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The L^2-Norm of the Euler Class for Foliations on Closed Irreducible Riemannian 3-Manifolds

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An upper bound for the L^2-norm of the Euler class $e(\mathcal{F})$ of an arbitrary transversely orientable foliation $\mathcal{F}$ of codimension one, defined on a three-dimensional closed irreducible orientable Riemannian 3-manifold M^3, is given in terms of constants bounding the volume, the radius of injectivity, the sectional curvature of M^3 and the modulus of mean curvature of the leaves. As a consequence, we get only finitely many cohomological classes of the group $H^2(M^3)$ that can be realized by the Euler class $e(\mathcal{F})$ of a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of mean curvature which is bounded above by the fixed constant H_0.

Key words: 3-manifold, foliation, Euler class, mean curvature

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1. Introduction

Let (M^3, g) be a closed oriented three-dimensional Riemannian manifold and $\mathcal{F}$ be a transversely oriented C^∞-smooth foliation of codimension one on M^3. Recall that a foliation $\mathcal{F}$ is *taut* if its leaves are minimal submanifolds of M^3 for some Riemannian metric on M^3. D. Sullivan [24] gave a description of taut foliations, namely, he proved that a foliation is taut if and only if each leaf of $\mathcal{F}$ is intersected by a transversal closed curve, which is equivalent to the condition that $\mathcal{F}$ does not contain generalized Reeb components (see bellow).

We previously proved the following result [2].

Theorem 1.1. *Let $V_0 > 0, i_0 > 0, K_0 \geq 0$ be fixed constants, and M^3 be a closed oriented three-dimensional Riemannian manifold with the following properties:*

1. *the volume* $\mathrm{Vol}(M^3) \leq V_0$;
2. *the sectional curvature K of M satisfies the inequality $K \leq K_0$;*
3. $\min\left\{\mathrm{inj}\left(M^3\right), \frac{\pi}{2\sqrt{K_0}}\right\} \geq i_0$, *where $inj\left(M^3\right)$ is the injectivity radius of M^3.*

Let us set

$$H_0 = \begin{cases} \min\left\{\frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}}\right\} & \textit{if } K_0 = 0, \\ \min\left\{\frac{2\sqrt{3}i_0^2}{V_0}, x_0\right\} & \textit{if } K_0 > 0, \end{cases}$$

where x_0 is the root of the equation

$$\frac{1}{K_0}\operatorname{arccot}^2\frac{x}{\sqrt{K_0}}-\frac{V_0}{2\sqrt{3}}x=0.$$

Then any smooth transversely oriented foliation $\mathcal{F}$ of codimension one on M^3 such that the modulus of the mean curvature H of its leaves satisfies the inequality $|H|<H_0$, should be taut, in particular, have minimal leaves for some Riemannian metric on M^3.

Notice that if M^3 admits a taut foliation, then M^3 is irreducible [18]. Let us recall that a 3-manifold M^3 is called *irreducible* if each embedded sphere bounds a ball in M^3. In particular, $\pi_2(M^3)=0$ (see [12]).

W. Thurston proved in [27] (see also [10]) that if $M^2\subset M^3$ is a closed embedded orientable surface which is different from S^2, then the Euler class $e(\mathcal{F})$ of a transversely oriented taut foliation $\mathcal{F}$ on M^3 satisfies

$$\left|e(\mathcal{F})\left[M^2\right]\right|\le-\chi\left(M^2\right). \tag{1.1}$$

Here, by the Euler class of the foliation $\mathcal{F}$, we mean the Euler class of the distribution tangent to $\mathcal{F}$.

Since any integer homology class $H_2\left(M^3;\mathbb{Z}\right)$ can be represented by a closed oriented surface (see subsection 2.2), the inequality above bounds the possible values of the cohomology class $e(\mathcal{F})$ on the generators of $H_2\left(M^3;\mathbb{Z}\right)$, and therefore the number of cohomological classes $H^2\left(M^3;\mathbb{Z}\right)$, realized as Euler classes $e(\mathcal{F})$, is finite.

In this paper, we estimate from above the L^2-norm of the Euler class of foliations on closed Riemannian 3-manifolds with leaves having a mean curvature bounded in absolute value by some positive constant. Below we prove the main theorem.

Theorem 1.2. *Let $V_0>0$, $i_0>0$, $H_0>0$, $k_0\le K_0$ be fixed constants. Suppose $\left(M^3,\mathcal{F}\right)$ to be a closed oriented irreducible three-dimensional Riemannian manifold equipped by a two-dimensional transversely oriented foliation $\mathcal{F}$, whose leaves have the modulus of the mean curvature H bounded above by the constant H_0, and M^3 satisfies the following conditions:*

1. *the volume* $\operatorname{Vol}\left(M^3\right)\le V_0$;
2. *the sectional curvature K of M satisfies the inequality $k_0\le K\le K_0$;*
3. *if $K_0>0$, then*

$$\min\left\{\operatorname{inj}\left(M^3\right),\frac{\pi}{2\sqrt{K_0}}\right\}\ge i_0,$$

if $K_0\le 0$, then

$$\operatorname{inj}\left(M^3\right)\ge i_0,$$

where $\operatorname{inj}\left(M^3\right)$ *is the injectivity radius of M^3.*

Then there exists a constant $C_1(V_0,i_0,k_0,K_0,H_0)$ such that the L^2-norm

$$\|e(\mathcal{F})\|_{L^2}\le C_1.$$

Corollary 1.3. *For any closed oriented Riemannian 3-manifold M^3 there are only finitely many cohomological classes of the group $H^2(M^3;\mathbb{R})$ that can be realized by the Euler class $e(\mathcal{F})$ of a two-dimensional transversely oriented foliation $\mathcal{F}$ whose leaves have the modulus of the mean curvature bounded above by the fixed constant H_0.*

Remark 1.4. In Theorem 1.2, the Euler class $e(\mathcal{F})$ is assumed to be real, i.e., the image of the integer Euler class via the homomorphism $H^2(M^3;\mathbb{Z}) \to H^2(M^3;\mathbb{R})$ is induced by the embedding of the coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$. Clearly, $e(\mathcal{F}) \in H^2(M^3;\mathbb{Z})_{\mathbb{R}} \subset H^2(M^3;\mathbb{R})$, where $H^2(M^3;\mathbb{Z})_{\mathbb{R}}$ is an integer lattice in $H^2(M^3;\mathbb{R})$. Recall also that the real cohomology groups are isomorphic to the de Rham cohomology groups and we can represent the real Euler class through a closed differential form, in particular, the harmonic form (see subsection 2.2).

Remark 1.5. As follows from Myers's theorem [17], if $k_0 > 0$, then $\pi_1(M^3)$ is finite and $H_1(M^3;\mathbb{R}) \cong H^2(M^3;\mathbb{R}) \equiv 0$, which implies $e(\mathcal{F}) = 0$. Thus we can suppose that $k_0 \leq 0$.

Remark 1.6. The foliation $\mathcal{F}$ does not contain a sphere as a leaf since in this case, by Reeb's stability theorem (see [26]), $M^3 \simeq S^2 \times S^1$, which contradicts the irreducibility of M^3.

2. Background material

2.1. Geometrical inequalities

2.1.1. Inequalities associated with a generalized Reeb component.

A subset of the foliated manifold $(M, \mathcal{F})$ is called *a saturated set* if it consists of leaves of the folation $\mathcal{F}$. A saturated set A of a three-dimensional compact orientable manifold M^3 with a given transversely orientable foliation $\mathcal{F}$ of codimension one is called a *generalized Reeb component* if A is a connected three-dimensional manifold with a boundary ∂A and any transversal to $\mathcal{F}$ vector field restricted to ∂A is directed either everywhere inwards or everywhere outwards of the generalized Reeb component A. In particular, the Reeb component R (see [26]) is a generalized Reeb component. It is clear that ∂A consists of a finite set of compact leaves of the foliation $\mathcal{F}$. It is not difficult to show that ∂A is a family of tori (see [11]).

The next result is due to G. Reeb.

Theorem 2.1 ([22])**.** *Let (M^3, g) be a closed oriented three-dimensional Riemannian manifold and $\mathcal{F}$ be a smooth transversely oriented foliation of codimension one on M. Then*

$$d\chi = 2H\mu, \tag{2.1}$$

where χ is the volume form of the foliation $\mathcal{F}$, and μ is the volume form on M^3.

Corollary 2.2. *Let M^3 be a closed oriented three-dimensional Riemannian manifold with a given transversely oriented smooth foliation $\mathcal{F}$ of codimension* *one. Suppose that $\mathcal{F}$ contains a generalized Reeb component A and the modulus of the mean curvature H of the foliation $\mathcal{F}$ is bounded above by $|H| \leq H_0$. Then*

$$\operatorname{Area}(\partial A) \leq 2H_0 \operatorname{Vol}(A) \quad \text{and} \quad \operatorname{Area}(\partial A) \leq H_0 \operatorname{Vol}(M^3). \tag{2.2}$$

Proof. According to the Stokes theorem and (2.1), we get

$$0 < \operatorname{Area}(\partial A) = \left|\int_{\partial A} \chi\right| = \left|\int_A d\chi\right| = 2\left|\int_A H\mu\right| \leq 2\int_A H_0\mu = 2H_0 \operatorname{Vol}(A).$$

Let $B = M^3 \setminus \operatorname{int} A$. Then B is also a generalized Reeb component and we have

$$\operatorname{Area}(\partial B = \partial A) = \left|\int_{\partial B} \chi\right| = \left|\int_B d\chi\right| = 2\left|\int_B H\mu\right| \leq 2\int_B H_0\mu = 2H_0 \operatorname{Vol}(B).$$

It follows that

$$2\operatorname{Area}(\partial A) \leq 2H_0(\operatorname{Vol}(A) + \operatorname{Vol}(B)) \leq 2H_0 \operatorname{Vol}(M^3),$$

which implies the result. □

Corollary 2.3. *The generalized Reeb component A is an obstruction to the foliation $\mathcal{F}$ being taut.*

Remark 2.4. The converse is also true. If the foliation is not taut, then it contains a generalized Reeb component (see [11]).

2.1.2. Systolic inequalities. Recall that the systole, denoted by *sys*, in a Riemannian manifold M with non-trivial fundamental group is the length of the smallest loop in M that is not null-homotopic in M. Under the condition of closeness M, such a loop exists and is necessary a closed geodesic. The proof does not differ from the proof of the existence of a closed geodesic in its free homotopy class (see [7, Chapter 12]).

The Loewner theorem below gives an upper bound on the systole in a Riemannian two-dimensional torus.

Theorem 2.5 (Loewner [21]). *Let T^2 be a two-dimensional torus with an arbitrary Riemannian metric on it. Then*

$$\operatorname{sys}^2 \leq \frac{2}{\sqrt{3}} \operatorname{Area}\left(T^2\right), \tag{2.3}$$

where sys *(abbreviated from systole) is the length of the shortest closed noncontractible geodesic on T^2.*

Due to Gromov, the generalization of this theorem is the following:

Theorem 2.6 ([15, Chap. 6]). *Let T^2 be a two-dimensional torus with an arbitrary Riemannian metric on it. Then there exists a pair of closed geodesics on T^2 of respective length λ_1, λ_2 such that*

$$\lambda_1\lambda_2 \leq \frac{2}{\sqrt{3}} \operatorname{Area}\left(T^2\right), \tag{2.4}$$

and whose homotopy classes form a generating set of $\pi_1\left(T^2\right) = \mathbb{Z}^2$.

Corollary 2.7. *Let T^2 be a Riemannian torus for which*

$$\text{sys} \geq C_0, \quad \text{Area}\left(T^2\right) \leq S_0$$

for some positive constants C_0, S_0. Then there exists a pair of closed geodesics on T^2 whose homotopy classes form a generating set of $\pi_1\left(T^2\right) = \mathbb{Z}^2$ and whose lengths λ_1, λ_2 do not exceed some constant $C\left(C_0, S_0\right)$.

Proof. From (2.4), it immediately follows that

$$\lambda_i \leq \frac{2}{\sqrt{3}} \frac{\text{Area}\left(T^2\right)}{\text{sys}} \leq C := \frac{2S_0}{\sqrt{3}C_0}, \quad i = 1, 2. \tag{2.5}$$

The corollary is proved. □

The concept of systole can be generalized to foliations.

Definition 2.8. Let $\left(M, \mathcal{F}\right)$ be a foliated manifold. Following [13, Chapter VII], we call a loop $f : S^1 \to M$ *integral* for $\mathcal{F}$ if $f\left(S^1\right)$ is contained in some leaf $\mathcal{L}$ of $\mathcal{F}$. In this case, $\mathcal{L}$ is referred to as the *support* of f.

Definition 2.9. The integral loop supported by $\mathcal{L}$ is referred to as *essential* if the loop $f : S^1 \to \mathcal{L}$ represents nontrivial element of the fundamental group $\pi_1\left(\mathcal{L}\right)$ and *inessential* otherwise.

We recently proved the following theorem.

Theorem 2.10 ([3]). *Let $\left(M, \mathcal{F}\right)$ be a foliated closed Riemannian manifold containing a leaf with a nontrivial fundamental group. Then there is an integral essential loop l_{sys} in M with smallest length among all integral essential loops in $\left(M, \mathcal{F}\right)$, which is necessary a closed geodesic in its support.*

Definition 2.11. Denote by $\text{sys}\left(\mathcal{F}\right)$ the length of the geodesic l_{sys} from Proposition 2.10.

2.1.3. Comparison inequalities. Recall the following comparison theorem for normal curvatures.

Theorem 2.12 ([4, 22.3.2.]). *Let $p \in M$ and $\beta : [0, r] \to M$ be a radial geodesic of the ball $B\left(p, r\right)$ of radius r centered at the point p of the Riemannian manifold M. Let $\beta\left(r\right)$ be a point not conjugate with p along β. Let the radius r be such that there are no conjugate points in the space of constant curvature K_0 within the radius of length r. Then if at each point $\beta\left(t\right)$ the sectional curvatures K of the manifold M do not exceed K_0, then the normal curvature k_n^S of the sphere $S\left(p, r\right)$ at the point $\beta\left(r\right)$ with respect to the normal $-\beta'$ is not less than the normal curvature k_n^0 of the sphere of radius r in the space of constant curvature K_0.*

Let M^3 be a 3-manifold satisfying the condition of Theorem 1.2. Notice that all normal curvatures of the sphere $S\left(r\right) \subset M^3$ of radius r are positive, provided that $r < i_0$ and the normal to the sphere $S\left(r\right)$ is directed inside the ball $B\left(r\right)$ which it bounds. (The sphere $S\left(r\right)$ indeed bounds the ball since $r < inj\left(M^3\right)$ by definition.) We will call such a normal *inward*.

Definition 2.13. We call a hypersurface $S \subset M^3$ of the Riemannian manifold M^3 the *supporting hypersurface* to the subset $A \subset M^3$ at the point $p \in \partial A \cap S$ with respect to the normal $n_p \perp T_pS$ if S cuts some spherical neighborhood B_p of the point p into two components, and $A \cap B_p$ is contained in that component to which the normal n_p is directed. We call the sphere $S(r) \subset M^3$ $(r < i_0)$ the *supporting sphere* to the set $A \subset M^3$ at the point $q \in A \cap S(r)$ if it is the supporting sphere to A at the point q with respect to the inward normal.

The following lemma is obvious.

Lemma 2.14 ([2, Lemma 4])**.** *Assume that the sphere $S(r_0)$ $(r_0 < i_0)$ is the supporting sphere to the surface $F \subset M^3$ at the point q. Then $k_n^S(v) \leq k_n^F(v)$ $\forall v \in T_qS(r_0)$, where $k_n^S(v)$ and $k_n^F(v)$ denote corresponding normal curvatures of $S(r_0)$ and F at the point q in the direction v.*

As a consequence of Lemma 2.14 and Theorem 2.12, we obtain the following inequalities at the touching point q:

$$0 < H_r^0 \leq H_r(q) \leq H(q), \tag{2.6}$$

where H_r^0 and H_r are mean curvatures of the spheres $S(r)$ bounding the ball of radius r, $r < i_0$, in the space of constant curvature K_0 and the manifold M^3 respectively, and H is the mean curvature of the surface F.

2.2. Harmonic maps to the circle and harmonic forms. Let M^3 be a closed oriented Riemannian 3-manifold. Recall that

$$H^1(M^3;\mathbb{Z}) \cong [M^3, S^1], \tag{2.7}$$

and each cohomological class $a \in H^1(M^3;\mathbb{Z})$ can be obtained as an image of the generator $[S^1]^* \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ under the homomorphism $f^* : H^1(S^1;\mathbb{Z}) \to H^1(M^3;\mathbb{Z})$ induced by the mapping $f : M^3 \to S^1$ uniquely defined up to homotopy. Recall also that the group $H_2(M^3;\mathbb{Z}) \overset{PD}{\cong} H^1(M^3;\mathbb{Z})$ does not contain a torsion and we can identify $H^1(M^3;\mathbb{Z})$ with the integer lattice $H^1(M^3;\mathbb{Z})_{\mathbb{R}} \subset H^1(M^3;\mathbb{R})$ and $H_2(M^3;\mathbb{Z})$ with $H_2(M^3;\mathbb{Z})_{\mathbb{R}} \subset H_2(M^3;\mathbb{R})$. Observe that the Poincaré duality $H^1(M^3;\mathbb{R}) \overset{PD}{\cong} H_2(M^3;\mathbb{R})$ induces the Poincaré duality of integer lattices $H^1(M^3;\mathbb{Z})_{\mathbb{R}} \overset{PD}{\cong} H_2(M^3;\mathbb{Z})_{\mathbb{R}}$.

Let us identify S^1 with the unit-length circle $\mathbb{R}/\mathbb{Z}$ with natural parameter θ. If f is a smooth function, then the preimage $f^{-1}(\theta)$ of a regular value $\theta \in S^1$ is a smooth (not necessarily connected) oriented submanifold $M^2 \subset M^3$, which we identify with the image of the embedding $i : M^2 \hookrightarrow M^3$. The singular homology class $[M^2, i] := i_*[M^2] \in H_2(M^3;\mathbb{Z})_{\mathbb{R}}$ corresponding to the singular cycle (M^2, i) is Poincaré dual to the cohomology class $a \in H^1(M^3;\mathbb{Z})_{\mathbb{R}}$, where $[M^2] \in H_2(M^2;\mathbb{R})$ denotes a fundamental class of M^2 which is the generator of the group $\mathbb{Z} \cong H_2(M^2;\mathbb{Z})_{\mathbb{R}} \subset H_2(M^2;\mathbb{R}) \cong \mathbb{R}$.

Remark 2.15. Note that by Sard's theorem, the set of regular values of f has a full measure in S^1 and it is also an open set in S^1 since M^3 is compact. The same is true for any smooth map $g : N \to L$ of the smooth compact manifolds N and L [19].

Now we should recall that each homotopy class in $[M^3, S^1]$ can be represented by the harmonic mapping [9]. Let $u : M^3 \to S^1$ be a harmonic map representing the nontrivial class $[u] \in [M^3, S^1] \cong H^1(M^3; \mathbb{Z})$. Observe that $\alpha = u^* d\theta,\ \theta \in S^1$, is a harmonic 1-form $\big($i.e., $d\alpha = \delta\alpha = 0\big)$ on M^3 corresponding to the integer lattice class $[u] \in H^1(M^3; \mathbb{Z})_{\mathbb{R}}$.

On the space of differential k-forms $\Omega^k(M^3)$, $k \in \{0, 1, 2, 3\}$, one can introduce the L^2-norm:

$$\|\alpha\|_{L^2} = \sqrt{\int_{M^3} \alpha \wedge *\alpha} = \sqrt{\int_{M^3} |\alpha|^2}, \tag{2.8}$$

where $*$ denotes the Hodge star operator, and $|\alpha_p| = \sqrt{*(\alpha_p \wedge *\alpha_p)},\ \ p \in M^3$. In the 3-dimensional vector space T_pM^3 each k-form α_p is simple and $|\alpha_p|$ coincides with the comass norm

$$|\alpha_p| = \max \alpha_p(e_1, \ldots, e_k),$$

where the maximum is taken over all orthogonal frames of vectors $(e_1, \ldots, e_k)$ in T_pM^3.

We also use the L^∞-norm on $\Omega^*(M^3)$ defined as follows:

$$\|\alpha\|_{L^\infty} = \max_{p \in M^3} |\alpha_p|.$$

The norm (2.8) induces the L^2-norm on the de Rham cohomology of M^3 as follows. Let $a \in H^k(M^3; \mathbb{R})$, then we set

$$\|a\|_{L^2} := \inf_{\alpha} \big\{\|\alpha\|_{L^2} : \alpha \in \Omega^k(M^3) \text{ is a smooth closed } k\text{-form representing } a\big\}.$$

From de Rham - Hodge theory, it follows that $\|a\|_{L^2} = \|\alpha\|_{L^2}$, where α is the unique harmonic form $(d\alpha = \delta\alpha = 0)$ representing the class $a \in H^k(M^3; \mathbb{R})$.

Using Poincaré duality $H_i(M^3; \mathbb{R}) \overset{PD}{\cong} H^{3-i}(M^3; \mathbb{R})$, we can introduce the L^2-norm on $H_2(M^3; \mathbb{R})$ setting

$$\|b\|_{L^2} = \|PD(b)\|_{L^2},\ \ b \in H_i(M^3; \mathbb{R}).$$

On the other hand, the non-degenerate Kronecker pairing

$$\langle \cdot, \cdot \rangle : H^k(M^3; \mathbb{R}) \times H_k(M^3; \mathbb{R}) \to \mathbb{R},$$

induced by integration of closed forms over cycles, allows us to define the L^2-norm $\|\cdot\|^*_{L^2}$ on $H_k(M^3; \mathbb{R}) \cong \big(H^k(M^3; \mathbb{R})\big)^*$ dual to the L^2-norm $\|\cdot\|_{L^2}$ on $H^k(M^3; \mathbb{R})$. As was shown in [1],

$$PD : \big(H^i(M^3; \mathbb{R}), \|\cdot\|_{L^2}\big) \to \big(H_{3-i}(M^3; \mathbb{R}), \|\cdot\|^*_{L^2}\big)$$

is an isometry for $i = 1, 2$.

Notice that

$$PD\big([\alpha \wedge \beta]\big) = PD\big([\beta \wedge \alpha]\big) = \big\langle [\alpha], PD\big([\beta]\big)\big\rangle = \big\langle [\beta], PD\big([\alpha]\big)\big\rangle,$$

where $\alpha \in \Omega^1\big(M^3\big)$ and $\beta \in \Omega^2\big(M^3\big)$ are closed forms. Since the set of integer-directed rays from $0 \in H^1\big(M^3;\mathbb{R}\big)$ is everywhere dense set in $H^1\big(M^3;\mathbb{R}\big)$, we have

$$\|b\|_{L^2} = \|PD(b)\|^*_{L^2} = \sup_{a\neq 0} \frac{\big\langle a, PD(b)\big\rangle}{\|a\|_{L^2}} = \sup_{[\Sigma]\neq 0} \frac{\langle b, [\Sigma]\rangle}{\|[\Sigma]\|_{L^2}}, \tag{2.9}$$

where $b \in H^2\big(M^3, \mathbb{R}\big)$, $a \in H^1\big(M^3, \mathbb{Z}\big)_{\mathbb{R}}$ and Σ is a compact oriented surface embedded in M^3 such that $PD(a) = [\Sigma]$.

Let us recall the following inequality (see [20, 7.1.13,7.1.17, 9.2.7, 9.2.8]). If α is a harmonic 1-form on closed Riemannian manifold M^n, then

$$\|\alpha\|_{L^\infty} \leq \Lambda_n(k, D)\|\alpha\|_2. \tag{2.10}$$

Here, $\|\alpha\|_2 = \frac{\|\alpha\|_{L^2}}{\sqrt{\mathrm{Vol}\,(M^n)}}$, $D > 0$ is the constant satisfying the inequality $\mathrm{Diam}\,\big(M^n\big) \leq D$, and $k \leq 0$ is the constant satisfying the inequality $\mathrm{Ric}\,\big(M^3\big) \geq \big(n-1\big)k$.

In the three-dimensional case, we have $n = 3$. In addition, we can put $\nu = 3$ (see [20, 7.1.13,7.1.17, 9.2.7]).

Remark 2.16. In [8], C.B. Croke gave an estimate for the diameter of a closed Riemannian manifold, which we adapt to the three-dimensional case:

$$\mathrm{Diam}\,\big(M^3\big) \leq \frac{27\pi\,\mathrm{Vol}\,\big(M^3\big)}{8\,\mathrm{inj}\,\big(M^3\big)^2}.$$

In particular, if M^3 satisfies the conditions of Theorem 1.2, we can take

$$D = \frac{27}{8}\pi\frac{V_0}{i_0^2}.$$

Moreover, we can put $k = k_0$ (see Remark 1.5), and thus we have

$$\Lambda_3\big(k, D\big) = \Lambda\big(V_0, i_0, k_0\big). \tag{2.11}$$

The following Stern's theorem estimates an average Euler characteristic of a surface dual to the harmonic mapping of M^3 into the circle.

Theorem 2.17 ([25]). *Let $u : M^3 \to S^1$ be a harmonic map to the unit-length circle representing the nontrivial class $[u] \in [M^3, S^1] \cong H^1\big(M^3;\mathbb{Z}\big) \overset{PD}{\cong} H_2\big(M^3;\mathbb{Z}\big)$. Then*

$$2\pi \int_{\theta\in S^1} \chi\big(\Sigma_\theta\big) \geq \frac{1}{2}\int_{\theta\in S^1}\int_{\Sigma_\theta} \big(|du|^{-2}|\,\mathrm{Hess}\,\big(u\big)|^2 + R_{M^3}\big), \tag{2.12}$$

where $\Sigma_\theta = u^{-1}\theta$, $\theta \in S^1$, and R_{M^3} is the scalar curvature of M^3.

Remark 2.18. For a regular value $\theta \in S^1$ of $u : M^3 \to S^1$, each connected component Σ_θ^i of Σ_θ represents a non-trivial homology class in $H_2\big(M^3\big)$ $\big($see [25]$\big)$, and since M^3 is assumed to be irreducible, $\chi\big(\Sigma_\theta^i\big) \leq 0$.

As a corollary, Stern obtained the following useful estimate.

Corollary 2.19 ([25]).

$$\int_{\theta \in S^1} \chi\big(\Sigma_\theta\big) \geq -\frac{1}{4\pi} \|\alpha\|_{L^2} \|R^-\|_{L^2}, \tag{2.13}$$

where $R^- := \min\{0, R\}$ is a non-positive part of the scalar curvature R and $\alpha = u^* d\theta$.

2.3. Novikov's theorem and a vanishing cycle. Let $\big(M^3, \mathcal{F}\big)$ be a foliated closed 3-manifold. An integral loop $\alpha : S^1 \to M^3$ is a *vanishing cycle* if there exists a homotopy $A : S^1 \times I \to M^3$ through integral loops $A_t := A|_{S^1 \times t}$ for $\mathcal{F}$ such that $A_0 = \alpha$ and A_t is inessential for $0 < t \leq 1$. A vanishing cycle α is *non-trivial* if α is essential.

The following well-known Novikov's theorem gives us topological obstructions to the existence of taut foliations.

Theorem 2.20 ([18]).

1. *For a closed orientable smooth 3-manifold M^3 and a transversely orientable C^2-smooth foliation $\mathcal{F}$ of codimension one on M^3, the following are equivalent.*
   a) *The foliation $\mathcal{F}$ has a Reeb component.*
   b) *There is a leaf L of $\mathcal{F}$ that is not π_1-injective. That is, the inclusion $i : L \to M^3$ induces a homomorphism $i_* : \pi_1\big(L\big) \to \pi_1\big(M^3\big)$ with nontrivial kernel.*
   c) *Some leaf of $\mathcal{F}$ contains a nontrivial vanishing cycle.*
2. *The support of the nontrivial vanishing cycle is a torus bounding a Reeb component.*

The construction underlying the proof of Novikov's theorem is as follows. Let a simple closed integral regular curve $\alpha : S^1 \to M^3$ belongs to the leaf $L \in \mathcal{F}$ and represents the nontrivial element of $Ker\big(i_* : \pi_1\big(L\big) \to \pi_1\big(M^3\big)\big)$. We can find an immersion $p : D \to M^3$ of the two-dimensional disk D such that $p\big(\partial D\big) = \alpha$. This immersion can be brought to a general position by a small perturbation $\big($modulo $\partial D\big)$. It means that the induced foliation $\mathcal{F}' := p^{-1}\big(\mathcal{F} \cap p\big(D\big)\big)$ has only Morse singularities $\big($saddles and centers$\big)$. Moreover, by a small perturbation, we can obtain not more than one singular point on a single leaf $\big($see [6, Lemma 9.2.1.]$\big)$. The resulting foliation outside the singular points on D can be oriented $\big($see Subsection 2.4$\big)$. Therefore, there is a smooth vector field X tangent to $\mathcal{F}'$ with zeros corresponding to the singular points of $\mathcal{F}'$. Recall that a separatrix coming out of a singular point and returning to it, together with the singular point $\big($a saddle), is called a *separatrix loop*. By the construction, a saddle singular point of $\mathcal{F}'$ belongs to at most two separatrix loops.

The idea of general position described above can be extended to arbitrary immersed compact surfaces. In particular, the following theorem holds.

Theorem 2.21 ([5, Theorem 7.1.10], [6, 9.2.A]). *Let M^3 be an oriented closed 3-manifold with a smooth transversely oriented foliation $\mathcal{F}$ on it. Then for any C^q-mapping $f : N^2 \to M^3$ of a compact oriented surface N^2 such that in the case of $\partial N^2 \neq \varnothing$, we have $f|_{\partial N^2}$ either is transverse to $\mathcal{F}$ or has an image in a leaf L of $\mathcal{F}$, and for any $\delta > 0$ there exists a δ-close to f C^q -immersion $p : N^2 \to M^3$ in $C^q(N^2, M^3)$ -topology, $q \geq 2$, such that:*

I. *The induced foliation $\mathcal{F}' := p^{-1}(\mathcal{F} \cap p(N^2))$ has only Morse singularities.*

II. *There is at most one singular point on one leaf.*

III. *In the case of $\partial N^2 \neq \varnothing$, we have $p|_{\partial N^2}$ either is transverse to $\mathcal{F}$ or has image in a leaf L of $\mathcal{F}$.*

An immersion p satisfying the properties I–III of Theorem 2.21 will be referred to as an immersion of general position.

Definition 2.22. Let us identify the closed orbits and separatrix loops of $\mathcal{F}'$ with the images of the corresponding loops $f : S^1 \to N^2$ which bypass them once along the trajectories of the vector field X. The loops $f : S^1 \to N^2$ are referred to as *essential* if the integral loop $p \circ f$ is essential and *inessential* otherwise. Note that due to Reeb's stability theorem, inessential closed orbits have a "good neighborhood", i.e., a neighborhood consisting of inessential closed orbits.

Fig. 2.1: Pinched annulus $\mathcal{P}$.

Definition 2.23. Let $p : N^2 \to M^3$ be an immersion of general position described above. Let us denote by $\mathcal{P}$ a subset of N^2, which is topologically a disk with a boundary that is either a closed orbit or a separatrix loop of $\mathcal{F}'$, or it is a pinched annulus (see Fig. 2.1) consisting of two separatrix loops with a common saddle point. Suppose that $\partial\mathcal{P}$ has a "good collar" in $\mathcal{P}$, i.e., a collar consisting of inessential closed orbits of $\mathcal{F}'$. Clearly, the p-image of $\partial\mathcal{P}$ represents a vanishing cycle. We call $\mathcal{O} := \partial\mathcal{P}$ the vanishing cycle too.

One of S.P. Novikov's key observations in [18] was the proof of the existence of a nontrivial vanishing cycle $\mathcal{O}$ inside of $(D, \mathcal{F}')$ (see above).

2.4. Euler class of foliations. Here we describe Thurston's construction for calculating the Euler class $e(\mathcal{F})$ of a transversely oriented codimension one foliation $\mathcal{F}$ on a closed oriented 3-manifold M^3 [27]. Let $p : N^2 \to (M^3, \mathcal{F})$ be an immersion of general position of a closed oriented surface N^2. The induced foliation $\mathcal{F}' = p^{-1}(\mathcal{F} \cap p(N^2))$ on N^2 can be oriented outside the singular points. To verify this, let us take a normal vector field n to the foliation $\mathcal{F}$, and for all $x = p(z) \in p(N^2)$ consider the orthogonal projection $n'(x)$ of the normal $n(x)$ to $\mathcal{F}$ on the tangent plane $p_*(T_z(N^2))$, which in the case where z is not a singular point uniquely determines the unit tangent vector e' to the leaf $\mathcal{L}'_z \in \mathcal{F}'$, $z \in \mathcal{L}'_z$, such that the frame $\left\{e', p_*^{-1}\frac{n'}{|n'|}\right\}$ defines a positive orientation of $T_z(N^2)$. Now we can define a smooth vector field X on N^2 tangent to $\mathcal{F}'$ whose zeros correspond to the singular points of $\mathcal{F}'$ putting

$$X = |n'|e'. \tag{2.14}$$

Remark 2.24. It is easy to define a vector field $X^\perp$ orthogonal to $\mathcal{F}'$ with respect to the induced Riemannian metric on N^2. The vector field $X^\perp$ has the same singular points as X and the integral curves of $X^\perp$ define a foliation $\mathcal{F}'^\perp$ orthogonal to $\mathcal{F}'$ on N^2.

The pair (N^2, p) can be understood as a singular cycle if we fix some triangulation on N^2. Let the singular homology class $[N^2, p] := p_*[N^2] \in H_2(M^3; \mathbb{Z})_\mathbb{R} \subset H_2(M^3; \mathbb{R})$ correspond to the singular cycle (N^2, p), where $[N^2]$ denotes a fundamental class of N^2. As W. Thurston showed in [27], to calculate the value of the Euler class $e(T\mathcal{F}) \in H^2(M^3, \mathbb{Z})_\mathbb{R}$ of the foliation $\mathcal{F}$ on the singular homology class $[N^2, p] \in H_2(M^3; \mathbb{Z})_\mathbb{R}$, it suffices to calculate the total index of singular points of the vector field X on N^2 taking into account the orientation of $p_*(T_q(N^2))$ at singular points. (We apply Thurston's results to immersed submanifolds rather than embedded ones, where the same ideas work automatically.) Since M^3 is oriented, we can uniquely choose a unit normal vector $m \in T_{p(q)}M^3$ to the plane $p_*(T_q(N^2)$, $q \in N^2$, which defines the orientation of $p_*(T_q(N^2))$ coming from the orientation of $T_q(N^2)$.

We say that a singular point $q \in N^2$ is of *negative* type if $m(p(q)) = -n(p(q))$. If $m(p(q)) = n(p(q))$, then the type of a singular point is called *positive*.

We denote by I_N the sum of indices of singular points of negative type, and by I_P the sum of indices of singular points of positive type. The value of the Euler class $e(T\mathcal{F})$ on the singular homology class $[N^2, p]$ is calculated as follows:

$$e(T\mathcal{F})([N^2, p]) = e(p^*(T\mathcal{F}))([N^2]) = I_P - I_N. \tag{2.15}$$

Recall that the Poincaré–Hopf theorem states that

$$\chi(N^2) = I_P + I_N. \tag{2.16}$$

3. Preliminary results

3.1. An upper bound for the number of Reeb components of a bounded mean curvature foliation. The results of these subsections are represented in [3]. For the sake of completeness, we give them in a slightly more general form.

Let us prove the following theorem.

Theorem 3.1. *Let M^3 be a closed oriented three-dimensional Riemannian manifold satisfying the conditions 1–3 of Theorem 1.2. Let $\mathcal{F}$ be a codimension one transversely oriented foliation on M^3, whose leaves have a modulus of mean curvature bounded above by the fixed constant H_0.*

Then

$$\operatorname{sys}\big(\mathcal{F}\big) \geq C_0 := \begin{cases} 2\min\left\{i_0, \frac{1}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}\right\} & \text{if } K_0 > 0, \\ 2\min\left\{i_0, \frac{1}{H_0}\right\} & \text{if } K_0 = 0, \\ 2\min\left\{i_0, \frac{1}{\sqrt{-K_0}} \operatorname{arccoth} \frac{H_0}{\sqrt{-K_0}}\right\} & \text{if } K_0 < 0 \\ & \quad \text{and } H_0 > \sqrt{-K_0}, \\ 2i_0 & \text{if } K_0 < 0 \\ & \quad \text{and } H_0 \leq \sqrt{-K_0}. \end{cases} \tag{3.1}$$

Proof. Case 1: $\frac{\operatorname{sys}(\mathcal{F})}{2} \geq i_0$. The result follows immediately.

Case 2: $\frac{\operatorname{sys}(\mathcal{F})}{2} < i_0$. Let l_{sys} be an integral closed geodesic which is not null-homotopic in its support and whose length $\operatorname{sys} = \operatorname{sys}\big(\mathcal{F}\big) < 2i_0$. Then there is an immersion

$$p : D \to \operatorname{int} B(r), \quad r \in \left(\frac{\operatorname{sys}}{2}, i_0\right)$$

of a disk D which is in general position with respect to $\mathcal{F}$ and such that $p\big(\partial D\big) = l_{\text{sys}}$. As noted in subsection 2.3, there is a vanishing cycle which belongs to

$$T^2 \cap p\big(D\big) \subset \operatorname{int} B(r),$$

where $T^2 \in \mathcal{F}$ is a torus bounding a Reeb component R.

Let $r \in \big(\frac{\operatorname{sys}}{2}, i_0\big)$ be a regular value of the mapping

$$pr_r|_{\ (\operatorname{int} B(i_0)) \cap T^2} : \big(\operatorname{int} B(i_0)\big) \cap T^2 \to \mathbb{R} \tag{3.2}$$

such that $pr_r(r, \phi_1, \phi_2) = r$, where (r, ϕ_1, ϕ_2) is a normal coordinate system in the ball $B(i_0)$.

In the case $S(r) \cap T^2 \neq \varnothing$, from [2, Proposition 2] it follows that the sphere $S(r)$ is a supporting sphere with respect to the inward normal at the tangent point q for some inner leaf of the Reeb component R.

It should be noticed that due to Sard's theorem , the set of regular values of the mapping (3.2) has a full measure in the interval $\big(\frac{\operatorname{sys}}{2}, i_0\big)$ and the value r can be taken arbitrarily close to $\frac{\operatorname{sys}}{2}$.

In the case $S(r) \cap T^2 = \varnothing$, we achieve the tangency of the sphere $S(r)$ and T^2 by decreasing the radius r, and the sphere $S(r)$ becomes supporting for the torus T^2.

It follows from (2.6) that

$$H_r^0 \leq H_0,$$

where

$$H_r^0 = \begin{cases} \sqrt{K_0} \cot\left(r\sqrt{K_0}\right) & \text{if } K_0 > 0, \\ \frac{1}{r}, & \text{if } K_0 = 0, \\ \sqrt{-K_0} \coth\left(r\sqrt{-K_0}\right) & \text{if } K_0 < 0. \end{cases}$$

Observe that H_0 must satisfy $\sqrt{-K_0} < H_0$ if $K_0 < 0$.

Hence we conclude that $sys(\mathcal{F})$ must satisfy the inequality

$$\operatorname{sys}(\mathcal{F}) \geq \begin{cases} \frac{2}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}} & \text{if } K_0 > 0, \\ \frac{2}{H_0} & \text{if } K_0 = 0, \\ \frac{2}{\sqrt{-K_0}} \operatorname{arccoth} \frac{H_0}{\sqrt{-K_0}} & \text{if } K_0 < 0. \end{cases}$$

Combining Case 1 and Case 2, we obtain the result. □

From Theorem 3.1 it follows:

Corollary 3.2. *The number of Reeb components of the foliation $\mathcal{F}$ does not exceed* $\frac{4H_0 \operatorname{Vol}\left(M^3\right)}{\sqrt{3}C_0^2}$.

Proof. From Theorem 2.5 and Corollary 2.2, we have

$$\frac{\sqrt{3}}{2} C_0^2 \leq \operatorname{Area}\left(\partial R\right) \leq 2H_0 \operatorname{Vol}\left(R\right). \tag{3.3}$$

It follows from (3.3) that $\operatorname{Vol}(R) \geq \frac{\sqrt{3}C_0^2}{4H_0}$. Since the interiors of Reeb components do not intersect, the number of Reeb components does not exceed $\frac{4H_0 \operatorname{Vol}(M^3)}{\sqrt{3}C_0^2}$. □

3.2. Choosing a regular value of the harmonic mapping $u : M^3 \to S^1$

Lemma 3.3. *Let M^3 from Theorem 1.2 and $u : M^3 \to S^1$ be a harmonic map to the unit-length circle S^1 representing the nontrivial class $[u] \in [M^3, S^1] \cong H^1(M^3; \mathbb{Z})$. Let μ be the standard Lebesgue measure on S^1. Let us denote*

$$\mathbf{A} = \left\{ \theta \in S^1 \;\middle|\; -\chi(\Sigma_\theta) \leq \frac{1}{2\pi} \|\alpha\|_{L^2} \|R^-\|_{L^2} \right\}, \tag{3.4}$$

where $\alpha = u^* d\theta$ and $\Sigma_\theta = u^{-1}\theta$, $\theta \in S^1$. Then $\mu(\mathbf{A}) > \frac{1}{2}$.

Proof. If we assume that the statement of Lemma 3.3 is not true, then, taking into account Remark 2.18, we get

$$\mu\left(\left\{\theta \in S^1 \;\middle|\; -\chi(\Sigma_\theta) > \frac{1}{2\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2}\right\}\right) \geq \frac{1}{2}$$

and

$$\int_{\theta\in S^1} -\chi(\Sigma_\theta) > \frac{1}{4\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2},$$

which contradicts to (2.13). □

It follows from Corollaries 2.7 and 2.2 that every torus T_j^2 bounding the Reeb component of $R_j \in \mathcal{F}$ contains a simple closed smooth curve γ_j which is non-homologous to zero in R_j and has a length bounded above by the constant $C = \frac{2H_0 \operatorname{Vol}\left(M^3\right)}{\sqrt{3}C_0}$. For convenience, we introduce the following notations:

$$\mathbf{\Gamma} := \bigsqcup_j \gamma_j, \quad \mathbf{T} := \bigsqcup_j T_j^2, \quad \mathbf{R} := \bigsqcup_j R_j.$$

By Corollary 3.2, we obtain the following upper bound on the length of $\mathbf{\Gamma}$:

$$l(\mathbf{\Gamma}) \leq C_{\mathbf{\Gamma}} := C\frac{4H_0 \operatorname{Vol}\left(M^3\right)}{\sqrt{3}C_0^2} = \frac{8H_0^2 \operatorname{Vol}\left(M^3\right)^2}{3C_0^3}. \tag{3.5}$$

Lemma 3.4. *Let $u : M^3 \to S^1$ be a harmonic map to the unit-length circle S^1, and μ denote the standard measure length on a curve. Let us denote*

$$\mathbf{B} := \left\{\theta \in S^1 \;\middle|\; \operatorname{card}\left(u|_{\mathbf{\Gamma}}\right)^{-1}(\theta) \leq 2C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}\right\}, \tag{3.6}$$

where $\alpha = u^*d\theta$. Then $\mu(\mathbf{B}) > \frac{1}{2}$.

Proof. First, note that $\|\alpha\|_{L^\infty}$ is equal to the norm $\|du\|_{L^\infty} = \max_{p\in M^3}|du|_p$. Assume that the statement of Lemma 3.4 is not true. Then we have

$$\mu\left(\left\{\theta \in S^1 \;\middle|\; \operatorname{card}\left(u|_{\mathbf{\Gamma}}\right)^{-1}(\theta) > 2C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}\right\}\right) \geq \frac{1}{2}. \tag{3.7}$$

Since $\mathbf{\Gamma}$ is compact, it follows from Remark 2.15 that the set of regular values $\operatorname{reg}\left(u|_{\mathbf{\Gamma}}\right)$ of the function $u|_{\mathbf{\Gamma}}$ is an open and everywhere dense set in S^1. (A value is considered regular if its preimage is empty.) Recall that nonempty open sets in S^1 are either all S^1 or a finite or countable disjoint union of open intervals in S^1:

$$\operatorname{reg}\left(u|_{\mathbf{\Gamma}}\right) = \bigsqcup_{\omega\in\Omega} J_\omega, \tag{3.8}$$

where Ω is either a finite or a countable indexing set, and J_ω either is an open interval in S^1 for each $\omega \in \Omega$ or is the entire circle S^1. Clearly, in the last case, $\Omega = \{\omega\}$.

Since the mapping $u|_{\mathbf{\Gamma}} : \mathbf{\Gamma} \to S^1$ is a covering map on each preimage $(u|_{\mathbf{\Gamma}})^{-1}(J_\omega)$, then, by assumption (3.7), there is a subset $\Omega' \subset \Omega$ such that the cardinality of the covering $(u|_{\mathbf{\Gamma}})^{-1}(J_\omega) \to J_\omega$, $\omega \in \Omega'$, is greater than $2C_{\mathbf{\Gamma}}\|du\|_{L^\infty}$ and

$$\mu\Big(\bigsqcup_{\omega\in\Omega'} J_\omega\Big) \geq \frac{1}{2}. \tag{3.9}$$

Due to (3.7) and (3.9), the additivity of μ implies

$$\begin{aligned} l(\mathbf{\Gamma}) = \mu(\mathbf{\Gamma}) \geq \mu\Big(\sum_{\omega\in\Omega'} (u|_{\mathbf{\Gamma}})^{-1}(J_\omega)\Big) = \sum_{\omega\in\Omega'} \mu\big((u|_{\mathbf{\Gamma}})^{-1}(J_\omega)\big) \\ > 2C_{\mathbf{\Gamma}}\|du\|_{L^\infty} \sum_{\omega\in\Omega'} \frac{1}{\|du\|_{L^\infty}}\mu(J_\omega) = 2C_{\mathbf{\Gamma}} \sum_{\omega\in\Omega'} \mu(J_\omega) \geq C_{\mathbf{\Gamma}}, \end{aligned} \tag{3.10}$$

which contradicts to (3.5) and proves Lemma 3.4. □

From Lemmas 3.3 and 3.4, we immediately obtain the following corollary.

Corollary 3.5. *Let $u : M^3 \to S^1$ be a harmonic map to the unit-length circle S^1. Then we can find the value $\theta_0 \in \mathbf{A} \cap \mathbf{B}$ such that θ_0 is a regular value for $u, u|_{\mathbf{T}}, u|_{\mathbf{\Gamma}}$.*

Proof. Since $\mu(S^1) = 1$, by the measure property, we have

$$\mu(\mathbf{A} \cup \mathbf{B}) = \mu(\mathbf{A}) + \mu(\mathbf{B}) - \mu(\mathbf{A} \cap \mathbf{B}) \leq \mu(S^1) \leq 1,$$

which implies $\mu(\mathbf{A} \cap \mathbf{B}) > 0$. The rest follows from Remark 2.15. □

Let us emphasize the following properties of Σ_{θ_0}:

- $-\chi(\Sigma_{\theta_0}) \leq \frac{1}{2\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2}$.
- If $x \in \Sigma_{\theta_0} \cap \mathbf{\Gamma}$, then $\mathbf{\Gamma} \pitchfork \Sigma_{\theta_0}$ at the point x.
- If $\Sigma_{\theta_0} \cap \mathbf{T} \neq \varnothing$, then $\Sigma_{\theta_0} \pitchfork \mathbf{T}$.

Definition 3.6. Denote by $\mathcal{C} = \{C_j\}$ the disjoint finite family (possibly empty) of circles such that $\Sigma_{\theta_0} \cap \mathbf{T} = \bigsqcup_j C_j$.

Corollary 3.7. *The number of those circles of the family $\mathcal{C}$ that represent the nontrivial kernel* $\ker\big(\mathbf{i}_* : H_1(\mathbf{T};\mathbb{Z}) \to H_1(\mathbf{R};\mathbb{Z})\big)$ *does not exceed $2C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}$, where $\mathbf{i}_*$ is a homomorphism induced by the embedding $\mathbf{i} : \mathbf{T} \hookrightarrow \mathbf{R}$.*

Proof. The proof follows immediately from the definition of the set $\mathbf{B}$ (see Lemma 3.4) and the fact that $\mathbf{\Gamma}$ necessarily intersects each of the circles in the family $\mathcal{C}$, which represents the nontrivial kernel $\ker\big(\mathbf{i}_* : H_1(\mathbf{T};\mathbb{Z}) \to H_1(\mathbf{R};\mathbb{Z})\big)$. The corollary is proved. □

Proposition 3.8. *Let $i : M^2 \hookrightarrow M^3$ be an embedding such that $i(M^2) = \Sigma_{\theta_0} = u^{-1}(\theta_0)$, where $\theta_0 \in S^1$ from Corollary 3.5. Then there is an embedding of general position $i' : M^2 \hookrightarrow M^3$ with the image $\Sigma'_{\theta_0} := i'(M^2)$ satisfying the following properties:*

1) $\Sigma'_{\theta_0} \simeq M^2$, *in particular,* $-\chi\big(\Sigma'_{\theta_0}\big) \leq \frac{1}{2\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2}$*;*
2) *if* $\Sigma'_{\theta_0} \cap \mathbf{T} \neq \varnothing$*, then* $\Sigma'_{\theta_0} \pitchfork \mathbf{T}$ *and the intersection* $\Sigma'_{\theta_0} \cap \mathbf{T}$ *is a disjoint union of circles* $\mathcal{C}' = \bigsqcup_j C'_j$*;*
3) *the number of those circles of the family* $\mathcal{C}'$ *that represent the nontrivial kernel* $\ker\big(\mathbf{i}_* : H_1(\mathbf{T};\mathbb{Z}) \to H_1(\mathbf{R};\mathbb{Z})\big)$ *does not exceed* $2C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}$*, where* $\alpha = u^*d\theta$*;*
4) $\big[M^2, i'\big] = \big[M^2, i\big] \in H_2\big(M^3;\mathbb{Z}\big)$.

Proof. For simplicity, we identify M^2 with $i\big(M^2\big)$. Let us consider a tubular neighborhood $W \subset M^3$ of the submanifold M^2 such that $W \cap \mathbf{T}$ consists of disjoint tubular neighborhoods $\{W_j \simeq C_j \times \mathbb{R}\}$ in $\mathbf{T}$ of the finite family of circles $\mathcal{C} = \{C_j\}$ defined in Definition 3.6. Since M^2 and M are orientable, W is diffeomorphic to the trivial normal bundle νM^2 over M^2. We can identify W with the direct product $M^2 \times \mathbb{R}$, where M^2 corresponds to the zero section $M^2 \simeq M^2 \times 0 \overset{i_W}{\hookrightarrow} M^2 \times \mathbb{R} \simeq W$. Identify the pair $\big(W, \bigsqcup_j W_j\big)$ with the pair of linear bundles $\big(\nu M^2, \nu M^2|_{\sqcup_j C_j}\big)$.

Let $p : W \to M^2$ be a projection along the fibers of W. Recall that the identity component $Diff_0^2\big(M^2, M^2\big)$ of C^2-diffeomorphisms $Diff^2\big(M^2, M^2\big)$ is open in $C^2\big(M^2, M^2\big)$ $\big($see [14]$\big)$ and its preimage under the continuous mapping $C^2\big(M^2, W\big) \overset{p_*}{\to} C^2\big(M^2, M^2\big)$, which is defined by $p_*\big(f\big) = p \circ f$, is an open neighborhood V_1 of the zero section $i_W : M^2 \to W$ $\big($see [19]$\big)$. Clearly, V_1 consists of some family of embeddings $M^2 \to W$ transversal to the fibers of W.

Since $\mathbf{T} \cap W$ is a closed subset of W, the subset of $C^2\big(M^2, W\big)$ transversal to $\mathbf{T} \cap W$ is open in $C^2\big(M^2, W\big)$ - topology $\big($see [19]$\big)$. Denote it by V_2. Let $i'_W : M^2 \to W$ satisfy the conditions I and II of Theorem 2.21 and $i'_W \in V_1 \cap V_2$. Let us put $i' := i^W \circ i'_W$, where $i^W : W \hookrightarrow M^3$ is a natural embedding. Denote by Σ'_{θ_0} the image $i'\big(M^2\big) \subset M^3$. From the properties of V_1 and V_2, it follows that each fiber of W transversely intersects the embedded submanifold Σ'_{θ_0} exactly at one point, and thus the parts 1 and 2 immediately follow. Since the fibers of the bundle W_j are the fibers of W, then $\Sigma'_{\theta_0} \pitchfork W_j$, and $\Sigma'_{\theta_0} \cap W_j$ is a circle C'_j transversal to the fibers of W_j for each j. Therefore C'_j is homotopic to C_j in W_j. If the circles C_j and C'_j are equipped with the corresponding orientations, then $[C_j] = [C'_j] \in H_1\big(\mathbf{T};\mathbb{Z}\big)$. Now the statement of part 3 immediately follows from Corollary 3.7. Since an arbitrary diffeomorphism belonging to $Diff_0^2\big(M^2, M^2\big)$ induces the identity isomorphism of $H_2\big(M^2;\mathbb{Z}\big)$ and the embeddings i and i', up to such a diffeomorphism differ in deformation along the fibers W, part 4 is proved. □

3.3. Surgeries. Let $i' : M^2 \hookrightarrow M^3$ be an embedding of general position from Proposition 3.8 and $l_1 \in M^2$ be an inessential closed orbit of $\mathcal{F}' = i'^{-1}\big(\mathcal{F} \cap i'\big(M^2\big)\big)$ such that $0 \neq [l_1] \in \pi_1\big(M^2, y_1\big)$, $y_1 \in l_1$. Since l_1 is inessential, due to the Jordan-Schönflies theorem, $i'\big(l_1\big)$ bounds a disk in its support $L \in \mathcal{F}$. Moreover, due to Reeb's stability theorem, there is a *good neighborhood* $V_{l_1} \simeq l_1 \times \big(-\varepsilon, \varepsilon\big)$ in M^2, i.e., a neighborhood fibered by the inessential closed orbits $l_1 \times t$, $t \in \big(-\varepsilon, \varepsilon\big)$.

Let us choose a nonzero value $\varepsilon_1 \in (0, \varepsilon)$ and produce a surgery on M^2 cutting out $V_1 \simeq l_1 \times (-\varepsilon_1, \varepsilon_1) \subset l \times (-\varepsilon, \varepsilon) \simeq V_{l_1}$ and gluing the disks $\mathcal{D}_1 \bigsqcup \mathcal{D}_{-1}$ instead. Denote by M_1^2 the obtained manifold. Then we find the next inessential closed orbit $l_2 \subset M_1^2$ (if it exists) with the good collar $V_{l_2} \simeq l_2 \times (-\varepsilon, \varepsilon)$ such that $0 \neq [l_2] \in \pi_1(M_1^2, y_2)$, $y_2 \in l_2$. Choosing a nonzero value $\varepsilon_2 \in (0, \varepsilon)$, we make a surgery cutting out $V_2 \simeq l_2 \times (-\varepsilon_2, \varepsilon_2) \subset l_2 \times (-\varepsilon, \varepsilon) \simeq V_{l_2}$ and gluing the disks $\mathcal{D}_2 \bigsqcup \mathcal{D}_{-2}$ instead. We obtain a new manifold M_2^2. Then we select the next curve $l_3 \subset M_2^2$ with the same properties and follow the same steps as above up to getting a manifold M_i^2.

Let $\{\mathcal{D}_{\pm i}\}$, $i \in \{1, \dots, \rho\}$, be a family of the disjoint disks surgically pasted instead of the cut out annuli $V_i \simeq l_i \times (-\varepsilon_i, \varepsilon_i) \subset l_i \times (-\varepsilon, \varepsilon)$, where $l_i \subset M_{i-1}^2$ is an inessential closed orbit such that $0 \neq [l_i] \in \pi_1(M_{i-1}^2, y_i)$, $y_i \in l_i$. Denote $l_{\pm i} = \partial \mathcal{D}_{\pm i}$. Let us endow M_ρ^2 with the structure of an differentiable oriented manifold joining the differentiable structures and corresponding orientations of disks $\bigsqcup_{i=1}^\rho \mathcal{D}_{\pm i}$ and $M^2 \setminus \bigsqcup_{i=1}^\rho V_i$ with a differentiable structure and an agreed orientation of a tubular neighborhood of the boundary $\partial(M^2 \setminus \bigsqcup_{i=1}^\rho V_i)$ (see [14]).

Let us extend $i'|_{M_\rho^2 \setminus \mathrm{int} \bigsqcup_{i=1}^\rho \mathcal{D}_{\pm i}} = i'|_{M^2 \setminus \bigsqcup_{i=1}^\rho V_i}$ to all of M_ρ^2 by embeddings $h_{\pm i} : \mathcal{D}_{\pm i} \to M^3$ such that $h_{\pm i}(\mathcal{D}_{\pm i}) = D_{\pm i}$, where $D_{\pm i} \subset L_{\pm i} \in \mathcal{F}$ are disks in the corresponding leaves of $\mathcal{F}$ such that $i'(l_{\pm i}) = \partial D_{\pm i}$, $i \in \{1, \dots, \rho\}$.

Let us consider arbitrarily small disjoint foliated neighborhoods $U_{\pm i}$ of $D_{\pm i}$. Applying an isotopy to $h_{\pm i}$ that is supported in $\mathcal{D}_{\pm i}$ and has a value in $U_{\pm i}$, which pushes out $D_{\pm i}$ to the side to which $i'(V_i)$ belongs to, we can obtain a smooth immersion $i'_\rho : M_\rho^2 \to M^3$ of general position that is a continuation of $i'|_{M_\rho^2 \setminus \mathrm{int} \bigsqcup_{i=1}^\rho \mathcal{D}_{\pm i}}$ such that the induced foliation ${i'_\rho}^{-1}(\mathcal{F} \cap i'_\rho(\mathcal{D}_{\pm i}))$ on each $\mathcal{D}_{\pm i}$ consists of inessential closed orbits surrounding a center, and the immersion i'_ρ is still transversal to $\mathbf{T}$.

Lemma 3.9. *We have* $[M_\rho^2, i'_\rho] = [M^2, i'] \in H_2(M^3; \mathbb{Z})$.

Proof. The singular cycles (M^2, i') and (M_ρ^2, i'_ρ) differ by the sum of spherical cycles $\oplus_{i=1}^\rho (S_i^2, g_i)$, where S_i^2 is identified with an annulus $A_i \cong \bar{V}_i$ to which two disks $\mathcal{D}_{\pm i}$ are glued by identifying the boundaries. Put $g_i|_{A_i} = i'$ and $g_i|_{\mathcal{D}_{\pm i}} = i'_\rho$. From the irreducibility of M^3 it follows that g_i can be extended to a mapping of the ball $G_i : D_i^3 \to M^3$. Taking into account the orientation coming from M^2 and M_ρ^2, on the level of singular chains we have $\partial(\oplus_{i=1}^\rho (D_i^3, G_i)) = \oplus_{i=1}^\rho (S_i^2, g_i)$, which implies the result. □

Remark 3.10. To estimate the number ρ of necessary surgeries, we note that if an inessential closed orbit l_k belongs to the toric component $T^2 \subset M_{k-1}^2$ and represents a nontrivial element of $\pi_1(T^2)$, then the surgery of T^2 along l_k results in a sphere S^2 and the singular cycles (T^2, i'_{k-1}), and (S^2, i'_k) are homologous. But M is supposed to be irreducible and therefore (S^2, i'_k) and (T^2, i'_{k-1}) are homologous to zero which is impossible (see Remark 2.18). Thus, we conclude that

$$\rho \leq g(M^2) - 1, \tag{3.11}$$

where $g(M^2)$ is the sum of the genera of the connected components of M^2.

Definition 3.11. Denote by (N^2, p) the singular cycle (M^2_ρ, i'_ρ), where ρ is the maximal number of surgeries described above. As usual, let $\mathcal{F}'$ denote the induced foliation $p^{-1}(\mathcal{F} \cap p(N^2))$.

Remark 3.12. By the construction, taking into account the Jordan-Schönflies theorem, each inessential closed orbit of $\mathcal{F}'$ must bound a disk in N^2.

Everywhere below, let $N^2, \mathcal{F}'$ and p satisfy Definition 3.11.

3.4. Maximal vanishing cycles. Let $\mathcal{O} = \partial\mathcal{P} \subset N^2$ be a vanishing cycle (see Definition 2.23). Notice that $\mathcal{P}$ is uniquely defined by $\mathcal{O}$ because an ambiguity can arise only when $\mathcal{O}$ is a closed orbit of $\mathcal{F}'$ and the connected component of N^2 containing $\mathcal{O}$ is a sphere, which is impossible. In this case, we will understand by $\mathcal{P}(\mathcal{O})$ the set $\mathcal{P}$ from Definition 2.23 bounded by the vanishing cycle $\mathcal{O}$.

Let us introduce the notion of the *maximal vanishing cycle.*

Definition 3.13. A vanishing cycle $\mathcal{O}_{\max} \subset N^2$ is called *maximal* if

$$\mathcal{P}(\mathcal{O}_{\max}) \subset \mathcal{P}(\mathcal{O}) \text{ implies } \mathcal{O}_{\max} = \mathcal{O}.$$

From Definition 3.13 there immediately follows:

Lemma 3.14. *$\mathcal{O}_{\max}$ is either an essential closed orbit of $\mathcal{F}'$, whose p-image is a nontrivial vanishing cycle, or it is singular, i.e., consisting of separatrix loops.*

Proof. Indeed, otherwise due to Reeb's stability theorem, $\mathcal{O}_{\max}$ is an inessential closed orbit having a good collar consisting of inessential closed orbits containing a vanishing cycle $\mathcal{O} = \partial\mathcal{P}(\mathcal{O})$ different from $\mathcal{O}_{\max}$ such that $\mathcal{P}(\mathcal{O}_{\max}) \subset \mathcal{P}(\mathcal{O})$, which is impossible. □

Remark 3.15. If $\mathcal{O}_{\max}$ is essential, then by Theorem 2.20, $p(\mathcal{O}) \in T^2$, where T^2 is the boundary torus of a Reeb component R and $p_*[\mathcal{O}_{\max}] \in \ker\big(i_* : \pi_1(T^2) \to \pi_1(R)\big)$. (By the class $[\mathcal{O}]$, we mean the class of the loop $f : S^1 \to N^2$ which bypasses $\mathcal{O}$ once along the trajectories of the vector field tangent to $\mathcal{F}'$.) Since the immersion p is transverse to $\mathbf{T}$ by the construction, then $\mathcal{O}_{\max}$ must be a regular vanishing cycle, i.e., a closed orbit of $\mathcal{F}'$. Therefore, when $\mathcal{O}_{\max}$ is singular, it must be inessential. In particular, if $\mathcal{O}_{\max}$ consists of two separatrix loops $\mathcal{O}_1$ and $\mathcal{O}_2$, i.e., $\mathcal{P}(\mathcal{O}_{\max})$ is a pinched annulus, then $\mathcal{O}_{\max}$ can be of two types:

A) Both $\mathcal{O}_1$ and $\mathcal{O}_2$ are inessential.

B) Both $\mathcal{O}_1$ and $\mathcal{O}_2$ are essential and $p_*[\mathcal{O}_1] = -p_*[\mathcal{O}_2] \in \pi_1(\mathcal{L})$, where $\mathcal{L} \in \mathcal{F}$ is a a support of $p(\mathcal{O}_{\max})$. Using the Jordan–Schönflies theorem, one can see that $p(\mathcal{O}_{\max})$ must bound a pinched annulus in $\mathcal{L}$.

Lemma 3.16. *Let $B \subset N^2$ be a disk of N^2 bounded by an inessential closed orbit of $\mathcal{F}'$. Then $B \subset \mathcal{P}(\mathcal{O}_{\max})$ for some maximal vanishing cycle $\mathcal{O}_{\max}$.*

Proof. Due to Reeb's stability theorem, each inessential closed orbit l_0 of $\mathcal{F}'$ has a good neighborhood homeomorphic to $\big(-\varepsilon,\varepsilon\big)\times l_0$, where $l_s = s\times l_0$ is an inessential closed orbit of $\mathcal{F}'$. Let $U=\bigcup_t B_t,\ t\in\mathcal{T}$, be the union of disks containing B, obtained by adding to B annuli consisting of the union of inessential closed orbits. Let $l_t=\partial B_t$. Clearly, the family of disks $\{B_t\}$ is linearly ordered by the inclusion $t_1<t_2\Leftrightarrow B_{t_1}\subset B_{t_2}$.

It should be noticed that $\partial\bar{U}$ cannot be a center since N^2 does not contain a connected component homeomorphic to S^2. Observe also that $\partial\bar{U}$ consists of orbits of $\mathcal{F}'$ which are not inessential closed orbits because such closed orbits have good neighborhoods and cannot belong to $\partial\bar{U}$.

Notice also that $\partial\bar{U}$ is a saturated set, i.e., it consists of leaves of $\mathcal{F}'$ $\big($see [23]$\big)$. If the closure $\partial\bar{U}$ contains a regular leaf $r\in\mathcal{F}'$ to which other leaves of $\partial\bar{U}$ are accumulated, then there exists a small transversal τ to $\mathcal{F}'$ through r that contains the interval J connecting two points $a\in l_{t_1},\ b\in l_{t_2}$, between which there are points of $\partial\bar{U}$. But it is impossible because $\mathcal{F}'^{\perp}$ is not degenerated on l_t and l_t separates N^2. Therefore, if τ leaves B_t, it never returns to B_t.

We conclude that $\partial\bar{U}$ consists at most of a finite union $\mathbf{O}:=\bigsqcup_i\mathcal{O}_i$ of essential closed orbits or separatrix loops of $\mathcal{F}'$. The claim is to show that $\mathbf{O}$ is connected and is a vanishing cycle. Denote by $\mathbf{U}:=\bigsqcup_i U_i$ a disjoint union of tube neighborhoods U_i of O_i. Clearly, $\bar{U}\setminus\mathbf{U}$ is compact and is contained inside of B_{t_0} for some $t_0\in\mathcal{T}$. Since $U\setminus B_{t_0}$ is connected, we immediately conclude that $\mathbf{O}$ is connected. From the orientability of N^2, it follows that $\mathbf{O}$ divides $\mathbf{U}$ into connected components, the closure of each of them in N^2 has a nonempty boundary consisting of orbits of $\mathbf{O}$. Since $U\setminus B_{t_0}$ is connected, it can belong only to one of these connected components and thus, by the definition, $\mathbf{O}$ is a vanishing cycle. The result follows from the finiteness of both the set of singular points and the number of essential regular vanishing cycles of $\mathcal{F}'$. □

Lemma 3.17. *Let $\mathcal{P}_{\max}=\mathcal{P}\big(\mathcal{O}_{\max}\big)$ and $\mathcal{P}'_{\max}=\mathcal{P}\big(\mathcal{O}'_{\max}\big)$, where $O_{\max}$ and $O'_{\max}$ are maximal vanishing cycles. Then either $\mathcal{P}_{\max}=\mathcal{P}'_{\max}$ or $\mathcal{P}_{\max}\cap\mathcal{P}'_{\max}=\varnothing$. In particular, $\mathcal{P}_{\max}$ in Lemma 3.16 is unique.*

Proof. It is enough to suppose that $\mathcal{O}_{\max}$ and $\mathcal{O}'_{\max}$ are different. Otherwise we obtain a contradiction since N^2 does not contain S^2 as a connected component. One of the following cases takes place for $\mathcal{O}_{\max}\cap\mathcal{O}'_{\max}$:

(i) $\varnothing$;

(ii) a saddle point;

(iii) a separatrix loop.

In the case (i) or (ii), at least one of $\mathcal{P}_{\max}$ or $\mathcal{P}'_{\max}$ must be a disk and $\mathcal{O}_{\max}\cup\mathcal{O}'_{\max}$ must be two separatrix loops with a common saddle point s. By Remark 3.15, $\mathcal{O}$ and $\mathcal{O}'$ are inessential and therefore, due to Reeb's stability theorem, there exists an external good collar V of $\mathcal{P}_{\max}\cup\mathcal{P}'_{\max}$. $\big($Note that $\mathcal{P}_{\max}\cup\mathcal{P}'_{\max}$ is homeomorphic to either a disk or a bouquet of two disks.$\big)$ Let $l\subset V$ be an inessential closed orbit. Clearly, l bounds a disk B containing $\mathcal{P}_{\max}\cup\mathcal{P}'_{\max}$.

Applying Lemma 3.16, we find a vanishing cycle $\mathcal{O}$ such that $\mathcal{P}_{\max} \cup \mathcal{P}'_{\max} \subset \mathcal{P}(\mathcal{O})$, which contradicts the maximality of both $\mathcal{O}_{\max}$ and $\mathcal{O}'_{\max}$.

Let us consider the case (i). We suppose that there exists $a \in \operatorname{int} \mathcal{P} \cap \operatorname{int} \mathcal{P}'$. Since $\mathcal{P}$ and $\mathcal{P}'$ are connected, $\mathcal{P}'_{\max} \not\subset \mathcal{P}_{\max}$ and $\mathcal{P}_{\max} \not\subset \mathcal{P}'_{\max}$, we have $\mathcal{O}_{\max} \cap \mathcal{P}'_{\max} \neq \varnothing$ and $\mathcal{O}'{}_{\max} \cap \mathcal{P}_{\max} \neq \varnothing$. Taking into account the condition (i) and the connectivity of $\mathcal{O}_{\max}$ and $\mathcal{O}'{}_{\max}$, we obtain

$$\mathcal{O}_{\max} \subset \operatorname{int} \mathcal{P}'_{\max} \quad \text{and} \quad \mathcal{O}'{}_{\max} \subset \operatorname{int} \mathcal{P}_{\max}.$$

Let $l \subset \mathcal{P}_{\max}$ be an inessential closed orbit of a good collar of $\mathcal{O}_{\max}$, which bounds a disk B inside of $\mathcal{P}_{\max}$ such that $a \cup \mathcal{O}'{}_{\max} \subset \operatorname{int} B$. Since $\mathcal{P}'_{\max} \not\subset B$, for reasons similar to the above, we conclude that $l \subset \operatorname{int} \mathcal{P}'_{\max}$. By the Jordan–Schönflies theorem, l bounds a disk $B' \subset \operatorname{int} \mathcal{P}'_{\max}$. On the other hand, l bounds $B \subset \mathcal{P}_{\max}$. Since $\mathcal{O}'{}_{\max} \subset \operatorname{int} B$, we conclude that $B \neq B'$ which implies that $B \cup B' \simeq S^2$. But this contradicts to the fact that N^2 does not contain connected components homeomorphic to the sphere. Thus, it follows that $\operatorname{int} \mathcal{P} \cap \operatorname{int} \mathcal{P}' = \varnothing$ which implies the result. □

Corollary 3.18. *Each center of $\mathcal{F}'$ belongs to the unique $\mathcal{P}_{\max} = \mathcal{P}(\mathcal{O}_{\max})$.*

Proof. A center $\mathcal{F}'$ has a punctured neighborhood consisting of inessential closed orbits and the result immediately follows from Lemmas 3.16 and 3.17. □

Lemma 3.19. *Let $\mathcal{P}_{\max} = \mathcal{P}(\mathcal{O}_{\max}) \subset N^2$ be a pinched annulus. Then the separatrix loops of $\mathcal{O}_{\max}$ are essential and their p-images bound a pinched annulus in the leaf $\mathcal{L} \in \mathcal{F}$ containing $p(\mathcal{O}_{\max})$.*

Proof. According to Remark 3.15, it is enough to show that there is no maximal vanishing cycle $\mathcal{O}_{\max}$ consisting of inessential separatrix loops.

Suppose that the separatrix loops of $\mathcal{O}_{\max}$ are inessential. Then, due to Reeb's stability theorem, they have good exterior collars with respect to the pinched annulus $\mathcal{P}_{\max}$. By Remark 3.12, each closed orbit of this collar must bound a disk in N^2. Since there are no connected components of N^2 homeomorphic to S^2, one of such disks contains $\mathcal{O}_{\max}$. We conclude that $\mathcal{O}_{\max} \subset \operatorname{int} \mathcal{P}(\mathcal{O})$ for some vanishing cycle $\mathcal{O}$, which contradicts the maximality of $\mathcal{O}_{\max}$. □

4. Proof of main theorem

4.1. The reducing of the number of singular points. Assume that $\{\mathcal{P}^k_{\max} = \mathcal{P}(\mathcal{O}^k_{\max}),\ k \in \mathbf{K}\}$ is a family of disks and pinched annuli in N^2 bounded by maximal vanishing cycles of $\mathcal{F}'$, where $\mathbf{K}$ denotes a finite (possibly empty) indexing set. Let $\{V_k \subset \mathcal{P}^k_{\max},\ k \in \mathbf{K}\}$ denote good collars of $\mathcal{O}^k_{\max}$ and $\{l_k \subset V_k, k \in \mathbf{K}\}$ be fixed inessential closed orbits of $\mathcal{F}'$ inside of good collars. Suppose that V_k is small enough for $p|_{V_k}$ to be an embedding. By Remark 3.12, Definition 2.9 and the Jordan-Schönflies theorem, each l_k bounds a disk B_k in N^2, and $p(l_k)$ bounds a disk $D_k \subset L_k \in \mathcal{F}$ in the supporting leaf $L_k \in \mathcal{F}$. We redefine the mapping $p|_{B_k}$ by the embedding $h_k : B_k \to M^3$ such that $h_k|_{l_k} = p|_{l_k}$ and $h_k(B_k) = D_k$.

Let us consider arbitrarily small foliated neighborhoods U_k of D_k. Applying an isotopy to h_k that is supported in B_k and has values in U_k, which pushes out D_k to the side inverse to $p(V_k) \cap U_k$, we can obtain a smooth immersion $p' : N^2 \to M^3$ of general position which is a continuation of $p|_{N^2 \setminus \mathrm{int} \bigsqcup_k B_k}$ such that the induced foliation $p'^{-1}(\mathcal{F} \cap p'(B_k))$ on each B_k consists of inessential closed orbits surrounding a center c_k.

Lemma 4.1. *We have* $[N^2, p] = [N^2, p'] \in H_2(M^3; \mathbb{Z})$.

Proof. For each $k \in \mathbf{K}$, let $S_k^2 := (B_k^1 \bigsqcup B_k^2)/(\partial B_k^1 \sim \partial B_k^2) \simeq S^2$ be two copies of B_k with naturally identified boundaries. Let us define a spheroid $g_k : S_k^2 \to M^3$, where $g_k|_{B_k^1} = p|_{B_k^1}$ and $g_k|_{B_k^2} = p'_{B_k^2}$. Since M is irreducible, g_k can be extended to a mapping of the ball $\Phi_k : D_k^3 \to M^3$ such that $S_k^2 = \partial D_k^3$. Taking into account the orientations of B_k^i, $i = 1, 2$, coming from the orientation of B_k, on the level of singular chains we obtain $\partial(D_k^3, \Phi_k) = (S_k^2, g_k)$. It means that $(N^2, p) - (N^2, p') = \partial(\oplus_k (D_k^3, \Phi_k))$ which implies the result. □

Definition 4.2. Let us denote $\mathcal{F}'' := p'^{-1}(\mathcal{F} \cap p'(N^2))$.

Let $\mathbf{K}' \subset \mathbf{K}$ be such that

$$\{\mathcal{P}_{\max}^k = \mathcal{P}(\mathcal{O}_{\max}^k),\ k \in \mathbf{K}' \subset \mathbf{K}\}$$

is a family of disks or pinched annuli such that each $\mathcal{O}_{\max}^k$ is singular with a saddle s_k. Let $(\mathcal{P}_{\max}, \mathcal{O}_{\max}, V, l, L, D, B, U, h, c, s)$ be an arbitrary element of $\{(\mathcal{P}_{\max}^k, \mathcal{O}_{\max}^k, V_k, l_k, L_k, D_k, B_k, U_k, h_k, c_k, s_k),\ k \in \mathbf{K}'\}$. From Remark 3.15 and Lemma 3.19, it follows that $p'(\mathcal{O}_{\max})$ also bounds respectively a disk or a pinched annulus in its support $L \in \mathcal{F}$, which we denote by $D_{\max}$.

Suppose that $D_{\max}$ is a pinched annulus. Then $D_{\max} \subset A \subset L$, where $A \simeq S^1 \times (0, 1)$ is an annular neighborhood of $D_{\max}$ in the leaf L and $D_{\max}$ is a deformation retract of A. Since the collar V of $\mathcal{O}_{\max}$ can be taken arbitrarily small, we can assume that the normal relative to $\mathcal{F}$ collar $N \simeq A \times [0, 1)$ of $A = A \times 0$ contains $p'(V)$ and the foliation $\mathcal{F} \cap N$ is transversal to the interval fibers $\{* \times [0, 1)\}$. The embedding

$$S^1 := S^1 \times 1/2 \hookrightarrow S^1 \times (0, 1) \simeq A$$

extends to the embedding $S^1 \times [0, 1) \hookrightarrow A \times [0, 1) \simeq N$ transversal to $\mathcal{F} \cap N$. The image of this embedding we also denote by $S^1 \times [0, 1)$. Clearly, the foliation $\mathcal{F} \cap N$ is obtained from the foliation $\mathcal{F} \cap (S^1 \times [0, 1))$ by multiplying it by the interval $(0, 1)$. Since leaves of $\mathcal{F} \cap (S^1 \times [0, 1))$ are homeomorphic to intervals or circles representing the generator of $\pi_1(S^1 \times [0, 1)) \cong \mathbb{Z}$, the foliation $\mathcal{F} \cap N$ consists of leaves that are either homeomorphic to annuli, which are a deformation retract of N, or contractible. It follows that each leaf $\mathcal{L}$ of $\mathcal{F} \cap N$ induces a monomorphism of fundamental groups with respect to the embedding $\mathcal{L} \longrightarrow N$. Therefore, since the loop $p'(l)$ is free homotopic to the loop $p'(\mathcal{O}_{\max})$ inside of N, and the loop $p'(\mathcal{O}_{\max})$ is null-homotopic in A, the loop $p'(l)$ is null-homotopic in N and therefore it is null-homotopic in its support $\mathcal{L} \in \mathcal{F} \cap N$. $(L \cap$

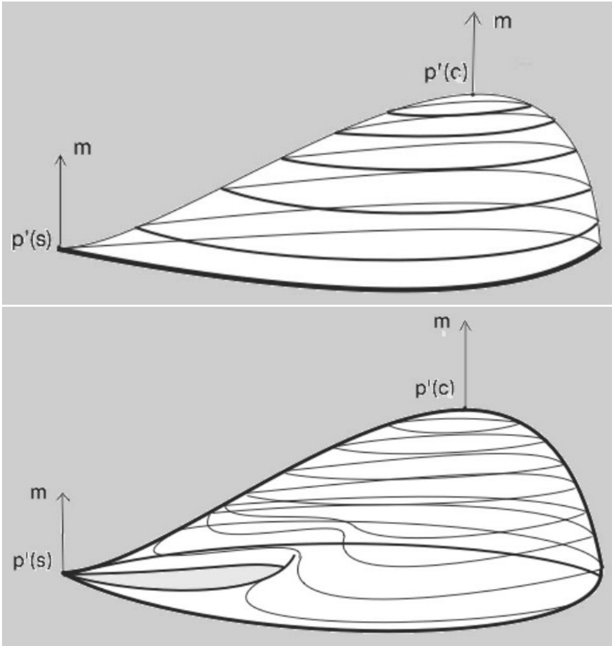

Fig. 4.1: The foliated ball B^3 and the pinched ball Q^3

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N can be disconnected.) Thus, by the Jordan–Schönflies theorem, $p'(l)$ bounds a disc in $\mathcal{L}$. Since there is no leaves of $\mathcal{F}$ homeomorphic to the sphere, this disc should coincide with the disk D.

For the case when $D_{\max}$ is a disk, we denote by A an open disk in L containing $D_{\max}$. Then, due to Reeb's stability theorem, the induced foliation $\mathcal{F} \cap N$ of the normal collar $N \simeq A \times [0,1)$ containing $p'(V)$ is homeomorphic to the product foliation $\{A \times *,\ * \in [0,1)\}$, i.e., is a foliation by disks and, by the Jordan–Schönflies theorem, $p'(l)$ also bounds the disc D in its support $\mathcal{L} \in \mathcal{F} \cap N$.

Since U is an arbitrarily small neighborhood of D, we can assume that $p'(B) \subset N$. Let us denote $B_{\max} := p'(B \cup V)$.

By the construction, in the case when $D_{\max}$ is a pinched annulus, $D_{\max} \cup B_{\max}$ bounds a ball Q^3 with two identified points, which we call a pinched ball. Using the same reasoning as for the disk D, we can show that the foliation $\mathcal{F} \cap Q^3 = \{D_t,\ t \in [0,1]\}$ is a foliation by disks excepting the cases $t = 0$, $D_0 = D_{\max}$, and $t = 1$, $D_1 = p'(c)$. By the diffeomorphism, we can represent $(N, \mathcal{F} \cap N)$ in $\mathbb{R}^3$ in such a way that the foliation $\mathcal{F} \cap N$ becomes transverse to the vertical direction and $D_{\max}$ belongs to the horizontal plane (see Fig. 4.1). (Recall that $\mathcal{F}$ is transversely oriented.)

If $D_{\max}$ is homeomorphic to a disk, then $D_{\max} \cup B_{\max}$ bounds the ball B^3. By the diffeomorphism, we can represent $(N, \mathcal{F} \cap N)$ in $\mathbb{R}^3$ in such a way that the foliation $\mathcal{F} \cap N$ becomes the level set of the height function and is a foliation by disks that degenerate to a point (see Fig. 4.1).

Taking into account the form of a surface in general position with respect to the foliation in the neighborhood of singular points, in both cases we can see that the directions of the normal vector field n to the foliation $\mathcal{F}$ and the normal vector field m to $B_{\max}$ at the singular points $p'(s)$ and $p'(c)$ either simultaneously coincide or are simultaneously opposite (see Fig. 4.1). Thus the types of the singular points s and c coincide. Since, by Lemma 3.17, the saddle point s belongs to only one $\mathcal{P}_{\max}$. Hence we conclude that when calculating the Euler class, the pair of singular points s and c can be eliminated because their total index in the sum (2.15) is equal to zero.

4.2. Estimation of the L^2-norm of the Euler class $e(\mathcal{F})$. Notice that the surgeries made in Section 3.3 do not generate new (i.e., not coming from $(M^2, \mathcal{F}')$) essential closed orbits of $(N^2, \mathcal{F}')$. Moreover, the surgeries increase the Euler characteristic. Taking into account Proposition 3.8, Remark 3.15 and Corollary 3.18, we conclude that the number of centers of $\mathcal{F}''$ which are not eliminated above (see subsection 4.1), i.e., centers corresponding to maximal regular vanishing cycles, does not exceed $2C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}$. Since

$$-\chi(N^2) \leq -\chi(M^2) \leq \frac{1}{2\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2},$$

using (2.15) and (2.16), considering the singularities eliminated above, we get the following estimate:

$$|e(T\mathcal{F})([N^2, p'])| \leq \frac{1}{2\pi}\|\alpha\|_{L^2}\|R^-\|_{L^2} + 4C_{\mathbf{\Gamma}}\|\alpha\|_{L^\infty}. \tag{4.1}$$

Taking into account (2.9), (2.10), and (2.11), we obtain

$$\|e(T\mathcal{F})\|_{L^2} \leq \frac{1}{2\pi}\|R^-\|_{L^2} + 4C_{\mathbf{\Gamma}}\frac{\Lambda}{\sqrt{\mathrm{Vol}\,(M^3)}}.$$

Since $R^- \geq 6k_0$, together with (3.5) this implies

$$\|e(T\mathcal{F})\|_{L^2} \leq -\frac{3}{\pi}k_0\sqrt{V_0} + \frac{32H_0^2V_0^{\frac{3}{2}}}{3C_0^3}\Lambda,$$

where the constant C_0 is defined in (3.1). Thus, putting

$$C_1 := -\frac{3}{\pi}k_0\sqrt{V_0} + \frac{32H_0^2V_0^{\frac{3}{2}}}{3C_0^3}\Lambda,$$

we obtain the statement of Theorem 1.2. □

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L^2-норма класу Ейлера шарувань на замкнених незвідних ріманових 3-многовидах

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Через сталі, що обмежують об'єм, радіус ін'єктивності, секційну кривизну многовиду та модуль середньої кривини шарів, наведено верхню межу L^2-норми класу Ейлера $e(\mathcal{F})$ довільного трансверсально орієнтованого шарування $\mathcal{F}$ ковимірності один, визначеного на тривимірному замкненому незвідному орієнтованому рімановому тривимірному многовиді M^3. Як наслідок, маємо тільки скінченну кількість когомологічних класів групи $H^2(M^3)$, які можуть бути реалізовані класом Ейлера $e(\mathcal{F})$ двовимірного трансверсально орієнтованого шарування $\mathcal{F}$, шари якого мають модуль середньої кривини, обмежений зверху фіксованою константою H_0.

Ключові слова: 3-вимірний многовид, шарування, клас Ейлера, середня кривина