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Positive Solutions of a Nonlinear Elliptic Equation Involving a Singular Term

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In this work, we study the existence and uniqueness of positive solutions to the equation $\Delta_p u = f(|x|)/u(x)$, $x \in \mathbb{R}^N$, where $N > p > 2$. More precisely, under certain assumptions concerning the function f, we provide an answer to the question of global existence formulated in [14] by using the theory of invariant manifolds in dynamical systems and the energy method. In addition, we perform a detailed analysis of the asymptotic behavior of solutions by using logarithmic transformations.

Key words: existence, uniqueness, positive solution, asymptotic behavior, Pohozaev identity, dynamical system

Mathematical Subject Classification 2020: 35A01, 35A02, 35B08, 35B09, 35B40, 35J60, 35J65

1. Introduction

The objective of this paper is to study the positive solutions of the following equation:

$$\Delta_p u = \frac{f(|x|)}{u}, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N > p > 2$.

Throughout our study, we contemplate the subsequent radial equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) = \frac{f(r)}{u(r)}, \quad r > 0, \tag{1.2}$$

where $N > p > 2$ and f is a continuous and strictly positive function on $[0, +\infty)$.

In the context, where $p = 2$ and $f(r) = 1$, equation (1.2) pertains to the study of singular minimal hypersurfaces exhibiting symmetry. For further details, refer to [21, 23] and the corresponding citations therein. In general, equations with negative exponents have garnered substantial attention in recent years, particularly in numerous studies within the realm of micro-electromechanical systems (MEMS), as evidenced by the works [4, 5, 8, 12, 15–20] and references therein.

In the case of a non-constant function for f, we encounter a model relevant to micro-electromechanical systems with variable dielectric permittivity. This specific model has attracted considerable attention in recent years. For a comprehensive explanation of its derivation, we direct the interested reader to [6, 7, 9–11, 22, 24], as well as the citations therein for further references.

In [14], H.X. Guo, Z.M. Guo, and K. Li studied the existence of the solutions of the equation

$$u''(r) + \frac{N-1}{r}\, u'(r) = f(r)u^{-1}(r), \quad r > 0, \tag{1.3}$$

where $N \geq 3$.

Furthermore, they provided explicit equivalents to regular solutions u which satisfy $\lim_{r\to 0} u(r) > 0$. However, they left open the question of the global existence of singular solutions u, that is, those that satisfy $\lim_{r\to 0} u(r) = 0$, which is one of the objectives of this work.

If $p > 2$ and $f(r) = K$, with K being a strictly positive constant, then (1.2) has an explicit solution in the following form:

$$u(r) = \left(\frac{N-1}{K}\right)^{-1/p} r, \ \ u'(r) = \left(\frac{N-1}{K}\right)^{-1/p}.$$

In this paper, we generalize equation (1.3) to a p-Laplace equation with $p > 2$ and f being not constant. We study the existence and uniqueness of regular and singular solutions. We also describe their behavior near the origin and at infinity. So, we are interested in the following limit problem:

$$(\mathbf{Q}) \qquad \begin{cases} (|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) = \dfrac{f(r)}{u(r)}, \quad r > 0, \\ \lim\limits_{r\to 0} u(r) = \lambda, \end{cases}$$

where $N > p > 2$, $\lambda \geq 0$ and f is a continuous and strictly positive function on $[0, +\infty)$.

During our study, we separately approach the two cases, $\lambda > 0$ and $\lambda = 0$. In other words, we address two distinct problems.

Problem ($\mathbf{P_1}$): Determine a function u defined on $[0, +\infty)$ such that $|u'|^{p-2}u' \in C^1([0, +\infty))$ and verify

$$(\mathbf{P_1}) \qquad \begin{cases} (|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) = \dfrac{f(r)}{u(r)}, \quad r > 0, \\ \lim\limits_{r\to 0} u(r) = \lambda, \end{cases}$$

where $N > p > 2$, $\lambda > 0$ and f is a strictly positive continuous function on $[0, +\infty)$.

Problem ($\mathbf{P_2}$): Determine a function u defined on $(0, +\infty)$ such that $u \in C^1(0, +\infty)$,$|u'|^{p-2}u' \in C^1(0, +\infty)$ and verify

$$(\mathbf{P_2}) \qquad \begin{cases} (|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) = \dfrac{f(r)}{u(r)}, \quad r > 0, \\ \lim\limits_{r\to 0} u(r) = 0, \end{cases}$$

where $N > p > 2$ and f is a strictly positive continuous function on $[0, +\infty)$.

Due to some preliminary results, we show that if $\lambda > 0$, then we have $\lim_{r\to 0} u'(r) = 0$ and if $\lambda = 0$, then we have $\lim_{r\to 0} r\,u'(r) = 0$.

The structure of this article is as follows. In Section 2, we establish certain preliminary results. In Section 3, we delve into the question of the existence and behavior of positive solutions for problem (P_1) at infinity. In Section 4, we undertake a more in-depth analysis of singular positive solutions of (1.2), in other words, we address problem (P_2). Finally, in Section 5, we present a conclusion for this paper and discuss a future direction.

2. Preliminaries

First, we introduce the lemma, which provides evidence that if problem **(Q)** possesses a solution, then it must exhibit strict monotonicity on $(0, +\infty)$.

Lemma 2.1. *Let u be a solution of problem* (**Q**). *Then*

$$\lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0.$$

Additionally, $u'(r) > 0$ for all $r > 0$.

Proof. Recalling equation (1.2), we have

$$\left(r^{N-1}|u'|^{p-2}u'\right)'(r) = r^{N-1}\frac{f(r)}{u(r)}. \tag{2.1}$$

Given that $f > 0$ on $(0, +\infty)$, we can deduce that $r^{N-1}|u'|^{p-2}u'$ strictly increases for all $r > 0$. Thus, we have $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \in [-\infty, +\infty)$. Suppose now, by contradiction, that $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) \neq 0$. We will consider two cases.

Suppose that there is $\mathcal{M}_0 > 0$ such that

$$r^{N-1}|u'|^{p-2}u' < -\mathcal{M}_0 \quad \text{for small } r.$$

That is, there is $\mathcal{M}_1 > 0$ such that

$$u'(r) < -\mathcal{M}_1\, r^{(1-N)/(p-1)} \quad \text{for small } r.$$

We integrate the last inequality over (r, r_0) for small r_0 and let $r \to 0$. We obtain $u' \in L^1(0, r_0)$ (because $u \in C^1(0, +\infty)$ and $\lim_{r\to 0} u(r) = \lambda \geq 0$), but $r^{(1-N)/(p-1)} \notin L^1(0, r_0)$ for small r_0, leading to a contradiction.

If there is $\mathcal{M}_2 > 0$ such that

$$r^{N-1}|u'|^{p-2}u' > \mathcal{M}_2 \quad \text{for small } r \ ,$$

then there is $\mathcal{M}_3 > 0$ such that

$$u'(r) > \mathcal{M}_3\, r^{(1-N)/(p-1)} \quad \text{for small } r.$$

We can continue in the same manner as in the previous case to arrive easily at a contradiction.

We deduce that $\lim_{r\to 0} r^{N-1}|u'|^{p-2}u'(r) = 0$. Moreover, integrating inequality (2.1) over $(0, r)$, we conclude that $u'(r) > 0$ for all $r > 0$. □

Furthermore, we present the next proposition.

Proposition 2.2. *Suppose that* $\inf_{[0,\infty)} f(r) > 0$. *Let* u *be a solution of problem* (**Q**). *Then*

$$u(r) \geq r\left(\frac{\inf_{[0,\infty)} f(r)}{N}\right)^{1/p} \quad \textit{for each } r > 0. \tag{2.2}$$

Proof. By integrating relation (2.1), we obtain

$$|u'|^{p-2}u'(r) = r^{1-N}\int_0^r s^{N-1}f(s)u^{-1}(s)\,ds. \tag{2.3}$$

Since $u' > 0$ on $(0, +\infty)$, then (2.3) implies

$$u'(r) > \left(\frac{\inf_{[0,\infty)} f(r)}{N}\right)^{1/(p-1)} r^{1/(p-1)}\,u^{-1/(p-1)}(r) \quad \text{for any } r > 0. \tag{2.4}$$

Thus we get

$$\left(u^{p/(p-1)}\right)' > \left(\frac{\inf_{[0,\infty)} f(r)}{N}\right)^{1/(p-1)} \left(r^{p/(p-1)}\right)' \quad \text{for any } r > 0. \tag{2.5}$$

Upon integrating inequality (2.5) over the interval $(0, r)$, for each $r > 0$, we derive (2.2). □

Proposition 2.3. *Suppose that* f *is bounded for large* r. *Let* u *be a solution of problem* (**Q**) *such that*

$$\lim_{r\to+\infty}\frac{u(r)}{r} = +\infty.$$

Then the function $\frac{u(r)}{r}$ *is strictly increasing for large* r.

Proof. Let

$$S(r) = ru'(r) - u(r) = r^2\left(\frac{u(r)}{r}\right)', \quad r > 0. \tag{2.6}$$

Since $\lim_{r\to+\infty}\frac{u(r)}{r} = +\infty$, it is enough to demonstrate that it is monotone for large r, that is to say, according to (2.6) that $S(r) \neq 0$ for large r.

Since $u'(r) > 0$ for any $r > 0$, then $u''(r)$ exists for any $r > 0$. Thus, by equation (1.2), we have

$$(p-1)\,u'^{p-2}(r)S'(r) = -(N-1)u'^{p-1}(r) + \frac{r\,f(r)}{u(r)}. \tag{2.7}$$

Assume by absurd that there is a large r_0 such that $S(r_0) = 0$. Then

$$(p-1)\,u'^{p-2}(r_0)S'(r_0) = \frac{r_0}{u(r_0)}\left\{f(r_0) - (N-1)\,r_0^{-p}\,u^p(r_0)\right\}. \tag{2.8}$$

Given that $\lim_{r\to+\infty}\frac{u(r)}{r} = +\infty$, we obtain that $S'(r_0) < 0$. This implies that $S(r) \neq 0$ for large r. Therefore $\frac{u(r)}{r}$ is increasing for large r. □

Proposition 2.4. *Suppose that* $\inf_{[0,\infty)} f(r) > 0$ *and* f *is bounded for large* r. *Let* u *be a solution of problem* (**Q**). *We have the following:*

(i) $u'(r)$ *is bounded for* r *large enough.*

(ii) *There is a constant* $K_2 > 0$ *such that*

$$u(r) \leq K_2\, r \quad \textit{for large } r. \tag{2.9}$$

(iii) *There is a constant* $K_3 > 0$ *such that*

$$u'(r) \geq K_3 \quad \textit{for large } r. \tag{2.10}$$

Before proving the last proposition, we introduce the following logarithmic transformation. Define for each $r > 0$,

$$\varphi(t) = \frac{u(r)}{r}, \tag{2.11}$$

where $t = \ln r$. Hence, φ verifies the equation

$$Y'(t) + (N-1)\, Y(t) - \frac{f\left(e^t\right)}{\varphi(t)} = 0, \tag{2.12}$$

where

$$Y(t) = |K|^{p-2} K(t) \tag{2.13}$$

and

$$K(t) = \varphi'(t) + \varphi(t) = u'(r). \tag{2.14}$$

Now we give the proof of Proposition 2.4.

Proof of Proposition 2.4. (i) By employing the transformation (2.11) and considering relation (2.14), we show that $K(t) = u'(r)$ is bounded when t is large enough. Assume, for contradiction, that $K(t)$ is unbounded when t is large, which means that $Y(t)$ is unbounded when t is large enough. Then we distinguish two cases.

Case 1. Let $\lim_{t\to+\infty} Y(t) = +\infty$. In this case, we have $\lim_{t\to+\infty} K(t) = +\infty$, which yields that $\lim_{r\to+\infty} u'(r) = +\infty$. Since $\lim_{r\to+\infty} u(r) = +\infty$ by Proposition 2.2, then we get by the Hôpital's rule,

$$\lim_{r\to+\infty} \frac{u(r)}{r} = \lim_{r\to+\infty} u'(r) = +\infty.$$

Recalling Proposition 2.3, we know that $S(r) > 0$ for large r. Therefore, we have $r\, u'(r) > u(r)$ for large r. This implies

$$u'^{p-1}(r)\, u(r) > r^{1-p} u^p(r) \quad \text{for large } r. \tag{2.15}$$

On the other hand, using relation (2.7) and taking into account the fact that $\lim_{r\to+\infty} \frac{u(r)}{r} = +\infty$ and (2.15), we conclude that $\lim_{r\to+\infty} \frac{u'^{p-1}(r)\, u(r)}{r} = +\infty$. Therefore, $S'(r) = ru''(r)$ becomes negative for large r. As a result, we conclude that $\lim_{r\to+\infty} u'(r)$ exists and it is finite (due to the positivity of $u'(r)$). This, however, contradicts the assumption that $\lim_{r\to+\infty} u'(r) = +\infty$.

Case 2. Let $\limsup_{t\to+\infty} Y(t) = +\infty$. Then there is a sequence $\{\xi_i\}$ that tends to $+\infty$ when i goes to infinity, where $Y'(\xi_i) = 0$ and $\lim_{i\to+\infty} Y(\xi_i) = +\infty$. Referring to (2.12), we get

$$(N-1)\,Y(\xi_i) = \frac{f\left(e^{\xi_i}\right)}{\varphi(\xi_i)}. \tag{2.16}$$

Taking the limit as $i \to +\infty$ in relation (2.16) and considering the boundedness of f and $\varphi^{-1}(t)$ when t is large enough (Proposition 2.2), we arrive at a contradiction. We deduce that $Y(t)$ is bounded for large t, and thus $u'(r)$ is bounded for large t.

(ii) Through (i), there is $K_1 > 0$ such that

$$0 < u'(r) \le K_1 \quad \text{for large } r. \tag{2.17}$$

By integrating inequality (2.17) on (R, r) for large R, we get

$$u(r) - u(R) \le K_1\,(r - R) \quad \text{for large } r. \tag{2.18}$$

As a result, there is another positive constant $K_2 > 0$ such that

$$0 < u(r) \le K_2\, r \quad \text{for large } r. \tag{2.19}$$

(iii) Utilizing (2.4) taking into consideration that $\frac{u(r)}{r}$ remains bounded for large r enough, we can easily deduce (2.10). □

Remark 2.5. Suppose that $\inf\limits_{[0,\infty)} f(r) > 0$ and f is bounded for large r. Bringing together the two Propositions 2.2 and 2.4, we have

$$0 < \left(\frac{\inf_{[0,\infty)} f(r)}{N}\right)^{1/p} \le \frac{u(r)}{r} \le K_2 \quad \text{for large } r \tag{2.20}$$

and

$$0 < K_3 \le u'(r) \le K_4 \quad \text{for large } r, \tag{2.21}$$

where K_2, K_3 and K_4 are strictly positive constants.

Additionally, we bring to mind this classical result.

Lemma 2.6 ([13]). *Let Z be a positive differentiable function verifying*

(i) $\int_{t_0}^{+\infty} Z(t)\,dt$ *is finite when t_0 is large enough.*

(ii) $Z'(t)$ *is bounded when t is large enough.*

Then $Z(t)$ goes to 0 as $t \to +\infty$.

3. Study of problem ($\mathbf{P_1}$)

In this section, we are focused on solving problem (P_1). More precisely, we establish that (P_1) has a unique solution and examine its behavior as it approaches infinity. For this, we assume that f satisfies the following conditions:

($\mathcal{F}_1$) $f \in C^1[0,\infty)$, $f'(r) \geq 0$ for any $r > 0$ and $\lim_{r\to+\infty} f(r) = f_\infty > 0$.

($\mathcal{F}_2$) There is $\rho > 1$ such that $\lim_{r\to+\infty} r^\rho f'(r) = 0$.

Theorem 3.1. *Assume that ($\mathcal{F}_1$) and ($\mathcal{F}_2$) hold. It follows that for any $\lambda > 0$, problem ($\mathbf{P_1}$) possesses a unique solution $u = u_\lambda$ that satisfies*

$$\lim_{r\to+\infty} \frac{u(r)}{r} = \lim_{r\to+\infty} u'(r) = \left(\frac{N-1}{f_\infty}\right)^{-1/p}. \tag{3.1}$$

We begin with the following proposition which establishes that for all $r > 0$, problem ($\mathbf{P_1}$) has a unique global solution. This proposition draws upon principles and concepts originating from [2].

Proposition 3.2. *Assume that $\inf_{[0,\infty)} f(r) > 0$. Then, for any $\lambda > 0$, problem ($\mathbf{P_1}$) possesses a unique entire solution denoted as $u = u_\lambda$. In addition, we have*

(i) $\lim_{r\to 0} u'(r) = 0$.

(ii) $\left(|u'|^{p-2}u'\right)'(0) = \dfrac{f(0)}{\lambda N}$.

Proof. The proof will be conducted in three steps.

Step 1. We establish the existence of a maximal solution u for($\mathbf{P_1}$). Let $r_1 \geq 0$ be an arbitrary value such that $u(r_1) = A_1$ and $u'(r_1) = A_2$. If $r_1 > 0$ and as $A_2 > 0$ (because u is strictly increasing), then, by transforming equation (1.2) into a first-order system, we readily obtain the existence and the uniqueness of solution for equation (1.2) in the vicinity of r_1 from the theory of ordinary differential equations [1]. Now, if r belongs to the interval $(0, \eta)$ for a small η, then, by integrating equation (2.1), we can derive

$$u(r) = \lambda + \int_0^r h(j(u)(s))\,\mathrm{d}s, \tag{3.2}$$

where

$$h(s) = |s|^{\frac{2-p}{p-1}} s, \quad s \in \mathbb{R} \tag{3.3}$$

and j is defined as follows:

$$j(u)(s) = s^{1-N} \int_0^s z^{N-1} f(z) u^{-1}(z)\,\mathrm{d}z. \tag{3.4}$$

Next, for $\eta > 0$ and $\lambda > \mathcal{K} > 0$, we define the complete metric space

$$E_{\lambda,\mathcal{K},\eta} = \{g \in C([0,\eta]) : \|g - \lambda\|_0 \leq \mathcal{K}\}. \tag{3.5}$$

In this context, $C([0,\eta])$ is a Banach space of continuous functions defined on $[0,\eta]$. We define the mapping χ on the set $E_{\lambda,\mathcal{K},\eta}$ as follows:

$$\chi(g)(r) = \lambda + \int_0^r h(j(g)(s))\,\mathrm{d}s. \tag{3.6}$$

To begin, we demonstrate that χ maps $E_{\lambda,\mathcal{K},\eta}$ into itself. Let $g \in E_{\lambda,\mathcal{K},\eta}$ and $r \in [0,\eta]$. Then $\chi(g)(r) \in C([0,\eta])$. Consequently, we obtain the following:

$$|\chi(g)(r) - \lambda| \le \int_0^r |j(g)(s)|^{\frac{2-p}{p-1}}\, j(g)(s)\,\mathrm{d}s. \tag{3.7}$$

Given that $g(r) \in [\lambda - \mathcal{K}, \lambda + \mathcal{K}]$ and based on our assumptions regarding f, we establish that there exist two constants, d^* and d_*, such that $0 < d_* \le f(r) \le d^*$ for all $r \in [0,\eta]$. Consequently, it becomes apparent that for any $s \in [0,\eta]$,

$$d_1\, s \le j(g)(s) \le d_2\, s, \tag{3.8}$$

where $d_1 = \frac{1}{N}\,(\lambda + \mathcal{K})^{-1}\, d_*$ and $d_2 = \frac{1}{N}\,(\lambda - \mathcal{K})^{-1}\, d^*$.

Thus, for each $r \in [0,\eta]$, we have

$$|\chi(g)(r) - \lambda| \le \frac{p-1}{p} d_2 \times d_1^{(2-p)/(p-1)} r^{p/(p-1)}. \tag{3.9}$$

Therefore, by choosing η small enough, we can ensure that $|\chi(g)(r) - \lambda| \le \mathcal{K}$.

Next, we establish that χ is a contraction. Consider two functions, g and ϑ, both belonging to $E_{\lambda,\mathcal{K},\eta}$,

$$|\chi(g)(r) - \chi(\vartheta)(r)| \leqslant \int_0^r |h(j(g)(s)) - h(j(\vartheta)(s))|\, ds, \tag{3.10}$$

where $j(g)$ is defined by (3.4). Let us denote $\phi = \min(j(g)(s), j(\vartheta)(s))$. Then we have

$$|\chi(g)(r) - \chi(\vartheta)(r)| \leqslant \int_0^r \phi^{\frac{2-p}{p-1}} |j(g)(s) - j(\vartheta)(s)|\, ds. \tag{3.11}$$

But, by combining the expression of j with (3.11), we deduce that

$$|\chi(g)(r) - \chi(\vartheta)(r)| \le \frac{(p-1)(\lambda - \mathcal{K})^{-2}}{p\,N}\, d_1^{(2-p)/(p-1)}\, d^*\, r^{p/(p-1)} \|g - \vartheta\|_0. \tag{3.12}$$

In summary, we can choose a suitably small value for r to ensure that χ is a contraction. Subsequently, by using the Banach fixed point theorem, we confirm that χ has a unique fixed point, which corresponds to a unique solution to problem $(\mathbf{P_1})$.

Step 2. Let $R_{\max} = +\infty$. Assume, for the sake of contradiction, that $R_{\max} < +\infty$. Integrating relation (2.1) over the interval $(0,r)$ and using the fact that $\lim_{r\to 0} r^{(N-1)/(p-1)}\, u'(r) = 0$, we obtain

$$r^{N-1}|u'|^{p-2}u' = \int_0^r s^{N-1} f(s) u^{-1}(s)\, ds. \tag{3.13}$$

As u is strictly increasing and f is bounded on the interval $(0, R_{\max})$, we can conclude that there is $d_3 > 0$ such that

$$|u'|^{p-1} \leq \frac{\lambda^{-1} d_3}{N} r. \tag{3.14}$$

In other words, for every r within the interval $(0, R_{\max})$,

$$|u'_\lambda(r)| \leq \left(\frac{\lambda^{-1} d_3}{N}\right)^{1/(p-1)} R_{\max}^{1/(p-1)}. \tag{3.15}$$

Hence we obtain

$$\lim_{r \to R_{\max}^-} |u'(r)| < +\infty.$$

However, this implies a contradiction.

Step 3. Let $\lim_{r\to 0} u'(r) = 0$ and $\left(|u'|^{p-2}u'\right)'(0) = \frac{f(0)}{\lambda N}$. By employing relation (3.14) and the fact that $u'(r) > 0$, we get easily $\lim_{r\to 0} u'(r) = 0$. Returning to equations (1.2) and (2.3), we derive

$$\lim_{r\to 0} \left(|u'|^{p-2}u'\right)'(r) = \lim_{r\to 0} \frac{|u'|^{p-2}u'(r)}{r} = \frac{f(0)}{\lambda N}.$$

Thus we have completed the proof. □

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. First, notice that the hypothesis $(\mathcal{F}_1)$ implies that $\inf_{[0,\infty)} f(r) > 0$ and f is bounded for large r. Drawing upon Proposition 3.2, we have proven the first portion of Theorem 3.1. Consequently, our focus now shifts towards understanding the behavior of the solution to $\mathbf{P_1}$ at infinity, a task for which we will draw upon insights from [3].

To facilitate our analysis, we define the following energy function related to equation (2.12):

$$J_1(t) = \tfrac{p-1}{p} |K(t)|^p - Y(t)\varphi(t) + \tfrac{N}{p}\varphi^p(t) - f(e^t)\ln(\varphi(t)). \tag{3.16}$$

Because φ, K and Y are bounded when t is large enough (as indicated by Proposition 2.4), it follows that J_1 is also bounded for large t. The remaining portion of the proof will be done in three distinct steps.

Step 1. Let$J_1(t)$ converge as $t \to +\infty$. If we derive J_1, we get

$$J_1'(t) = -N Z_1(t) - e^t f'(e^t) \ln(\varphi(t)), \tag{3.17}$$

where

$$Z_1(t) = \left[|K(t)|^{p-1} - \varphi^{p-1}(t)\right]\left[|K(t)| - \varphi(t)\right]. \tag{3.18}$$

Subsequently, by integrating (3.17) over the interval (t_0, t) for large t_0, we obtain

$$J_1(t) = J_1(t_0) - N R_1(t) - \int_{t_0}^{t} e^s f'(e^s) \ln(\varphi(s))\, ds, \tag{3.19}$$

where

$$R_1(t) = \int_{t_0}^{t} Z_1(s)\, ds. \tag{3.20}$$

Exploiting the property that the function $z \to z^{p-1}$ is increasing, we deduce that $Z_1(t) \geq 0$. Consequently, we can deduce that the function $R_1(t)$ is positive and increasing. Furthermore, based on relation (3.19), $R_1(t)$ is equivalent to

$$R_1(t) = \frac{J_1(t_0)}{N} - \frac{J_1(t)}{N} - \frac{1}{N}\int_{t_0}^{t} e^s f'(e^s) \ln(\varphi(s))\, ds. \tag{3.21}$$

Given that φ is bounded between two strictly positive constants for large t (as indicated by Propositions 2.2 and 2.4) and considering the hypotheses $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$, we can conclude that there is a constant $\epsilon > 0$ such that

$$\int_{t_0}^{t} e^s\, f'(e^s) \ln(\varphi(s))\, ds \leq \epsilon \int_{t_0}^{t} e^{(1-\rho)s}\, ds < +\infty,$$

which gives that $R_1(t)$ is bounded for large t. Thus, $\int_{t_0}^{+\infty} Z_1(s)\, ds$ exists, which is equivalent to $R_1(t)$ that converges as $t \to +\infty$. Consequently, by t tending to $+\infty$ in (3.19), we show that $J_1(t)$ converges as $t \to +\infty$ and we note $J_1 = \lim_{t\to+\infty} J_1(t)$.

Step 2. Let $\lim_{t\to+\infty} \varphi'(t) = 0$. Since $K(t) > 0$ if t is large, then based on relations (2.14) and (3.18), it is enough to show that $\lim_{t\to+\infty} Z_1(t) = 0$. Furthermore, recalling Lemma 2.6, it is also necessary to show that $Z_1'(t)$ is bounded for large t.

Notice that $Z_1(t)$ can be rewritten as

$$Z_1(t) \quad = Y^{p/(p-1)}(t) - Y(t)\varphi(t) - \varphi^{p-1}(t)\varphi'(t). \tag{3.22}$$

Hence we get

$$\begin{aligned} Z_1'(t) = \frac{p}{p-1} K(t)Y'(t) - Y(t)\varphi'(t) - \varphi(t)Y'(t) \\ - (p-1)\varphi^{p-2}(t)\varphi'^{2}(t) - \varphi^{p-1}(t)\varphi''(t). \end{aligned} \tag{3.23}$$

As $\varphi(t)$, $K(t)$, and $Y(t)$ are bounded when t is large enough, then combining this information with (2.14) and (2.12), we can conclude that both $\varphi'(t)$ and $Y'(t)$ are bounded if t is large. It still remains to establish that $\varphi''(t)$ is bounded when t is large enough. Using (2.14), we have

$$\varphi''(t) = K'(t) - \varphi'(t). \tag{3.24}$$

Since $\varphi'(t)$ is bounded for large t, it remains to show that $K'(t)$ is bounded for large t. By (2.13), we have

$$K'(t) = \frac{1}{p-1}\, (K(t))^{2-p}\, Y'(t). \tag{3.25}$$

Thus, by (2.10), we have $K(t) > K_3$ for large t. In other words, $(K(t))^{2-p}$ is bounded for large values of t. Consequently, $K'(t)$ is bounded for large t, and based on (3.24) and (3.23), we can conclude that $Z_1'(t)$ is bounded for large t. This yields that $\lim_{t\to+\infty} Z_1(t) = 0$ as stated in Lemma 2.6.

Step 3. Let $\varphi(t)$ converge as $t \to +\infty$. Since $\varphi(t)$ is bounded for large t, we consider the assumption that $\varphi(t)$ is oscillating for large t. In this case, there exist two sequences, α_i and β_i, that go to $+\infty$ as i tends to $+\infty$ such that α_i and β_i are respectively local minimum and local maximum of φ. These sequences satisfy the conditions $\alpha_i < \beta_i < \alpha_{i+1}$ and

$$0 < \liminf_{t\to+\infty} \varphi(t) = \lim_{i\to+\infty} \varphi(\alpha_i) = \alpha \tag{3.26}$$

$$< \limsup_{t\to+\infty} \varphi(t) = \lim_{i\to+\infty} \varphi(\beta_i) = \beta < +\infty. \tag{3.27}$$

Let

$$\chi_1(s) = \frac{N-1}{p} s^p - f_\infty \ln(s), \quad s > 0. \tag{3.28}$$

Given that $\varphi'(\alpha_i) = \varphi'(\beta_i) = 0$, we can use (3.26), (3.16), (2.13), and (2.14) to derive

$$\lim_{i\to+\infty} J_1(\alpha_i) = \chi_1(\alpha) \text{ et } \lim_{i\to+\infty} J_1(\beta_i) = \chi_1(\beta). \tag{3.29}$$

As $\lim_{t\to+\infty} J_1(t) = J_1$ (by Step 1), then

$$\chi_1(\alpha) = \chi_1(\beta) = J_1. \tag{3.30}$$

Hence, we can find $\nu_1 \in (\alpha, \beta)$ and $t_i \in (\alpha_i, \beta_i)$ such that $\varphi(t_i) = \nu_1$, $\chi_1'(\nu_1) = 0$, and $\chi_1(\nu_1) \neq J_1$. However, by Step 2, $\lim_{i\to+\infty} \varphi'(t_i) = 0$. This in turn implies, based on (2.14), that $\lim_{i\to+\infty} K(t_i) = \nu_1$. Consequently, we have $\lim_{i\to+\infty} J_1(t_i) = \chi_1(\nu_1) = J_1$, which leads to a contradiction. Therefore, we deduce that $\varphi(t)$ converges as $t \to +\infty$. If we denote $\lim_{t\to+\infty} \varphi(t) = l$, then using the fact that $\lim_{t\to+\infty} \varphi'(t) = 0$ and relations (2.14) and (2.13), we have respectively $\lim_{t\to+\infty} K(t) = l$ and $\lim_{t\to+\infty} Y(t) = l^{p-1}$. Since $\lim_{t\to+\infty} \varphi(t) = l > 0$ by (2.11), (2.20) and hypothesis $(\mathcal{F}_1)$, then, by equation (2.12), $Y'(t)$ converges necessarily to 0 when $t \to +\infty$. Hence, $(N-1)\, l^{p-1} - \frac{f_\infty}{l} = 0$, which implies that $l = \left(\frac{N-1}{f_\infty}\right)^{-1/p}$. This concludes the proof. □

4. Study of problem $(\mathbf{P_2})$

In this section, we focus on studying the singular solutions u of (1.2) that satisfy $\lim_{r\to 0} u(r) = 0$. Specifically, we aim to establish both the uniqueness and the existence of solutions defined over the interval $(0, +\infty)$ to problem (P_2), while also describing their behavior in the vicinity of both infinity and the origin.

Theorem 4.1. *Suppose that $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ hold. Then problem $(\mathbf{P_2})$ has a unique solution, denoted as $u = u_0$, which verifies (3.1) and*

$$\lim_{r\to 0} \frac{u(r)}{r} = \lim_{r\to 0} u'(r) = \left(\frac{N-1}{f(0)}\right)^{-1/p}. \tag{4.1}$$

Before giving the proof of this theorem, we need the following proposition.

Proposition 4.2. *Suppose that* $\inf_{[0,\infty)} f(r) > 0$. *Let* u *be a solution of problem* ($\mathbf{P_2}$). *Then:*

(i) $\lim_{r\to 0} r\, u'(r) = 0$,

(ii) $u'(r)$ *is bounded if* r *is sufficiently small,*

(iii) $\dfrac{u(r)}{r}$ *is bounded if* r *is sufficiently small.*

Proof. (i) Let

$$S_{(N-p)/(p-1)}(r) = \frac{N-p}{p-1}u(r) + ru'(r), \quad r > 0. \tag{4.2}$$

Since $u'(r) > 0$, then by equation (1.2), we have

$$(p-1)u'^{p-2}S'_{(N-p)/(p-1)}(r) = r\,\frac{f(r)}{u(r)}, \quad r > 0. \tag{4.3}$$

Since $N > p$, $u(r) > 0$, $u'(r) > 0$ and $f(r) > 0$ for any $r > 0$, then we have $S_{(N-p)/(p-1)}(r) > 0$ and $S'_{(N-p)/(p-1)}(r) > 0$ for any $r > 0$, which yields $\lim_{r\to 0} S_{(N-p)/(p-1)}(r) \in [0, +\infty[$. Since $\lim_{r\to 0} u(r) = 0$, then, by (4.2), $r\,u'(r)$ converges when r tends to 0 and necessarily $\lim_{r\to 0} r\,u'(r) = 0$.

(ii) We introduce the following change for any $r > 0$:

$$\varphi_0(t) = \frac{u(r)}{r}, \quad t = -\ln r. \tag{4.4}$$

Then equation (1.2) is equivalent to

$$w'(t) - (N-1)\,w(t) - \frac{f\left(e^{-t}\right)}{\varphi_0(t)} = 0, \tag{4.5}$$

where

$$w(t) = |h(t)|^{p-2}h(t) \quad \text{and} \quad h(t) = \varphi_0'(t) - \varphi_0(t) = -u'(r). \tag{4.6}$$

Assume, for the sake of contradiction, that u' is unbounded for small r. Then, by relation (4.6), we have that $h(t)$ is unbounded for large t. It follows that $w(t)$ is unbounded for large t. Two cases arise.

Case 1. $\lim_{t\to+\infty} |w(t)| = +\infty$. Therefore, using relation (4.5) and considering the boundedness of φ_0^{-1} for large t (as per Proposition 2.2), we can derive

$$\lim_{t\to+\infty} \frac{w'(t)}{w(t)} = \lim_{t\to+\infty} (\ln|w(t)|)' = N-1.$$

Then, by Hôspital's rule, we obtain

$$\lim_{t\to+\infty} \frac{\ln|w(t)|}{t} = N-1.$$

By choosing $p-1<\delta<N-1$ (since $N>p$), we get

$$\ln|w(t)|>\delta t \quad \text{for large } t.$$

Hence,

$$|w(t)|>e^{\delta t} \quad \text{for large } t.$$

As a result, using relation (4.6), we can deduce that

$$e^{-\delta t/(p-1)}|h(t)|>1 \quad \text{for large } \ t.$$

However, this contradicts the fact that

$$\lim_{t\to+\infty} e^{-\delta t/(p-1)}|h(t)| = \lim_{r\to 0} r^{\delta/(p-1)-1} r\, u'(r) = 0,$$

because $\frac{\delta}{p-1}>1$ and $\lim_{r\to 0} r\, u'(r)=0$.

Case 2. $\limsup_{t\to+\infty}|w(t)|=+\infty$. This gives the existence of a sequence s_i that tends to $+\infty$ as $i\to+\infty$, such that $w'(s_i)=0$ and $\lim_{i\to+\infty}|w(s_i)|=+\infty$. Using equation (4.5), we obtain

$$(N-1)|w(s_i)| = \frac{f(e^{-s_i})}{\varphi_0(s_i)} \underset{i\to+\infty}{\longrightarrow} +\infty.$$

However, this cannot occur as $\varphi_0^{-1}(t)$ is bounded if t is large enough (as stated in Proposition 2.2). Then there exists a constant $C>0$ such that

$$0<u'(r)\le C_1 \quad \text{for small } r. \tag{4.7}$$

(iii) Integrating (4.7) on (r_0,r) for small r and letting $r_0\to 0$, we get easily that $\frac{u(r)}{r}$ is bounded when r is small enough. This ends the proof. □

Inspired by the previous proposition, we will study now the asymptotic behavior of the connecting orbits of a 1-dimensional ordinary differential equation (ODE). This analysis will help us deduce the asymptotic behavior of $\frac{u(r)}{r}$ and provide the proof for Theorem 4.1.

Proof of Theorem 4.1. Interestingly, the behavior of solutions of problem $(\mathbf{P_2})$ at infinity is similar to that of the solutions of problem $(\mathbf{P_1})$, as shown in Theorem 3.1. Thus, to complete the proof, we need to examine the behavior of these solutions near the origin. To do this, we introduce the following change:

$$\begin{aligned} &A(t)=\frac{u(r)}{r}, \quad B(t)=|u'|^{p-2}u'(r),\\ &C(t)=r, \quad t=-\ln r. \end{aligned} \tag{4.8}$$

From equation (1.2), we obtain the system

$$\begin{cases} A' = A - B^{1/(p-1)},\\ B' = (N-1)\,B - f(C)\,A^{-1},\\ C' = -C, \end{cases} \tag{4.9}$$

where $A' = \frac{dA}{dt}$, $B' = \frac{dB}{dt}$ and $C' = \frac{dC}{dt}$.

Since $t = -\ln(r)$ and t tends to $+\infty$ as r approaches 0, our analysis now focuses on examining the behavior of (4.9) as $t \to +\infty$ to understand the behavior of solutions of $(\mathbf{P_2})$ near the origin. In the set $(A \geq 0,\ B \geq 0,\ C \geq 0)$, we discover that (4.9) possesses a unique fixed point

$$E = \left(\left(\frac{N-1}{f(0)}\right)^{-1/p}, \left(\frac{N-1}{f(0)}\right)^{-(p-1)/p}, 0 \right).$$

Furthermore, at the point E, the linearized equation for (4.9) yields

$$\begin{cases} X' = X - \dfrac{1}{p-1} \left(\dfrac{N-1}{f(0)}\right)^{(p-2)/p} Y, \\ Y' = f(0) \left(\dfrac{N-1}{f(0)}\right)^{2/p} X + (N-1)\, Y, \\ Z' = -Z. \end{cases} \tag{4.10}$$

A straightforward phase-plane analysis shows that system (4.10) has three eigenvalues: $\mu_1 = -1$, μ_2, and μ_3, which satisfy quadratic equation

$$-\mu^2 + N\,\mu - \frac{p(N-1)}{p-1} = 0.$$

Since -1 is a strictly negative eigenvalue of system (4.10), equation (4.9) exhibits a one-dimensional stable manifold denoted as W_E^S. This manifold is tangent to the eigenvector $(0, 0, 1)$ associated with the negative eigenvalue of -1. Because W_E^S is 1-dimensional, according to the theory of dynamical systems, we can deduce that it forms an orbit of system (4.9). Denoting $W_E^S(t) = (A(t),\, B(t),\, C(t))$, and using (4.8), we can derive a positive function denoted as $u = u_0$. This function serves as a solution to (1.2) for small r. Since $W_E^S(t)$ converges to the fixed point E as $t \to +\infty$, we obtain (4.1).

Now we demonstrate that the local solution of (1.2) can be extended to each $r > 0$. Let u be a maximal solution defined in $(0, r_{\max})$ such that $u(r) > 0$ for any $r \in (0, r_{\max})$. Assume by contradiction that $r_{\max} < +\infty$, then

$$\lim_{r \to r_{\max}} |u(r)| = \lim_{r \to r_{\max}} |u^{'}(r)| = +\infty.$$

By using Pohozaev's identity, we have

$$G(r) = r^N \left(\frac{p-1}{p} |u'|^p + \frac{N-p}{p} \frac{|u'|^{p-2} u'\, u}{r} - f(r) \ln(u) \right). \tag{4.11}$$

Since $u'(r) > 0$ in $(0, r_{\max})$ by Lemma 2.1, then

$$G(r) \geq r^N\, u(r) \left(\frac{N-p}{p} \frac{|u'|^{p-1}}{r} - f(r) \frac{\ln(u)}{u} \right),$$

which yields that $\lim_{r \to r_{\max}} G(r) = +\infty$ (because $N > p$).

On the other hand, for any $r \in (0, r_{\max})$, we have

$$G'(r) = r^{N-1}\left[\frac{N-p}{p} f(r) - N f(r)\,\ln(u) - r f'(r)\,\ln(u)\right], \tag{4.12}$$

Since f is positive and increasing by hypothesis $(\mathcal{F}_1)$, then $\lim_{r \to r_{\max}} G'(r) = -\infty$, which is impossible.

Finally, we have to show the uniqueness of the solution u. Suppose that U is another solution of $(\mathbf{P_2})$. Replacing u by U in (4.8) and referring to Proposition 4.2, we have $A_U(t)$, $B_U(t)$ and $C_U(t)$ are bounded for large t. Therefore, according to the Poincaré–Bendixson theory, we can conclude that $(A_U(t), B_U(t), C_U(t))$ converges to E as $t \to +\infty$. Consequently, $(A_U(t), B_U(t), C_U(t))$ lies on the stable manifold W_E^S. Since W_E^S is a one-dimensional manifold and considering the uniqueness of solutions to (4.9), we can deduce that there is some $T \geq 0$ such that

$$A_U(t) = A(t+T), \quad B_U(t) = B(t+T), \quad C_U(t) = C(t+T),$$

which gives that

$$U(r) = \frac{u(\theta r)}{\theta}, \tag{4.13}$$

where $\theta = e^{-T}$. To prove that $U = u$, it suffices to show that $T = 0$. Suppose this is not the case, then $\theta \neq 1$. Now, using the expression (4.13), we see that u satisfies the following equation:

$$\left(|u'(\theta r)|^{p-2}u'(\theta r)\right)' + \frac{N-1}{\theta r}|u'(\theta r)|^{p-2}u'(\theta r) = \frac{f(r)}{u(\theta r)}. \tag{4.14}$$

As u is a singular solution of equation (1.2), then

$$\left(|u'(\theta r)|^{p-2}u'(\theta r)\right)' + \frac{N-1}{\theta r}|u'(\theta r)|^{p-2}u'(\theta r) = \frac{f(\theta r)}{u(\theta r)}. \tag{4.15}$$

By comparing (4.14) and (4.15), it becomes clear that we must have

$$f(\theta r) = f(r), \quad r > 0. \tag{4.16}$$

By taking $r = \theta\,\tilde{r}$ and by iteration, we deduce from (4.16) that $f(\theta^m r) = f(r)$ for any $r > 0$ and $m \in \mathbb{N}$. Hence, if $0 < \theta < 1$, then letting $m \to +\infty$, we obtain $f(r) = f(0)$ for any $r \geq 0$ due to the continuity of f at 0. On the other hand, if $\theta > 1$, we get that $f(r) = f_\infty$ when $m \to +\infty$. Therefore, we obtain that $f(r) = f(0) = f_\infty$ and (1.2) can be expressed in the following form:

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) = \frac{f(0)}{u(r)}. \tag{4.17}$$

We show that (4.17) has a unique solution

$$u(r) = \left(\frac{N-1}{f(0)}\right)^{-1/p} r. \tag{4.18}$$

To establish this, we require the transformation (2.11). Then φ satisfies the equation

$$Y'(t) + (N-1)\,Y(t) - \frac{f(0)}{\varphi(t)} = 0. \tag{4.19}$$

Furthermore, we introduce the following energy function associated with (4.19):

$$F(t) = \frac{p-1}{p}\,|K(t)|^p - Y(t)\varphi(t) + \frac{N}{p}\varphi^p(t) - f(0)\,\ln(\varphi(t)). \tag{4.20}$$

Hence,

$$F'(t) = -NZ_1(t), \tag{4.21}$$

where $Z_1(t) \geq 0$ is given by (3.18).

According to relations (2.11), (4.1) and (2.14), we get

$$\lim_{t\to-\infty} \varphi(t) = \left(\frac{N-1}{f(0)}\right)^{-1/p} \quad \text{and} \quad \lim_{t\to-\infty} K(t) = \left(\frac{N-1}{f(0)}\right)^{-1/p}.$$

Also, by (3.1), we have

$$\lim_{t\to+\infty} \varphi(t) = \left(\frac{N-1}{f_\infty}\right)^{-1/p} \quad \text{and} \quad \lim_{t\to+\infty} K(t) = \left(\frac{N-1}{f_\infty}\right)^{-1/p}.$$

Since $f(r) = f(0) = f_\infty$, then

$$\lim_{t\to-\infty} \varphi(t) = \lim_{t\to+\infty} \varphi(t) = \left(\frac{N-1}{f(0)}\right)^{-1/p}$$

and

$$\lim_{t\to-\infty} K(t) = \lim_{t\to+\infty} K(t) = \left(\frac{N-1}{f(0)}\right)^{-1/p}.$$

It follows that

$$\lim_{t\to-\infty} F(t) = \lim_{t\to+\infty} F(t).$$

Then we get that $F(t)$ is a constant function since F is decreasing by (4.21) and the fact that $Z_1 \geq 0$. This implies that $F(t) = 0$ for any $t \in (-\infty, +\infty)$, that is, $Z_1(t) = 0$ for any $t \in (-\infty, +\infty)$ and it follows that $\varphi'(t) = 0$ for any $t \in (-\infty, +\infty)$. Consequently, we necessarily have $\varphi(t) = ((N-1)/f(0))^{-1/p}$ for any $t \in (-\infty, +\infty)$. In other words,

$$u(r) = \left(\frac{N-1}{f(0)}\right)^{-1/p} r \quad \text{for any } r > 0.$$

Therefore, $U(r) = u(r)$ for any $r > 0$, which contradicts the assumption that the period $T \neq 0$. Thus the solution u is unique, which finishes the proof. □

5. Conclusion

In this paper, we proved the existence and uniqueness of the solution of equation (1.2) under specific assumptions about the function f. Furthermore, we conducted a detailed analysis of the asymptotic behavior of regular and singular solutions by employing the theory of invariant manifolds in dynamical systems and the energy method. As a prospect for future research, we propose delving into the same equation exploring its dynamics by reducing the assumptions on the function f.

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Додатні розв'язки нелінійного еліптичного рівняння, що містить сингулярний член

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У цій роботі ми досліджуємо існування та єдиність додатних розв'язків рівняння $\Delta_p u = f(|x|)/u(x)$, $x \in \mathbb{R}^N$, де $N > p > 2$. Точніше, за певних припущень щодо функції f ми даємо відповідь на питання про глобальне існування, сформульоване в роботі [14], використовуючи теорію інваріантних многовидів у динамічних системах та енергетичний метод. Крім того, ми проводимо детальний аналіз асимптотичної поведінки розв'язків за допомогою логарифмічних перетворень.

Ключові слова: існування, єдиність, додатний розв'язок, асимптотична поведінка, тотожність Похожаєва, динамічна система