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Partial Differential Equations in Module of Copolynomials over a Commutative Ring

S.L. Gefter and A.L. Piven'

Let K be an arbitrary commutative integral domain with identity We study the copolynomials of n variables, i.e., K-linear mappings from the ring of polynomials $K[x_1, \ldots, x_n]$ into K. We prove an existence and uniqueness theorem for a linear differential equation of infinite order which can be considered as an algebraic version of the classical Malgrange–Ehrenpreis theorem for the existence of the fundamental solution of a linear differential operator with constant coefficients. We find the fundamental solutions of linear differential operators of infinite order and show that the unique solution of the corresponding inhomogeneous equation can be represented as a convolution of the fundamental solution of this operator and the right-hand side. We also prove the existence and uniqueness theorem of the Cauchy problem for some linear differential equations in the module of formal power series with copolynomial coefficients.

Key words: copolynomial, fundamental solution, convolution, δ-function, differential operator of infinite order, Cauchy problem, Laplace transform

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1. Introduction

The Poisson formula

$$u(t,x) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4a^2 t}} Q(x-y)\, dy$$

for the solution of the Cauchy problem for the one-dimensional heat equation

$$\frac{\partial u(t,x)}{\partial t} = a^2 \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad u(0,x) = Q(x)$$

is very interesting. At first sight, this formula seems to be rather "transcendental". However, if the initial condition $Q(x)$ is a polynomial of degree m with integer coefficients and $a \in \mathbb{Z}$, then the considered Cauchy problem has the unique polynomial solution with integer coefficients

$$u(t,x) = \sum_{k=0}^{[m/2]} a^{2k} \frac{Q^{(2k)}(x)}{k!} t^k.$$

The form of this solution shows that it is defined over the ring $\mathbb{Z}$, i.e., to find it we use only addition and multiplication (unlike the Poisson formula, where the coefficient a^2 is in a denominator of an exponent of power). Thus, in some sense the Poisson formula has an arithmetic origin. The analogue of the Poisson formula in the case of the non-invertible operator coefficient a^2 and the convergent power series $Q(x)$ was considered in [9].

Now let K be an arbitrary commutative integral domain with identity. In the present paper, we consider a purely algebraic version of the Poisson formula (see Example 6.11) and other similar formulas in the case when the initial condition Q is a copolynomial over K, that is, a K-linear functional on the ring $K[x_1, \dots, x_n]$ of polynomials of n variables. General properties of copolynomials of n variables are considered in Section 2. Recently, the case $n = 1$ was partially studied in [6, 7, 14]. In these papers, copolynomials were called formal generalized functions (see also [8]). Notice that given properties of copolynomials connected with the convolution are consequences of general constructions of the theory of Hopf algebras (see, for example, [20, 26]). In Section 3, differential operators of infinite order on the module of copolynomials are studied. In Section 4, with the help of the Laplace transform a connection between copolynomials and formal power series is established (see Propositions 4.2, 4.5 and Theorem 4.3). The main results of the present paper are contained in Sections 5 and 6. Theorem 5.1 and Corollary 5.2 can be considered as an algebraic version of the classical Malgrange-Ehrenpreis theorem for the existence of the fundamental solution of a linear differential operator with constant coefficients (see, for example, [16, Theorem 7.3.10], [17, Section 10.2])). Moreover, in Theorem 5.1 and Corollary 5.4, it was shown that the unique solution of the inhomogeneous equation $\mathcal{F}u = T$ with a linear differential operator $\mathcal{F}$ of infinite order can be represented as a convolution of the fundamental solution of this operator and the right-hand side T from the module of copolynomials. It should be noticed that unlike the classical theory (see, for example, [17]), the solution of the inhomogeneous equation $\mathcal{F}u = T$, and in particular its fundamental solution in the module of copolynomials are defined uniquely. In Section 6, the concept of a fundamental solution of the Cauchy problem for the equation $\frac{\partial u}{\partial t} = \mathcal{F}u$ is introduced and studied. The main result of this section is Theorem 6.9 which states that under the fulfillment of additional restrictions on the ring K the Cauchy problem $\frac{\partial u}{\partial t} = (\mathcal{F}u)(t, x),\ u(0, x) = Q(x)$ with a copolynomial $Q(x)$ has a unique solution and furthermore this solution is a convolution of the fundamental solution of the Cauchy problem and the initial condition. In Sections 5 and 6, we present meaningful examples which illustrate the constructed theory.

Notice that differential operators of infinite order on various spaces were studied in numerous works (see, for example, [2, 10, 11, 19, 22–25]). In the classical scalar case, the series with respect to the derivatives of the δ-function are intensively studied because of their applications to differential and functional-differential equations and the theory of orthogonal polynomials (see, for example, [3, 15]).

We are planning to continue our research, in particular, to introduce and study a multiplication of copolynomials for the investigation of some nonlinear partial differential equations in our further paper.

2. Preliminaries

Let K be an arbitrary commutative integral domain with identity and let $K[x_1,\dots,x_n]$ be a ring of polynomials with coefficients in K.

Definition 2.1. By a copolynomial over the ring K we mean a K-linear functional defined on the ring $K[x_1,\dots,x_n]$, i.e., a homomorphism from the module $K[x_1,\dots,x_n]$ into the ring K.

We denote the module of copolynomials over K by $K[x_1,..,x_n]'$. Thus, $T\in K[x_1,\dots,x_n]'$ if and only if $T: K[x_1,\dots,x_n]\to K$ and T has the property of K-linearity: $T(ap+bq)=aT(p)+bT(q)$ for all $p,q\in K[x_1,\dots,x_n]$ and $a,b\in K$. If $T\in K[x_1,..,x_n]'$ and $p\in K[x_1,\dots,x_n]$, then for the value of T on p we use the notation (T,p). We also write a copolynomial $T\in K[x_1,\dots,x_n]'$ in the form $T(x)$, where $x=(x_1,\dots,x_n)$ is regarded as the argument of polynomials $p(x)\in K[x_1,\dots,x_n]$ subjected to the action of the K-linear mapping T. In this case, the result of action of T upon p can be represented in the form $(T(x),p(x))$.

Let $\mathbb{N}_0$ be the set of nonnegative integers. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^n$ we put [22, Chap. 1, §1–2]

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}},\quad |\alpha|=\sum_{j=1}^n\alpha_j,$$
$$x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},\quad \alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!.$$

For multi-indexes $\alpha,\beta\in\mathbb{N}_0^n$, the relation $\alpha\le\beta$ means that $\alpha_j\le\beta_j$ for all $j=1,\dots,n$. If $\alpha\le\beta$, then we will use the notation $\binom{\beta}{\alpha}=\prod_{j=1}^n\binom{\beta_j}{\alpha_j}$.

Let $p(x)=\sum_{|\alpha|\le m}a_\alpha x^\alpha\in K[x_1,\dots,x_n]$. If $h=(h_1,\dots,h_n)$, then the polynomial $p(x+h)\in K[x_1,\dots,x_n][h_1,\dots,h_n]$ can be represented in the form

$$p(x+h)=\sum_{|\alpha|\le m}p_\alpha(x)h^\alpha,$$

where $p_\alpha(x)\in K[x_1,\dots,x_n]$. Since in the case of a field with zero characteristic $p_\alpha(x)=\frac{D^\alpha p(x)}{\alpha!}$, we also assume that, by definition, $\frac{D^\alpha p(x)}{\alpha!}=p_\alpha(x),\quad |\alpha|\le m$ is true for any commutative ring K. For $m<|\alpha|$, we assume that $\frac{D^\alpha p(x)}{\alpha!}=0$.

We now introduce the notion of shift for a copolynomial [7, 8]. For $T\in K[x_1,\dots,x_n]'$ and fixed $h=(h_1,\dots,h_n)\in K^n$, we define a copolynomial $T(x+h)$ by the formula

$$(T(x+h),p)=(T,p(x-h)),\quad p\in K[x_1,\dots,x_n].$$

Definition 2.2. The partial derivative $\frac{\partial T}{\partial x_j}$ of a copolynomial $T \in K[x_1, \dots, x_n]'$ with respect to the variable x_j $(j = 1, \dots, n)$ is defined as in the classical case by the formula

$$\left(\frac{\partial T}{\partial x_j}, p\right) = -\left(T, \frac{\partial p}{\partial x_j}\right), \quad p \in K[x_1, \dots, x_n]. \tag{2.1}$$

By using (2.1), we arrive at the following expression for the derivative $D^\alpha T$:

$$(D^\alpha T, p) = (-1)^{|\alpha|}(T, D^\alpha p), \quad p \in K[x_1, \dots, x_n].$$

Therefore,

$$(D^\alpha T, p) = 0, \text{ where } p \in K[x_1, \dots, x_n] \text{ and } |\alpha| > \deg p.$$

By virtue of the equality

$$\left(\frac{D^\alpha T}{\alpha!}, p\right) = (-1)^{|\alpha|}\left(T, \frac{D^\alpha p}{\alpha!}\right), \quad p \in K[x_1, \dots, x_n], \tag{2.2}$$

the copolynomials $\frac{D^\alpha T}{\alpha!}$ are well-defined for any $T \in K[x_1, \dots, x_n]'$ and $\alpha \in \mathbb{N}_0^n$.

Example 2.3. The copolynomial δ-function is given by the formula

$$(\delta, p) = p(0), \quad p \in K[x_1, \dots, x_n].$$

Therefore,

$$(D^\alpha \delta, p) = (-1)^{|\alpha|}(\delta, D^\alpha p) = (-1)^{|\alpha|} D^\alpha p(0), \quad \alpha \in \mathbb{N}_0^n.$$

Example 2.4. Let $K = \mathbb{R}$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function such that

$$\int_{\mathbb{R}^n} |x^\alpha f(x)| dx < +\infty, \quad \alpha \in \mathbb{N}_0^n. \tag{2.3}$$

Then f generates the regular copolynomial T_f:

$$(T_f, p) = \int_{\mathbb{R}^n} p(x) f(x) dx, \quad p \in \mathbb{R}[x_1, \dots, x_n].$$

In this case, unlike the classical theory, all copolynomials are regular [3, Theorem 7.3.4], although a nonzero function f can generate the zero copolynomial (see [7, Example 2.2] and [8, Remark 1]). Moreover, if $f \in C^1(\mathbb{R}^n)$ and the conditions (2.3) are satisfied for $\frac{\partial f}{\partial x_j}$ $(j = 1, \dots, n)$, then it can be shown that

$$\left(\frac{\partial T_f}{\partial x_j}, p\right) = \int_{\mathbb{R}^n} p(x) \frac{\partial f}{\partial x_j} dx, \quad p \in \mathbb{R}[x_1, \dots, x_n], \ j = 1, \dots, n.$$

Remark 2.5. The notion of a copolynomial differs from that of a formal distribution used in the theory of vertex operator algebras (see [4,18]), although there are some natural connections between these notions.

We now consider the problem of convergence in the space $K[x_1,\dots,x_n]'$. In the ring K, we consider the discrete topology. Further, in the module of copolynomials $K[x_1,\dots,x_n]'$, we consider the topology of pointwise convergence. It is easy to show that the last topology is generated by the following metric:

$$d(T_1,T_2)=\sum_{|\alpha|=0}^{\infty}\frac{d_0((T_1,x^\alpha),(T_2,x^\alpha))}{2^{|\alpha|}},$$

where d_0 is the discrete metric on K. The convergence of a sequence $\{T_k\}_{k=1}^{\infty}$ to T in $K[x_1,\dots,x_n]'$ means that for every polynomial $p\in K[x_1,\dots,x_n]$ there exists a number $k_0\in\mathbb{N}$ such that

$$(T_k,p)=(T,p),\quad k=k_0,k_0+1,k_0+2,\dots$$

The series $\sum_{k=0}^{\infty}T_k$ converges in $K[x_1,\dots,x_n]'$ if a sequence of its partial sums $\sum_{k=0}^{N}T_k$ converges in $K[x_1,\dots,x_n]'$.

The following lemma shows the possibility of the decomposition of an arbitrary copolynomial in series about the system $\frac{D^\alpha\delta}{\alpha!}$, $\alpha\in\mathbb{N}_0^n$.

Lemma 2.6. *Let $T\in K[x_1,\dots,x_n]'$. Then*

$$T=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}(T,x^\alpha)\frac{D^\alpha\delta}{\alpha!}.$$

Proof. For any multi-index $\beta\in\mathbb{N}_0^n$, we have

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}(T,x^\alpha)\left(\frac{D^\alpha\delta}{\alpha!},x^\beta\right)=(-1)^{|\beta|}(T,x^\beta)\left(\frac{D^\beta\delta}{\beta!},x^\beta\right)=\big(T,x^\beta\big).\qquad\square$$

Definition 2.7. If $p\in K[x_1,\dots,x_n]$ and $T\in K[x_1,\dots,x_n]'$ is a copolynomial, then their *convolution* $T*p$ is defined naturally as follows:

$$(T*p)(x)=(T(y),p(x-y))=\sum_{|\alpha|\le m}(-1)^{|\alpha|}(T,y^\alpha)\frac{D^\alpha p(x)}{\alpha!},$$

where $m=\deg p$. Thus $T*p$ is a polynomial with coefficients in K, i.e., $T*p\in K[x_1,\dots,x_n]$.

Remark 2.8. By Definition 2.7, we have $\delta*p=p$.

Definition 2.9. The *tensor product* $T_1\otimes T_2\in K[x_1,\dots,x_n,y_1,\dots,y_m]'$ of copolynomials $T_1\in K[x_1,\dots,x_n]'$ and $T_2\in K[y_1,\dots,y_m]'$ is defined by the equality

$$\big(T_1\otimes T_2,x^\alpha y^\beta\big)=\big(T_1,x^\alpha\big)\big(T_2,y^\beta\big),\quad \alpha\in\mathbb{N}_0^n,\ \beta\in\mathbb{N}_0^m.$$

Further, with the help of K-linearity, the result of the action $(T_1\otimes T_2,p)$ of the copolynomial $T_1\otimes T_2$ to an arbitrary polynomial $p\in K[x_1,\dots,x_n,y_1,\dots,y_m]$ is defined.

Definition 2.10. Let $T_1, T_2 \in K[x_1,\dots,x_n]'$. We remind the definition of their *convolution* (see [20] and [26, Section 2.1]). If $p \in K[x_1,\dots,x_n]$ and $p(x+y) = \sum_{|\alpha|\le m} \frac{D^\alpha p(x)}{\alpha!} y^\alpha$, then

$$(T_1 * T_2, p) = (T_1 \otimes T_2, p(x+y)) = \sum_{|\alpha|\le m} \left(T_1(x), \frac{D^\alpha p(x)}{\alpha!}\right)(T_2(y), y^\alpha). \quad (2.4)$$

Notice that $T_1 * T_2 \in K[x_1,\dots,x_n]'$.

The following assertion establishes the commutativity and associativity for the convolution of copolynomials.

Proposition 2.11. *Let $T_1, T_2, T_3 \in K[x_1,\dots,x_n]'$. Then*

$$T_1 * T_2 = T_2 * T_1,$$
$$(T_1 * T_2) * T_3 = T_1 * (T_2 * T_3).$$

Example 2.12. Let $T \in K[x_1,\dots,x_n]'$. We find the convolution $\delta * T$. For $p \in K[x_1,\dots,x_n]$, $\deg p = m$, we obtain

$$(\delta * T, p) = \sum_{|\alpha|\le m} \left(\delta, \frac{D^\alpha p(x)}{\alpha!}\right)(T, y^\alpha) = \sum_{|\alpha|\le m} \left(\frac{D^\alpha p}{\alpha!}\right)(0)(T, y^\alpha) = (T, p).$$

Hence $\delta * T = T$.

Corollary 2.13. *The module $K[x_1,\dots,x_n]'$ under the convolution operation is an associative commutative algebra with identity over the ring K.*

The following theorem establishes the property of continuity for the convolution.

Theorem 2.14. *Let $T_k \in K[x_1,\dots,x_n]'$, $k \in \mathbb{N}$ and $T_k \to 0$, $k \to \infty$ in the topology of $K[x_1,\dots,x_n]'$. Then $T_k * S \to 0$, $k \to \infty$ in the topology of $K[x_1,\dots,x_n]'$ for every copolynomial $S \in K[x_1,\dots,x_n]'$.*

Proof. Indeed, by (2.4), for every polynomial $p \in K[x_1,\dots,x_n]$ of degree m, we have

$$(T_k * S, p) = \sum_{|\alpha|\le m} (T_k(y), y^\alpha)\left(S(x), \frac{D^\alpha p(x)}{\alpha!}\right) \to 0, \quad k \to \infty. \qquad \square$$

Corollary 2.15. *Assume that $T_k \in K[x_1,\dots,x_n]'$, $k \in \mathbb{N}$, and the series $\sum_{k=1}^\infty T_k$ converges in the topology of $K[x_1,\dots,x_n]'$. Then, for every copolynomial $S \in K[x_1,\dots,x_n]'$, the series $\sum_{k=1}^\infty (T_k * S)$ converges in the same topology and*

$$\sum_{k=1}^\infty (T_k * S) = \left(\sum_{k=1}^\infty T_k\right) * S.$$

3. Linear differential operators of infinite order on the module of copolynomials

We now consider the linear differential operator of infinite order on $K[x_1,\dots,x_n]'$:

$$\mathcal{F}=\sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha,$$

where $a_\alpha\in K$. This operator acts upon a copolynomial $T\in K[x_1,\dots,x_n]'$ by the following rule: if $p\in K[x_1,\dots,x_n]$ and $m=\deg p$, then

$$(\mathcal{F}T,p)=\left(\sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha T,p\right)=\sum_{|\alpha|\le m}(-1)^{|\alpha|}a_\alpha(T,D^\alpha p)=\sum_{|\alpha|\le m}a_\alpha(D^\alpha T,p).$$

Thus, the differential operator $\mathcal{F}:K[x_1,\dots,x_n]'\to K[x_1,\dots,x_n]'$ is well-defined and for any polynomial p of degree at most m the equality

$$(\mathcal{F}T,p)=\sum_{|\alpha|\le m}a_\alpha(D^\alpha T,p) \tag{3.1}$$

is true.

Lemma 3.1. *The differential operator $\mathcal{F}:K[x_1,\dots,x_n]'\to K[x_1,\dots,x_n]'$ is a continuous K-linear mapping.*

Proof. Assume that a sequence of copolynomials $\{T_k\}_{k=0}^{\infty}$ converges to T in $K[x_1,\dots,x_n]'$. Then there exists $k_0=k_0(p)\in\mathbb{N}$ such that the equality $(T_k,p)=(T,p)$ is true for all $k\ge k_0(p)$. Let $m=\deg p$ and $s(p)=\max\{k_0(D^\alpha p):|\alpha|\le m\}$. Then, by using (3.1), we obtain

$$(\mathcal{F}T_k,p)=\left(\sum_{|\alpha|\le m}a_\alpha D^\alpha T_k,p\right)=\sum_{|\alpha|\le m}a_\alpha(-1)^\alpha(T_k,D^\alpha p)$$
$$=\sum_{|\alpha|\le m}a_\alpha(-1)^\alpha(T,D^\alpha p)=\left(\sum_{|\alpha|\le m}a_\alpha D^\alpha T,p\right)=(\mathcal{F}T,p),\quad k\ge s(p),$$

i.e., the sequence $\{\mathcal{F}T_k\}_{k=0}^{\infty}$ converges to $\mathcal{F}T$. The lemma is proved. □

The following assertion shows that the convolution operation and the differential operator of infinite order commute. We also show that every such differential operator is a convolution operator.

Theorem 3.2. *Let $T_1,T_2\in K[x_1,\dots,x_n]'$ and let $\mathcal{F}=\sum_{|\alpha|=0}^{\infty}a_\alpha D^\alpha$ be a differential operator of infinite order on $K[x_1,\dots,x_n]'$ with coefficients $a_\alpha\in K$. Then*

$$\mathcal{F}(T_1*T_2)=(\mathcal{F}T_1)*T_2.$$

Therefore $\mathcal{F}(T)=\mathcal{F}(\delta)*T$ for all $T\in K[x_1,\dots,x_n]'$.

Proof. Let $p \in K[x_1, \ldots, x_n]$, $m = \deg p$ and let $\beta \in \mathbb{N}_0^n$ be an arbitrary multi-index. By the definition of the convolution,

$$(D^\beta(T_1 * T_2), p) = (-1)^{|\beta|}(T_1 * T_2, D^\beta p) = (-1)^{|\beta|} \sum_{|\alpha| \le m} \left(T_1, \frac{D^{\alpha+\beta} p}{\alpha!}\right)(T_2, y^\alpha)$$
$$= \sum_{|\alpha| \le m} \left(D^\beta T_1, \frac{D^\alpha p}{\alpha!}\right)(T_2, y^\alpha) = \big((D^\beta T_1) * T_2, p\big).$$

Therefore,

$$D^\beta(T_1 * T_2) = \big(D^\beta T_1\big) * T_2. \tag{3.2}$$

By (3.1), for every $T \in K[x_1, \ldots, x_n]'$, the series $\sum_{|\beta|=0}^\infty a_\beta D^\beta T$ converges in the topology of $K[x_1, \ldots, x_n]'$. Therefore, by Corollary 2.15 and equality (3.2),

$$(\mathcal{F}(T_1 * T_2)) = \sum_{|\beta|=0}^\infty a_\beta D^\beta(T_1 * T_2) = \sum_{|\beta|=0}^\infty a_\beta\big(\big(D^\beta T_1\big) * T_2\big) = (\mathcal{F}T_1) * T_2. \tag{3.3}$$

Now, substituting $T_1 = \delta$, $T_2 = T$ into (3.3), we get $\mathcal{F}(T) = \mathcal{F}(\delta) * T$. The theorem is proved. □

4. The Laplace transform in the module of copolynomials

Let $z = (z_1, \ldots, z_n)$ and let $K\left[\left[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}\right]\right]$ be the module of formal Laurent series with coefficients in K. For the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we put $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. For $g \in K\left[\left[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}\right]\right]$, $g(z) = \sum_{\alpha \in \mathbb{Z}^n} g_\alpha z^\alpha$, we naturally define the formal residue

$$\operatorname{Res}(g(z)) = g_{(-1,\ldots,-1)}.$$

Now we define a Laplace transform of a copolynomial $T \in K[x_1, \ldots, x_n]'$.

Definition 4.1. Let $T \in K[x_1, \ldots, x_n]'$. Assume that the ring K contains the field of rational numbers $\mathbb{Q}$. Consider the following formal power series from $K[[z_1, \ldots, z_n]]$:

$$L(T)(z) = \widetilde{T}(z) = \sum_{|\alpha|=0}^\infty \frac{(T, x^\alpha)}{\alpha!} z^\alpha.$$

A power series $\widetilde{T}(z)$ will be called the *Laplace transform* of the copolynomial T.

We can write informally as follows: $\widetilde{T}(z) = \big(T, e^{\langle z, x\rangle}\big)$. It is obvious that the mapping $L : K[x_1, \ldots, x_n]' \to K[[z_1, \ldots, z_n]]$, $L(T) = \widetilde{T}$ is a continuous isomorphism of K-modules if we consider the standard Krull topology on $K[[z_1, \ldots, z_n]]$ [12, Section 1, §3, Section 4], i.e., the topology of coefficient-wise stabilization.

Proposition 4.2 (The inversion formula or the Parseval identity)**.** *Let* $K \supset \mathbb{Q}$*,* $T \in K[x_1, \dots, x_n]'$ *and* $p(x) = \sum_{|\alpha| \le m} c_\alpha x^\alpha \in K[x_1, \dots, x_n]$*. Then*

$$(T(x), p(x)) = \operatorname{Res}(\widetilde{T}(z)\widetilde{p}(z)),$$

where $\widetilde{p}(z) = \sum_{|\alpha| \le m} \frac{\alpha! c_\alpha}{z^{\alpha+\iota}}$ *is the Laplace transform of the polynomial* $p(x)$*.*

Proof. It is sufficient to consider the case $p(x) = x^\beta$ for some multi-index $\beta \in \mathbb{N}_0^n$. We have $\widetilde{p}(z) = \frac{\beta!}{z^{\beta+\iota}}$. Therefore,

$$\widetilde{T}(z)\widetilde{p}(z) = \sum_{|\alpha|=0}^{\infty} \frac{(T, x^\alpha)}{\alpha!} z^\alpha \frac{\beta!}{z^{\beta+\iota}}$$

and $\operatorname{Res}(\widetilde{T}(z)\widetilde{p}(z)) = (T, x^\beta)$. □

The following theorem asserts that the Laplace transform sends the convolution of copolynomials to the product of their Laplace transforms.

Theorem 4.3. *Let* $T_1, T_2 \in K[x_1, \dots, x_n]'$*. Assume that the ring* K *contains the field of rational numbers* $\mathbb{Q}$*. Then*

$$(\widetilde{T_1 * T_2})(z) = \widetilde{T_1}(z)\widetilde{T_2}(z).$$

Proof. Since for any $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}(T_1 * T_2, x^\alpha) &= \sum_{|\beta| \le |\alpha|} \left(T_1(x), \frac{D^\beta x^\alpha}{\beta!}\right) (T_2(y), y^\beta) \\ &= \sum_{\beta \le \alpha} \left(T_1(x), \binom{\alpha}{\beta} x^{\alpha-\beta}\right) (T_2(y), y^\beta),\end{aligned}$$

we have

$$\begin{aligned}(\widetilde{T_1 * T_2})(z) &= \sum_{|\alpha|=0}^{\infty} \frac{(T_1 * T_2, x^\alpha)}{\alpha!} z^\alpha \\ &= \sum_{|\alpha|=0}^{\infty} \sum_{\beta \le \alpha} \left(T_1(x), \frac{\binom{\alpha}{\beta}}{\alpha!} x^{\alpha-\beta}\right) (T_2(y), y^\beta) z^\alpha \\ &= \sum_{|\alpha|=0}^{\infty} \sum_{\beta \le \alpha} \left(T_1(x), \frac{x^{\alpha-\beta}}{(\alpha-\beta)!}\right) \left(T_2(y), \frac{y^\beta}{\beta!}\right) z^\alpha = \widetilde{T_1}(z)\widetilde{T_2}(z).\end{aligned}$$

The theorem is proved. □

Example 4.4. Suppose that $n = 1$, K is an arbitrary commutative integral domain, $a \in K$, and

$$(\mathcal{E}_a, p) = \sum_{k=0}^{m} a^k p^{(k)}(0),$$

where $p \in K[x]$, $m = \deg p$. Then $\mathcal{E}_a \in K[x]'$ and

$$\mathcal{E}_a = \sum_{j=0}^{\infty} (-1)^j a^j \delta^{(j)}.$$

If $K = \mathbb{R}$ and $a > 0$, then

$$(\mathcal{E}_a, p) = \int_{-\infty}^{\infty} p(x) f_a(x)\, dx,$$

where

$$f_a(x) = \begin{cases} \frac{1}{a} e^{-\frac{x}{a}}, & x > 0 \\ 0, & x < 0. \end{cases}$$

(see Example 4 in [8]). Let K be again a commutative integral domain such that $K \supset \mathbb{Q}$. If $a, b \in K$, then $\widetilde{\mathcal{E}_a}(z) = \sum_{j=0}^{\infty} a^j z^j$ and $(a-b)\widetilde{\mathcal{E}_a}(z)\widetilde{\mathcal{E}_b}(z) = a\widetilde{\mathcal{E}_a}(z) - b\widetilde{\mathcal{E}_b}(z)$. With the help of Theorem 4.3 we obtain $L((a-b)(\mathcal{E}_a * \mathcal{E}_b)) = a\widetilde{\mathcal{E}_a - b\mathcal{E}_b}$ and

$$(a-b)(\mathcal{E}_a * \mathcal{E}_b) = a\mathcal{E}_a - b\mathcal{E}_b. \tag{4.1}$$

Now the convolution equation (4.1) can be checked for an arbitrary commutative integral domain K (see also Example 1.1 in [5], where the similar equation for formal Laurent series was considered). Indeed, by Theorem 3.2 and Corollary 2.15, we obtain

$$\begin{aligned}
(a-b)(\mathcal{E}_a * \mathcal{E}_b) &= (a-b)\left(\sum_{j=0}^{\infty} (-1)^j a^j \delta^{(j)}\right) * \left(\sum_{k=0}^{\infty} (-1)^k b^k \delta^{(k)}\right) \\
&= (a-b) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} a^j b^k (\delta^{(j)} * \delta^{(k)}) \\
&= (a-b) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} a^j b^k \delta^{(j+k)} \\
&= (a-b) \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} (-1)^l a^j b^{l-j} \delta^{(l)} = (a-b) \sum_{l=0}^{\infty} (-1)^l \sum_{j=0}^{l} a^j b^{l-j} \delta^{(l)} \\
&= \sum_{l=0}^{\infty} (-1)^l (a^{l+1} - b^{l+1}) \delta^{(l)} = a\mathcal{E}_a - b\mathcal{E}_b.
\end{aligned}$$

We established a connection between Laplace transform of the copolynomial $\mathcal{F}T = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha T$, where $T \in K[x_1, \ldots, x_n]'$, and the symbol $\varphi(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ of the differential operator $\mathcal{F}$.

Proposition 4.5. *Let $K \supset \mathbb{Q}$ and let $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ be a linear differential operator of infinite order on $K[x_1, \ldots, x_n]'$ with coefficients $a_\alpha \in K$. Then, for every $T \in K[x_1, \ldots, x_n]'$, the equality*

$$\widetilde{\mathcal{F}T}(z) = \varphi(-z)\widetilde{T}(z) \tag{4.2}$$

holds.

Proof. By the definition of a Laplace transform, for any multi-index $\alpha \in \mathbb{N}_0^n$, we have

$$\widetilde{D^\alpha T}(z) = \sum_{|\beta|=0}^{\infty} \frac{(D^\alpha T, x^\beta)}{\beta!} z^\beta = \sum_{\beta \geq \alpha} (-1)^{|\alpha|} \frac{(T, x^{\beta-\alpha})}{(\beta-\alpha)!} z^\beta$$

$$= (-z)^\alpha \sum_{|\beta|=0}^{\infty} \frac{(T, x^\beta)}{\beta!} z^\beta = (-z)^\alpha \widetilde{T}(z).$$

Multiplying this equality by a_α and summing all $\alpha \in \mathbb{N}_0^n$, we obtain (4.2). □

5. Fundamental solution of a linear differential operator of infinite order

Let $T \in K[x_1, \dots, x_n]'$ be a copolynomial and let $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ be a linear differential operator of infinite order on $K[x_1, \dots, x_n]'$ with coefficients $a_\alpha \in K$. Consider the following differential equation:

$$\mathcal{F}u = T. \tag{5.1}$$

We prove an existence and uniqueness theorem for equation (5.1) and continuous dependence for the unique solution of this equation on T. By I, denote the identity mapping of $K[x_1, \dots, x_n]'$.

Theorem 5.1. *Let a_0 be an invertible element of the ring K. Then the linear differential operator $\mathcal{F}$ of infinite order is bijective and its inverse operator $\mathcal{F}^{-1}$ is a continuous mapping. Moreover,*

$$\mathcal{F}^{-1} = a_0^{-1} \sum_{k=0}^{\infty} (I - a_0^{-1}\mathcal{F})^k, \tag{5.2}$$

where the series in the right-hand side of (5.2) converges in the topology of $K[x_1, \dots, x_n]'$. In particular, for any copolynomial $T \in K[x_1, \dots, x_n]'$, there exists a unique solution $u \in K[x_1, \dots, x_n]'$ of equation (5.1). This solution admits representations

$$u = \mathcal{F}^{-1}(T) = \mathcal{F}^{-1}(\delta) * T$$

and continuously depends on T in the topology of $K[x_1, \dots, x_n]'$.

Proof. We have the following representation of the operator $\mathcal{F}$:

$$\mathcal{F} = a_0 \left(I - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right), \tag{5.3}$$

where $\mathcal{G}_j$ $(j = 1, \dots, n)$ are some linear differential operators. For every $k \in \mathbb{N}$, we have

$$\left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n}.$$

Now, for every copolynomial $T \in K[x_1, \dots, x_n]'$ and polynomial $p \in K[x_1, \dots, x_n]$ of degree m, we have

$$\sum_{k=0}^{\infty} \left(\left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k T, p \right) = \left(\sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} T, p \right)$$
$$= \sum_{|\alpha|=0}^{\infty} \left(\frac{|\alpha|!}{\alpha!} D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} T, p \right) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \left(\frac{|\alpha|!}{\alpha!} \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} T, D^\alpha p \right).$$

Therefore the series $\sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k T$ converges for any copolynomial $T \in K[x_1, \dots, x_n]'$, the operator $I - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j$ is bijective and its inverse operator has the form

$$\left(I - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^{-1} = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k.$$

Now (5.3) implies the bijectivity of the operator $\mathcal{F}$ and

$$\mathcal{F}^{-1} = a_0^{-1} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k = a_0^{-1} \sum_{k=0}^{\infty} (I - a_0^{-1} \mathcal{F})^k. \tag{5.4}$$

Thus the representation (5.2) is true for the inverse operator $\mathcal{F}^{-1}$. Hence, the differential equation (5.1) has a unique solution $u \in K[x_1, \dots, x_n]'$ and, moreover, $u = \mathcal{F}^{-1} T$. By equality (5.4) and Corollary 2.15, we obtain the following representation for this solution:

$$u = \mathcal{F}^{-1} T = a_0^{-1} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k T = a_0^{-1} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k (\delta * T)$$
$$= \left(a_0^{-1} \sum_{k=0}^{\infty} \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j \right)^k \delta \right) * T = \mathcal{F}^{-1}(\delta) * T$$

(see also Example 2.12).

The continuity of the operator $\mathcal{F}^{-1}$ follows from the continuity of the convolution (see Theorem 2.14). The theorem is proved. □

Corollary 5.2. *Let a_0 be an invertible element of the ring K. Then the differential equation $\mathcal{F} u = \delta$ has the unique solution*

$$\mathcal{E} = \mathcal{F}^{-1} \delta. \tag{5.5}$$

Definition 5.3. The copolynomial defined by (5.5) is called the *fundamental solution of the linear differential operator $\mathcal{F}$.*

Theorem 5.1 implies the following assertion.

Corollary 5.4. *Let a_0 be an invertible element of the ring K. Then the unique solution of equation (5.1) is the convolution of the fundamental solution $\mathcal{E}$ and the copolynomial T: $u = \mathcal{E} * T$.*

Remark 5.5. In all previous results a_0 was supposed to be an invertible element of the ring K. It should be noticed that this condition is necessary for the existence of the fundamental solution of the differential operator $\mathcal{F}$. Indeed, if $\mathcal{E}$ is the fundamental solution of this operator, then applying the left- and right-hand sides of equation $\mathcal{F}\mathcal{E} = \delta$ to 1, we get that $a_0(\mathcal{E}, 1) = 1$, i.e., a_0 is an invertible element of the ring K.

Remark 5.6. Let a_0 be an invertible element of K. Then the differential operator $\mathcal{F} : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ is bijective and for any polynomial $p \in K[x_1, \dots, x_n]$ the differential equation

$$\mathcal{F}u = p$$

has a unique solution $u \in K[x_1, \dots, x_n]$, moreover, $\deg u \le \deg p$. Furthermore, this solution is a convolution of the fundamental solution $\mathcal{E}$ of the differential operator $\mathcal{F}$ and the polynomial p: $u = \mathcal{E} * p$. The proof of this assertion is similar to that of Theorem 5.1.

Example 5.7. Let $n = 1$ and $a \in K$. Consider the linear differential operator $\mathcal{F} = a\frac{d}{dx} + I$ on $K[x]'$. By Corollary 5.2, the operator $\mathcal{F}$ has the fundamental solution

$$\mathcal{E}_a = \mathcal{F}^{-1}\delta = \left(I + a\frac{d}{dx}\right)^{-1}\delta = \sum_{j=0}^{\infty}(-1)^j a^j \delta^{(j)},$$

i.e., $a\frac{d\mathcal{E}_a}{dx} + \mathcal{E}_a = \delta$ (see Example 4.4). If $K = \mathbb{R}$ and $a > 0$, then we obtain that the fundamental solution $\mathcal{E}_a$ regarded as a regular copolynomial coincides with the classical fundamental solution $\frac{1}{a}\,\theta(x)e^{-x/a}$ of the differential operator $\mathcal{F}$, where $\theta(x)$ is the Heaviside function (see also Examples 4 and 5 in [8]).

Example 5.8. The linear differential operator of infinite order $\mathcal{F} = \sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\right)^k$ has the inverse operator $I - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}$. Therefore $\mathcal{E} = \delta - \sum_{j=1}^{n}\frac{\partial\delta}{\partial x_j}$ is the fundamental solution of the operator $\mathcal{F}$. By Theorem 5.1, for every copolynomial $T \in K[x_1, \dots, x_n]'$, the differential equation of infinite order

$$\sum_{k=0}^{\infty}\left(\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\right)^k u = T$$

has a unique solution $u = \mathcal{F}^{-1}T = T - \sum_{j=1}^{n}\frac{\partial T}{\partial x_j}$.

Example 5.9. Let c be an invertible element of the ring K. In the module $K[x_1, x_2, x_3]'$, we consider the Helmholtz equation

$$\triangle\mathcal{E} + c\mathcal{E} = \delta, \tag{5.6}$$

where $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator. By Theorem 5.1, equation (5.6) has the unique solution (see (5.2)):

$$\mathcal{E} = (cI + \triangle)^{-1}\delta = \sum_{k=0}^{\infty}(-1)^k c^{-k-1}\triangle^k\delta. \tag{5.7}$$

For any $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$ and $k \in \mathbb{N}_0$, we have

$$\triangle^k x^\beta = \sum_{|\alpha|=k} \frac{k!}{\alpha!} D^{2\alpha} x^\beta, \quad x^\beta = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}.$$

Therefore,

$$(\triangle^{|\alpha|}\delta, x^\beta) = (\delta, \triangle^{|\alpha|}x^\beta) = \begin{cases} \frac{|\alpha|!(2\alpha)!}{\alpha!}, & \beta = 2\alpha, \\ 0, & \beta \neq 2\alpha. \end{cases}$$

Substituting this expression into (5.7), we obtain

$$(\mathcal{E}, x^\beta) = \begin{cases} (-1)^{|\alpha|} \frac{|\alpha|!(2\alpha)!}{\alpha! c^{|\alpha|+1}}, & \beta = 2\alpha, \\ 0, & \beta \neq 2\alpha. \end{cases}$$

This formula gives the fundamental solution of the Helmholtz operator $\triangle + cI$. In the case $K = \mathbb{R}$ and $c > 0$, this solution is connected with classical fundamental solutions $-\frac{1}{4\pi}\frac{e^{\pm i\sqrt{c}|x|}}{|x|}$, $(|x| = \sqrt{x_1^2 + x_2^2 + x_3^2})$ of the Helmholtz operator by the equalities

$$(\mathcal{E}, x^\beta) = \lim_{b\to+0} \int_{\mathbb{R}^3} e^{-b|x|} \left(-\frac{\cos(\sqrt{c}|x|)}{4\pi|x|} \right) x^\beta \, dx, \quad \beta \in \mathbb{N}_0^3, \tag{5.8}$$

where the integral in the right-hand side of (5.8) is calculated with the help of converting it to the spherical coordinates.

Example 5.10. Let $a, c \in K$ and let c be an invertible element of the ring K. We find the fundamental solution of the linear differential operator $\mathcal{F} = \frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2} + cI$. We have

$$\mathcal{F} = c\left(I - \left(ac^{-1}\frac{\partial^2}{\partial x^2} - c^{-1}\frac{\partial}{\partial t} \right) \right).$$

Taking into account (5.2) and (5.5), we obtain the following expression for the fundamental solution of the operator $\mathcal{F}$:

$$\mathcal{E} = \mathcal{F}^{-1}\delta = \sum_{k=0}^{\infty} c^{-k-1} \left(a\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t} \right)^k \delta$$

$$= \sum_{k=0}^{\infty} c^{-k-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j a^{k-j} \frac{\partial^{2k-j}\delta}{\partial t^j \partial x^{2k-2j}}.$$

This implies that for every $s, l \in \mathbb{N}_0$,

$$(\mathcal{E}, x^m t^l) = \begin{cases} \frac{(2s)!(l+s)!}{s!} a^s c^{-l-s-1}, & m = 2s, \\ 0, & m = 2s+1. \end{cases}$$

This result can also be obtained with the help of the Laplace transform. Assume that the ring K contains the field of rational numbers $\mathbb{Q}$. We apply the Laplace transform to both sides of the equation

$$\frac{\partial \mathcal{E}}{\partial t} - a\frac{\partial^2 \mathcal{E}}{\partial x^2} + c\mathcal{E} = \delta(t,x).$$

By Proposition 4.5, we obtain

$$(c - az_2^2 - z_1)\widetilde{\mathcal{E}}(z_1, z_2) = 1.$$

Since c is an invertible element of the ring K, the polynomial $c - az_2^2 - z_1$ is an invertible element of the ring $K[[z_1, z_2]]$. Then

$$\widetilde{\mathcal{E}}(z_1, z_2) = \frac{1}{c - az_2^2 - z_1} = \sum_{k=0}^{\infty} c^{-k-1} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} z_1^j z_2^{2k-2j}.$$

Let $p(t,x) = x^m t^l$. Then $\widetilde{p}(z_1, z_2) = \frac{m!l!}{z_1^{l+1} z_2^{m+1}}$ and

$$\widetilde{\mathcal{E}}(z_1, z_2)\widetilde{p}(z_1, z_2) = \sum_{k=0}^{\infty} c^{-k-1} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} m!l! z_1^{j-l-1} z_2^{2k-2j-m-1}.$$

Thus, by Proposition 4.2, we obtain

$$(\mathcal{E}(t,x), p(t,x)) = \operatorname{Res}(\widetilde{\mathcal{E}}(z_1, z_2)\widetilde{p}(z_1, z_2)) = \begin{cases} \frac{(2s)!(l+s)!}{s!} a^s c^{-l-s-1}, & m = 2s, \\ 0, & m = 2s+1. \end{cases}$$

Now, let $K = \mathbb{R}$, $a > 0$ and $c > 0$. Notice that

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} t^l e^{-ct} x^m \frac{e^{-\frac{x^2}{4at}}}{\sqrt{4\pi at}}\, dx = \begin{cases} \frac{(2s)!(l+s)!}{s!} a^s c^{-l-s-1}, & m = 2s, \\ 0, & m = 2s+1. \end{cases}$$

Thus, in the space $\mathbb{R}[t,x]'$, the fundamental solution of the differential operator $\mathcal{F}$, regarded as a regular copolynomial, coincides with the function $\frac{\theta(t)e^{-ct}}{\sqrt{4\pi at}} e^{-\frac{x^2}{4at}}$.

Example 5.11. We find the fundamental solution of the differential operator $\mathcal{F} = \frac{\partial^2}{\partial x \partial t} + \frac{\partial}{\partial x} - \frac{\partial}{\partial t} - I$. By Definition 5.3, it is a solution of the differential equation

$$\frac{\partial^2 \mathcal{E}}{\partial x \partial t} + \frac{\partial \mathcal{E}}{\partial x} - \frac{\partial \mathcal{E}}{\partial t} - \mathcal{E} = \delta.$$

Then the sequence $C_{sl} = (\mathcal{E}, x^s t^l)$ $(l, s \in \mathbb{N}_0)$ is a solution of the following problem for the difference equation:

$$C_{sl} = slC_{s-1,l-1} - sC_{s-1,l} + lC_{s,l-1}, \quad s, l \in \mathbb{N},$$
$$C_{s0} = (-1)^{s+1} s!, \qquad C_{0l} = -l!, \qquad s, l \in \mathbb{N}_0.$$

This problem has the unique solution

$$C_{sl} = (-1)^{s+1} l! s!, \quad s, l = 0, 1, 2, \dots$$

We notice that

$$-\int_0^\infty dt \int_{-\infty}^0 e^{-t+x} t^l x^s \, dx = (-1)^{s+1} l! s!.$$

Therefore, in the space $\mathbb{R}[t, x]'$, the fundamental solution of the differential operator $\mathcal{F}$, regarded as a regular copolynomial, coincides with the function $-\theta(t)\theta(-x)e^{x-t}$.

Example 5.12. We find the fundamental solution of the transport operator $\mathcal{F} = \frac{\partial}{\partial t} + \sum_{i=1}^n s_i \frac{\partial}{\partial x_i} + I$, where $s_i \in K$. Taking into account (5.2) and (5.5), we obtain the expression for the fundamental solution of the differential operator $\mathcal{F}$:

$$\begin{aligned}
\mathcal{E}(t, x) = (\mathcal{F}^{-1}\delta)(t, x) &= \sum_{k=0}^{\infty} (-1)^k \left(\frac{\partial}{\partial t} + \sum_{i=1}^n s_i \frac{\partial}{\partial x_i} \right)^k \delta(t, x) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^k \binom{k}{j} \frac{\partial^j}{\partial t^j} \left(\sum_{i=1}^n s_i \frac{\partial}{\partial x_i} \right)^{k-j} \delta(t, x) \\
&= \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^k \binom{k}{j} \frac{\partial^j}{\partial t^j} \sum_{|\alpha|=k-j} \frac{|\alpha|!}{\alpha!} s^\alpha D^\alpha \delta(t, x), \quad s = (s_1, \dots, s_n).
\end{aligned}$$

Then, for every $l \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^n$, we have

$$\begin{aligned}
(\mathcal{E}, t^l x^\beta) &= \sum_{k=0}^{\infty} (-1)^k \sum_{j=0}^k \binom{k}{j} \sum_{|\alpha|=k-j} \frac{|\alpha|!}{\alpha!} s^\alpha (-1)^{|\alpha|+j} \frac{\partial^j}{\partial t^j} D^\alpha (t^l x^\beta) \Big|_{\substack{t=0 \\ x=0}} \\
&= s^\beta (|\beta| + l)!.
\end{aligned}$$

Now, let $K = \mathbb{R}$. Notice that

$$\int_0^\infty e^{-t} t^l (\delta(x - ts), x^\beta) \, dt = s^\beta (|\beta| + l)!, \quad l \in \mathbb{N}_0, \ \beta \in \mathbb{N}_0^n, \ s \in \mathbb{R}^n.$$

Thus a connection between the fundamental solution of the transport operator and the classical fundamental solution $\theta(t)e^{-t}\delta(x - ts)$ of this operator is established:

$$(\mathcal{E}, t^l x^\beta) = \int_0^\infty e^{-t} t^l \big(\delta(x - ts), x^\beta\big) \, dt, \quad l \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n.$$

Example 5.13. Now we consider the m-th order ordinary linear differential equation in the module $K[x]'$ of copolynomials of one variable

$$\sum_{j=0}^{m} a_j \frac{d^j u}{dx^j} = T, \tag{5.9}$$

where $a_j \in K, j = 0, \dots, m$, $a_0 \neq 0$, $a_m \neq 0$, and $T, u \in K[x]'$ are known and unknown copolynomials of one variable. This equation is a particular case of equation (5.1) with the differential operator $\mathcal{F} = \sum_{j=0}^{m} a_j \frac{d^j}{dx^j}$. Assume that a_0 is an invertible element of the ring K. Then, by Theorem 5.1, equation (5.9) has a unique solution. This solution has the form

$$u = a_0^{-1} \sum_{k=0}^{\infty} (I - a_0^{-1}\mathcal{F})^k T. \tag{5.10}$$

Now, for every $k \in \mathbb{N}_0$, we have

$$\begin{aligned}(I - a_0^{-1}\mathcal{F})^k &= (-1)^k \frac{d^k}{dx^k} \left(\sum_{j=1}^{m} a_0^{-1} a_j \frac{d^{j-1}}{dx^{j-1}} \right)^k \\ &= (-1)^k \sum_{|\gamma|=k} \frac{k!}{\gamma!} a_0^{-k} a_1^{\gamma_1} \cdots a_m^{\gamma_m} \frac{d^{k+\sum_{j=1}^m (j-1)\gamma_j}}{dx^{k+\sum_{j=1}^m (j-1)\gamma_j}}.\end{aligned}$$

Substituting this expression into (5.10), we obtain the following representation for the unique solution of equation (5.9):

$$u(x) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} (-1)^k \frac{k!}{\gamma!} a_0^{-k-1} a_1^{\gamma_1} \cdots a_m^{\gamma_m} T^{\left(k+\sum_{j=1}^m (j-1)\gamma_j\right)}(x).$$

(see [13, Section 4], where a similar formula was obtained in another situation). In particular, the first-order equation $a_1 u'(x) + a_0 u(x) = T(x)$ has the unique solution

$$u(x) = \sum_{k=0}^{\infty} (-1)^k a_0^{-k-1} a_1^k T^{(k)}(x),$$

and the second-order equation $a_2 u''(x) + a_1 u'(x) + a_0 u(x) = T(x)$ has the unique solution

$$\begin{aligned}u(x) &= \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^k \binom{k}{j} a_0^{-k-1} a_1^{k-j} a_2^j T^{(k+j)}(x) \\ &= \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} (-1)^k \binom{k}{j} a_0^{-k-1} a_1^{k-j} a_2^j T^{(k+j)}(x) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+j} \binom{k+j}{j} a_0^{-j-k-1} a_1^k a_2^j T^{(k+2j)}(x)\end{aligned}$$

$$= \sum_{s=0}^{\infty} \left(\sum_{j=0}^{[s/2]} (-1)^{s-j} \binom{s-j}{j} a_0^{j-s-1} a_1^{s-2j} a_2^{j} \right) T^{(s)}(x)$$

(see Formula (4.10) in [7]).

6. Fundamental solution of the Cauchy problem for a linear differential equation in the module of copolynomials

6.1. Formal power series over the module of copolynomials. The module of formal power series of the form $u(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k$ with coefficients $u_k(x) \in K[x_1, \dots, x_n]'$ will be denoted by $K[x_1, \dots, x_n]'[[t]]$.

The partial derivative with respect to t of the series $u(t,x) \in K[x_1, \dots, x_n]'[[t]]$ is defined by the formula

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} k u_k(x) t^{k-1}.$$

The partial derivatives D^α with respect to variables $x_1, \dots, x_n$ of the series $u(t,x) \in K[x_1, \dots, x_n]'[[t]]$ are defined as follows:

$$D^\alpha u(t,x) = \sum_{k=0}^{\infty} (D^\alpha u_k)(x) t^k.$$

The action of the K-linear operator $\mathcal{A} : K[x_1, \dots, x_n]' \to K[x_1, \dots, x_n]'$ on a formal power series $u(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k \in K[x_1, \dots, x_n]'[[t]]$ is defined coefficient-wisely:

$$(\mathcal{A}u)(t,x) = \sum_{k=0}^{\infty} (\mathcal{A}u_k)(x) t^k.$$

It is obvious that if $\mathcal{A}$ is an invertible K-linear operator on the module $K[x_1, \dots, x_n]'$, then its extension on the module $K[x_1, \dots, x_n]'[[t]]$ is also invertible.

We denote by $(u(t,x), p(x))$ the action of $u(t,x) \in K[x_1, \dots, x_n]'[[t]]$ on $p(x) \in K[x_1, \dots, x_n]$, which is defined coefficient-wisely:

$$(u(t,x), p(x)) = \sum_{k=0}^{\infty} (u_k(x), p(x)) t^k.$$

Thus, $(u(t,x), p(x)) \in K[[t]]$.

Definition 6.1. Let $u(t,x) = \sum_{k=0}^{\infty} u_k(x)t^k \in K[x_1, \dots, x_n]'[[t]]$. The *convolution* of a copolynomial $T \in K[x_1, \dots, x_n]'$ and a formal power series $u(t,x)$ is also defined coefficient-wisely:

$$(T * u)(t,x) = \sum_{k=0}^{\infty} (T * u_k(x)) t^k,$$

Thus, $(T * u)(t,x) \in K[x_1, \dots, x_n]'[[t]]$.

6.2. The Cauchy problem for a linear partial differential equation in the module of copolynomials. Let $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ be a linear differential equation of infinite order on $K[x_1, \dots, x_n]'$ with coefficients $a_\alpha \in K$. In the module $K[x_1, \dots, x_n]'[[t]]$, we consider the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = (\mathcal{F}u)(t,x), \tag{6.1}$$

$$u(0,x) = Q(x) \in K[x_1, \dots, x_n]'. \tag{6.2}$$

The following example shows that if a_0 is invertible, then this Cauchy problem may have no solutions.

Example 6.2. Let $K = \mathbb{Z}$, $\mathcal{F} = I$ and $Q(x) = \delta(x)$. Then the Cauchy problem (6.1), (6.2) is written in the form

$$\frac{\partial u(t,x)}{\partial t} = u(t,x), \tag{6.3}$$

$$u(0,x) = \delta(x). \tag{6.4}$$

Any solution of this problem can be represented in the form of a formal power series $u(t,x) = \sum_{k=0}^{\infty} u_k(x) t^k$ with coefficients $u_k(x) \in \mathbb{Z}[x_1, \dots, x_n]'$. Substituting this representation into (6.3), (6.4), we get

$$u_0(x) = \delta(x), \quad (k+1)u_{k+1}(x) = u_k(x), \quad k = 0, 1, 2, \dots$$

This implies $2(u_2, 1) = 1$, which contradicts the condition $(u_2, 1) \in \mathbb{Z}$.

The following theorem shows that in the case $a_0 = 0$ the Cauchy problem (6.1), (6.2) has a unique solution.

Theorem 6.3. *Let $a_0 = 0$ and let the ring K be of characteristic 0. Then, for any copolynomial $Q \in K[x_1, \dots, x_n]'$, the formal power series*

$$u(t,x) = \sum_{k=0}^{\infty} \frac{(\mathcal{F}^k Q)(x)}{k!} t^k \tag{6.5}$$

is well-defined and it is a unique solution of the Cauchy problem (6.1), (6.2). Furthermore, for every $t \in K$, the series (6.5) converges in the topology of the module $K[x_1, \dots, x_n]'$.

Proof. First, we show that $K[x_1, \dots, x_n]'$ is a torsion-free $\mathbb{Z}$-module (see the definition in [1, Section VII, §2]). Suppose that an element $T \in K[x_1, \dots, x_n]'$ satisfies the equality $kT = 0$ for some natural k. Then $(kT, p) = k(T, p) = 0$ for every polynomial $p \in K[x_1, \dots, x_n]$. Since the integral domain K is of characteristic 0, we have $(T, p) = 0$, i.e., $T = 0$.

Further, we prove that the formal power series (6.5) is well-defined and it is a unique solution of the Cauchy problem (6.1), (6.2). Since $a_0 = 0$, we obtain the following representation for the operator $\mathcal{F}$:

$$\mathcal{F} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \mathcal{G}_j,$$

where $\mathcal{G}_j$ $(j = 1, \dots, n)$ are some differential operators.Therefore,

$$\mathcal{F}^k = k! \sum_{|\alpha|=k} \frac{D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n}}{\alpha!}, \quad k \in \mathbb{N}.$$

Since copolynomials $\frac{D^\alpha \mathcal{G}_1^{\alpha_1} \ldots \mathcal{G}_n^{\alpha_n} Q}{\alpha!}$ are well-defined (see (2.2)), the element $\mathcal{F}^k Q$ in the module $K[x_1, \dots, x_n]'$ is divided by $k!$. In its turn, the module $K[x_1, \dots, x_n]'$ is a torsion-free $\mathbb{Z}$-module and we get

$$\frac{\mathcal{F}^k Q}{k!} = \sum_{|\alpha|=k} \frac{D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} Q}{\alpha!}, \quad k \in \mathbb{N}. \tag{6.6}$$

By Theorem 2.3 [6], the series (6.5) is well-defined, the Cauchy problem (6.1), (6.2) has a unique solution and this solution has the form (6.5). Now we show that for any $t \in K$ the series (6.5) converges in $K[x_1, \dots, x_n]'$. We consider the partial sums $u_N(t, x) = \sum_{k=0}^{N} \frac{(\mathcal{F}^k Q)(x)}{k!} t^k$ of this series and show that they are stabilized on every polynomial. By equalities (2.2) and (6.6), for any $p \in K[x_1, \dots, x_n]$ we have

$$\begin{aligned}
(u_N(t, x), p(x)) &= \sum_{k=0}^{N} \sum_{|\alpha|=k} \left(\frac{D^\alpha \mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} Q}{\alpha!}, p \right) t^k \\
&= \sum_{k=0}^{N} \sum_{|\alpha|=k} (-1)^{|\alpha|} \left(\mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} Q, \frac{D^\alpha p}{\alpha!} \right) t^k \\
&= \sum_{k=0}^{m} \sum_{|\alpha|=k} (-1)^{|\alpha|} \left(\mathcal{G}_1^{\alpha_1} \cdots \mathcal{G}_n^{\alpha_n} Q, \frac{D^\alpha p}{\alpha!} \right) t^k \\
&= \sum_{k=0}^{m} \left(\frac{(\mathcal{F}^k Q)(x)}{k!}, p(x) \right) t^k, \quad N \geq m,
\end{aligned}$$

where $m = \deg p$. The theorem is proved. □

Remark 6.4. The condition that K has the characteristic 0 is essential for a uniqueness of the solution of the Cauchy problem (6.1), (6.2) even for $a_0 = 0$. Indeed, let $K = \mathbb{Z}/2\mathbb{Z}$. This is a field of characteristic 2. Then the Cauchy problem (6.1), (6.2) for $Q(x) = 0$ has a solution $u(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k$, where $u_0(x) = u_1(x) = 0$, $u_{2k}(x)$ is an arbitrary element of $K[x_1, \dots, x_n]'$ and $u_{2k+1}(x) = (\mathcal{F} u_{2k})(x)$ for any $k \in \mathbb{N}$. Therefore the considered Cauchy problem has non-trivial solutions.

Example 6.2 shows that the Cauchy problem (6.1), (6.2) may have no solutions when $a_0 \neq 0$. The following theorem shows that under an additional restriction on the ring K there exists a solution of this problem even when $a_0 \neq 0$.

Theorem 6.5. *The following conditions are equivalent:*

1. *For any linear differential operator* $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ *of infinite order with coefficients* $a_\alpha \in K$ *and for any copolynomial* $Q \in K[x_1, \dots, x_n]'$ *there exists a solution of the Cauchy problem* (6.1), (6.2).
2. *The ring* K *contains the field of rational numbers.*

Furthermore, a solution of this Cauchy problem is unique and it has the form (6.5).

Proof. 2⇒1. Since the ring K contains the field of rational numbers, this ring is of characteristic 0 and $\frac{1}{k!} \in K$ for any $k \in \mathbb{N}$. Arguing as in the proof of Theorem 6.3, we obtain that the module $K[x_1, \dots, x_n]'$ is also a torsion-free $\mathbb{Z}$-module and the element $\mathcal{F}^k Q$ is divided by $k!$ in the module $K[x_1, \dots, x_n]'$ for any $k \in \mathbb{N}$. By Theorem 2.3 [6], the series (6.5) is well-defined, the Cauchy problem (6.1), (6.2) has a unique solution and this solution has the form (6.5).

1⇒2. Suppose that for any copolynomial $Q \in K[x_1, \dots, x_n]'$ and for any differential operator $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ of infinite order with coefficients $a_\alpha \in K$ there exists a solution $u(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k$ of the Cauchy problem (6.1), (6.2). We put $a_0 = 1$ and $Q(x) = \delta(x)$. Then coefficients $u_k(x)$ of the corresponding solution $u(t, x)$ satisfy the equalities

$$u_0(x) = \delta(x), \quad (k+1)u_{k+1}(x) = (\mathcal{F}u_k)(x), \quad k = 0, 1, 2, \dots$$

Therefore, $(k+1)(u_{k+1}, 1) = (u_k, 1)$ and $(k+1)!(u_{k+1}, 1) = k!(u_k, 1) = 1$ for any $k \in \mathbb{N}_0$. This implies that elements $k! \in \mathbb{N}$, regarded as elements of K, are invertible. Therefore $\frac{1}{k!} \in K$, $k \in \mathbb{N}$. Then K contains the field of rational numbers. □

Theorems 6.3 and 6.5 lead to the following assertion.

Corollary 6.6. *Assume that one of the following two conditions is satisfied:*

1. *The ring* K *is of characteristic 0 and* $a_0 = 0$.
2. *The ring* K *contains the field of rational numbers.*

Then the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = (\mathcal{F}u)(t, x), \quad u(0, x) = \delta(x),$$

has a unique solution in the module $K[x_1, \dots, x_n]'[[t]]$. *This solution has the form*

$$\mathcal{E}_C(t, x) = \sum_{k=0}^{\infty} \frac{(\mathcal{F}^k \delta)(x)}{k!} t^k. \tag{6.7}$$

Definition 6.7. The formal power series $\mathcal{E}_C(t, x) \in K[x_1, \dots, x_n]'[[t]]$ defined in (6.7) is called the *fundamental solution of the Cauchy problem* (6.1), (6.2).

Arguing as in the proof of Theorem 6.5, we obtain the following criterion of the existence of a fundamental solution of the Cauchy problem (6.1), (6.2) for any differential operator $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ of infinite order.

Theorem 6.8. *The following conditions are equivalent:*

1. *There exists a fundamental solution of the Cauchy problem* (6.1), (6.2) *for any linear differential operator* $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ *of infinite order with coefficients* $a_\alpha \in K$.
2. *The ring* K *contains the field of rational numbers.*

Moreover, a fundamental solution of this Cauchy problem is unique and it has the form (6.7).

The following assertion shows that, under the assumptions of Corollary 6.6, the unique solution of the Cauchy problem (6.1), (6.2) is represented as the convolution of the fundamental solution $\mathcal{E}_C(t,x)$ and the copolynomial Q.

Theorem 6.9. *Let the assumptions of Corollary* 6.6 *hold. Then a unique solution of the Cauchy problem* (6.1), (6.2) *can be represented in the form*

$$u(t,x) = \mathcal{E}_C(t,x) * Q.$$

Proof. Indeed, a unique solution of the Cauchy problem (6.1), (6.2) is defined by (6.5). On the other hand, in view of Definition 6.1 and Theorem 3.2, we have

$$\mathcal{E}_C(t,x) * Q = \sum_{k=0}^{\infty} \frac{(\mathcal{F}^k\delta) * Q}{k!} t^k = \sum_{k=0}^{\infty} \frac{\mathcal{F}^k(\delta * Q)}{k!} t^k = \sum_{k=0}^{\infty} \frac{\mathcal{F}^k Q}{k!} t^k = u(t,x)$$

(see also Example 2.12). □

Corollary 6.10. *Let the assumptions of Theorem* 6.3 *hold. Then, for every fixed* $t \in K$, *the sum of the series* (6.5) *continuously depends on* Q *in the topology of the module* $K[x_1,\dots,x_n]'$.

Proof. Indeed, for every $t \in K$, the fundamental solution $\mathcal{E}_C(t,x)$ is a copolynomial and by Theorem 6.9 the sum of the series (6.5) can be represented as a convolution of copolynomials $\mathcal{E}_C(t,x)$ and $Q(x)$. Now the assertion of the corollary follows from Theorem 2.14. □

Example 6.11. Let the ring K be of characteristic 0 and let $a \in K$. In the module $K[x_1,\dots,x_n]'[[t]]$, we consider the heat equation

$$\frac{\partial u(t,x)}{\partial t} = a\triangle u(t,x), \qquad \triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \tag{6.8}$$

which is a particular case of equation (6.1) with the differential operator $\mathcal{F} = a\triangle$. The assumptions of Theorems 6.3, 6.9 and Corollary 6.6 are satisfied. By Theorem 6.3, for any $Q \in K[x_1,\dots,x_n]'$, the Cauchy problem (6.8), (6.2) has a unique solution and this solution has the form (see (6.5)):

$$u(t,x) = \sum_{k=0}^{\infty} a^k \frac{\triangle^k Q}{k!} t^k. \tag{6.9}$$

By Corollary 6.6, the fundamental solution of this Cauchy problem exists and has the form

$$\mathcal{E}_C(t,x) = \sum_{k=0}^{\infty} a^k \frac{\triangle^k \delta}{k!} t^k. \tag{6.10}$$

As in Example 5.9, for any multi-index $\beta \in \mathbb{N}_0^n$ and $k \in \mathbb{N}$, we obtain

$$\frac{\triangle^k x^\beta}{k!} = \sum_{|\alpha|=k} \frac{D^{2\alpha} x^\beta}{\alpha!}.$$

Therefore,

$$\left(\frac{\triangle^{|\alpha|}\delta}{|\alpha|!}, x^\beta\right) = \left(\delta, \frac{\triangle^{|\alpha|} x^\beta}{|\alpha|!}\right) = \begin{cases} \frac{(2\alpha)!}{\alpha!}, & \beta = 2\alpha, \\ 0, & \beta \neq 2\alpha. \end{cases}$$

Substituting this expression into (6.10), we obtain the following representation for the fundamental solution of the Cauchy problem (6.8), (6.2):

$$(\mathcal{E}_C(t,x), x^\beta) = \begin{cases} \frac{(2\alpha)!}{\alpha!}(at)^{|\alpha|}, & \beta = 2\alpha, \\ 0, & \beta \neq 2\alpha. \end{cases} \tag{6.11}$$

Now, let $K = \mathbb{R}$ and $a > 0$. We show that in the space $\mathbb{R}[x_1, \dots, x_n]'$ for every $t > 0$ the sum of the series (6.10), regarded as a regular copolynomial, has the form

$$\sum_{k=0}^{\infty} a^k \frac{\triangle^k \delta}{k!} t^k = \frac{1}{(\sqrt{4\pi a t})^n} e^{-\frac{|x|^2}{4at}}, \quad |x|^2 = \sum_{j=1}^{n} x_j^2.$$

For this purpose, we first note that

$$\frac{1}{\sqrt{4\pi a t}} \int_{-\infty}^{\infty} y^k e^{-\frac{y^2}{4at}}\, dy = \begin{cases} \frac{(2l)!}{l!} a^l t^l, & k = 2l, \\ 0, & k = 2l+1. \end{cases}$$

Now, taking into account (6.11), we obtain

$$(\mathcal{E}_C(t,x), x^\beta) = \begin{cases} \frac{(2\alpha)!}{\alpha!}(at)^{|\alpha|}, & \beta = 2\alpha \\ 0, & \beta \neq 2\alpha \end{cases} = \frac{1}{(\sqrt{4\pi a t})^n} \int_{\mathbb{R}^n} x^\beta e^{-\frac{|x|^2}{4at}}\, dx.$$

Example 6.12. Assume that the ring K contains the field of rational numbers, $a \in K$ and $Q(x) \in K[x_1, \dots, x_n]'$. In the module $K[x_1, \dots, x_n]'[[t]]$, we consider the Cauchy problem for the inhomogeneous heat equation

$$\frac{\partial v(t,x)}{\partial t} = a \triangle v(t,x) + Q(x), \tag{6.12}$$

$$v(0,x) = 0. \tag{6.13}$$

It is easy to see that if $v(t,x) \in K[x_1, \dots, x_n]'[[t]]$ is a solution of the Cauchy problem (6.12), (6.13), then the formal power series

$$u(t,x) = \frac{\partial v(t,x)}{\partial t} \tag{6.14}$$

is a solution of the Cauchy problem (6.8), (6.2). The unique solution of the Cauchy problem (6.8), (6.2) has the form (6.9) (see Example 6.11). From (6.9), (6.14) and (6.13), we uniquely restore the formal power series $v(t,x)$,

$$v(t,x) = \sum_{k=0}^{\infty} a^k \frac{\triangle^k Q}{(k+1)!} t^{k+1}. \tag{6.15}$$

Since

$$\frac{\triangle^k Q}{(k+1)!} = \sum_{|\alpha|=k} \frac{D^{2\alpha} Q}{\alpha!(k+1)} \in K[x_1,\dots,x_n]', \quad k \in \mathbb{N},$$

this series is well-defined. Substituting (6.15) into (6.12) and (6.13), we see that the series $v(t,x)$ is a solution of the Cauchy problem (6.12), (6.13). The uniqueness of a solution of the Cauchy problem (6.12), (6.13) follows from Theorem 6.5.

Example 6.13. Assume that K is of characteristic 0 and $s_1,\dots,s_n \in K$. We find the fundamental solution of the Cauchy problem for the transport equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} s_j \frac{\partial u}{\partial x_j}. \tag{6.16}$$

Equation (6.16) is a particular case of equation (6.1) with the differential operator $\mathcal{F} = \sum_{j=1}^n s_j \frac{\partial}{\partial x_j}$. By Corollary 6.6, the fundamental solution of the Cauchy problem (6.16),(6.2) has the form

$$\mathcal{E}_C(t,x) = \sum_{k=0}^{\infty} \frac{\left(\sum_{j=1}^n s_j \frac{\partial}{\partial x_j}\right)^k \delta}{k!} t^k = \sum_{k=0}^{\infty} t^k \sum_{|\alpha|=k} \frac{s^\alpha D^\alpha \delta}{\alpha!},$$

where $s = (s_1,\dots,s_n)$. Since for any $\beta \in \mathbb{N}_0^n$,

$$\left(\frac{D^\alpha \delta}{\alpha!}, x^\beta\right) = \begin{cases} (-1)^{|\beta|}, & \beta = \alpha, \\ 0, & \beta \neq \alpha, \end{cases}$$

we obtain by virtue the definition of the shift of a copolynomial

$$\big(\mathcal{E}_C(t,x), x^\beta\big) = (-1)^{|\beta|} t^{|\beta|} s^\beta = \big(\delta(x+ts), x^\beta\big).$$

Hence the fundamental solution of the Cauchy problem for equation (6.16) coincides with the copolynomial $\delta(x+ts)$.

6.3. Connections between fundamental solutions. We assume that $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ is a linear differential operator of infinite order on $K[x_1,\dots,x_n]'$ with coefficients $a_\alpha \in K$ and a_0 is an invertible element of the ring K. We also assume that the ring K contains the field of rational numbers. By Corollaries 5.2 and 6.6, the differential operator $\mathcal{F}$ and the Cauchy problem (6.1), (6.2) have the fundamental solutions $\mathcal{E}(x)$ and $\mathcal{E}_C(t,x)$. Furthermore, by

Theorem 5.1, the operator $\mathcal{F}$ is invertible. Theorem 5.1 and Corollary 5.2 imply that the differential operator $\frac{\partial}{\partial t} - \mathcal{F} : K[t, x_1, \dots, x_n]' \to K[t, x_1, \dots, x_n]'$ is also invertible and this operator also has a fundamental solution which will be denoted by $\tilde{\mathcal{E}}(t, x)$.

At first, we give the connections between fundamental solutions $\mathcal{E}(x)$ and $\mathcal{E}_C(t, x)$. By definitions of the fundamental solutions of an operator and a Cauchy problem (see equalities (5.5) and (6.7)), we obtain the formula

$$\mathcal{E}(x) = (\mathcal{F}^{-1}\mathcal{E}_C)(0, x),$$

which expresses the fundamental solution of the operator $\mathcal{F}$ through the fundamental solution of the Cauchy problem (6.1), (6.2). With the help of (5.5) and (6.7), we obtain the formula

$$\mathcal{E}_C(t, x) = \sum_{k=0}^{\infty} \frac{(\mathcal{F}^{k+1}\mathcal{E})(x)}{k!} t^k,$$

which expresses the fundamental solution of the Cauchy problem (6.1), (6.2) through the fundamental solution of the operator $\mathcal{F}$.

Now we establish the connections between fundamental solutions $\tilde{\mathcal{E}}(t, x)$ and $\mathcal{E}(x)$. We have

$$\frac{\partial}{\partial t} - \mathcal{F} = \mathcal{F}\left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right).$$

Then the operator $\left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right) : K[t, x_1, \dots, x_n]' \to K[t, x_1, \dots, x_n]'$ is invertible and we have the operator equality

$$\left(\frac{\partial}{\partial t} - \mathcal{F}\right)^{-1} = \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right)^{-1} \mathcal{F}^{-1}. \tag{6.17}$$

Applying (6.17) to the copolynomial $\delta(t, x) = \delta(t) \otimes \delta(x) \in K[t, x_1, \dots, x_n]'$, we obtain the formulas

$$\begin{aligned}
\tilde{\mathcal{E}}(t, x) &= \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right)^{-1} \mathcal{F}^{-1}(\delta(t) \otimes \delta(x)) \\
&= \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right)^{-1} (\delta(t) \otimes (\mathcal{F}^{-1}\delta)(x)) \\
&= \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right)^{-1} (\delta(t) \otimes \mathcal{E}(x)) \\
&= \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right)^{-1} (\delta(t) \otimes (\mathcal{F}^{-1}\mathcal{E}_C)(0, x)),
\end{aligned}$$

which express the fundamental solution of the operator $\frac{\partial}{\partial t} - \mathcal{F}$ through either the fundamental solution of the operator $\mathcal{F}$ or the fundamental solution of the Cauchy problem (6.1), (6.2). Moreover, we obtain the formula

$$\delta(t) \otimes \mathcal{E}(x) = \left(\mathcal{F}^{-1}\frac{\partial}{\partial t} - I\right) \tilde{\mathcal{E}}(t, x),$$

which implicitly expresses the fundamental solution of the operator $\mathcal{F}$ through the fundamental solution of the operator $\frac{\partial}{\partial t} - \mathcal{F}$.

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S.L. Gefter,
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: gefter@karazin.ua

A.L. Piven',
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: aleksei.piven@karazin.ua

Диференціальні рівняння з частинними похідними у модулі кополіномів над комутативним кільцем

S.L. Gefter and A.L. Piven'

Нехай K є довільною комутативною областю цілісності з одиницею. Досліджуються кополіноми n змінних, тобто K-лінійні відображення з кільця поліномів $K[x_1, \ldots, x_n]$ у кільце K. Доводиться теорема існування та єдиності розв'язку для лінійного диференціального рівняння нескінченного порядку, яку можна розглядати як алгебраїчну версію класичної теореми Мальгранжа– Separate Еренпрайса існування фундаментального розв'язку лінійного диференціального оператора зі сталими коефіцієнтами. Знайдено фундаментальні розв'язки лінійних диференціальних операторів нескінченного порядку та показано, що єдиний розв'язок відповідного неоднорідного рівняння може бути поданий як згортка фундаментального розв'язку цього оператора та правої частини. Також доведено теорему існування та єдиності розв'язку задачі Коші для деяких лінійних диференціальних рівнянь у модулі формальних степеневих рядів із кополіноміальними коефіцієнтами.

Ключові слова: кополіном, фундаментальний розв'язок, згортка, δ-функція, диференціальний оператор нескінченного порядку, задача Коші, перетворення Лапласа