Fourier Uniqueness and Non-Uniqueness Pairs

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To Vladimir Marchenko on the occasion of his 102nd birthday

Motivated by recent works by Radchenko and Viazovska and by Ramos and Sousa, we find sufficient conditions for a pair of discrete subsets of the real line to be a uniqueness or a non-uniqueness pair for the Fourier transform. These conditions are close to each other. The uniqueness result can be upgraded to an interpolation formula, which in turn produces an abundance of discrete measures with discrete Fourier transform.

Key words: Fourier uniqueness pairs, Fourier interpolation

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1. Introduction and main results

1.1. Uniqueness and non-uniqueness pairs. Let H be a space of functions on $\mathbb{R}$. For instance, H can be $L^2(\mathbb{R})$ or the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$. Later we will also consider other spaces H. We define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi \mathrm{i} \xi x} \, \mathrm{d}x.$$

Then its inverse is given by

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi \mathrm{i} \xi x} \, \mathrm{d}\xi.$$

Definition 1.1 (uniqueness and non-uniqueness pairs)**.** We call a pair (Λ, M) of sets $\Lambda, M \subset \mathbb{R}$ a uniqueness pair (UP) for H if

$$f \in \mathsf{H}, \ f|_{\Lambda} = 0, \ \widehat{f}|_M = 0 \implies f = 0.$$

Otherwise, (Λ, M) is a non-uniqueness pair (NUP) for H.

This definition is invariant under the transformations

$$(\Lambda, M) \mapsto (t\Lambda, t^{-1}M), \quad t \neq 0,$$

and

$$(\Lambda, M) \mapsto (\Lambda + r, M + s), \quad r, s \in \mathbb{R},$$

provided that, for every $f \in \mathsf{H}$ and every $t \neq 0, r, s \in \mathbb{R}$, the space H also contains the functions $x \mapsto f(tx)$, $x \mapsto f(x + r)$, and $x \mapsto e^{2\pi \mathrm{i} s x} f(x)$.

The most classical examples of uniqueness and non-uniqueness pairs (for instance, in the space $\mathcal{S}$) are

(UP) $\Lambda = \mathbb{Z}$, $M = \mathbb{R} \setminus \left(-\frac{1}{2}, \frac{1}{2}\right)$,

(NUP) $\Lambda = c\mathbb{Z}$, $M = \mathbb{R} \setminus \left(-\frac{1}{2}, \frac{1}{2}\right)$ with any $c > 1$.

The condition $M = \mathbb{R} \setminus \left(-\frac{1}{2}, \frac{1}{2}\right)$ implies that the spectrum of f is contained in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, that is, f is an entire function of exponential type at most π. The famous Beurling–Malliavin theorem [4, 5] can be viewed as a description of the sets Λ such that $(\Lambda, \mathbb{R} \setminus (-a, a))$ is a UP for each $a < \frac{1}{2}$ and a NUP for each $a > \frac{1}{2}$.

Another classical example of a uniqueness pair in $L^2(\mathbb{R})$ is due to Benedicks [3] and Amrein and Berthier [2]:

(UP) Λ, M have complements of finite Lebesgue measure.

Later, the second-named author [19, Section 2.5] found "a one-sided version" of this result:

(UP) $\Lambda \cap \mathbb{R}_-$, $M \cap \mathbb{R}_-$ have complements of finite Lebesgue measure.

In all these examples, at least one of the sets Λ, M in the uniqueness pair was not discrete.

The situation changes if we assume that the functions in H, as well as their Fourier transforms, are well-defined pointwise. In 2017 Radchenko and Viazovska [23] found a surprising example of a uniqueness pair for the set of *even* functions in $\mathcal{S}$:

(UP) $\Lambda = M = \{0, \pm\sqrt{1}, \pm\sqrt{2}, \pm\sqrt{3}, \ldots\}$.

This follows from their remarkable interpolation formula, which expresses any function $f \in \mathcal{S}$ by the values of f and $\widehat{f}$ on this set and two more values $f'(0)$ and $\widehat{f}'(0)$. Radchenko and Viazovska used an arithmetic approach based on the theory of modular forms. This makes their construction quite rigid.

Our work was inspired by a work of Ramos and Sousa [24], in which they developed a purely analytic approach to the construction of discrete uniqueness pairs. Starting with a function $f \in \mathcal{S}$ such that $f|_\Lambda$ and $\widehat{f}|_M$ vanish, their idea was, first, to estimate the size of the derivatives $f^{(j)}$ and $\widehat{f}^{(j)}$, $j = 1, 2, \ldots$, to show that f and $\widehat{f}$ have an analytic continuation to $\mathbb{C}$ with certain bounds, and then, using entire function theory, that f is a zero function. Their approach used a repeated application of the Rolle's theorem, and required unnecessarily restrictive assumptions on the denseness of Λ and M. For instance, they were able to show that the pair of the sets $\Lambda = M = \{\pm n^\alpha \colon n \in \mathbb{N}\}$ is a UP for the Schwartz space $\mathcal{S}$ if $\alpha < 1 - \frac{\sqrt{2}}{2}$, but their techniques were not strong enough to cover the full range $\alpha < \frac{1}{2}$.

1.2. Supercritical and subcritical pairs.

Definition 1.2 (supercritical and subcritical pairs)**.** Let the sequences $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ and $M = (\mu_j)_{j \in \mathbb{Z}}$ be ordered so that

$$\ldots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} \ldots, \quad \lim_{j \to \pm\infty} \lambda_j = \pm\infty,$$

$$\ldots < \mu_{j-1} < \mu_j < \mu_{j+1} \ldots, \quad \lim_{j \to \pm\infty} \mu_j = \pm\infty.$$

Given $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we call a pair (Λ, M) supercritical if

$$\limsup_{|j| \to \infty} |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) < \frac{1}{2}, \quad \limsup_{|j| \to \infty} |\mu_j|^{q-1}(\mu_{j+1} - \mu_j) < \frac{1}{2},$$

and subcritical if

$$\liminf_{|j| \to \infty} |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) > \frac{1}{2}, \quad \liminf_{|j| \to \infty} |\mu_j|^{q-1}(\mu_{j+1} - \mu_j) > \frac{1}{2}.$$

Example. Consider the sequences

$$\Lambda = \left\{ \text{sign}(j) \left(\frac{p}{2} a|j| \right)^{1/p} \right\}_{j \in \mathbb{Z}}, \quad M = \left\{ \text{sign}(j) \left(\frac{q}{2} a|j| \right)^{1/q} \right\}_{j \in \mathbb{Z}},$$

$\frac{1}{p} + \frac{1}{q} = 1$, $a > 0$, and note that

$$\lim_{|j| \to \infty} |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) = \frac{1}{2}a, \quad \lim_{|j| \to \infty} |\mu_j|^{q-1}(\mu_{j+1} - \mu_j) = \frac{1}{2}a.$$

Hence, for $a < 1$, the pair (Λ, M) is supercritical, while for $a > 1$, the pair (Λ, M) becomes subcritical. The case $p = q = 2$ and $a = 1$ corresponds to the pair considered by Radchenko and Viazovska.

Theorem 1.3. *Suppose that $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(i) *any supercritical pair (Λ, M) is a uniqueness pair for the Schwartz space $\mathcal{S}$;*
(ii) *any subcritical pair (Λ, M) is a non-uniqueness pair for the Schwartz space $\mathcal{S}$.*

Actually, we prove more. We show that the uniqueness part holds for functions from a much larger Hilbert space

$$\mathcal{H} = \left\{ f \colon f, \widehat{f} \in H_1, \ \|f\|_{\mathcal{H}}^2 = \|f\|_{H_1}^2 + \|\widehat{f}\|_{H_1}^2 \right\},$$

where H_1 is the Sobolev space of functions $f \in L^2(\mathbb{R})$ such that

$$\|f\|_{H_1}^2 = \int_{\mathbb{R}} (1 + \xi^2)|\widehat{f}(\xi)|^2 \, \mathrm{d}\xi < \infty.$$

Note that the space $\mathcal{H}$ is complete and that in this space the point evaluation is bounded for both f and $\widehat{f}$. To see that $\mathcal{H}$ is complete, we let (f_n) be a Cauchy sequence in $\mathcal{H}$. Then (f_n) and $(\widehat{f}_n)$ are Cauchy sequences in H_1. Hence, $f_n \to f$, $\widehat{f}_n \to g$ in H_1, and therefore, in $L^2(\mathbb{R})$. Thus, $\widehat{f} = g$, and $\widehat{f}_n \to \widehat{f}$ in H_1. We conclude that $f \in \mathcal{H}$ and $f_n \to f$ in $\mathcal{H}$. The boundedness of the point evaluation in $\mathcal{H}$ is also straightforward: using the Fourier inversion and the Cauchy–Schwarz inequality, we get

$$|f(x)| \le \int_{\mathbb{R}} \sqrt{1 + \xi^2} \, |\widehat{f}(\xi)| \, \frac{\mathrm{d}\xi}{\sqrt{1 + \xi^2}} \le \sqrt{\pi} \|f\|_{H_1}, \quad x \in \mathbb{R},$$

and similarly, $|\widehat{f}(\xi)| \leq \sqrt{\pi}\|\widehat{f}\|_{H_1}$, $\xi \in \mathbb{R}$.

In its turn, the non-uniqueness part of Theorem 1.3 requires less stringent assumptions on the pair (Λ, M) (see the remark in the beginning of Section 7) and holds for a much smaller Gelfand–Shilov space

$$\mathcal{S}(p,q) = \left\{f\colon \exists c = c_f > 0 \text{ s.t. } \int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p}\,\mathrm{d}x < \infty, \ \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 e^{c|\xi|^q}\,\mathrm{d}\xi < \infty\right\}.$$

The elements of the Gelfand–Shilov space are entire functions of finite order of growth (f has order p and $\widehat{f}$ has order q).

Remark 1.4. Taking into account the invariance of uniqueness and non-uniqueness pairs under dilations, one can formally generalize Theorem 1.3 as follows.

Suppose that, for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\limsup_{|j| \to \infty} |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) = a,$$

and

$$\limsup_{|j| \to \infty} |\mu_j|^{q-1}(\mu_{j+1} - \mu_j) = b,$$

with

$$a^{1/p}b^{1/q} < \frac{1}{2}.$$

Then (Λ, M) is a uniqueness pair.

Similarly, if

$$\liminf_{|j| \to \infty} |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) = a,$$

and

$$\liminf_{|j| \to \infty} |\mu_j|^{q-1}(\mu_{j+1} - \mu_j) = b,$$

with

$$a^{1/p}b^{1/q} > \frac{1}{2},$$

then (Λ, M) is a non-uniqueness pair.

1.3. Quantitative uniqueness. To state the rest of our results, we introduce a scale of Hilbert spaces $\mathcal{H}_{s,p,q}$.

Definition 1.5 (Hilbert spaces $\mathcal{H}_{s,p,q}$). For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $s > 0$, let $\mathcal{H}_{s,p,q}$ be the Hilbert space

$$\mathcal{H}_{s,p,q} = \left\{f\colon f \in H_{qs}, \widehat{f} \in H_{ps},\ \|f\|_{\mathcal{H}_{s,p,q}}^2 = \|f\|_{H_{ps}}^2 + \|\widehat{f}\|_{H_{qs}}^2\right\},$$

where H_t is the Sobolev space of functions $f \in L^2(\mathbb{R})$ such that

$$\|f\|_{H_t}^2 = \int_{\mathbb{R}} (1 + |\xi|^{2t})|\widehat{f}(\xi)|^2\,\mathrm{d}\xi < \infty.$$

Note that $\mathcal{H}_{\frac{1}{2},2,2} = \mathcal{H}$.

We will use the spaces $\mathcal{H}_{s,p,q}$ only for the values of s such that $s \min(p, q) \geq 1$ so that $\mathcal{H}_{s,p,q} \subset \mathcal{H}$.

We also need to introduce the notion of p-separated sequences, that is, the sequences for which the consecutive elements cannot be too close.

Definition 1.6 (p-separated sequences)**.** Let $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ be a sequence ordered so that

$$\ldots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} \ldots, \quad \lim_{j \to \pm\infty} \lambda_j = \pm\infty.$$

We call Λ a p-separated sequence if there exists $c > 0$ such that for all $j \in \mathbb{Z}$ we have

$$\lambda_{j+1} - \lambda_j \geq \frac{c}{(1 + \min(|\lambda_j|, |\lambda_{j+1}|))^{p-1}}.$$

We will tacitly use the following simple claim:

Claim 1.7. *For any supercritical pair* (Λ, M)*, there exists a supercritical pair* (Λ', M') *such that* $\Lambda' \subset \Lambda$*,* $M' \subset M$*, and* Λ' *is p-separated while M' is q-separated.*

Proof. We only show how to select the needed subsequence Λ'_+ from $\Lambda \cap (0, +\infty) = \{\lambda_k, \lambda_{k+1}, \ldots\}$, the other three cases are handled similarly. Let

$$a = \limsup |\lambda_j|^{p-1}(\lambda_{j+1} - \lambda_j) < \frac{1}{2}$$

and consider a small number $\delta > 0$. We build the set Λ'_+ iteratively, starting with $\Lambda'_+ = \{\lambda_k\}$. We will be looking at the elements of $\Lambda \cap (0, +\infty)$ in order. Assume that we are currently looking at λ_m and let λ_n be the last element that we added to Λ'_+. We add λ_m to Λ'_+ if and only if $\lambda_n^{p-1}(\lambda_m - \lambda_n) \geq \delta$. Clearly, the resulting set Λ'_+ is p-separated. We show that if $a + 2\delta < \frac{1}{2}$, then this set remains p-supercritical.

Let us enumerate $\Lambda'_+ = \{\gamma_1, \gamma_2, \ldots\}$ and show that, for sufficiently large l, we have $\gamma_l^{p-1}(\gamma_{l+1} - \gamma_l) < a + 2\delta$. Assume that in the original sequence $\gamma_l = \lambda_n$, $\gamma_{l+1} = \lambda_m$. If $m = n + 1$ then $\gamma_l^{p-1}(\gamma_{l+1} - \gamma_l) < a + \delta$, provided that l is large enough.

Now, assume that $m > n + 1$, that is, $\lambda_{m-1} \notin \Lambda'_+$. Then

$$\gamma_l^{p-1}(\gamma_{l+1} - \gamma_l) = \lambda_n^{p-1}(\lambda_m - \lambda_{m-1}) + \lambda_n^{p-1}(\lambda_{m-1} - \lambda_n)$$
$$< \lambda_{m-1}^{p-1}(\lambda_m - \lambda_{m-1}) + \lambda_n^{p-1}(\lambda_{m-1} - \lambda_n) < (a + \delta) + \delta = a + 2\delta,$$

provided again that l, and therefore m, are large enough. $\qquad\square$

Thus, there is no loss in generality to assume in Theorem 1.3 that Λ and M are p- and q-separated, respectively. We will call such pairs separated with parameters p and q.

Our next result is a quantitative version of the uniqueness part of Theorem 1.3.

Theorem 1.8. *Let (Λ, M) be a separated supercritical pair with parameters $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $s > 0$ be such that $s \min(p, q) \geq 1$. Then there exist numbers $c, C > 0$, such that, for every $f \in \mathcal{H}$,*

$$c\|f\|^2_{\mathcal{H}_{s,p,q}} \leq \left[\sum_{\lambda \in \Lambda} (1 + |\lambda|)^{(2s-1)p+1} |f(\lambda)|^2 \right.$$

$$\left. + \sum_{\mu \in M} (1 + |\mu|)^{(2s-1)q+1} |\widehat{f}(\mu)|^2 \right] \leq C\|f\|^2_{\mathcal{H}_{s,p,q}}.$$

Note that the sums in the middle are comparable to the Riemann sums for the integrals that define $\|f\|^2_{\mathcal{H}_{s,p,q}}$. By the left-hand inequality, if the expression in the brackets is finite for a function $f \in \mathcal{H}$, then f belongs to the space $\mathcal{H}_{s,p,q}$.

2. Corollaries and related work

2.1. Fast decay along a supercritical pair yields membership in $\mathcal{S}$. Combining Theorem 1.8 and Claim 1.7 with a simple fact that $\bigcap_{s>0} \mathcal{H}_{s,p,q} = \mathcal{S}$, we get the following corollary:

Corollary 2.1. *Let (Λ, M) be a supercritical pair. The function $f \in \mathcal{H}$ belongs to the Schwartz space $\mathcal{S}$ if and only if for all $r < \infty$ we have*

$$|f(\lambda_j)| = O\left(|\lambda_j|^{-r}\right), \quad |\widehat{f}(\mu_j)| = O\left(|\mu_j|^{-r}\right), \quad |j| \to \infty.$$

2.2. Interpolation formula. Theorem 1.8 essentially says that suitably normalized reproducing kernels at the sets Λ and M form a frame for the Hilbert space $\mathcal{H}_{s,p,q}$. It is well-known that any such frame bound gives rise to an interpolation formula, see e.g. [10, Section 5.1] or [22, Section 1.8]. For the reader's convenience and to keep this text self-contained, we will outline the argument in our case following [22].

Corollary 2.2. *Let (Λ, M) be a supercritical pair with parameters $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $s > 0$ satisfy $s \min(p, q) \geq 1$. Then there exist functions $(a_\lambda)_{\lambda \in \Lambda}$ and $(b_\mu)_{\mu \in M}$ in $\mathcal{H}_{s,p,q}$, such that, for every $f \in \mathcal{H}_{s,p,q}$, we have*

$$f = \sum_{\lambda \in \Lambda} f(\lambda) a_\lambda + \sum_{\mu \in M} \widehat{f}(\mu) b_\mu \tag{2.1}$$

and the series on the RHS converges in $\mathcal{H}_{s,p,q}$.

Furthermore, the functions a_λ and b_μ can be chosen so that

$$\|a_\lambda\|_{\mathcal{H}_{s,p,q}} \leq C(1 + |\lambda|)^{(s-\frac{1}{2})p+\frac{1}{2}}, \quad \lambda \in \Lambda,$$

and

$$\|b_\mu\|_{\mathcal{H}_{s,p,q}} \leq C(1 + |\mu|)^{(s-\frac{1}{2})q+\frac{1}{2}}, \quad \mu \in M.$$

Since $|f(x)| \le \sqrt{\pi} \|f\|_{H_1}$ and $\|f\|_{H_1} \le \|f\|_{\mathcal{H}} \le \sqrt{2} \|f\|_{\mathcal{H}_{s,p,q}}$, we have $|f(x)| \le \sqrt{2\pi} \|f\|_{\mathcal{H}_{s,p,q}}$, uniformly in $x \in \mathbb{R}$, and similarly $|\widehat{f}(\xi)| \le \sqrt{2\pi} \|f\|_{\mathcal{H}_{s,p,q}}$ uniformly in $\xi \in \mathbb{R}$, so the series on the RHS of (2.1) converges uniformly both on the space and the Fourier sides. Similarly, if $s \min(p,q) \ge n+1$, $n \in \mathbb{N}$, then the series in (2.1) converges uniformly together with the first n derivatives both on the space and the Fourier sides.

On the other hand, it is important to note that the functions a_λ, b_μ depend on s and that we do not know whether there exists a similar interpolation formula convergent in the topology of the Schwartz space $\mathcal{S}$.

Proof of Corollary 2.2. By Claim 1.7, we can assume that the pair (Λ, M) is separated.

Let $\ell^2(\Lambda, M)$ be the Hilbert space of (complex-valued) sequences x indexed by $\Lambda \cup M$ such that

$$\|x\|^2 \stackrel{\text{def}}{=} \sum_{\lambda \in \Lambda} |x(\lambda)|^2 (1+|\lambda|)^{(2s-1)p+1} + \sum_{\mu \in M} |x(\mu)|^2 (1+|\mu|)^{(2s-1)q+1} < \infty.$$

Then the map

$$f \mapsto \big((f(\lambda))_{\lambda \in \Lambda}, (\widehat{f}(\mu))_{\mu \in M} \big)$$

defines a linear operator

$$T \colon \mathcal{H}_{s,p,q} \to \ell^2(\Lambda, M),$$

and, by Theorem 1.8, there exists a positive C such that, for each $f \in \mathcal{H}_{s,p,q}$,

$$C^{-1} \|f\|_{\mathcal{H}_{s,p,q}} \le \|Tf\|_{\ell^2(\Lambda,M)} \le C \|f\|_{\mathcal{H}_{s,p,q}}. \tag{2.2}$$

Let $X = T\mathcal{H}_{s,p,q}$ be the image of the space $\mathcal{H}_{s,p,q}$. By (2.2), this is a closed subspace of $\ell^2(\Lambda, M)$. Denote by $P \colon \ell^2(\Lambda, M) \to X$ the orthogonal projection to X.

Let δ_λ, $\lambda \in \Lambda$, and δ_μ, $\mu \in M$, be the "delta-functions" in $\ell^2(\Lambda, M)$, i.e., $\delta_\lambda(\lambda) = 1$, $\delta_\lambda(\lambda') = \delta_\lambda(\mu) = 0$ for $\lambda' \in \Lambda \setminus \{\lambda\}$, $\mu \in M$, and similarly for δ_μ. Let $e_\lambda = P\delta_\lambda$, $e_\mu = P\delta_\mu$. Then, for each $x = Tf$ in X, we have

$$x = \sum_{\lambda \in \Lambda} x(\lambda) e_\lambda + \sum_{\mu \in M} x(\mu) e_\mu = \sum_{\lambda \in \Lambda} f(\lambda) e_\lambda + \sum_{\mu \in M} \widehat{f}(\mu) e_\mu,$$

with the series convergent in $\ell^2(\Lambda, M)$. Applying the inverse operator T^{-1} (which is, by (2.2), a bounded operator from X to $\mathcal{H}_{s,p,q}$), letting $a_\lambda = T^{-1} e_\lambda$, $b_\mu = T^{-1} e_\mu$, and noting that

$$\|a_\lambda\|_{\mathcal{H}_{s,p,q}} \le \|T^{-1}\|_{X \to \mathcal{H}_{s,p,q}} \|e_\lambda\|_{\ell^2(\Lambda,M)}$$
$$\le C \|\delta_\lambda\|_{\ell^2(\Lambda,M)} = C(1+|\lambda|)^{(s-\frac{1}{2})p+\frac{1}{2}}, \quad \lambda \in \Lambda,$$

and, similarly, $\|b_\mu\|_{\mathcal{H}_{s,p,q}} \le C(1+|\mu|)^{(s-\frac{1}{2})q+\frac{1}{2}}$, $\mu \in M$, we conclude the proof. $\quad\square$

2.3. Crystalline measures. Keeping $x \in \mathbb{R} \setminus \Lambda$ fixed, we can rewrite the interpolation formula (2.1) in the form

$$f(x) - \sum_{\lambda \in \Lambda} f(\lambda) a_\lambda(x) = \sum_{\mu \in M} \widehat{f}(\mu) b_\mu(x), \quad f \in \mathcal{S}.$$

Then (keeping in mind that $\sup_{\mathbb{R}} |a_\lambda| \leq C \|a_\lambda\|_{\mathcal{H}_{s,p,q}}$ and $\sup_{\mathbb{R}} |b_\mu| \leq C \|b_\mu\|_{\mathcal{H}_{s,p,q}}$), we see that the locally finite measure $\nu_x = \sum_{\mu \in M} b_\mu(x) \delta_\mu$ is a tempered distribution supported on the set M, and its distributional Fourier transform equals to $\widehat{\nu}_x = \delta_x - \sum_{\lambda \in \Lambda} a_\lambda(x) \delta_\lambda$, which is a locally finite measure supported on the set $\Lambda \bigcup \{x\}$. Applying this argument with some $\lambda' \in \Lambda$, $\Lambda' = \Lambda \setminus \{\lambda'\}$ in place of Λ, and $x = \lambda'$, we arrive at a curious conclusion:

Corollary 2.3. *For any supercritical pair (Λ, M), there exists a non-trivial locally finite measure ν supported by M that is a tempered distribution, whose distributional Fourier transform $\widehat{\nu}$ is a locally finite measure supported by Λ.*

Such measures ν are called *one-dimensional quasicrystals* or *crystalline measures* according to Dyson [9] and Meyer [17, 18].

In fact, we can prove slightly more. For a pair of discrete sets Λ, M, we denote by $\mathcal{N}(\Lambda, M)$ the vector space of locally finite measures ν supported on M which are tempered distributions, whose distributional Fourier transforms $\widehat{\nu}$ are locally finite measures supported on Λ. Then, since removing finitely many points from a supercritical pair leaves a supercritical pair, the space $\mathcal{N}(\Lambda, M)$ is not only non-trivial, but also infinite-dimensional for any supercritical pair (Λ, M).

On the other hand, for the subcritical pair, this space is always finite-dimensional, see Remark 7.5 in the end of Section 7.3.

2.4. Related work. There are several recent results inspired by the aforementioned Radchenko–Viazovska paper [23]. Bondarenko, Radchenko and Seip [6] revealed a collection of similar interpolation formulas, the most curious one corresponding to $\Lambda = \{(4\pi)^{-1} \log n \colon n \in \mathbb{N}\}$ and $M = \{i(\rho - \frac{1}{2})\}$, where ρ runs through all non-trivial zeroes of the Riemann zeta function with positive imaginary part (assuming the Riemann hypothesis and simplicity of zeroes). Similarly to [23], they used an arithmetic approach based on the theory of modular forms and Dirichlet series. In [25], Ramos and Sousa managed to slightly perturb the nodes of the interpolation formula found in [23].

Recently, in [1] Adve found conditions for the pair of discrete set $(\Lambda, M) \subset \mathbb{R}^d \times \mathbb{R}^d$ to be a UP/NUP for the Schwartz space $S(R^d)$. His conditions are less precise than the ones supplied by Theorem 1.3 in the case $d = 1$ (apart from that, they are of uniform nature, like in our toy-model Theorem 1.3A, rather than asymptotic). We note that his construction of NUPs does not appeal to complex analysis. It should also be mentioned that other, denser than in [1], multi-dimensional discrete UP were recently constructed by Ramos and Stoller [26] and by Viazovska [27].

In [11], the first-named author proved that interpolation formulas like (2.1) can only exist if the following inequality for the counting functions of the sets

Λ, M is satisfied for all $W, T > 1$:

$$|\Lambda \cap [-T, T]| + |M \cap [-W, W]| \geq 4WT - C \log^2(4WT).$$

One can easily see that the supercriticality condition in Theorem 2.2 implies this inequality (even with 4 replaced by some bigger constant), while, for subcritical pairs in the non-uniqueness part of Theorem 1.3, there exist values of T, W that violate it regardless of the value of the constant C.

For the reader interested in one-dimensional quasicrystals, we mention recent works by Kurasov and Sarnak [12], Lev and Reti [13, 14], Olevskii and Ulanovskii [20, 21], and Meyer [17, 18], which also contain further references.

3. Poincaré–Wirtinger-type inequalities

The central role in our arguments will be played by the Poincaré–Wirtinger inequality and its extensions. The most classical version states that if $f \in C^1[0, 1]$ and $f(0) = f(1) = 0$, then[1]

$$\int_0^1 |f|^2 \leq \left(\frac{1}{\pi}\right)^2 \int_0^1 |f'|^2.$$

We start with a "stable version" of this inequality, which does not assume that f vanishes at the endpoints.

Lemma 3.1. *For any $\varepsilon > 0$, all $a < b$ and all functions $f \in H_1(\mathbb{R})$, we have*

$$\int_a^b |f|^2 \leq (1 + \varepsilon) \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'|^2 + \left(1 + \varepsilon^{-1}\right)(b - a)\left(|f(a)|^2 + |f(b)|^2\right).$$

Proof. Since smooth functions are dense in $H_1(\mathbb{R})$ and the point evaluation is bounded in $H_1(\mathbb{R})$, it suffices to prove the estimate for a smooth function f on $[a, b]$.

By the scale- and shift-invariance we can assume that $a = 0$ and $b = 1$. Consider the function $g = f - \ell$, where the linear function ℓ is taken so that $\ell(0) = f(0)$ and $\ell(1) = f(1)$, i.e., $g(0) = g(1) = 0$. Then, by the classical Poincaré–Wirtinger inequality,

$$\int_0^1 |g|^2 \leq \frac{1}{\pi^2} \int_0^1 |g'|^2.$$

Furthermore, by the Cauchy–Schwarz inequality,

$$\int_0^1 |f|^2 = \int_0^1 |g + \ell|^2 \leq (1 + \varepsilon) \int_0^1 |g|^2 + \left(1 + \varepsilon^{-1}\right) \int_0^1 |\ell|^2,$$

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[1]For the proof, extend the function f to the odd 2-periodic function $\widetilde{f}$ and let $g(t) = \widetilde{f}(2t)$. The function g is a 1-periodic C^1-function with zero mean. Expanding g and g' into the Fourier series and using that $\widehat{g'}(k) = 2\pi \mathrm{i} k \widehat{g}(k)$, $k \in \mathbb{Z}$, we get

$$4 \int_0^1 |f'|^2 = \int_{\mathbb{R}/\mathbb{Z}} |g'|^2 = 4\pi^2 \sum_{k \neq 0} k^2 |\widehat{g}(k)|^2 \geq 4\pi^2 \sum_{k \neq 0} |\widehat{g}(k)|^2 = 4\pi^2 \int_{\mathbb{R}/\mathbb{Z}} |g|^2 = 4\pi^2 \int_0^1 |f|^2.$$

and

$$\int_0^1 |f'|^2 = \int_0^1 |g'|^2 + \int_0^1 |\ell'|^2 \geq \int_0^1 |g'|^2$$

(since the function g' has zero mean and, therefore, is orthogonal to constant functions). At last, $|\ell| \leq \max(|\ell(0)|, |\ell(1)|) = \max(|f(0)|, |f(1)|)$ everywhere on $[0, 1]$, whence

$$\int_0^1 |\ell|^2 \leq \max\left(|f(0)|^2, |f(1)|^2\right) \leq |f(0)|^2 + |f(1)|^2.$$

Combining these estimates, we complete the proof. $\square$

We also need the following trace-type bound:

Lemma 3.2. *For all $a < b$ and all $f \in H_1(\mathbb{R})$, we have*

$$\frac{1}{b-a} |f(a)|^2 \leq \frac{2}{(b-a)^2} \int_a^b |f|^2 + \frac{2}{3} \int_a^b |f'|^2.$$

Proof. As above, we assume that $a = 0$, $b = 1$, and that the function f is smooth. Integrating the identity

$$f(0) = f(x) - \int_0^x f'$$

over $x \in [0, 1]$, we get

$$f(0) = \int_0^1 f - \int_0^1 (1-x) f',$$

whence,

$$|f(0)| \leq \int_0^1 |f| + \int_0^1 (1-x)|f'|,$$

and

$$|f(0)|^2 \leq 2 \left[\left(\int_0^1 |f| \right)^2 + \left(\int_0^1 (1-x)|f'| \right)^2 \right]$$

$$\leq 2 \int_0^1 |f|^2 + 2 \int_0^1 (1-x)^2 \cdot \int_0^1 |f'|^2 = 2 \int_0^1 |f|^2 + \frac{2}{3} \int_0^1 |f'|^2,$$

proving the lemma. $\square$

Definition 3.3 (*l*-dense sets). A discrete set $\Gamma \subset \mathbb{R}$ is *l*-dense if $\mathbb{R} \setminus \Gamma$ contains no interval of length greater than l.

Lemma 3.4.

(i) *Let $t > 0$, $0 < \varepsilon < 1$, and let Γ be a $(1-\varepsilon)(2t)^{-1}$-dense discrete subset of $\mathbb{R}$. There exists a constant C_ε such that, for all convex increasing C^1-functions $\Phi \colon [0, \infty) \to \mathbb{R}$ and all functions $f \in \mathcal{H}$, we have*

$$\Phi(t^2) \int_{\mathbb{R}} |f(x)|^2 \, dx \leq \int_{\mathbb{R}} \Phi(\xi^2) |\widehat{f}(\xi)|^2 \, d\xi + C_\varepsilon t \Phi'(t^2) \sum_{\gamma \in \Gamma} |f(\gamma)|^2.$$

(ii) *Let $t > 0$, and let Γ be a $(2t)^{-1}$-dense discrete subset of $\mathbb{R}$. Then, for all convex increasing C^1-functions $\Phi \colon [0, \infty) \to \mathbb{R}$ and all functions $f \in \mathcal{H}$ vanishing on Γ, we have*

$$\Phi(t^2) \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x \le \int_{\mathbb{R}} \Phi(\xi^2) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

The case $\Phi(t) = t^\theta$, $\theta \ge 1$, will be of a particular interest for us later.

Proof. First, we consider the case $\Phi(t) = t$. Summing the estimates from Lemma 3.1 over all gaps between the points of the set Γ and using that the length of the gaps cannot exceed $(1 - \varepsilon)/(2t)$, we get

$$\int_{\mathbb{R}} |f|^2 \le \frac{(1+\varepsilon)(1-\varepsilon)^2}{(2\pi t)^2} \int_{\mathbb{R}} |f'|^2 + \frac{(1+\varepsilon^{-1})(1-\varepsilon)}{t} \sum_{\gamma \in \Gamma} |f(\gamma)|^2,$$

whence (using the Plancherel theorem),

$$t^2 \int_{\mathbb{R}} |f|^2 \le \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi + C_\varepsilon t \sum_{\gamma \in \Gamma} |f(\gamma)|^2.$$

For the general case, we rewrite the case $\Phi(t) = t$ as

$$\int_{|\xi| \le t} (t^2 - \xi^2) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \le \int_{|\xi| > t} (\xi^2 - t^2) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi + C_\varepsilon t \sum_{\gamma \in \Gamma} |f(\gamma)|^2.$$

Multiplying this inequality by $\Phi'(t^2)$, we get

$$\Phi'(t^2) \int_{|\xi| \le t} (t^2 - \xi^2) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi$$
$$\le \Phi'(t^2) \int_{|\xi| > t} (\xi^2 - t^2) |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi + C_\varepsilon t \Phi'(t^2) \sum_{\gamma \in \Gamma} |f(\gamma)|^2.$$

Noting that, for $|\xi| > t$, we have $\Phi(\xi^2) - \Phi(t^2) \ge \Phi'(t^2)(\xi^2 - t^2)$, and that, for $|\xi| \le t$, we have $\Phi(t^2) - \Phi(\xi^2) \le \Phi'(t^2)(t^2 - \xi^2)$, we get

$$\int_{|\xi| \le t} |\widehat{f}(\xi)|^2 (\Phi(t^2) - \Phi(\xi^2)) \, \mathrm{d}\xi$$
$$\le \int_{|\xi| > t} |\widehat{f}(\xi)|^2 (\Phi(\xi^2) - \Phi(t^2)) \, \mathrm{d}\xi + C_\varepsilon t \Phi'(t^2) \sum_{\gamma \in \Gamma} |f(\gamma)|^2,$$

which after rearranging gives us part 3.4(i) of the lemma.

To prove the second part, we notice that the set Γ is $(1 - \varepsilon)(2t')^{-1}$-dense with any $t' < t$ and $\varepsilon < 1 - t^{-1}t'$. Applying part 3.4(i), recalling that f vanishes on Γ, and letting $t' \uparrow t$, we get part 3.4(ii). $\qquad\square$

4. Toy model: the Poincaré–Wirtinger inequality in action

Here we prove the uniqueness part of Theorem 1.3, as well as Theorem 1.8, in the case $p = q = 2$, $s = \frac{1}{2}$, under a more restrictive *uniform supercriticality assumption*. In this case, the proofs of these results become short and free of cumbersome technical details.

4.1. Uniformly supercritical uniqueness pairs. As before, we order the sequences $\Lambda = (\lambda_j)_{j \in \mathbb{Z}}$ and $M = (\mu_j)_{j \in \mathbb{Z}}$, so that

$$\ldots < \lambda_{j-1} < \lambda_j < \lambda_{j+1} \ldots, \quad \lim_{j \to \pm\infty} \lambda_j = \pm\infty,$$

$$\ldots < \mu_{j-1} < \mu_j < \mu_{j+1} \ldots, \quad \lim_{j \to \pm\infty} \mu_j = \pm\infty.$$

Definition 4.1 (uniform supercriticality)**.** We call a pair (Λ, M) uniformly supercritical, if

$$\sup_{j \in \mathbb{Z}} \max\{|\lambda_j|, |\lambda_{j+1}|\} \cdot (\lambda_{j+1} - \lambda_j), \ \sup_{j \in \mathbb{Z}} \max\{|\mu_j|, |\mu_{j+1}|\} \cdot (\mu_{j+1} - \mu_j) < \frac{1}{2}. \quad (4.1)$$

Theorem 1.3A. *Let* (Λ, M) *be a uniformly supercritical pair. Then the only function* $f \in \mathcal{H}$ *such that* $f(\lambda_j) = \widehat{f}(\mu_j) = 0$, $j \in \mathbb{Z}$, *is* $f = 0$.

Proof. Choose $a < 1$ so that the LHS of (4.1) is less than $\frac{1}{2}a$. Then, we have

$$\int_{\mathbb{R}} x^2 |f(x)|^2 \, dx = \sum_{j \in \mathbb{Z}} \int_{\lambda_j}^{\lambda_{j+1}} x^2 |f(x)|^2 \, dx$$

$$\leq \sum_{j \in \mathbb{Z}} \max(|\lambda_j|, |\lambda_{j+1}|)^2 \int_{\lambda_j}^{\lambda_{j+1}} |f(x)|^2 \, dx$$

$$\leq \sum_{j \in \mathbb{Z}} \max(|\lambda_j|, |\lambda_{j+1}|)^2 \left(\frac{\lambda_{j+1} - \lambda_j}{\pi} \right)^2 \int_{\lambda_j}^{\lambda_{j+1}} |f'(x)|^2 \, dx$$

$$\leq \frac{a^2}{(2\pi)^2} \sum_{j \in \mathbb{Z}} \int_{\lambda_j}^{\lambda_{j+1}} |f'(x)|^2 \, dx$$

$$= \frac{a^2}{(2\pi)^2} \int_{\mathbb{R}} |f'(x)|^2 \, dx = a^2 \int_{\mathbb{R}} y^2 |\widehat{f}(y)|^2 \, dy,$$

where in the second inequality we used the classical Poincaré–Wirtinger inequality.

Similarly, we get

$$\int_{\mathbb{R}} y^2 |\widehat{f}(y)|^2 \, dy \leq a^2 \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx.$$

Since $a < 1$, this implies that $\int_{\mathbb{R}} x^2 |f(x)|^2 \, dx = 0$, i.e., $f = 0$ almost everywhere. $\square$

By being slightly more careful, we can assume in Theorem 1.3A that the LHS of (4.1) are only at most $\frac{1}{2}$ and not strictly smaller than $\frac{1}{2}$. As the Radchenko–Viazovska construction [23] shows, this is no longer true in the case of more general Theorem 1.3.

Note that it is enough to add finitely many points to each of the sets Λ and M, to turn any supercritical pair (Λ, M) (with $p = q = 2$) into the uniformly supercritical one. Therefore, if (Λ, M) is a supercritical pair with $p = q = 2$, the linear space $\{f \in \mathcal{H} : f|_\Lambda = 0, \widehat{f}|_M = 0\}$ can be only finite-dimensional. Unfortunately, we did not succeed in finding a short way to conclude from here that this linear space contains only the identically vanishing function. Nevertheless, the same idea based on the Poincaré–Wirtinger inequality will play a crucial role in the proof of the uniqueness part of Theorem 1.3.

4.2. Quantitative uniqueness. The next result is a uniform version of Theorem 1.8.

Theorem 1.8A. *Let (Λ, M) be a separated uniformly supercritical pair with $p = q = 2$. There exist $A, B > 0$ such that, for all $f \in \mathcal{H}$, we have*

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{\lambda \in \Lambda}(1 + |\lambda|)|f(\lambda)|^2 + \sum_{\mu \in M}(1 + |\mu|)|\widehat{f}(\mu)|^2 \leq B\|f\|_{\mathcal{H}}^2. \tag{4.2}$$

We start with the left-hand part of (4.2), which does not need the separation assumption.

Proof of the left-hand part of (4.2). Choose $a < 1$ so that the LHS of (4.1) is less than $\frac{1}{2}a$. Repeating the proof of Theorem 1.3A, with Lemma 3.1 in place of the Poincaré–Wirtinger inequality, we get

$$\int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x \leq a^2(1 + \varepsilon) \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi$$
$$+ 2(1 + \varepsilon^{-1}) \sum_{j \in \mathbb{Z}} \max(|\lambda_{j-1}|, |\lambda_j|, |\lambda_{j+1}|)^2 (\lambda_{j+1} - \lambda_j)|f(\lambda_j)|^2. \tag{4.3}$$

By the uniform supercriticality, $\max(|\lambda_{j-1}|, |\lambda_j|, |\lambda_{j+1}|) \leq C(1 + |\lambda_j|)$, and therefore, the second term on the RHS of (4.3) is bounded by

$$C_\varepsilon \sum_{j \in \mathbb{Z}} \max(|\lambda_{j-1}|, |\lambda_j|, |\lambda_{j+1}|)|f(\lambda_j)|^2 \leq C_\varepsilon \sum_{\lambda \in \Lambda}(1 + |\lambda|)|f(\lambda)|^2.$$

Hence,

$$\int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x \leq a^2(1 + \varepsilon) \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi + C_\varepsilon \sum_{\lambda \in \Lambda}(1 + |\lambda|)|f(\lambda)|^2.$$

Similarly,

$$\int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \leq a^2(1 + \varepsilon) \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x + C_\varepsilon \sum_{\mu \in M}(1 + |\mu|)|\widehat{f}(\mu)|^2.$$

Adding these estimates, and choosing $\varepsilon > 0$ so that $a^2(1 + \varepsilon) < 1 - \delta$, $\delta > 0$, we get

$$\delta \left[\int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right]$$
$$\leq C_\varepsilon \left[\sum_{\lambda \in \Lambda} (1 + |\lambda|) |f(\lambda)|^2 + \sum_{\mu \in M} (1 + |\mu|) |\widehat{f}(\mu)|^2 \right].$$

Recalling that $\|f\|_{\mathcal{H}}^2$ is dominated by $\int_{\mathbb{R}} x^2 |f|^2 + \int_{\mathbb{R}} \xi^2 |\widehat{f}|^2$ (for instance, by the classical Heisenberg inequality[2] [8, Section 2.9]), we get

$$\delta \|f\|_{\mathcal{H}}^2 \leq \widetilde{C}_\varepsilon \left[\sum_{\lambda \in \Lambda} (1 + |\lambda|) |f(\lambda)|^2 + \sum_{\mu \in M} (1 + |\mu|) |\widehat{f}(\mu)|^2 \right],$$

which implies the left-hand part of estimate (4.2) with $A = \delta \widetilde{C}_\varepsilon^{-1}$. $\qquad\square$

Proof of the right-hand part of (4.2). Here we will use only the separation condition, choosing c so that $\lambda_{j+1} - \lambda_j \geq c(1 + \min(|\lambda_j|, |\lambda_{j+1}|))^{-1}$, $j \in \mathbb{Z}$. For $\lambda_j \geq 0$, applying Lemma 3.2 with $a = \lambda_j$, $b = \lambda_j + c(1 + \lambda_j)^{-1}$ so that $[a, b] \subset [\lambda_j, \lambda_{j+1}]$, we get

$$c^{-1}(1 + \lambda_j) |f(\lambda_j)|^2 \leq 2c^{-2}(1 + \lambda_j)^2 \int_{\lambda_j}^{\lambda_j + c(1+\lambda_j)^{-1}} |f|^2 + \frac{2}{3} \int_{\lambda_j}^{\lambda_j + c(1+\lambda_j)^{-1}} |f'|^2$$
$$\leq 2c^{-2}(1 + \lambda_j)^2 \int_{\lambda_j}^{\lambda_{j+1}} |f|^2 + \frac{2}{3} \int_{\lambda_j}^{\lambda_{j+1}} |f'|^2$$
$$\leq 2c^{-2} \int_{\lambda_j}^{\lambda_{j+1}} (1 + x^2) |f|^2 + \frac{2}{3} \int_{\lambda_j}^{\lambda_{j+1}} |f'|^2.$$

For $\lambda_j < 0$, we apply Lemma 3.2 to the function $f(-x)$ with $a = -\lambda_j$, $b = -\lambda_j + c(1 + |\lambda_j|)^{-1}$ (note that now $[a, b] \subset [-\lambda_j, -\lambda_{j-1}]$) and get

$$c^{-1}(1 + |\lambda_j|) |f(\lambda_j)|^2 \leq 2c^{-2} \int_{\lambda_{j-1}}^{\lambda_j} (1 + x^2) |f|^2 + \frac{2}{3} \int_{\lambda_{j-1}}^{\lambda_j} |f'|^2.$$

Summing these estimates and recalling that $\int_{\mathbb{R}} |f'|^2 = 4\pi^2 \int_{\mathbb{R}} \xi^2 |\widehat{f}|^2$, we get the claimed upper bound for the sum over $\lambda \in \Lambda$. The proof of the upper bound for the sum over $\mu \in M$ is similar. $\qquad\square$

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[2] We use it in the form

$$(2\pi)^{-1} \|f\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} x^2 |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

5. Uniqueness Pairs

In this section we prove the following result, which somewhat extends the uniqueness part of Theorem 1.3:

Theorem 1.3-UP. *Let (Λ, M) be a supercritical pair with parameters p and q. Then the only function $f \in \mathcal{H}$ such that $f(\lambda) = \widehat{f}(\mu) = 0$, $\lambda \in \Lambda$, $\mu \in M$, is $f = 0$.*

The proof splits into three parts — two parts that deal with smooth functions on $\mathbb{R}$, and a complex-analytic part. First, we prove that any function $f \in \mathcal{H}$ that vanishes on Λ and whose Fourier transform vanishes on M must be a Schwartz function. Then we show that this function actually has an even faster decay together with its Fourier transform and belongs to the Gelfand–Shilov space $\mathcal{S}(p, q)$, i.e., for some $c > 0$,

$$\int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p} \, \mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 e^{c|\xi|^q} \, \mathrm{d}\xi < \infty.$$

In particular, f and $\widehat{f}$ are entire functions of finite order. The main tool in both steps is a modification of the Poincaré–Wirtinger inequality for fractional derivatives, which will be our starting point, and its slowly varying local version.

To finish the proof, we use some Phragmén–Lindelöf type bounds from the entire function theory, working simultaneously on the space and the Fourier sides.

5.1. From Sobolev to Schwartz spaces. Our goal now is to prove that, under the assumptions of Theorem 1.3-UP, for every $\theta > 1$, we have

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\theta} \, \mathrm{d}x < \infty \quad \text{and} \quad \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta} \, \mathrm{d}\xi < \infty$$

(recall that a function $f \in L^2(\mathbb{R})$ belongs to Schwartz class $\mathcal{S}$ if and only if both of these integrals are finite for all $\theta > 1$).

For convenience, we set $a = p - 1$, $b = q - 1$, where p and q are the parameters from the supercriticality assumption in Theorem 1.3-UP. Then $ab = 1$. Without loss of generality, we assume that $a \geq 1 \geq b$.

Next, observe that, by the supercriticality of the pair (Λ, M), there exist $\sigma > 1$ and $X_0, \Xi_0 > 0$ such that, for $x, \xi \in \mathbb{R}$, the intervals of $\mathbb{R} \setminus \Lambda$ and $\mathbb{R} \setminus M$ containing x and ξ respectively have lengths at most $(2\sigma|x|^a)^{-1}$ and $(2\sigma|\xi|^b)^{-1}$, provided that $|x| > X_0$, $|\xi| > \Xi_0$. In order to apply Lemma 3.4, we take a non-negative cutoff function $F \in C_0^\infty(-1, 1)$ with $\int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}x = 1$, and consider the function $g = fF_v$, where $F_v(x) = F(x - v)$, $v \in \mathbb{R}$. If $|v| > X_0 + 1$, the function g vanishes on a $(2\sigma)^{-1}(|v| - 1)^{-a}$-dense set, so, applying Lemma 3.4(ii) to the function g, we get

$$\Phi(\sigma^2(|v| - 1)^{2a}) \int_{\mathbb{R}} |f(x)F_v(x)|^2 \, \mathrm{d}x \leq \int_{\mathbb{R}} |(\widehat{f} * \widehat{F_v})(\xi)|^2 \Phi(\xi^2) \, \mathrm{d}\xi \tag{5.1}$$

with an arbitrary C^1-convex increasing function Φ vanishing at the origin.

Choosing $\Upsilon_0 > X_0 + 2$ so large that

$$\left(\frac{\Upsilon_0 + 1}{\Upsilon_0 - 1}\right)^{2a} < \sigma,$$

and noting that $|x - \upsilon| \le 1$ for every x in the support of F_υ, we estimate the LHS of (5.1) from below by

$$\int_{\mathbb{R}} |f(x)|^2 |F_\upsilon(x)|^2 \Phi(\sigma|x|^{2a}) \, dx \ge \sigma \int_{\mathbb{R}} |f(x)|^2 |F_\upsilon(x)|^2 \Phi(|x|^{2a}) \, dx,$$

provided that $|\upsilon| > \Upsilon_0$. Plugging this into (5.1) and integrating both parts over such υ, we obtain

$$\sigma \int_{|x| > \Upsilon_0 + 1} |f(x)|^2 \Phi(|x|^{2a}) \, dx \le \iint_{\mathbb{R} \times \mathbb{R}} |(\widehat{f} * \widehat{F_\upsilon})(\xi)|^2 \Phi(\xi^2) \, d\xi \, d\upsilon. \qquad (5.2)$$

Now, we are going to bound the RHS from above. Recalling that $\widehat{F_\upsilon}(\eta) = \widehat{F}(\eta)e^{-2\pi i \upsilon \eta}$ and using the Plancherel theorem, we have

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \widehat{f}(\xi - \eta)\widehat{F}(\eta)e^{-2\pi i \upsilon \eta} \, d\eta \right|^2 d\upsilon = \int_{\mathbb{R}} |\widehat{f}(\xi - \eta)|^2 |\widehat{F}(\eta)|^2 \, d\eta.$$

Thus, after a change of variables, the RHS of (5.2) is equal to

$$\iint_{\mathbb{R} \times \mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 \Phi\big((\xi + \eta)^2\big) \, d\xi \, d\eta. \qquad (5.3)$$

So far Φ was an arbitrary convex increasing C^1-function. Now it is time to choose it. We fix $\theta > 1$ and a large number T, and put $\Phi_T(t) = t^\theta$, $0 \le t \le T$, and $\Phi_T(t) = T^\theta + \theta T^{\theta-1}(t - T)$, $t > T$. Note that we have

$$\Phi_T\big((\xi + \eta)^2\big) \le \Phi_T\big((|\xi| + |\eta|)^2\big) \le \Phi_T\big(\xi^2\big) \left(1 + \frac{|\eta|}{|\xi|}\right)^{2\theta}, \qquad (5.4)$$

which follows from the fact that $\Phi_T(t)t^{-\theta}$ is a non-increasing function.

Since $|\widehat{F}(\eta)|^2$ decays faster than any power of η, for $|\xi| \to \infty$, we have

$$\int_{\mathbb{R}} |\widehat{F}(\eta)|^2 \left(1 + \frac{|\eta|}{|\xi|}\right)^{2\theta} d\eta \longrightarrow \int_{\mathbb{R}} |\widehat{F}(\eta)|^2 \, d\eta = \int_{\mathbb{R}} |F(x)|^2 \, dx = 1$$

by the dominated convergence theorem, so we obtain

$$\int_{\mathbb{R}} |\widehat{F}(\eta)|^2 \left(1 + \frac{|\eta|}{|\xi|}\right)^{2\theta} d\eta < \sqrt{\sigma},$$

as long as $|\xi| \ge \Xi_1 = \Xi_1(\sigma, p, \theta)$ (it is crucial that Ξ_1 does not depend on T). Thus,

$$\iint_{\{|\xi| \ge \Xi_1\} \times \mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 \Phi_T\big((\xi + \eta)^2\big) \, d\xi \, d\eta \le \sqrt{\sigma} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Phi_T\big(\xi^2\big) \, d\xi. \qquad (5.5)$$

Note that the integral on the RHS is finite since $\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 (1+\xi^2)\, \mathrm{d}\xi < \infty$ (recall that, for every T, we have $\Phi_T(\xi^2) = O(\xi^2)$, as $|\xi| \to \infty$).

In (5.3), the integral over $\{|\xi| \leq \Xi_1\} \times \mathbb{R}$ is dominated by

$$\iint_{[-\Xi_1, \Xi_1] \times \mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 (\Xi_1 + |\eta|)^{2\theta}\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

which is some finite constant (independent of T) due to the fast decay of $\widehat{F}$.

At last, trivially, we have

$$\int_{|x| \leq \Upsilon_0 + 1} |f(x)|^2 \Phi_T\big(|x|^{2a}\big)\, \mathrm{d}x \leq (\Upsilon_0 + 1)^{2a\theta} \int_{\mathbb{R}} |f(x)|^2\, \mathrm{d}x.$$

Putting all the estimates together we get

$$\int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(|x|^{2a}\big)\, \mathrm{d}x \leq \sigma^{-1/2} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Phi_T\big(\xi^2\big)\, \mathrm{d}\xi + K, \qquad (5.6)$$

where $K = K_f > 0$ is independent of T.

Now, let us switch the roles of f and $\widehat{f}$ and consider the function $\Psi_T(t) = \Phi_T(t^a)$ (which is still convex, C^1-smooth, and increasing) instead of Φ_T. Now, the function $\Psi_T(t) t^{-a\theta}$ is non-increasing, the whole argument goes through, and we get

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(|\xi|^{2b}\big)\, \mathrm{d}\xi \leq \sigma^{-1/2} \int_{\mathbb{R}} |f(x)|^2 \Psi_T\big(x^2\big)\, \mathrm{d}x + K. \qquad (5.7)$$

The integral on the RHS of (5.7) coincides with the one on the LHS of (5.6), and hence, is also finite.

Now, we add inequalities (5.6) and (5.7). Recalling that $\Psi_T\big(|\xi|^{2b}\big) = \Phi_T\big(\xi^2\big)$ and $\Phi_T\big(|x|^{2a}\big) = \Psi_T\big(x^2\big)$ and using the finiteness of the RHS, we obtain

$$\int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(|x|^{2a}\big)\, \mathrm{d}x + \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Phi_T\big(\xi^2\big)\, \mathrm{d}\xi \leq \frac{2K}{1 - \sigma^{-1/2}},$$

independently of the choice of $T > 0$. It remains to let $T \to \infty$ and use the monotone convergence theorem to get the bound

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2a\theta}\, \mathrm{d}x + \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta}\, \mathrm{d}\xi \leq \frac{2K}{1 - \sigma^{-1/2}} < \infty.$$

5.2. From Schwartz to Gelfand–Shilov spaces. Here, we show that if a Schwartz function f vanishes on Λ and its Fourier transform $\widehat{f}$ vanishes on M, and the pair (Λ, M) is supercritical with parameters p and q, then, for every sufficiently big positive integer $\ell \geq \ell_0$, we have

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{p\ell}\, \mathrm{d}x, \quad \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{q\ell}\, \mathrm{d}\xi \leq C^\ell \ell! \int_{\mathbb{R}} |f(x)|^2\, \mathrm{d}x. \qquad (5.8)$$

It is easy to see that these bounds yield convergence of the integrals [3]

$$\int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p}\, \mathrm{d}x, \quad \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 e^{c|\xi|^q}\, \mathrm{d}\xi < \infty.$$

In this part, we again rely on our version of the Poincaré–Wirtinger inequality (Lemma 3.4). The main difference with the previous part is that now we need to exercise some care in choosing the cutoff function F: its choice has to be adjusted to the value of the parameter ℓ in (5.8).

As in the first step, we set $a = p - 1$, $b = q - 1$, and assume that Λ is a discrete set such that for $|x| \geq X_0$ the constituting interval of $\mathbb{R} \backslash \Lambda$ containing x has length at most $(2\sigma|x|^a)^{-1}$ for some $\sigma > 1$. Let $f \in \mathcal{S}$ vanish on Λ. Fix $\theta \geq 1$ and consider the function

$$F = \mathbb{1}_{[-u,u]} * \underbrace{\frac{k}{u}\mathbb{1}_{\left[-\frac{u}{2k},\frac{u}{2k}\right]} * \ldots * \frac{k}{u}\mathbb{1}_{\left[-\frac{u}{2k},\frac{u}{2k}\right]}}_{k\text{--fold convolution}},$$

where $\mathbb{1}_I$ is the indicator function of the interval I, and large parameters k and u are to be chosen later. We are going to apply Lemma 3.4(ii) with $\Phi(x) = x^\theta$ to the function fF_v, where $F_v(x) = F(x - v)$.

Note that $\operatorname{supp} F_v \subset \left[v - \frac{3}{2}u, v + \frac{3}{2}u\right]$. If $|v| - \frac{3}{2}u \geq X_0$, the function fF_v vanishes on a $(2\sigma)^{-1}\left(|v| - \frac{3}{2}u\right)^{-a}$-dense set, so Lemma 3.4(ii) applied with $t = \sigma\left(|v| - \frac{3}{2}u\right)^a$ gives us

$$\left(|v| - \frac{3}{2}u\right)^{2a\theta} \int_{\mathbb{R}} |f(x)|^2 |F_v(x)|^2\, \mathrm{d}x \leq \sigma^{-2\theta} \int_{\mathbb{R}} |\xi|^{2\theta} |(\widehat{f} * \widehat{F}_v)(\xi)|^2\, \mathrm{d}\xi. \quad (5.9)$$

Now, we integrate both sides over $|v| \geq \frac{3}{2}u + X_0$. The integral of the LHS is at least

$$\int_{|x| \geq 3u + X_0} (|x| - 3u)^{2a\theta} |f(x)|^2\, \mathrm{d}x \cdot \int_{\mathbb{R}} |F(x)|^2\, \mathrm{d}x,$$

because, as long as x with $|x| \geq 3u + X_0$ is contained in $\operatorname{supp} F_v$, we have $|v| \geq \frac{3}{2}u + X_0$ and $|v| - \frac{3}{2}u \geq |x| - 3u$. Assuming $u \geq X_0$ and $K > 4$, we can bound this quantity from below by

$$\left(\frac{K - 3}{K}\right)^{2a\theta} \int_{|x| \geq Ku} |x|^{2a\theta} |f(x)|^2\, \mathrm{d}x \cdot \int_{\mathbb{R}} |F(x)|^2\, \mathrm{d}x.$$

---

[3]Indeed, assume that $\|f\|_{L^2(\mathbb{R})} = 1$. Since the integrals $\int_{\mathbb{R}} |f(x)|^2 |x|^{p\ell}\, \mathrm{d}x$ are finite for all $\ell \geq 0$, we can choose a constant C_1 so big that, for $0 \leq \ell < \ell_0$,

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{p\ell}\, \mathrm{d}x \leq C_1^\ell \ell!.$$

Whence,

$$\int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p}\, \mathrm{d}x = \sum_{\ell \geq 0} \frac{c^\ell}{\ell!} \int_{\mathbb{R}} |f(x)|^2 |x|^{p\ell}\, \mathrm{d}x \leq \sum_{\ell \geq 0} \left(c \cdot \max(C, C_1)\right)^\ell < \infty,$$

provided that c is chosen so small that $c \cdot \max(C, C_1) < 1$.

Let us now estimate the integral $\int_{\mathbb{R}} \mathrm{d}\upsilon \int_{\mathbb{R}} |\xi|^{2\theta} |(\widehat{f} * \widehat{F_\upsilon})(\xi)|^2 \, \mathrm{d}\xi$, which appears when we integrate the RHS of (5.9) over υ. As in the previous step, noting that $\widehat{F_\upsilon}(\eta) = e^{-2\pi i \upsilon \eta} \widehat{F}(\eta)$ and using the Plancherel theorem, we find that it is equal to

$$\iint_{\mathbb{R} \times \mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 |\xi + \eta|^{2\theta} \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

Note that

$$\widehat{F}(\eta) = \widehat{\mathbb{1}}_{[-u,u]}(\eta) \cdot \left(\frac{\sin \pi \frac{u}{k} \eta}{\pi \frac{u}{k} \eta} \right)^k.$$

We claim that

$$(|\xi| + |\eta|) \cdot \left| \frac{\sin \pi \frac{u}{k} \eta}{\pi \frac{u}{k} \eta} \right| \leq |\xi| + \frac{k}{\pi u}.$$

If $|\eta| \leq k/(\pi u)$ the claim becomes obvious after we estimate the second factor on the LHS by 1. Otherwise, we estimate the absolute value of the sine on the LHS by 1, and note that

$$\pi \frac{u|\eta|}{k} \cdot \left(|\xi| + \frac{k}{\pi u} \right) \geq |\xi| + |\eta|$$

because in this case $\pi u |\eta|/k \geq 1$. Thus, if $k \geq \theta$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 |\xi + \eta|^{2\theta} \, \mathrm{d}\xi \, \mathrm{d}\eta \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} |\widehat{\mathbb{1}}_{[-u,u]}(\eta)|^2 \, \mathrm{d}\xi \, \mathrm{d}\eta$$

$$= 2u \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \, \mathrm{d}\xi,$$

where in the first step we again used that $|\sin t|/|t| \leq 1$.

Taking into account that $\int_{\mathbb{R}} |F(x)|^2 \, \mathrm{d}x \geq u$ (because $F(x) = 1$ if $|x| \leq u/2$) and juxtaposing the estimates for the LHS and RHS of (5.9), we arrive at the inequality

$$\int_{|x| \geq Ku} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x \leq 2 \left(\frac{1}{\sigma} \left(\frac{K}{K-3} \right)^a \right)^{2\theta} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \, \mathrm{d}\xi, \quad (5.10)$$

provided that $u \geq X_0$, $K > 4$, and $k \geq \theta$. We will be interested in large θ. So, we choose K so large that $\frac{1}{\sigma} \left(\frac{K}{K-3} \right)^a < 1$. Then we shall take $k \in [\theta, \pi\theta]$ and split the integral on the RHS into the sum of the integrals

$$\left(\int_{|\xi| > K \frac{k}{\pi u}} + \int_{|\xi| \leq K \frac{k}{\pi u}} \right) |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \, \mathrm{d}\xi$$

$$\leq \left(\frac{K+1}{K} \right)^{2\theta} \int_{|\xi| \geq K \frac{k}{\pi u}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta} \, \mathrm{d}\xi + \int_{|\xi| \leq K \frac{k}{\pi u}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \, \mathrm{d}\xi$$

$$\leq \left(\frac{K+1}{K} \right)^{2\theta} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta} \, \mathrm{d}\xi + \int_{|\xi| \leq K \frac{k}{\pi u}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \, \mathrm{d}\xi.$$

Adding the integral $\int_{|x| \leq Ku} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x$ to both sides of (5.10), we obtain

$$\int_{\mathbb{R}} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x \leq 2 \left(\frac{1}{\sigma} \frac{K+1}{K} \left(\frac{K}{K-3} \right)^a \right)^{2\theta} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta} \, \mathrm{d}\xi$$

$$+ 2 \int_{|\xi| \leq K \frac{k}{\pi u}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \mathrm{d}\xi + \int_{|x| \leq Ku} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x.$$

Bounding the second and the third terms on the RHS as

$$\int_{|\xi| \leq K \frac{k}{\pi u}} |\widehat{f}(\xi)|^2 \left(|\xi| + \frac{k}{\pi u} \right)^{2\theta} \mathrm{d}\xi \underset{k \leq \pi\theta}{\leq} (K+1)^{2\theta} \left(\frac{\theta}{u} \right)^{2\theta} \int_{\mathbb{R}} |f|^2$$

and

$$\int_{|x| \leq Ku} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x \leq K^{2a\theta} u^{2a\theta} \int_{\mathbb{R}} |f|^2,$$

and assuming further that $\frac{K+1}{K} \left(\frac{K}{K-3} \right)^a \sigma^{-1} < 1$, we conclude that, for large enough θ, we have

$$\int_{\mathbb{R}} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2\theta} \, \mathrm{d}\xi$$

$$+ \left(K^{2a\theta} u^{2a\theta} + 2(K+1)^{2\theta} \left(\frac{\theta}{u} \right)^{2\theta} \right) \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x.$$

At last, choosing $u = \theta^{\frac{1}{a+1}}$, which is consistent with $u \geq X_0$ as $\theta \to \infty$, we finally get

$$\int_{\mathbb{R}} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x \leq \frac{1}{2} \int_{\mathbb{R}} |\xi|^{2\theta} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi + C^{2\theta} \theta^{\frac{2a}{a+1}\theta} \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x. \qquad (5.11)$$

By the same argument, if $\widehat{f}$ vanishes on the set M such that, for $|\xi| \geq \Xi_0$, the constituting interval of $\mathbb{R} \setminus M$ containing ξ has length at most $(2\sigma |\xi|^b)^{-1}$ with $\sigma > 1$ and $b = a^{-1}$, we obtain

$$\int_{\mathbb{R}} |\xi|^{2b\kappa} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \leq \frac{1}{2} \int_{\mathbb{R}} |x|^{2\kappa} |f(x)|^2 \, \mathrm{d}x + C^{2\kappa} \kappa^{\frac{2b}{b+1}\kappa} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi, \qquad (5.12)$$

provided that κ is large enough.

We will choose θ and κ so that $\kappa = a\theta$, and, since $b = a^{-1}$, $\theta = b\kappa$. Note that in this case we have

$$\frac{a}{a+1} \theta = \frac{\kappa}{a+1} = \frac{b}{b+1} \kappa,$$

so the second terms in the RHS of (5.11) and (5.12) are the same up to the value of the constant C. Adding (5.11) and (5.12), we thus obtain, for big enough θ,

$$\int_{\mathbb{R}} |x|^{2a\theta} |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} |\xi|^{2b\kappa} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \leq C_1^{2\theta} \theta^{\frac{2a}{a+1}\theta} \int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x.$$

At last, we choose

$$\theta = \frac{a+1}{2a}\ell, \quad \ell = \ell_0, \ell_0 + 1, \ell_0 + 2, \ldots,$$

where ℓ_0 is a sufficiently large positive integer. Since $a = p - 1$, with this choice of the parameters, we have $2a\theta = p\ell$, $2\theta = q\ell$, and therefore, for $\ell \geq \ell_0$,

$$\max\left[\int_{\mathbb{R}} |x|^{p\ell}|f(x)|^2\,\mathrm{d}x, \int_{\mathbb{R}} |\xi|^{q\ell}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\right]$$
$$\leq C_2^\ell \ell^\ell \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x \leq C_3^\ell \ell! \int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x,$$

proving (5.8).

5.3. Phragmén–Lindelöf tricks. We know that, for some $c > 0$, the integrals $\int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p}\,\mathrm{d}x$ and $\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 e^{c|\xi|^q}\,\mathrm{d}\xi$ are finite. These estimates imply, in particular, that f' and $\widehat{f}'$ are bounded in absolute value by some constant A. Therefore, for every $x_0 \in \mathbb{R}$,

$$\int_{\mathbb{R}} |f(x)|^2 e^{c|x|^p}\,\mathrm{d}x \geq \int_{\mathbb{R}} \max(|f(x_0)| - A|x - x_0|, 0)^2 e^{c|x|^p}\,\mathrm{d}x$$
$$\geq \int_{x_0 - (2A)^{-1}|f(x_0)|}^{x_0 + (2A)^{-1}|f(x_0)|} (|f(x_0)| - A(x - x_0))^2 e^{c|x|^p}\,\mathrm{d}x$$
$$\geq \frac{1}{8A}|f(x_0)|^3 e^{c|x_0|^p},$$

whence, $|f(x)| = O\left(e^{-\frac{c}{3}|x|^p}\right)$ as $x \to \infty$. Similarly, $|\widehat{f}(y)| = O\left(e^{-\frac{c}{3}|y|^q}\right)$ as $y \to \infty$. We also note that, by standard estimates of the Fourier transform, f is an entire function of order p, and $\widehat{f}$ is an entire function of order q, and naturally, in this part we will use some complex analysis techniques.

We begin by defining the pair of the Phragmén–Lindelöf indicators

$$k_1(\theta) = \frac{1}{2\pi} \limsup_{r \to \infty} \frac{\log|f(re^{\mathrm{i}\theta})|}{r^p},$$

and

$$k_2(\varphi) = \frac{1}{2\pi} \limsup_{\rho \to \infty} \frac{\log|\widehat{f}(\rho e^{\mathrm{i}\varphi})|}{\rho^q}$$

(the factors $(2\pi)^{-1}$ are introduced to simplify some expressions). We note that both indicators are negative at 0 and $\pm\pi$, and set

$$\kappa_i = \min(|k_i(0)|, |k_i(\pm\pi)|), \quad i = 1, 2.$$

Claim 5.1. *We have*

$$k_1(\theta) \leq \frac{1}{p(q\kappa_2)^{p/q}}|\sin\theta|^p$$

and

$$k_2(\varphi) \leq \frac{1}{q(p\kappa_1)^{q/p}}|\sin\varphi|^q.$$

Proof. For every $\varepsilon > 0$, we have

$$
\begin{aligned}
|f(z)| &\le \int_{\mathbb{R}} |\widehat{f}(\xi)| e^{2\pi |\xi| |\operatorname{Im} z|} \, d\xi \\
&\le C_\varepsilon \int_{\mathbb{R}} e^{(-2\pi \kappa_2 + 2\varepsilon)|\xi|^q + 2\pi |\xi| |\operatorname{Im} z|} e^{-\varepsilon |\xi|^q} \, d\xi \\
&\le C'_\varepsilon e^{2\pi \max\limits_{\xi \in \mathbb{R}} (|\xi| |\operatorname{Im} z| - (\kappa_2 - \frac{\varepsilon}{\pi})|\xi|^q)}.
\end{aligned}
$$

By the Young inequality,

$$
|\xi| |\operatorname{Im} z| \le \left(\kappa_2 - \frac{\varepsilon}{\pi} \right) |\xi|^q + \frac{1}{p} \frac{|\operatorname{Im} z|^p}{\left(q \left(\kappa_2 - \frac{\varepsilon}{\pi} \right) \right)^{p/q}}.
$$

Hence, setting $z = r e^{i\theta}$ and letting $r \to \infty$, we get

$$
k_1(\theta) \le \frac{1}{p} \frac{|\sin \theta|^p}{\left(q \left(\kappa_2 - \frac{\varepsilon}{\pi} \right) \right)^{p/q}}.
$$

Since $\varepsilon > 0$ was arbitrary, this gives the first bound. The proof of the second estimate is the same. $\qquad\square$

Replacing $f(x)$ by $f(-x)$ if necessary, we may assume without loss of generality that $\kappa_1 = |k_1(0)|$. Replacing $f(x)$ by $\overline{f(x)}$ after that, if needed, we can also assume that $\kappa_2 = |k_2(0)|$ as well. We also note that both quantities κ_1, κ_2 are finite[4].

To use the density of zeroes of f and $\widehat{f}$, we need an observation about analytic functions of exponential type in an angle $\Gamma(\alpha) = \{z \colon |\arg(z)| \le \alpha\}$ with $\alpha < \pi/2$.

Claim 5.2. *Suppose that F is analytic in $\Gamma(\alpha)$, continuous up to the boundary, and satisfies $|F(z)| \le M e^{a\pi |\operatorname{Im} z|}$ with some $a < 1$. Let $\mathfrak{Z} = \{\mathfrak{z}_1, \mathfrak{z}_2, \dots\} \subset \mathbb{R}_+$ be a discrete set with $\mathfrak{z}_1 < \mathfrak{z}_2 < \dots$ and $\mathfrak{z}_{j+1} - \mathfrak{z}_j \le 1$. If $F|_{\mathfrak{Z}} = 0$, then*

$$
\limsup_{x \to +\infty} \frac{\log |F(x)|}{x} \le Q(a, \alpha) < 0.
$$

Note that we do not require $\mathfrak{z}_1$ to be close to the origin and that it does not appear as an argument of Q.

Proof. Take any point $x \in \mathbb{R}_+$ and consider the disk D of radius $r = x \sin \alpha$ centered at x. Note that $D \subset \Gamma(\alpha)$. We take x so large that $(1 - \sin \alpha) x > \mathfrak{z}_1$. Then the points of the set $\mathfrak{Z} \cap D$ form a 1-net on the diameter of D. By the classical Jensen formula, we have

$$
\log |F(x)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| F\left(x + r e^{i\eta} \right) \right| \, d\eta + \sum_{\mathfrak{z} \in \mathfrak{Z} \cap D} \log \frac{|x - \mathfrak{z}|}{r}.
$$

---

[4]The Phragmén–Lindelöf principle yields that if the indicator of an entire function of finite order and bounded type equals to $-\infty$ at one point, then the function vanishes identically, see [15, Ch.I, §15] or [16, Remark 2 in Section 8.2]

The first term on the RHS is bounded from above by

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} a\pi r |\sin \eta| \, d\eta + \log M = 2ar + O(1),$$

while the sum can be estimated from above by

$$2 \sum_{1 \le k \le r} \log \frac{k}{r} \le 2 \sum_{1 \le k \le r} \int_{k}^{k+1} \log \frac{t}{r} \, dt \le 2 \int_{1}^{r+1} \log \frac{t}{r} \, dt = (-2 + o(1))r$$

as $r \to \infty$. Thus,

$$\frac{1}{x} \log |F(x)| \le (2(a-1) + o(1)) \sin \alpha, \quad x \to \infty,$$

which implies the claim with $Q(a, \alpha) = -2(1-a) \sin \alpha$. □

Note that if elements of $\mathfrak{Z}$ are ℓ-dense instead of being 1-dense, we can consider $F(\ell z)$ instead of F, assume that $a\ell < 1$, and arrive at essentially the same conclusion.

Now we claim that, for all $\theta \in \left(0, \frac{\pi}{2p}\right)$, we have

$$\kappa_1 \ge \frac{1}{p} \tan p\theta - \frac{1}{p(q\kappa_2)^{p/q}} \frac{(\sin \theta)^p}{\cos p\theta}. \tag{5.13}$$

Indeed, consider the function $F(z) = e^{(2\pi\kappa_1 - \varepsilon)z} f(z^{1/p})$ with small $\varepsilon > 0$ in the right half-plane $\mathrm{Re}(z) > 0$. By Claim 5.1, for $|\theta| < \frac{\pi}{2p}$, we have

$$\log |F(re^{ip\theta})| \le 2\pi (\kappa_1 \cos p\theta + k_1(\theta))r + O(1)$$
$$\le 2\pi \left(\kappa_1 \cos p\theta + \frac{1}{p(q\kappa_2)^{p/q}} |\sin \theta|^p \right) r + O(1).$$

Suppose that estimate (5.13) does not hold for some $\theta_0 \in \left(0, \frac{\pi}{2p}\right)$, that is,

$$\kappa_1 \cos p\theta_0 + \frac{1}{p(q\kappa_2)^{p/q}} (\sin \theta_0)^p < \frac{c}{p} \sin p\theta_0$$

with some $c < 1$. Consider the function F in the angle $\Gamma(\alpha)$ with $\alpha = p\theta_0$. On the boundary rays $|F(z)| = O\left(e^{\frac{2\pi c}{p} |z| \sin p\theta_0}\right) = O\left(e^{a\pi |\,\mathrm{Im}\, z|}\right)$ with $a = \frac{2c}{p}$ satisfying $a\frac{p}{2} < 1$. On the positive ray $|F|$ remains bounded (recall that $\kappa_1 = -k_1(0)$). At last, $|F(z)| = O\left(e^{C|z|}\right)$ everywhere in $\Gamma(\alpha)$. Therefore, by the Phragmén–Lindelöf principle, the function $F(z)e^{a\pi i z}$ is bounded in the angle $\{0 \le \arg(z) \le \alpha\}$ (the upper half of $\Gamma(\alpha)$). Thus, $\log |F(z)| \le a\pi \,\mathrm{Im}\, z + O(1)$ for $\mathrm{Im}\, z \ge 0$, $z \in \Gamma(\alpha)$. Arguing similarly in the lower half of $\Gamma(\alpha)$, we see that $|F(z)| \le O\left(e^{a\pi |\,\mathrm{Im}\, z|}\right)$ everywhere in $\Gamma(\alpha)$.

On the other hand, F vanishes at every point λ^p, $\lambda \in \Lambda \cap (0, \infty)$. By the supercriticality of Λ,

$$\lambda_{j+1}^p - \lambda_j^p \le p\lambda_{j+1}^{p-1}(\lambda_{j+1} - \lambda_j) < \frac{1}{2}p$$

for large enough j, and, applying Claim 5.2, we conclude that

$$\limsup_{x \to +\infty} \frac{\log |F(x)|}{x} \leq Q < 0$$

independently of ε. However, the definition of $\kappa_1 = -k_1(0)$ implies that it is impossible if $\varepsilon < |Q|$, proving estimate (5.13).

Thus, we have

$$\kappa_1 \geq \frac{1}{p} \tan p\theta - \frac{1}{(q\kappa_2)^{p/q}} \frac{|\sin \theta|^p}{p \cos p\theta}, \quad 0 < \theta < \frac{\pi}{2p}.$$

Next, we will need the trigonometric inequality

$$\frac{1}{p} \tan p\theta \geq \frac{\sin \theta}{(\cos p\theta)^{1/p}}, \quad 0 \leq \theta < \frac{\pi}{2p},$$

with strict inequality for $\theta > 0$. Indeed, both sides vanish at $\theta = 0$ and the derivative of the LHS is $\frac{1}{\cos^2 p\theta}$, while the derivative of the RHS is

$$\frac{\cos \theta \cos p\theta + \sin \theta \sin p\theta}{(\cos p\theta)^{1+1/p}} = \frac{\cos(p-1)\theta}{(\cos p\theta)^{1+1/p}},$$

which is clearly smaller for $\theta \in \left(0, \frac{\pi}{2p}\right)$.

Letting $\eta = \frac{\sin \theta}{(\cos p\theta)^{1/p}}$, we get

$$\kappa_1 > \eta - \frac{1}{(q\kappa_2)^{p/q}} \frac{\eta^p}{p}.$$

Maximizing the RHS with respect to η (note that η runs through the whole interval $(0, +\infty)$ when θ increases from 0 to $\pi/(2p)$), we let $\eta = q\kappa_2$, which ultimately gives us $\kappa_1 > \kappa_2$.

The same argument with the roles of p and q exchanged gives us $\kappa_2 > \kappa_1$. The contradiction we arrive at shows that f is a zero function, completing the proof of Theorem 1.3-UP. □

6. Frame bounds

In this Section we turn to Theorem 1.8, using ideas similar to the ones we used in the proofs of Theorem 1.8A dealing with uniformly supercritical sets and in the first step of the proof of Theorem 1.3-UP. *Throughout this section we always assume that* (Λ, M) *is a separated supercritical pair with parameters* (p, q), $\frac{1}{p} + \frac{1}{q} = 1$.

First, we show that the lower frame-type bound in the conclusion of Theorem 1.8 holds if we add to each of the sets Λ, M finitely many points. Then, using a classical Duffin–Schaeffer argument, we remove this extra assumption and prove this lower bound without it. Finally, we prove the upper frame-type bound and thus deduce Theorem 1.8 in full generality.

6.1. Adding finitely many points to Λ and M. Here, we show that for all $s > 0$ such that $s\min(p, q) \geq 1$ there exist sets $\Lambda' \supset \Lambda$, $M' \supset M$ and a constant $C > 0$, such that $\Lambda' \backslash \Lambda$ and $M' \backslash M$ are finite, and, for all functions f in $\mathcal{H}$, we have

$$\|f\|_{\mathcal{H}_{s,p,q}}^2 \leq C\left(\sum_{\lambda \in \Lambda'} (1 + |\lambda|)^{(2s-1)p+1} |f(\lambda)|^2 + \sum_{\mu \in M'} (1 + |\mu|)^{(2s-1)q+1} |\widehat{f}(\mu)|^2 \right).$$
(6.1)

Proof. We assume that both sums on the RHS are finite (otherwise, there is nothing to prove). We will use the same notation and conventions as in Section 5.1. Specifically, we assume without loss of generality that $p \geq 2 \geq q$ and let $a = p - 1$, $b = q - 1$. We consider the same convex increasing functions Φ_T and Ψ_T as in that section: for big number T, we put $\Phi_T(t) = t^\theta, 0 \leq t \leq T$ and $\Phi_T(t) = T^\theta + \theta T^{\theta-1}(t - T), t \geq T$, where $\theta = qs$, and $\Psi_T(t) = \Phi_T(t^a)$. Note that for $t \geq 0$ we have $\Phi_T(t) \leq t^\theta$, $\Phi_T'(t) \leq \theta t^{\theta-1}$, and $\Psi_T(t) \leq t^{a\theta}$, $\Psi_T'(t) \leq a\theta t^{a\theta-1}$.

Let $F \in C_0^\infty(-1, 1)$ with $\int_{\mathbb{R}} |F(x)|^2 \, dx = 1$. Put $F_\upsilon(x) = F(x - \upsilon)$. For big x, the constituting interval of $\mathbb{R}\backslash\Lambda$ containing x has length at most $(1 - \varepsilon)(2\sigma(1 + |x|)^{p-1})^{-1}$ for some $\varepsilon > 0$, $\sigma > 1$. The set Λ' will be Λ with a lot of points added near the origin so that all the constituting intervals of $\mathbb{R}\backslash\Lambda'$ have length bounded by a small numerical constant. Applying Lemma 3.4(i) to the function fF_υ and the function Φ_T, we get

$$\Phi_T V(\sigma^2 U(\upsilon)^2) \int_{\mathbb{R}} |f(x)|^2 |F_\upsilon(x)|^2 \, dx \leq \int_{\mathbb{R}} |(\widehat{f} * \widehat{F}_\upsilon)(\xi)|^2 \Phi_T(\xi^2) \, d\xi$$
$$+ C_\varepsilon \sigma U(\upsilon) \Phi_T'(\sigma^2 U(\upsilon)^2) \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 |F_\upsilon(\lambda)|^2, \quad (6.2)$$

where $U(\upsilon) = \max(K, |\upsilon|)^{p-1}$ and positive K can be chosen arbitrary large, provided that we add sufficiently many points to Λ.

To proceed, we need an auxiliary estimate.

Claim 6.1. *For $|x - \upsilon| \leq 1$, we have*

$$\Phi_T(\sigma^2 U(\upsilon)^2) \geq (1 - \varepsilon_1)\Phi_T(\sigma^2 U(x)^2),$$

where ε_1 does not depend on T and can be chosen arbitrarily small when K is big enough, while the parameters p, q and s remain fixed.

Proof. Since the function U is even, we may assume that x and υ are non-negative. If $\upsilon \geq x$ or $\upsilon \leq x \leq K$ the estimate is obvious with $\varepsilon_1 = 0$. In the remaining case, we have $K - 1 < x - 1 \leq \upsilon < x$, whence

$$U(\upsilon) \geq U(x - 1) \geq (x - 1)^{p-1} = (1 - x^{-1})^{p-1} U(x) \geq (1 - K^{-1})^{p-1} U(x).$$

Next, we note that $\Phi_T'(t) \leq \theta t^{-1}\Phi_T(t)$, whence (taking into account that Φ_T' does not decrease), for $t' < t$,

$$\Phi_T(t) \leq \Phi_T(t') + (t - t')\Phi_T(t) \leq \Phi_T'(t') + \theta\left(1 - \frac{t'}{t}\right)\Phi_T(t).$$

Using this with $t = \sigma^2 U(x)^2$ and $t' = \sigma^2 U(v)^2$ (and recalling that $\theta = qs$ is fixed), we get what we wanted. $\qquad \square$

Now, integrating inequality (6.2) over $v \in \mathbb{R}$, we obtain

$$\iint_{\mathbb{R} \times \mathbb{R}} |f(x)|^2 |F_v(x)|^2 \Phi_T\big(\sigma^2 U(v)^2\big)\, \mathrm{d}v\, \mathrm{d}x \leq \iint_{\mathbb{R} \times \mathbb{R}} |(\widehat{f} * \widehat{F}_v)(\xi)|^2 \Phi_T\big(\xi^2\big)\, \mathrm{d}\xi\, \mathrm{d}v$$

$$+ C_\varepsilon \sigma \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 \int_{|v - \lambda| \leq 1} U(v) \Phi_T'\big(\sigma^2 U(v)^2\big) |F_v(\lambda)|^2\, \mathrm{d}v. \quad (6.3)$$

For the LHS of (6.3), using Claim 6.1, we have

$$\iint_{\mathbb{R} \times \mathbb{R}} |f(x)|^2 |F_v(x)|^2 \Phi_T\big(\sigma^2 U(v)^2\big)\, \mathrm{d}v\, \mathrm{d}x \geq (1 - \varepsilon_1) \int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(\sigma^2 U(x)^2\big)\, \mathrm{d}x.$$

Since Φ_T is convex and $\Phi_T(0) = 0$, we have

$$(1 - \varepsilon_1) \int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(\sigma^2 U(x)^2\big)\, \mathrm{d}x \geq (1 - \varepsilon_1)\sigma^2 \int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(U(x)^2\big)\, \mathrm{d}x.$$

Using that $\Phi_T'\big(\sigma^2 U(v)^2\big) \leq \theta \sigma^{2\theta - 2} U(v)^{2\theta - 2}$ and that, for $|\lambda - v| \leq 1$, $U(v)^{2\theta - 1} \leq C_1(K) U(\lambda)^{2\theta - 1}$, we bound from above the sum term which appears on the RHS of (6.3) by

$$C_1(K) \cdot C_\varepsilon \theta \sigma^{2\theta - 1} \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 U(\lambda)^{2\theta - 1}.$$

As in Sections 5.1 and 5.2, by the Plancherel theorem, after integration over v the integral term on the RHS of (6.3) equals

$$\iint_{\mathbb{R} \times \mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 \Phi_T\big(|\xi + \eta|^2\big)\, \mathrm{d}\xi\, \mathrm{d}\eta. \quad (6.4)$$

To bound this integral, we need another auxiliary estimate for the function Φ_T.

Claim 6.2. *For all $\delta > 0$, there exists C_δ (possibly also depending on s, p, q but not on T) such that for all $\xi, \eta \in \mathbb{R}$, we have*

$$\Phi_T\big(|\xi + \eta|^2\big) \leq (1 + \delta)\Phi\big(\xi^2\big) + C_\delta |\eta|^{2\theta}. \quad (6.5)$$

Proof. We use the bound (5.4)

$$\Phi_T\big((\xi + \eta)^2\big) \leq \Phi_T\big(\xi^2\big) \left(1 + \frac{|\eta|}{|\xi|}\right)^{2\theta}.$$

If $|\eta| \leq c|\xi|$, this estimate gives us

$$\Phi_T\big((\xi + \eta)^2\big) \leq \Phi_T\big(\xi^2\big)(1 + c)^{2\theta} < (1 + \delta)\Phi_T\big(\xi^2\big),$$

provided that $(1 + c)^{2\theta} < 1 + \delta$. If $|\eta| \geq c|\xi|$, then

$$\Phi_T\big((\xi + \eta)^2\big) \leq \Phi_T\big((1 + c^{-1})^2 \eta^2\big) \leq \big(1 + c^{-1}\big)^{2\theta} |\eta|^{2\theta},$$

proving the claim. $\qquad \square$

Now, we return to the integral (6.4). Since $|\widehat{F}(\eta)|^2$ decays faster than any negative power and $\int_{\mathbb{R}} |\widehat{F}(\eta)|^2 \, d\eta = \int_{\mathbb{R}} |F(x)|^2 \, dx = 1$, using (6.5) and letting $V(\xi) = \max(K, |\xi|)^{q-1}$, we get

$$
\int_{\mathbb{R}} |\widehat{F}(\eta)|^2 \Phi_T\big(|\xi + \eta|^2\big) d\eta \leq (1 + \delta)\Phi_T\big(\xi^2\big) + C'_\delta
$$
$$
= (1 + \delta)\Psi_T\big(\xi^{2(q-1)}\big) + C'_\delta \leq (1 + 2\delta)\Psi_T\big(V(\xi)^2\big),
$$

if K is big enough. Then, the integral (6.4) does not exceed

$$
(1 + 2\delta) \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(V(\xi)^2\big) \, d\xi.
$$

Combining everything, we get, for big enough K,

$$
(1 - \varepsilon_1)\sigma^2 \int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(U(x)^2\big) \, dx \leq (1 + 2\delta) \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(V(\xi)^2\big) \, d\xi
$$
$$
+ C_1(K) \cdot C_\varepsilon \theta \sigma^{2\theta-1} \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 U(\lambda)^{2\theta-1}.
$$

Recalling that

- ε_1 and δ can be made arbitrarily small while $\sigma > 1$ remains fixed,
- $U(\lambda)^{2\theta-1} = \max(K, |\lambda|)^{(2\theta-1)(p-1)}$,
- $\theta = sq = sp/(p-1)$, whence, $(2\theta - 1)(p-1) = 2sp - p + 1 = (2s-1)p + 1$,

we get

$$
\int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(U(x)^2\big) \, dx \leq (1 - \rho) \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(V(\xi)^2\big) \, d\xi
$$
$$
+ C \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 \max(K, |\lambda|)^{(2s-1)p+1}
$$

with some $\rho > 0$. Note that the integral in the RHS is finite since $f \in \mathcal{H}$ and the sum is finite by our assumption. Thus the LHS is finite as well.

Similarly, adding enough points to M and repeating the same argument for the function $\widehat{f}$ with the function Ψ_T instead of Φ_T we get

$$
\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(V(\xi)^2\big) \, d\xi \leq (1 - \rho) \int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(U(x)^2\big) \, dx
$$
$$
+ C \sum_{\mu \in M'} |\widehat{f}(\mu)|^2 \max(K, |\mu|)^{(2s-1)q+1},
$$

where the RHS is finite as we just established.

Summing these estimates and dividing by ρ we get

$$
\int_{\mathbb{R}} |f(x)|^2 \Phi_T\big(U(x)^2\big) \, dx + \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \Psi_T\big(V(\xi)^2\big) \, d\xi
$$

$$\leq C \sum_{\lambda \in \Lambda'} |f(\lambda)|^2 \max(K, |\lambda|)^{(2s-1)p+1} + C \sum_{\mu \in M'} |\widehat{f}(\mu)|^2 \max(K, |\mu|)^{(2s-1)q+1}.$$

Finally, letting $T \to \infty$ and using the dominated convergence theorem we see that the LHS tends to

$$\int_{\mathbb{R}} |f(x)|^2 \max(K, |x|)^{2ps} \mathrm{d}x + \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \max(K, |\xi|)^{2qs} \, \mathrm{d}\xi,$$

which is at least $c\|f\|_{\mathcal{H}_{s,p,q}}^2$ for some $c > 0$. Bounding $\max(K, |y|)$ on the RHS by $C'(1 + |y|)$, we complete the proof of inequality (6.1). $\qquad\square$

6.2. Removing finitely many points. Our next goal is to show that the main estimate (6.1) holds for the pair (Λ, M). For this, we use the following lemma, cf. Duffin–Schaeffer [7, Lemma IX]:

Lemma 6.3 (Duffin–Schaeffer). *Let $\{v_n\}_{n=0}^{\infty}$ be a system of vectors in a Hilbert space H. Assume that there exists $C > 0$ such that, for all $v \in H$, we have*

$$\|v\|^2 \leq C \sum_{n=0}^{\infty} |\langle v, v_n \rangle|^2, \tag{6.6}$$

and assume that the system $\{v_n\}_{n=1}^{\infty}$ is complete in H. Then there exists $C' > 0$ such that, for all $v \in H$, we have

$$\|v\|^2 \leq C' \sum_{n=1}^{\infty} |\langle v, v_n \rangle|^2. \tag{6.7}$$

For the reader's convenience, we provide a simple proof.

Proof. We pick a small number $0 < \varepsilon < 1$ to be determined later and take a finite linear combination of the vectors $v_1, v_2, \ldots,$ approximating v_0 with error at most ε:

$$\left\| \sum_{n=1}^{M} a_n v_n - v_0 \right\| \leq \varepsilon.$$

By Cauchy–Schwarz inequality we have

$$\left\| \left\langle v, \sum_{n=1}^{M} a_n v_n \right\rangle \right| - |\langle v, v_0 \rangle| \right| \leq \varepsilon \|v\|,$$

therefore,

$$|\langle v, v_0 \rangle|^2 - \left| \left\langle v, \sum_{n=1}^{M} a_n v_n \right\rangle \right|^2 \leq (2\|v_0\| + \varepsilon)\varepsilon \|v\|^2 \leq \varepsilon(2\|v_0\| + 1)\|v\|^2. \tag{6.8}$$

We choose $\varepsilon < (2C(2\|v_0\| + 1))^{-1}$. Juxtaposing estimates (6.6) and (6.8), we get

$$\|v\|^2 \le 2C \left(\sum_{n=1}^{\infty} |\langle v, v_n \rangle|^2 + \left| \left\langle v, \sum_{n=1}^{M} a_n v_n \right\rangle \right|^2 \right).$$

Once again applying the Cauchy–Schwarz inequality, we get

$$\|v\|^2 \le 2C \left(\sum_{n=1}^{\infty} |\langle v, v_n \rangle|^2 + M \sum_{n=1}^{M} |a_n|^2 |\langle v, v_n \rangle|^2 \right),$$

which yields the estimate we need with $C' = 2C(1 + M \max |a_n|^2)$. $\qquad \square$

By induction, the same conclusion holds if we have an arbitrary finite set of vectors in place of v_0.

With this lemma at hand, our next goal is to prove that, for $f \in \mathcal{H}$,

$$\|f\|_{\mathcal{H}_{s,p,q}}^2 \le C \left[\sum_{\lambda \in \Lambda} (1 + |\lambda|)^{(2s-1)p+1} |f(\lambda)|^2 + \sum_{\mu \in M} (1 + |\mu|)^{(2s-1)q+1} |\widehat{f}(\mu)|^2 \right]. \quad (6.9)$$

First, we show this assuming that $f \in \mathcal{H}_{s,p,q}$. If $s \min(p, q) \ge 1$, the point evaluations are continuous functionals on the Hilbert space $\mathcal{H}_{s,p,q}$, therefore, there exist vectors $\varphi_x \in \mathcal{H}_{s,p,q}$ such that for all $f \in \mathcal{H}_{s,p,q}$ we have $f(x) = \langle f, \varphi_x \rangle$. Similarly, there exist vectors ψ_ξ such that $\widehat{f}(\xi) = \langle f, \psi_\xi \rangle$. Then, inequality (6.1) says that suitably normalized vectors $\varphi_\lambda, \lambda \in \Lambda'$, and $\psi_\mu, \mu \in M'$, satisfy assumption (6.6) in Lemma 6.3. On the other hand, since $\mathcal{H}_{s,p,q} \subset \mathcal{H}$, by Theorem 1.3-UP, the system $\{\varphi_\lambda\}_{\lambda \in \Lambda} \cup \{\psi_\mu\}_{\mu \in M}$ is complete in $\mathcal{H}_{s,p,q}$. Thus, by Lemma 6.3, this system satisfies (6.7), which gives us (6.9) for $f \in \mathcal{H}_{s,p,q}$.

Now, if $f \in \mathcal{H}$ and the sums over Λ and M in (6.9) are finite, then the same sums over Λ' and M' are finite as well (the sets $\Lambda' \setminus \Lambda$ and $M' \setminus M$ are finite, and, for $f \in \mathcal{H}$, the point evaluations are bounded on the space and the Fourier sides). Then, by Section 6.1, $f \in \mathcal{H}_{s,p,q}$, and we arrive at the already considered case.

6.3. Upper frame bound. In this section we will show that, if the sequences Λ and M are p- and q-separated, respectively, then the converse to the estimate (6.9) is also true. Our first step is the following extension of Lemma 3.2.

Claim 6.4. *Let Γ be a discrete set such that all gaps between consecutive elements of it have lengths at least δ for some $\delta > 0$. There exists an absolute constant $C > 0$ such that for all $\theta \ge 1$ and for all $f \in H_\theta$ we have*

$$\delta \sum_{\gamma \in \Gamma} |f(\gamma)|^2 \le C \left[\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x + \delta^{2\theta} \int_{\mathbb{R}} |\xi|^{2\theta} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right].$$

Proof. If $\theta = 1$ then this follows from summing suitably scaled estimates from the Lemma 3.2 on all the intervals between consecutive elements of Γ. For the general case, we have

$$\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x + \delta^2 \int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 = \int_{\mathbb{R}} \left(1 + (\delta|\xi|)^2\right) |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

$$\leq 2 \int_{\mathbb{R}} \left(1 + (\delta|\xi|)^{2\theta}\right) |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi = 2 \left(\int_{\mathbb{R}} |f(x)|^2 \mathrm{d}x + \delta^{2\theta} \int_{\mathbb{R}} |\xi|^{2\theta} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \right).$$

Thus we need to multiply the constant C by at most 2. $\qquad\square$

Now, we assume that the set Λ is p-separated, that is, for all $x \in \mathbb{R} \backslash \Lambda$, the interval $I \subset \mathbb{R} \backslash \Lambda$ containing x has length $\geq c(1 + |x|)^{1-p}$. As in the previous section, we fix $F \in C_0^\infty(-1, 1)$ with $\int_{\mathbb{R}} |F(x)|^2 \,\mathrm{d}x = 1$ and put $F_v(x) = F(x - v)$. Applying Claim 6.4 with $\theta = sq$ and $\delta = c(1 + |v|)^{1-p}$ to the function $f F_v$, and multiplying both sides of the resulting inequality by $(1 + |v|)^{2sp}$, we get

$$(1 + |v|)^{2sp+1-p} \sum_{\lambda \in \Lambda} |f(\lambda)|^2 |F_v(\lambda)|^2$$

$$\leq C \left[(1 + |v|)^{2sp} \int_{\mathbb{R}} |f(x)|^2 |F_v(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}} |\xi|^{2qs} |(\widehat{f} * \widehat{F}_v)(\xi)|^2 \,\mathrm{d}\xi \right], \quad (6.10)$$

where we used the fact that for $x \in \mathrm{supp}\, F_v$, $(1 + |x|)$ is comparable to $(1 + |v|)$. Now, we will integrate this inequality over $v \in \mathbb{R}$. For the LHS we get

$$\int_{\mathbb{R}} (1 + |v|)^{2sp+1-p} \sum_{\lambda \in \Lambda} |f(\lambda)|^2 |F_v(\lambda)|^2 \,\mathrm{d}v,$$

which is comparable to

$$\sum_{\lambda \in \Lambda} (1 + |\lambda|)^{(2s-1)p+1} |f(\lambda)|^2,$$

since again, for $\lambda \in \mathrm{supp}\, F_v$, $(1 + |\lambda|)$ is comparable to $(1 + |v|)$.

For the first term on the RHS of (6.10), by a similar computation, we get that its integral over $v \in \mathbb{R}$ is comparable to

$$\int_{\mathbb{R}} |f(x)|^2 (1 + |x|)^{2sp} \,\mathrm{d}x.$$

Finally, for the second term on the RHS, its integral is

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\xi|^{2qs} |(\widehat{f} * \widehat{F}_v)(\xi)|^2 \,\mathrm{d}\xi \,\mathrm{d}v = \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 |\xi + \eta|^{2qs} \,\mathrm{d}\xi \,\mathrm{d}\eta$$

$$\leq 2^{2qs} \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\widehat{F}(\eta)|^2 \left(|\xi|^{2qs} + \eta^{2qs}\right) \mathrm{d}\xi \,\mathrm{d}\eta \leq C \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \left(1 + |\xi|^{2qs}\right) \mathrm{d}\xi,$$

where in the first step, as before, we used the Plancherel theorem. Combining everything, we see that the RHS of (6.10) is bounded by $C \|f\|^2_{\mathcal{H}_{s,p,q}}$, that is,

$$\sum_{\lambda \in \Lambda} (1 + |\lambda|)^{(2s-1)p+1} |f(\lambda)|^2$$

is dominated by $\|f\|_{\mathcal{H}_{s,p,q}}^2$, as required.

Similarly, if M is q-separated, we can get that

$$\sum_{\mu \in M} (1 + |\mu|)^{(2s-1)q+1} |\widehat{f}(\mu)|^2$$

is dominated by $\|f\|_{\mathcal{H}_{s,p,q}}^2$.

Together, the results of this and two previous sections give us Theorem 1.8.

7. Non-uniqueness pairs

Here, we prove the second part of Theorem 1.3, which claims that any subcritical pair (Λ, M) is a non-uniqueness pair for the Schwartz space $\mathcal{S}$. As we have already mentioned, we will prove slightly more:

Theorem 1.3-NUP. *Any subcritical pair (Λ, M) is a non-uniqueness pair for the Gelfand–Shilov space $\mathcal{S}(p,q)$, where p and q are the parameters from the subcriticality assumption.*

It is worth mentioning that this theorem holds under much weaker assumptions on the pair (Λ, M). We say that a sequence $\Gamma \subset \mathbb{C}$ has a density with respect to the order p if there exists a limit

$$D(\Gamma, p) = \lim_{r \to \infty} \frac{n_\Gamma(r)}{r^p}.$$

Here and elsewhere, $n_\Gamma(r) = |\Gamma \cap r\overline{\mathbb{D}}|$ denotes the counting function of the sequence Γ. In Theorem 1.3-NUP it suffices to assume that the sets $\Lambda_\pm = \Lambda \cap \mathbb{R}_\pm$, $M_\pm = M \cap \mathbb{R}_\pm$ have densities

$$D(\Lambda_+, p), D(\Lambda_-, p) < \frac{2}{p},$$
$$D(M_+, q), D(M_-, q) < \frac{2}{q},$$

and are p- and q-separated (in the sense of Definition 1.6). Such a version would require the full strength of Levin's theorem [15, Ch. II, Theorem 5].

7.1. Preliminaries.

Definition 7.1 (p-smooth sequences). Let $p > 0$ and let $\Gamma = (\gamma_j)$ be a sequence of points in $\mathbb{C}$ lying on a ray $\arg(\gamma_j) = \theta$, $|\gamma_j| \uparrow \infty$. We call the sequence Γ p-smooth with a positive density $D = D(\Gamma, p)$ (with respect to the exponent p) if

a) $|n_\Gamma(r) - Dr^p| = O(1), \quad r \to \infty$;
b) there exists $d > 0$ such that $|\gamma_{j+1}| - |\gamma_j| \geq d(1 + |\gamma_j|)^{1-p}$, $j = 1, 2, \dots$

Claim 7.2. *Let $p > 0$, and let $\Gamma = (\gamma_j) \subset \mathbb{R}_+$, $\gamma_1 < \gamma_2 < \ldots$, $\gamma_j \uparrow \infty$. Suppose that*

$$\liminf_{j \to \infty} \gamma_j^{p-1}(\gamma_{j+1} - \gamma_j) > \delta > 0. \tag{7.1}$$

Then, for any $D > 1/(p\delta)$, there exists a p-smooth supsequence $\Gamma' \supset \Gamma$ with density D, and $|\Gamma' \setminus \Gamma| = \infty$.

Proof. First, we note that it suffices to prove the claim for $p = 1$. The general case follows by applying the case $p = 1$ to the sequence (γ_j^p).

Split $[0, +\infty)$ into intervals of length $1/D < \delta$. By our assumption, each of these intervals, except maybe finitely many ones, contains at most one point of the set Γ. At the same time, there are infinitely many intervals which do not contain points of Γ (otherwise, $\liminf r^{-1} n_\Gamma(r) \geq D$, while (7.1) yields that $\limsup r^{-1} n_\Gamma(r) \leq \delta^{-1}$). If the interval does not contain a point of Γ, we place a new point at its center. $\square$

In what follows, relying on Claim 7.2, we assume that $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and add to each of the sets $\Lambda_\pm = \Lambda \cap \mathbb{R}_\pm$, $M_\pm = M \cap \mathbb{R}_\pm$ infinitely many points so that the enlarged sets $\Lambda'_\pm$ and $M'_\pm$ become p- and q-smooth and satisfy

$$D(\Lambda'_+, p) = D(\Lambda'_-, p) < \frac{2}{p},$$

$$D(M'_+, q) = D(M'_-, q) < \frac{2}{q}.$$

7.2. The main lemma. The proof of Theorem 1.3-NUP relies on the construction of the sequences of entire functions $(\Phi_\lambda)_{\lambda \in \Lambda'}$ and $(\Psi_\mu)_{\mu \in M'}$ that interpolate the δ-functions δ_λ and δ_μ and satisfy certain bounds uniformly with respect to λ and μ.

Lemma 7.3. *Let $p > 1$, $q = p/(p-1)$. Let $\Lambda' \subset \mathbb{R}$, and let the sets $\Lambda'_\pm$ be p-smooth with*

$$D(\Lambda'_+, p) = D(\Lambda'_-, p) < \frac{2}{p}.$$

Then, for any sufficiently small $a > 0$, there exist $C < \infty$, $a' > a > a'' > 0$, and a sequence of entire functions $(\Phi_\lambda)_{\lambda \in \Lambda'}$ with the following properties

$$\Phi_\lambda(\lambda') = \begin{cases} 1, & \lambda' = \lambda, \\ 0, & \lambda' \in \Lambda' \setminus \{\lambda\}; \end{cases} \tag{7.2}$$

$$|\Phi_\lambda(x)| \leq C e^{-a''|x|^p + a|\lambda|^p}, \quad x \in \mathbb{R}; \tag{7.3}$$

$$|\widehat{\Phi}_\lambda(\xi)| \leq C e^{-a'|\xi|^q + a|\lambda|^p}, \quad \xi \in \mathbb{R}. \tag{7.4}$$

Note that simultaneously with the sequence $(\Phi_\lambda)_{\lambda \in \Lambda'}$ the lemma also provides us with the sequence of entire functions $(\Psi_\mu)_{\mu \in M'}$ such that

$$\Psi_\mu(\mu') = \begin{cases} 1, & \mu' = \mu, \\ 0, & \mu' \in M' \setminus \{\mu\}; \end{cases} \tag{7.5}$$

$$|\Psi_\mu(\xi)| \le Ce^{-a''|\xi|^q + a|\mu|^q}, \quad \xi \in \mathbb{R}; \tag{7.6}$$

$$|\check{\Psi}_\mu(x)| \le Ce^{-a'|x|^p + a|\mu|^q}, \quad x \in \mathbb{R}, \tag{7.7}$$

where $\check{\Psi}$ denotes the inverse Fourier transform. We use bounds (7.3) and (7.6) for the functions Φ and Ψ with the same positive constant a'' (otherwise, we take the minimum of two constants). Similarly, we assume that the positive constant a' in (7.4) and in (7.7) is the same.

The proof of Lemma 7.3 relies on several notions and results pertaining to the classical theory of entire functions, in particular, on the B. Ya. Levin estimates for entire functions with regular zeroes.

We proceed with the proof of Theorem 1.3-NUP, assuming the existence of the sequences (Φ_λ) and (Ψ_μ), and then will provide the proof of Lemma 7.3.

7.3. Free interpolation by Fourier pairs. We fix $\delta \le 1/(4C)$, where C is a constant from estimates (7.3), (7.4), (7.6), and (7.7), and choose L so large that

$$\sum_{\lambda \in \Lambda' \setminus [-L, L]} e^{-(a'-a)|\lambda|^p} + \sum_{\mu \in M' \setminus [-L, L]} e^{-(a'-a)|\mu|^q} < \delta.$$

Then, we set $\Lambda'_L = \Lambda' \setminus [-L, L]$, and $M'_L = M' \setminus [-L, L]$. By B_a we denote the Banach space of pairs of fast decaying sequences

$$\kappa = (\alpha, \beta), \quad \alpha \colon \Lambda'_L \to \mathbb{C}, \quad \beta \colon M'_L \to \mathbb{C},$$

endowed with the weighted ℓ^1 norm

$$\|\kappa\|_a \overset{\text{def}}{=} \sum_{\lambda \in \Lambda'_L} |\alpha(\lambda)| e^{a|\lambda|^p} + \sum_{\mu \in M'_L} |\beta(\mu)| e^{a|\mu|^q}.$$

Lemma 7.4. *For any sequence $\kappa \in \mathsf{B}_a$ with a sufficiently small $a > 0$, there exists an entire function f from the Gelfand–Shilov space $\mathcal{S}(p, q)$ that solves the interpolation problem*

$$\begin{cases} f(\lambda) = \alpha(\lambda), & \lambda \in \Lambda'_L, \\ \widehat{f}(\mu) = \beta(\mu), & \mu \in M'_L. \end{cases} \tag{7.8}$$

Remark 7.5. Note that there is no hope to solve such an interpolation problem (even for the Schwartz function f) with the full set of nodes $\Lambda' \cup M'$. Evident obstacles are classical, as well as non-classical, Poisson summation formulas. It is curious that removing only finitely many nodes from both sets, we get rid of the hurdles.

This remark can be rephrased in terms of crystalline measures. Lemma 7.4 yields that the only locally finite measure of tempered growth supported on Λ' such that its Fourier transform is a locally finite measure of tempered growth supported on M' is a zero measure. Since $\Lambda \setminus \Lambda'$ and $M \setminus M'$ are finite, by a simple linear algebra argument we can conclude that the vector space of such measures, but supported on Λ and M respectively, is finite-dimensional.

7.4. Proof of Lemma 7.4. We will be using a convenient operator formalism suggested by Adve [1] (in the arXiv version we used an equivalent iteration scheme). By S_b we denote the Banach space of functions from the Gelfand–Shilov space $\mathcal{S}(p,q)$ satisfying

$$\|f\|_b = \sup_{x\in\mathbb{R}} |f(x)|e^{b|x|^p} + \sup_{\xi\in\mathbb{R}} |\widehat{f}(\xi)|e^{b|\xi|^q} < \infty.$$

By T we denote the trace map $f \mapsto \left(f|_{\Lambda'_L}, \widehat{f}|_{M'_L}\right)$. We fix a sufficiently small positive a, take $a' > a > a'' > 0$ and sequences of entire functions $(\Phi_\lambda)_{\lambda\in\Lambda'_L}$, $(\Psi_\mu)_{\mu\in M'_L}$ as in Lemma 7.3, and consider the linear map

$$P\colon \kappa = (\alpha,\beta) \mapsto \sum_{\lambda\in\Lambda'_L} \alpha(\lambda)\Phi_\lambda + \sum_{\mu\in M'_L} \beta(\mu)\check{\Psi}_\mu.$$

We claim that

A) P is a bounded operator from B_a to $\mathsf{S}_{a''}$.
B) $\|I - TP\|_{\mathsf{B}_a\to\mathsf{B}_a} \leq \frac{1}{2}$.

This will readily yield Lemma 7.4. Indeed, by B), the operator $TP = I - (I - TP)$ is invertible on B_a by the Neumann series, and, by A), the operator $P(TP)^{-1}\colon \mathsf{B}_a \to \mathsf{S}_{a''}$ is bounded. Thus, given $\kappa \in \mathsf{B}_a$, the function $f = P(TP)^{-1}\kappa$ solves the interpolation problem (7.8).

Proof of A). Let $f = P\kappa$ with $\kappa = (\alpha,\beta) \in \mathsf{B}_a$. Then

$$\begin{aligned}
|f(x)| &\leq \sum_{\lambda\in\Lambda'_L} |\alpha(\lambda)||\Phi_\lambda(x)| + \sum_{\mu\in M'_L} |\beta(\mu)||\check{\Psi}_\mu(x)| \\
&\leq C\left[\sum_{\lambda\in\Lambda'_L} |\alpha(\lambda)|e^{-a''|x|^p+a|\lambda|^p} + \sum_{\mu\in M'_L} |\beta(\mu)|e^{-a'|x|^p+a|\mu|^q}\right] \quad \text{(by (7.3),(7.6))} \\
&\leq C\|\kappa\|_a e^{-a''|x|^p} \qquad \text{(since } a'' < a'),
\end{aligned}$$

and, similarly, $|\widehat{f}(\xi)| \leq C\|\kappa\|_a e^{-a''|\xi|^q}$. That is, $\|P\kappa\|_{a''} \leq C\|\kappa\|_a$, proving A). $\quad\square$

Proof of B). Let $\kappa = (\alpha,\beta) \in \mathsf{B}_a$. Then

$$\big[(I - TP)\kappa\big](\lambda) = -\sum_{\mu\in M'_L} \beta(\mu)\check{\Psi}_\mu(\lambda), \qquad \lambda \in \Lambda'_L,$$

and

$$\big[(I - TP)\kappa\big](\mu) = -\sum_{\lambda\in\Lambda'_L} \alpha(\lambda)\widehat{\Phi}_\lambda(\mu), \qquad \mu \in M'_L.$$

We have

$$\sum_{\lambda\in\Lambda'_L} \big|\big[(I-TP)\kappa\big](\lambda)\big|e^{a|\lambda|^p} \underset{(7.7)}{\leq} C \sum_{\mu\in M'_L} |\beta(\mu)|e^{a|\mu|^q} \cdot \sum_{\lambda\in\Lambda'_L} e^{(a-a')|\lambda|^p}$$

$$\leq C\delta \sum_{\mu \in M_L'} |\beta(\mu)| e^{a|\mu|^q} \qquad \text{(choice of } L\text{)}$$

$$\leq C\delta \|\kappa\|_a \leq \frac{1}{4}\|\kappa\|_a \qquad (C\delta \leq \frac{1}{4}).$$

Similarly,

$$\sum_{\mu \in M_L'} \big|\big[(I - TP)\kappa\big](\mu)\big| e^{a|\mu|^q} \leq \frac{1}{4}\|\kappa\|_a.$$

That is, $\|(I - TP)\kappa\|_a \leq \frac{1}{2}\|\kappa\|_a$, proving B). □

7.5. Completing the proof of Theorem 1.3-NUP. To finish the proof of the theorem, we consider sequences $\kappa = (\alpha, \beta)$ with α vanishing on $\Lambda \setminus [-L, L]$ and β vanishing on $M \setminus [-L, L]$. Since the set $\Lambda' \setminus (\Lambda \cup [-L, L])$ is infinite, Lemma 7.4 will provide us with an infinite-dimensional subspace of functions $f \in \mathcal{S}(p, q)$ such that $f\big|_{\Lambda \setminus [-L,L]} = 0$ and $\widehat{f}\big|_{M \setminus [-L,L]} = 0$. By a simple linear algebra argument, this subspace contains non-zero functions f satisfying finitely many linear equations

$$\begin{cases} f(\lambda) = 0, & \lambda \in \Lambda \cap [-L, L], \\ \widehat{f}(\mu) = 0, & \mu \in M \cap [-L, L]. \end{cases}$$

This completes the proof of Theorem 1.3-NUP modulo the proof of Lemma 7.3.

7.6. Proof of Lemma 7.3 We will be using several notions and results from the classical theory of entire functions, see Levin [15, 16].

7.6.1. The class K_p.

Definition 7.6 (The class K_p). A 2π-periodic continuous function k belongs to the class K_p (p is a positive constant) if

$$k'' + p^2 k = \sum_{j=1}^{n} m_j \delta_{\theta_j},$$

where δ_θ is a unit point measure at θ, the masses m_j are positive, and the differential operator on the LHS is understood distributionally.

Formally the latter means that, for any 2π-periodic C^∞-function φ,

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} k(\varphi'' + p^2\varphi) = \sum_{j=1}^{n} m_j \varphi(\theta_j). \tag{7.9}$$

We assume that the points $\theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}/2\pi\mathbb{Z}$ are arranged counterclockwise and mention two simple facts concerning functions $k \in \mathsf{K}_p$.

(i) On each interval (θ_j, θ_{j+1}), the function k is p-trigonometric, i.e.,

$$k(\theta) = a_j \cos p\theta + b_j \sin p\theta.$$

(ii) The masses m_j are equal to the jumps of the derivative of k, i.e.,

$$m_j = k'(\theta_j + 0) - k'(\theta_j - 0).$$

For the reader's convenience, we sketch the proofs of these facts[5].

(i) If the function k is C^∞-smooth on (θ_j, θ_{j+1}), then, taking an arbitrary $\varphi \in C_0^\infty(\theta_j, \theta_{j+1})$ and integrating by parts twice on the LHS of (7.9), we conclude that $k'' + p^2 k = 0$ on (θ_j, θ_{j+1}) and, therefore, k is p-trigonometric therein. If k is only continuous, then we take a mollifier $\psi \in C_0^\infty(-\pi, \pi)$ with integral 1, put $\psi_t = t\psi(t \cdot)$, apply the previous argument to the convolution $k * \psi_t$, conclude that $k * \psi_t$ is p-trigonometric on $\left(\theta_j + \pi t^{-1}, \theta_{j+1} - \pi t^{-1}\right)$ for each t, and then let $t \to \infty$, noting that if a sequence of p-trigonometric functions converges pointwise, then the limiting function is p-trigonometric as well.

(ii) The integral on the LHS of (7.9) equals

$$\sum_{j=1}^n \int_{\theta_j}^{\theta_{j+1}} k(\varphi'' + p^2 \varphi) = \sum_{j=1}^n (k'(\theta_j + 0) - k'(\theta_j - 0))\varphi(\theta_j)$$

(we integrated twice by parts and combined the terms that appeared with opposite signs).

7.6.2. Entire functions with smooth sequences of zeroes. With each function $k \in \mathsf{K}_p$ we associate a class of k-smooth discrete sets in the plane.

Definition 7.7 (k-smooth sets)**.** A discrete set of points

$$Z = \bigcup_{j=1}^n Z_j, \quad Z_j = \left\{z_{\ell,j} = |z_{\ell,j}|e^{\mathrm{i}\theta_j} : \ell \geq 1\right\},$$

is *k-smooth*, if it satisfies the following conditions:

(a) for each $1 \leq j \leq n$, the set Z_j is p-smooth on the ray $\{\arg(z) = \theta_j\}$ with density $D_j = (2\pi p)^{-1} m_j$, that is,

$$\left|Z_j \cap \left[0, re^{\mathrm{i}\theta_j}\right]\right| = \frac{m_j}{2\pi p} r^p + O(1), \quad r \to \infty;$$

(b) the disks $D_{\ell,j} = D\left(z_{\ell,j}, d(1 + |z_{\ell,j}|)^{1-p}\right)$, $\ell \geq 1$, $1 \leq j \leq n$, are disjoint for some $d > 0$.

The following theorem is a very special case of Levin's theorem [15, Ch. II, Theorem 5].

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[5]Functions from K_p are p-trigonometrically convex [15, Ch.I, §16], [16, Lecture 8], and these facts are special cases of the corresponding properties of p-trigonometrically convex functions.

Theorem L. *Let $k \in \mathsf{K}_p$. Then, for any k-smooth set Z, there exists an entire function S, whose zeroes are simple and coincide with Z, such that, for every $\varepsilon > 0$,*

$$|S(re^{i\theta})| \le C_\varepsilon e^{(k(\theta)+\varepsilon)r^p} \quad \text{everywhere in } \mathbb{C}, \tag{I}$$

$$|S(re^{i\theta})| \ge c_\varepsilon e^{(k(\theta)-\varepsilon)r^p} \quad \text{everywhere in } \mathbb{C} \setminus \bigcup_{j=1}^{n} \bigcup_{\ell \ge 1} D_{\ell,j}. \tag{II}$$

Note that estimate (II) yields the lower bound

$$|S'(z_{\ell,j})| \ge c_\varepsilon e^{(k(\theta_j)-2\varepsilon)|z_{\ell,j}|^p}. \tag{III}$$

To get it, one has to apply the minimum modulus principle to the harmonic function $\log |S(z)/(z - z_{\ell,j})|$ in the disk $D_{\ell,j}$.

For the reader's convenience, we will provide in Appendix a short self-contained proof of Theorem L, which does not use the machinery developed in [15, Ch. II].

7.6.3. Choice of the functions $k_p(\theta)$.

We will apply Theorem L to a special family of functions $k_p \in \mathsf{K}_p$. These functions will possess the following properties:

a) they are symmetric with respect to $\theta = 0$ and $\theta = \pm\frac{\pi}{2}$, that is, $k_p(-\theta) = k_p(\theta)$, and $k_p(\pi - |\theta|) = k_p(\theta)$ for $-\pi \le \theta \le \pi$;

b) $k_p(0) = k_p(\pm\pi) < 0$;

c) the functions k_p are increasing and p-trigonometric on $\left[0, \frac{\pi}{2p}\right]$.

For $1 < p \le 2$, we set

$$k_p(\theta) = \alpha \sin p\theta - \beta \cos p\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \tag{7.10}$$

with $\alpha, \beta > 0$, and then extend it to the whole interval $[-\pi, \pi]$ using the symmetries and letting

$$k_p(\pi - \theta) = k_p(-\pi + \theta) = k_p(-\theta) = k_p(\theta), \quad 0 \le \theta \le \frac{\pi}{2}.$$

To keep the jumps of k' at $\theta = \pm\pi/2$ positive, we require that $k_p'(\pi/2 - 0) < 0$, that is, for $1 < p < 2$, $\beta < \alpha|\cot(\pi p/2)|$ (for $p = 2$, this requirement is void).

For $p > 2$, we let $p = 2^n p'$ with $n \in \mathbb{N}$ and $1 < p' \le 2$, and define $k_p(\theta) = k_{p'}(2^n \theta)$. Note that relation (7.10) continues to hold for $0 \le \theta \le \pi/(2p)$.

To adjust the function k_p to the densities $D(\Lambda'_\pm, p)$, we choose $\sigma < 1$ so that

$$D(\Lambda'_+, p) = D(\Lambda'_-, p) = \frac{2\sigma}{p},$$

and let $\alpha = 2\pi\sigma/p$. Then,

$$k_p'(+0) - k_p'(-0) = k_p'(\pm\pi + 0) - k_p'(\pm\pi - 0) = 2\alpha p = 2\pi p \cdot \frac{2\sigma}{p} = 2\pi p D(\Lambda'_\pm, p),$$

which will be needed momentarily for the application of Levin's theorem. The small parameter β in the definition of the function k_p will be fixed a bit later.

7.6.4. The entire function Φ. We apply Theorem L with the function k_p and a set $Z \supset \Lambda'$.

We enlarge Λ', adding to it p-smooth sets lying on the rays corresponding to discontinuities of k_p' different from $\theta = 0$ and $\theta = \pm\pi$, and choosing their densities according to the jumps of k_p' at these discontinuities, so that the new set Z meets conditions (a) and (b) from the definition of the k-smooth set. Then Theorem L provides us with an entire function Φ with simple zeroes at Z, such that for every $\varepsilon > 0$,

$$|\Phi(re^{i\theta})| \leq C_\varepsilon e^{(k(\theta)+\varepsilon)r^p} \quad \text{everywhere in } \mathbb{C},$$

and

$$|\Phi'(\lambda)| \geq c_\varepsilon e^{-(\beta+\varepsilon)|\lambda|^p}, \qquad \lambda \in \Lambda',$$

where $\beta = -k_p(0) = -k_p(\pm\pi)$.

7.6.5. Decay of the Fourier transform. The next step is to estimate the decay of the Fourier transform of the function $F_\lambda(z) = \Phi(z)/(z - \lambda)$, uniformly in $\lambda \in \Lambda'$. First, we claim that, for every $\varepsilon > 0$,

$$|F_\lambda(re^{i\theta})| \leq C_\varepsilon e^{(k_p(\theta)+\varepsilon)r^p} \quad \text{everywhere in } \mathbb{C}. \tag{7.11}$$

Clearly, this bound holds in $\{z \colon |\Im(z)| \geq 1\}$. Then, by the Phragmén–Lindelöf theorem, it also holds in the strip $\{z \colon |\Im(z)| \leq 1\}$ (recall that the entire function F_λ has finite order of growth).

Shifting the integration line in the definition of the Fourier transform $\widehat{F}_\lambda$, we get

$$\widehat{F}_\lambda(\xi) = \int_{\mathbb{R}} F_\lambda(x + iy)e^{-2\pi i\xi(x+iy)}\,\mathrm{d}x,$$

with any $y \in \mathbb{R}$. The shift is justified since the function $k_p(\theta)$ stays negative in the angles $\{|\theta| \leq \delta\}$ and $\{|\pi - \theta| \leq \delta\}$ with some $\delta > 0$, and therefore, $|F_\lambda(z)|$ decays in these angles like $e^{-c|z|^p}$. Thus, for every $\varepsilon > 0$,

$$\left|\widehat{F}_\lambda(\xi)\right| \leq C_\varepsilon \inf_{y \in \mathbb{R}} \sup_{x \in \mathbb{R}} |F_\lambda(x+iy)|e^{\varepsilon|x|^p + 2\pi\xi \cdot y}. \tag{7.12}$$

We will choose y satisfying $\xi \cdot y \leq 0$, so that the last factor in (7.12) is always $e^{-2\pi|\xi|\cdot|y|}$.

Letting $x + iy = re^{i\theta}$ and taking into account estimate (7.11), in order to proceed, we need to bound the expression $(k_p(\theta)+2\varepsilon)r^p$ for $-\pi \leq \theta \leq \pi$. Because of the symmetries of k_p, we can restrict ourselves to the region $0 \leq \theta \leq \pi/2$.

Claim 7.8. *For all $s \in (\sigma^q, 1)$, there exists $b_0(s)$ such that, for all $\theta \in \left[0, \frac{\pi}{2}\right]$ and all $b > b_0(s)$, we have*

$$k_p(\theta) < 2\pi\frac{b^p}{p}\sin^p\theta, \tag{7.13}$$

provided that

$$\beta = 2\pi s\frac{b^{-q}}{q}.$$

Proof. Choose $\theta_0 \in \left(0, (2p)^{-1}\pi\right)$ so that $s \cos p\theta_0 = \sigma^q$. Note that, for $\theta_0 \leq \theta \leq \pi/2$, the inequality holds since, increasing b, we unboundedly increase the minimum of the RHS of (7.13), while the LHS of (7.13) stays bounded.

For $0 \leq \theta \leq \theta_0$, we have $k_p(\theta) = \frac{2\pi\sigma}{p}\sin\theta - \beta\cos\theta$, so we need to show that

$$\frac{2\pi\sigma}{p}\sin p\theta < \beta\cos p\theta + 2\pi\frac{b^p}{p}\sin^p\theta,$$

or (substituting the value of β) that

$$\frac{\sigma}{p}\sin p\theta < s\frac{b^{-q}}{q}\cos p\theta + \frac{b^p}{p}\sin^p\theta.$$

Then, since $\cos p\theta \geq \cos p\theta_0$, it suffices to show that

$$\frac{\sigma}{p}\sin p\theta < \sigma^q\frac{b^{-q}}{q} + \frac{b^p}{p}\sin^p\theta.$$

By the Young inequality, the RHS is not less than $\sigma\sin\theta$, while, by the concavity of the sine function on $\left[0, \frac{\pi}{2}\right]$, for $p > 1$, $\sin p\theta < p\sin\theta$ everywhere on $\left(0, \frac{\pi}{2p}\right]$. This proves estimate (7.13) in the range $0 < \theta \leq \theta_0$. For $\theta = 0$, the estimate holds because $k_p(0) < 0$. $\qquad\square$

From now on, we fix $s \in (\sigma^q, 1)$, and let $\beta = 2\pi sb^{-q}/q$, with some $b > b_0(s)$ to be specified later. Then, by the claim,

$$k_p(\theta) + 2\varepsilon \leq 2\pi\frac{b^p}{p}\sin^p\theta.$$

Note that $\varepsilon > 0$ can be chosen independent of θ by continuity of the functions k_p and sine. Hence,

$$(k_p(\theta) + 2\varepsilon)r^p - 2\pi|\xi|\cdot|y| \leq 2\pi\left[\frac{b^p}{p}|y|^p - |\xi|\cdot|y|\right].$$

Thus, for $x + \mathrm{i}y = re^{\mathrm{i}\theta}$,

$$|F_\lambda(x+\mathrm{i}y)|e^{\varepsilon|x|^p - 2\pi|\xi||y|} \leq C_\varepsilon e^{(k_p(\theta)+\varepsilon)|x+\mathrm{i}y|^p + \varepsilon|x|^p - 2\pi|\xi||y|}$$
$$\leq C_\varepsilon e^{(k_p(\theta)+2\varepsilon)|x+\mathrm{i}y|^p - 2\pi|\xi||y|} \leq C_\varepsilon e^{2\pi[b^p|y|^p/p - |\xi||y|]}.$$

Minimizing the exponent on the RHS, we choose

$$|y| = \left(\frac{|\xi|}{b^p}\right)^{1/(p-1)}.$$

Then,

$$2\pi\left[\frac{b^p}{p}|y|^p - |\xi|\cdot|y|\right] = -2\pi\frac{b^{-q}}{q}|\xi|^q,$$

and therefore,

$$\left|\widehat{F_\lambda}(\xi)\right| \leq C_\varepsilon' e^{-\beta'|\xi|^q}, \qquad \xi \in \mathbb{R}, \ \lambda \in \Lambda',$$

with

$$\beta' = 2\pi\frac{b^{-q}}{q} = \frac{\beta}{s}.$$

7.6.6. Completing the proof of Lemma 7.3. It remains to let

$$\Phi_\lambda(z) = \frac{\Phi(z)}{\Phi'(\lambda)(z-\lambda)} = \frac{F_\lambda(z)}{\Phi'(\lambda)}, \qquad \lambda \in \Lambda'.$$

Evidently,

$$\Phi_\lambda(\lambda') = \begin{cases} 1, & \lambda' = \lambda, \\ 0, & \lambda' \in \Lambda' \setminus \{\lambda\}. \end{cases}$$

Furthermore, for every $\varepsilon > 0$, we have

$$|\Phi_\lambda(x)| \le C_\varepsilon e^{-(\beta-\varepsilon)|x|^p + (\beta+\varepsilon)|\lambda|^p} \quad x \in \mathbb{R}, \ \lambda \in \Lambda',$$

and

$$|\widehat{\Phi}_\lambda(\xi)| \le C_\varepsilon e^{-\frac{\beta}{s}|\xi|^q + (\beta+\varepsilon)|\lambda|^p}, \qquad \xi \in \mathbb{R}, \ \lambda \in \Lambda'.$$

At last, given sufficiently small $a > 0$, we choose β so that $\beta < a < \beta/s$ (recall that we can choose β as small, as we please, choosing the parameter b in Claim 7.8 sufficiently large), and then choose $\varepsilon < \min(a - \beta, \beta)$. Letting $a' = \beta/s$ and $a'' = \beta - \varepsilon$, we obtain the needed upper bounds:

$$|\Phi_\lambda(x)| \le C e^{-(\beta-\varepsilon)|x|^p + (\beta+\varepsilon)|\lambda|^p} = C e^{-a''|x|^p + a|\lambda|^p}, \quad x \in \mathbb{R}, \ \lambda \in \Lambda',$$

and

$$|\widehat{\Phi}_\lambda(\xi)| \le C e^{-\frac{\beta}{s}|\xi|^q + (\beta+\varepsilon)|\lambda|^p} = C e^{-a'|\xi|^q + a|\lambda|^p}, \quad \xi \in \mathbb{R}, \ \lambda \in \Lambda',$$

thus completing the proof of Lemma 7.3.

A. Proof of Theorem L

A.1. Brelot–Hadamard representation of the function K. Consider the function $K(re^{i\theta}) = k(\theta)r^p$, where k is a function from the class K_p, which is p-trigonometric outside the points $\{\theta_j\}$, $1 \le j \le n$. As before, we assume that these points are arranged counterclockwise. The function K is a continuous subharmonic function in $\mathbb{C}$ with the Riesz measure

$$\mu = \frac{1}{2\pi}\Delta K = \frac{1}{2\pi p} \sum_{j=1}^n m_j \delta_{\theta_j} \otimes d(r^p),$$

where $m_j = k'(\theta_j + 0) - k'(\theta_j - 0) > 0$ are the jumps of k'. That is, for any compactly supported smooth test function f, we have[6]

$$\int_{\mathbb{C}} K\Delta f = \sum_{j=1}^n m_j \int_0^\infty f(re^{i\theta})r^{p-1}\,dr.$$

In particular, the function K is harmonic in each angle $\{\theta_j < \arg(z) < \theta_{j+1}\}$. In addition, if p is a positive integer, then

$$\sum_j m_j e^{-ip\theta_j} = 0. \tag{A.1}$$

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[6]To verify this, one can apply Green's formula to K and f in the truncated sectors $\Omega_{\rho,R,j} = \{re^{i\theta} : \rho < r < R, \theta_j < \theta < \theta_{j+1}\}$ with sufficiently large R, add the results, and then let $\rho \to 0$.

To see this, one can apply integration by parts in each interval (θ_j, θ_{j+1}) to the functions $k(\theta)$ and $e^{-\mathrm{i}p\theta}$ and add the results up.

Set $\mu_R = \mathbb{1}_{R\overline{\mathbb{D}}}\mu$. Let

$$\mathsf{E}_s(w) = (1 - w)e^{w + w^2/2 + \cdots + w^s/s}, \quad s \in \mathbb{N},$$

be the Weierstrass factor, and let $\mathsf{H}_s(w) = \log|\mathsf{E}_s(w)|$. Choose $s < p \le s + 1$ and set

$$U_R(z) = \int \mathsf{H}_s(z/\zeta)\,\mathrm{d}\mu_R(\zeta).$$

This is a subharmonic function with $\Delta U_R = (2\pi)^{-1}\mu_R$ (where the Laplacian is understood in the sense of distributions).

Lemma A.1 (A special case of the Brelot–Hadamard theorem).

1. *We have $U_R(z) \le C|z|^p$, $z \in \mathbb{C}$, uniformly in $R \ge 1$.*
2. *The family U_R converges as $R \to \infty$, and*

$$\lim_{R \to \infty} U_R(z) = \begin{cases} K(z), & p \notin \mathbb{N}, \\ K(z) + \operatorname{Re}(cz^p), & p \in \mathbb{N}, \end{cases}$$

   locally uniformly in $\mathbb{C}$.

For the reader's convenience, we outline the proof of this lemma, which follows the proof of the Hadamard representation of entire functions of finite order of growth.

Proof of Lemma A.1. In order to prove the first statement, we fix $z \ne 0$, write

$$U_R(z) = \left(\int_{|\zeta| \le 2|z|} + \int_{|\zeta| > 2|z|} \right) \mathsf{H}_s(z/\zeta)\,\mathrm{d}\mu_R(\zeta),$$

and estimate each of the integrals starting with the second one. For $|w| \le \frac{1}{2}$, we have

$$\mathsf{H}_s(w) = -\operatorname{Re}\frac{w^{s+1}}{s+1} + O(|w|^{s+2}).$$

Since

$$\int_{2|z|}^{\infty} \left(\frac{|z|}{t}\right)^{s+2} t^{p-1}\,\mathrm{d}t = \frac{2^{p-s-2}}{s+2-p}|z|^p,$$

the O-term presents no trouble. The same can be said about $-\operatorname{Re} w^{s+1}/(s+1)$ if $s + 1 > p$, that is, if p is non-integer.

If p is a positive integer and $s + 1 = p$, observing that by (A.1),

$$\int_{|\zeta| > 2|z|} \zeta^{-p}\,\mathrm{d}\mu_R(\zeta) = 0, \quad \text{for all } 0 < R < \infty,$$

we see that the term $-\operatorname{Re} w^{s+1}/(s+1)$ has no influence at all.

It remains to crudely estimate the integrals over $|\zeta| \le 2|z|$:

$$\int_0^{2|z|} \log\left|1 - \frac{z}{t}\right| t^{p-1}\, \mathrm{d}t \le \int_0^{2|z|} \log\left(1 + \frac{|z|}{t}\right) t^{p-1}\, \mathrm{d}t$$

$$= |z|^p \int_0^2 \log\left(1 + \frac{1}{t}\right) t^{p-1}\, \mathrm{d}t$$

and

$$\int_0^{2|z|} \left(\frac{|z|}{t}\right)^j t^{p-1}\, \mathrm{d}t = |z|^p \int_0^2 t^{p-1-j}\, \mathrm{d}t, \quad 1 \le j \le s < p.$$

Thus, we have our growth bound.

Similar estimates show that $U_{R_2}(z) - U_{R_1}(z)$ tends to zero locally uniformly in z, as $R_2 > R_1$ and $R_1 \to \infty$, which proves the locally uniform convergence of $U_R(z)$ as $R \to \infty$. The limiting function U is continuous and, for every $f \in C_0^\infty$,

$$\int U \Delta f = \lim_{R \to \infty} \int U_R \Delta f = \lim_{R \to \infty} \int f\, \mathrm{d}\mu_R = \int f\, \mathrm{d}\mu,$$

that is, $\Delta U = \mu$ in the sense of distributions. Thus, $\Delta(U - K) = 0$ in the sense of distributions and, by the classical Weyl lemma, the function $H = U - K$ is harmonic. Since it is bounded from above by $C|z|^p$ everywhere in $\mathbb{C}$, by a version of the Liouville theorem, $H(z)$ vanishes identically when p is non-integer, and $H(z) = \mathrm{Re}(cz^p)$ when p is a positive integer[7]. $\qquad\square$

A.2. Construction of the entire function S. Now we switch from the continuous measure μ to the discrete measure $\eta = \sum_{\zeta \in Z} \delta_\zeta$, where Z is a k-smooth set of points. We assume that $0 \notin Z$ (otherwise, we construct S for the set $Z \setminus \{0\}$ as below, and take the function zS).

We will again denote $\eta_R = \mathbb{1}_{R\overline{\mathbb{D}}}\eta$ and let s be an integer such that $s < p \le s + 1$. Set

$$S_R(z) = \prod_{\zeta \in Z \cap R\overline{\mathbb{D}}} \mathsf{E}_s\left(\frac{z}{\zeta}\right),$$

Then

$$\log|S_R(z)| = \int_{\mathbb{C}} \mathsf{H}_s\left(\frac{z}{\zeta}\right) \mathrm{d}\eta_R(\zeta).$$

The functions S_R are entire and their zero sets coincide with $Z \cap R\overline{\mathbb{D}}$. We want to show that they also converge uniformly on compact sets and that $\log|S_R|$ does not differ from U_R too much.

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[7]Indeed, since H is a real part of some entire function, for $z = re^{i\theta}$, we have a representation

$$H(z) = \sum_{j \ge 0} r^j H_j(\theta), \qquad H_j(\theta) = \alpha_j \cos j\theta + \beta_j \sin j\theta.$$

Since U and K vanish at the origin, $H_0 = H(0) = 0$. Then $Cr^p - \sum_{j \ge 1} r^j H_j(\theta) \ge 0$. Let $M_j = \max_\theta H_j(\theta)$. Multiplying the latter inequality by $M_k - H_k(\theta)$ and integrating over θ, we see that $CM_k r^p - r^k \|H_k\|_2^2 \ge 0$, i.e., $\|H_k\|_2^2 \le CM_k r^{p-k}$, for all $k \ge 1$ and all $r > 0$. Letting $r \to 0$, we see that $H_k = 0$ for all $k < p$, while letting $r \to \infty$, we see that $H_k = 0$ for all $k > p$.

For non-integer values of p, the convergence follows from the estimate

$$\mathsf{E}_s(w) = 1 + O\big(|w|^{s+1}\big), \quad |w| \le \frac{1}{2},$$

combined with convergence of the series $\sum_{\zeta \in Z} |\zeta|^{-t}$ for any $t > p$, in particular, for $t = s + 1$.

When p is an integer, we have $s + 1 = p$,

$$\mathsf{E}_s(w) = 1 - \frac{w^p}{p} + O\big(|w|^{p+1}\big), \quad |w| \le \frac{1}{2}.$$

Denote by n_j the counting measure of the portion of Z lying on the ray $\arg(\zeta) = \theta_j$, and let $\nu_j(r) = n_j(r\overline{\mathbb{D}}) - \frac{m_j}{2\pi p}r^p$. Denoting by Ω complex-valued constants, we have, for $R \ge 1$,

$$\sum_{\zeta \in Z \cap R\overline{\mathbb{D}}} \frac{1}{\zeta^p} \underset{\text{(A.1)}}{=} \sum_{j=1}^{n} e^{-\mathrm{i}p\theta_j} \int_{[1,R]} \frac{\mathrm{d}\nu_j(r)}{r^p} + \Omega.$$

Integrating by parts, and using that, by k-smoothness of Z, $|\nu_j(t)| = O(1)$ on $\mathbb{R}_+$, we see that the RHS equals $\Omega_1 + O(R^{-p})$ as $R \to \infty$, which proves the existence of the limit of $S_R(z)$.

It remains to show that the difference of the functions $V_R = \log|S_R|$ and U_R is close to a harmonic polynomial of degree at most p. We have

$$\big(U_R - V_R\big)(z) = -\sum_j \int_{[0,R]} \mathsf{H}_s\left(\frac{ze^{-\mathrm{i}\theta_j}}{t}\right) \mathrm{d}\nu_j(t)$$

We consider the contribution of the positive semi-axis, the contribution of other rays is treated similarly. Dropping the index j, we have

$$\int_{[0,R]} \mathsf{H}_s\Big(\frac{z}{t}\Big) \mathrm{d}\nu(t)$$

$$= \mathsf{H}_s\left(\frac{z}{R}\right)\nu(R) - \int_0^R \nu(t) \operatorname{Re}\left(\frac{1}{t-z} - \frac{1}{t}\right) \mathrm{d}t - \sum_{k=1}^{s} \operatorname{Re}(z^k) \int_0^R \frac{\nu(t)}{t^{k+1}} \mathrm{d}t.$$

The first term on the RHS tends to zero as $R \to \infty$ locally uniformly in z and can be discarded. Since $\nu(t) = -ct^p$ near the origin and is bounded near infinity, the coefficients

$$\int_0^\infty \nu(t)t^{-k-1} \mathrm{d}t, \quad 1 \le k \le s,$$

are convergent integrals, i.e., the third term on the RHS converges to the real part of a polynomial of degree s, as $R \to \infty$. It remains to bound

$$\int_0^R |\nu(t)| \left|\frac{1}{t-z} - \frac{1}{t}\right| \mathrm{d}t.$$

Let λ_1 be the smallest absolute value of the points from Z lying on the positive ray. Then, for $|z| \geq 1 + \lambda_1$, the integral we are estimating is bounded by

$$c \int_0^{\lambda_1} \frac{t^p}{t}\,\mathrm{d}t + \|\nu\|_\infty \left[\int_{\lambda_1}^{2|z|} \frac{\mathrm{d}t}{t} + \int_1^{2|z|} \frac{\mathrm{d}t}{|t-z|} + \int_{2|z|}^\infty \frac{|z|}{|t-z|t}\,\mathrm{d}t \right].$$

The third integral needs a little care. Let $z = x + \mathrm{i}y$. For $x < 0 < t$, we have $|t - z|^{-1} \leq t^{-1}$, and therefore, $\int_1^{2|z|} \frac{\mathrm{d}t}{|t-z|} \leq \log 2 + \log|z|$. For $x \geq 0$, we have

$$\int_1^{2|z|} \frac{\mathrm{d}t}{|t-z|} \leq \int_1^{\max(x-|y|,1)} \frac{\mathrm{d}t}{x-t} + \int_{x-|y|}^{x+|y|} \frac{\mathrm{d}t}{|y|} + \int_{x+|y|}^{2|z|} \frac{\mathrm{d}t}{t-x}$$

$$\leq 2 \int_{|y|}^{2|z|} \frac{\mathrm{d}s}{s} + 2 = 2\log|z| + 2\log\frac{1}{|y|} + 2\log 2 + 2.$$

Hence,

$$c \int_0^{\lambda_1} \frac{t^p}{t}\,\mathrm{d}t + \|\nu\|_\infty \left[\int_{\lambda_1}^{2|z|} \frac{\mathrm{d}t}{t} + \int_1^{2|z|} \frac{\mathrm{d}t}{|t-z|} + \int_{2|z|}^\infty \frac{|z|}{|t-z|t}\,\mathrm{d}t \right]$$

$$\leq C \left[1 + \log|z| + \log\left(1 + \frac{1}{|y|} \right) \right].$$

Correcting the functions S_R by e^P, where P is a polynomial of degree at most p, independent of R, we get the limiting entire function S with the zero set Z satisfying

$$\big| \log|S(z)| - K(z) \big| \leq C \left[\log(2 + |z|) + \log\left(1 + \frac{1}{d(z)} \right) \right], \qquad \text{(A.2)}$$

where $d(z)$ is the distance from z to the union of our rays.

Now choose some big power $\tau > p$ and consider a point z, $|z| \geq 1$, with $d(z) < |z|^{-\tau}$ (if $d(z) \geq |z|^{-\tau}$, then (A.2) directly yields the needed bounds of S). Without loss of generality, we may assume that the nearest ray to z is the positive real semi-axis. Consider the disk D centered at z of radius $\rho = 2|z|^{-\tau}$. Since $K(re^{\mathrm{i}\theta}) = k(\theta)r^p$ with a Lipschitz function k, and $\tau > p$, we have

$$\sup \big\{ |K(z') - K(z)| \colon z' \in D \big\} = O(1), \quad z \in \mathbb{C}. \qquad \text{(A.3)}$$

By subharmonicity of $\log|S|$,

$$\log|S(z)| \leq \frac{1}{2\pi} \int_{-\pi}^\pi \log\big| S(z + \rho e^{\mathrm{i}\theta}) \big|\,\mathrm{d}\theta$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^\pi K(z + \rho e^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

$$+ C \left[\frac{1}{2\pi} \int_{-\pi}^\pi \log\left(1 + \frac{1}{d(z + \rho e^{\mathrm{i}\theta})} \right)\,\mathrm{d}\theta + \log(1 + |z|) \right]$$

$$\underset{(A.3)}{\leq} K(z) + C \log(1 + |z|).$$

This gives us estimate (I) is Theorem L.

Furthermore, if this disk has no zeroes of S in it, then

$$
\begin{aligned}
\log |S(z)| &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |S(z + \rho e^{i\theta})| \, d\theta \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} K(z + \rho e^{i\theta}) d\theta \\
&\qquad\qquad - C\Big[\frac{1}{2\pi} \int_{-\pi}^{\pi} \log\Big(1 + \frac{1}{d(z + \rho e^{i\theta})}\Big) d\theta + \log(1 + |z|)\Big] \\
&\geq K(z) - C \log(1 + |z|)
\end{aligned}
$$

holds as well, which yields estimate (II).

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Єдиність і неєдиність пар Фур'є

Aleksei Kulikov, Fedor Nazarov, and Mikhail Sodin

Мотивовані недавніми роботами Радченка і В'язовської, а також Рамоса і Соузи, ми знаходимо умови на пари дискретних множин на дійсній осі достатні для єдиності цих пар для перетворення Фур'є. Ці умови близькі одна до одної. Результат про єдиність може бути посилений до інтерполяційної формули, яка в свою чергу дає багато прикладів дискретних мір з дискретним перетворенням Фур'є.

Ключові слова: єдиність, пари Фур'є, інтерполяція Фур'є