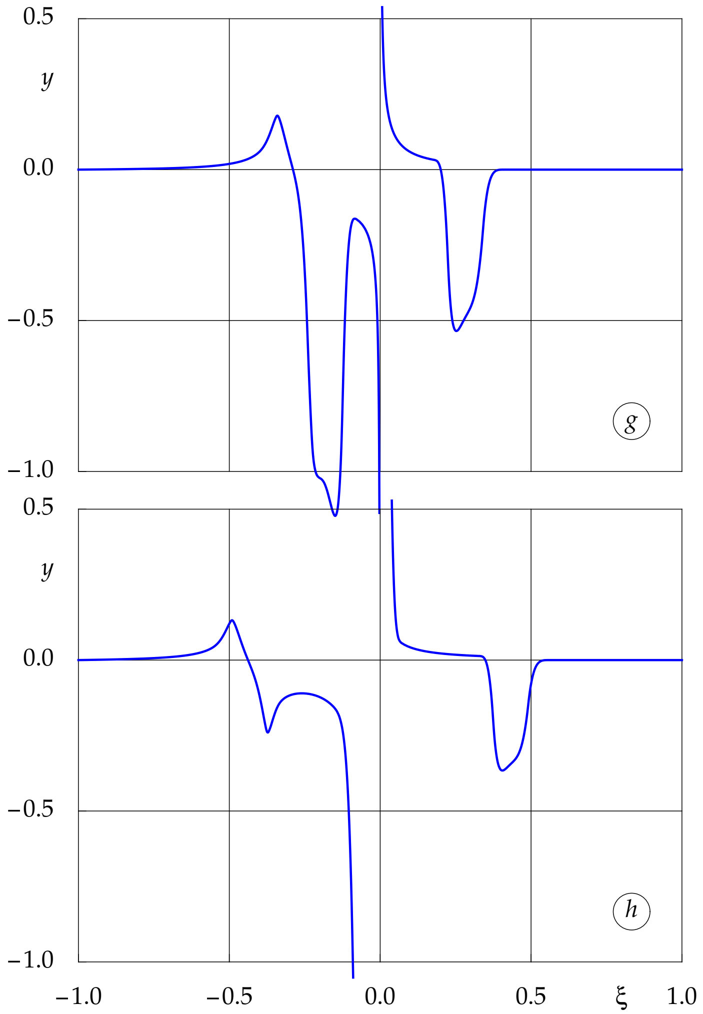

Fig. 5.4: The shape of ‘the string’ at time instants t: (g) 2.8, (h) 3.2

Thus, y^{II} is a non-variational solution to the IBVP (1.1)–(1.3). Since the set of all weak solutions is convex, we deduce that this problem admits infinitely many weak solutions and some of them can be represented as follows

$$y(t,x)=\sigma y^I(t,x)+(1-\sigma)y^{II}(t,x),\quad \sigma\in[0,1],$$

wherefrom

$$\left[y^I;y^{II}\right]:=\left\{y(t,x)=\sigma y^I(t,x)+(1-\sigma)y^{II}(t,x):\sigma\in[0,1]\right\}\subset\Xi_{(y_0,y_1)}.$$

For the specific case of the problem: 1) $\alpha = 1.25$; 2) the initial conditions are given so that 'the string' is at rest and has a step-wise shape

$$y_1(x) \equiv 0, \quad y_0(x) = \begin{cases} 0, & |x - x_0| > \delta, \ x \in \overline{\Omega}, \\ h, & |x - x_0| \leq \delta, \ x \in \overline{\Omega}, \end{cases}$$

where $x_0 = -0.5$, $\delta = 0.1$, $h = 0.5$, and a mollifier is appilied to $y_0(x)$, to avoid the Gibbs phenomenon in (5.1).

The solution to the problem (1.1)–(1.3) is presented in Figs. 5.3, 5.4. The initial shape (Figs. 5.3, a) immediately breaks into two travelling waves moving into the opposite directions (Figs. 5.3, b). The travelling wave moving to the right slows down and increases its amplitude when approaching the degeneracy point (Figs. 5.3, c,d,e). The wave loses the continuity of its shape, and the amplitude becomes unbounded when the wave approaches the degeneracy point (Figs. 5.3, f; Figs. 5.4, g,h). The traveling wave moving to the left, reaches the left end of 'the string', reflects with overturning and starts to run towards the right end, passing through the degeneracy point. Both traveling waves retain the memory concerning their original shapes during multiple passing the degeneracy point.

Trying to interpret the obtained solution, we could imagine 'the string' as a two-core wire. The first core (associated with the functions $X_{1,\mu}(x)$) being damaged retains its continuity and is able to act like a swivel, but not to transmit the tension. The second core (associated with the functions $X_{2,\mu}(x)$) being damaged losses its continuity, nevertheless the insulation of the core, even being stretched infinitely, is able to transmit non-zero tension between two parts of the damaged core.

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Про класи слабких розв'язків початково-крайової задачі для одновимірного лінійного виродженого хвильового рівняння

Vladimir Borsch and Peter Kogut

У роботі обговорюються питання існування та єдиності слабких, варіаційних та неваріаційних розв'язків початково-крайової задачі для одновимірного лінійного хвильового рівняння з сильним типом виродження в головній частині диференціального оператора. Мета полягає в аналізі коректності постановки такої задачі та дослідженні впливу нещільності множини гладких функцій у відповідному ваговому просторі Соболєва на неєдиність її слабких розв'язків. У роботі показується, що загалом єдиність слабких розв'язків може бути порушена, якщо "міра виродження" відповідає сильному випадку.

Ключові слова: сильно вироджене хвильове рівняння, існування та єдиність розв'язків, вагові простори Соболєва, слабкі розв'язки, варіаційні розв'язки, неваріаційні розв'язки