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Exploring the Properties of f-Harmonic Vector Fields

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In this paper, our objective is to explore specific characteristics of f-harmonic vector fields. Firstly, we delve into the properties of an f-harmonic Killing vector field when it acts as an f-harmonic map between a Riemannian manifold denoted as (M, g) and its tangent bundle (TM, g_S), which is equipped with the Sasaki metric. We emphasize this investigation when (M, g) takes the form of either an Einstein manifold or a space form. Secondly, we study the traits exhibited by an f-harmonic vector field between a Riemannian manifold (M, g) and its tangent bundle TM equipped with either a deformed Sasaki metric g_{DS} or a Mus–Sasaki metric g_{SF}. Lastly, we conclude this article by providing insightful examples of f-harmonic vector fields in the context of the Heisenberg group.

Key words: f-harmonic Killing vector field, Einstein manifold, tangent bundle

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1. Introduction

Harmonic maps between Riemannian manifolds act as critical points for the energy functional. They minimize the energy or mathematical action linked to the mapping process and provide solutions to the Laplace–Beltrami equation. Harmonic maps find applications in various areas, including physics, such as in the study of minimal surfaces. This concept has been the subject of extensive research. Additionally, mathematicians have developed variations of harmonic maps, including p-harmonic maps and exponentially harmonic maps. p-harmonic maps generalize harmonic maps by taking into account more general elliptic partial differential equations. The Laplace–Beltrami equation corresponds to the case when $p = 2$. p-harmonic maps are solutions to the p-Laplace equation, and they provide a broader framework for studying mappings between manifolds. Exponential harmonic maps introduce nonlinearity to the harmonic map equation using exponential functions. They find applications in geometric analysis and have been studied extensively in relation to geometric flows. f-harmonic maps represent a broader and more generalized category compared to harmonic maps, p-harmonic maps, or exponentially harmonic maps in the field of differential geometry and mathematical analysis. They are critical points of an energy functional that involves a general function denoted as f (hence, f-harmonic). This function f can introduce additional complexities and variations into the harmonic mapping problem. The specific properties and behavior of f-harmonic maps depend on the choice of this function and the underlying Riemannian manifolds. f-harmonic maps serve as a versatile tool for studying mappings between Riemannian manifolds with diverse and customizable properties. Researchers explore various facets of f-harmonic maps to gain insights into their behavior, properties, and applications across a wide array of mathematical contexts. Their versatility makes them a valuable subject of research in the realm of differential geometry and mathematical analysis.

f-harmonic vector fields are significant in the study of Riemannian manifolds and mappings between them. This paper explores various aspects of f-harmonic vector fields analyzing and drawing conclusions for each situation.

Given a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y).$$

Here, ∇ represents the Riemannian connection on M^m, and ∇^{ϕ} denotes the pull-back connection on the pull-back bundle $\phi^{-1}TN$. The f-tension field is defined as

$$\tau_f(\phi) = \operatorname{Tr}_g \nabla f d\phi = f\tau(\phi) + d\phi(\operatorname{grad}_g f),$$

where $f : M \to \mathbb{R}_+$ is a smooth positive function and $\tau(\phi) = \operatorname{Tr}_g \nabla d\phi$ (see [3,11,12,19]). Here, Tr_g represents the trace operation with respect to the metric tensor g. Also, ϕ is said to be f-harmonic if and only if $\tau_f(\phi) = 0$ or equivalent to

$$\tau(\phi) = -d\phi(\operatorname{grad}_g \ln f). \tag{1.1}$$

The existence and explicit construction of f-harmonic mappings between Riemannian manifolds (M^m, g) and (N^n, h) represent fundamental problems in the theory of f-harmonic mappings [5,6,9,13,14,18,21,24]. Nonetheless, the absence of a comprehensive existence theory for f-harmonic mappings adds intrigue to the quest for f-harmonic maps represented by vector fields functioning as mappings from the Riemannian manifold (M^m, g) to its tangent bundle TM.

The contributions of this article are as follows. Firstly, the paper explores the properties of f-harmonic vector fields under specific assumptions. It investigates the case where (M^m, g) is an Einstein manifold. Particularly, it focuses on f-harmonic Killing vector fields as f-harmonic maps from (M^m, g) to its tangent bundle (TM, g_S) equipped with the Sasaki metric. The results are presented in Theorem 3.2 to Theorem 3.4. Secondly, the paper extends its study to f-harmonic vector fields between (M^m, g) and its tangent bundle (TM, g_{DS}) (or (TM, g_{SF})), which are equipped with deformed Sasaki metrics (or Mus–Sasaki metrics). The properties and characteristics of these f-harmonic vector fields are examined and outlined in Theorem 4.4 to Theorem 4.6 (respectively, Theorem 5.3 to Theorem 5.4). The paper concludes by providing illustrative examples of f-harmonic vector fields on the Heisenberg group, offering practical insights into these concepts and their applications. This paper delves into the properties and characteristics of f-harmonic vector fields, focusing on specific scenarios such as

Einstein manifolds and Riemannian manifolds equipped with deformed Sasaki or Mus–Sasaki metrics. These investigations contribute to the understanding of f-harmonic mappings and provide valuable insights into their behavior in various mathematical contexts.

2. Preliminaries

Consider an m-dimensional Riemannian manifold M^m endowed with a Riemannian metric g and let TM represent its tangent bundle denoted by $\pi : TM \to M^m$. When a system of local coordinates (U, x^i) is established within M^m, it naturally gives rise to a system of local coordinates on TM. These coordinates can be expressed as $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = m + i = m + 1, ..., 2m$. Here, (u^i) represents Cartesian coordinates within each tangent space $T_P M$ at a point $P \in M^m$. These coordinates are defined in relation to the standard basis $\left\{ \frac{\partial}{\partial x^i}\Big|_P \right\}$, where P denotes an arbitrary point within U and is characterized by coordinates (x^i).

For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M^m, the vertical lift, denoted as ${}^V X$, and the horizontal lift, denoted as ${}^H X$, with respect to the induced coordinates, are expressed as follows:

$$ {}^V X = X^i \partial_{\bar{i}}, \qquad {}^H X = X^i \partial_i - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}}. $$

Here, ∂_i represents $\frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}}$ represents $\frac{\partial}{\partial u^i}$, while Γ^i_{sk} denotes the coefficients of the Levi-Civita connection ∇ associated with the Riemannian metric g [22].

Specifically, we define the vertical spray, denoted as ${}^V u$, and the horizontal spray, denoted as ${}^H u$, on TM as follows:

$$ {}^V u = u^{i\,V}(\partial_i) = u^i \partial_{\bar{i}}, \qquad {}^H u = u^{i\,H}(\partial_i) = u^i \delta_i, $$

where $\delta_i = \partial_i - u^s \Gamma^m_{si} \partial_{\overline{m}}$. ${}^V u$ is also known as the canonical or Liouville vector field on TM.

The bracket operation between vertical and horizontal vector fields is determined by the following formulas [8, 22]:

$$ \begin{cases} \left[{}^H X, {}^H Y\right] = {}^H [X, Y] - {}^V (R(X, Y)u), \\ \left[{}^H X, {}^V Y\right] = {}^V (\nabla_X Y), \\ \left[{}^V X, {}^V Y\right] = 0 \end{cases} $$

for all vector fields X and Y on M^m, where R represents the Riemannian curvature tensor associated with the metric g.

Lemma 2.1 ([12]). *Let $\psi : (\overline{N}, \overline{h}) \to (N, h)$ constitute a Riemannian immersion. Then, for any smooth map $\phi : (M^m, g) \to (\overline{N}, \overline{h})$, the tension field $\tau(\phi)$ corresponds to the projection of the tension $\psi \circ \phi$ onto $\overline{N}$.*

3. The tangent bundle with the Sasaki metric

First, we consider the vector field ζ as a map from the Riemannian manifold (M^m, g) into its tangent bundle (TM, g_S) equipped with the Sasaki metric. In this section, we will give the necessary and sufficient conditions for ζ to be the f-harmonic map. Also, some special cases (Killing vector field, constant norm, real space form) are considered.

The Sasaki metric is a well-established Riemannian metric for the tangent bundle of a Riemannian manifold. It was first introduced by mathematician Sasaki in 1958 and has since become a fundamental concept in Riemannian geometry. The metric provides a powerful framework for understanding the geometry and properties of tangent bundles associated with Riemannian manifolds. The Sasaki metric, denoted as g_S, on the tangent bundle TM of a Riemannian manifold (M^m, g) is uniquely determined by the following set of properties:

$$\begin{aligned} g_S({}^HX, {}^HY) &= g(X, Y), \\ g_S({}^VX, {}^HY) &= g_S({}^HX, {}^VY) = 0, \\ g_S({}^VX, {}^VY) &= g(X, Y) \end{aligned}$$

for all vector fields X, Y on M^m (see [10, 15]).

Lemma 3.1 ([10,15])**.** *Consider a Riemannian manifold (M^m, g). If we have vector fields X, Y on M^m, as well as a point (x, u) in the tangent bundle TM, with the property $Y_x = u$, then we can express this relationship as follows:*

$$\begin{aligned} d_xX(Y_x) &= {}^HY_{(x,u)} + {}^V(\nabla_Y X)_{(x,u)}, \\ \tau(X) &= -\operatorname{Tr}_g\left\{{}^HR(\nabla_* X, X)* - {}^V\nabla^2 X\right\}. \end{aligned}$$

From (1.1) and Lemma 3.1, we have the following theorem.

Theorem 3.2. *Suppose we have a Riemannian manifold (M^m, g) and its tangent bundle (TM, g_S) equipped with the Sasaki metric. In this context, the vector field $\zeta : M^m \to TM$ is considered as an f -harmonic vector field if and only if the following conditions are satisfied:*

$$\operatorname{Tr}_g R(\nabla_*\zeta, \zeta)* = \operatorname{grad}_g(\ln f), \tag{3.1}$$

$$\operatorname{Tr}_g \nabla^2\zeta = -\nabla_{\operatorname{grad}_g(\ln f)}\zeta. \tag{3.2}$$

As a partial case. If $\zeta : M^m \to TM$ is a Killing vector field, then ζ is an f-harmonic vector field if and only if the following conditions are satisfied:

$$\begin{aligned} \operatorname{Tr}_g R(\nabla_*\zeta, \zeta)* &= \operatorname{grad}_g(\ln f), \\ Q(\zeta) &= \nabla_{\operatorname{grad}_g(\ln f)}\zeta. \end{aligned}$$

Proof. The proof comes directly from the fact that if ζ is a Killing vector field, then $\operatorname{Tr}_g \nabla^2\zeta = -Q(\zeta)$ [1]. □

Corollary 3.3. *Consider a Riemannian manifold* (M^m, g) *and its tangent bundle* (TM, g_S) *equipped with the Sasaki metric. If* $\zeta : M^m \to TM$ *is a vector field with constant norm* ($|\zeta| = \text{const}$), *then* ζ *is an* f*-harmonic vector field if and only if* ζ *is parallel and* $f = \text{const}$.

Proof. Since $|g(\zeta, \zeta)| = \text{const}$, we can conclude that

$$g(\nabla^2\zeta, \zeta) = -|\nabla\zeta|^2$$

and

$$g(\nabla_{\text{grad}_g(\ln f)}\zeta, \zeta) = 0.$$

Thus, from (3.2), we deduce that $\nabla\zeta = 0$ and from (3.1), we can infer that f is a constant. □

Theorem 3.4. *Let* $(M^m(c), g)$ *represent a real space form manifold and* (TM, g_S) *denote its tangent bundle equipped with the Sasaki metric. In this context, the vector field* $\zeta : M^m \to TM$ *is an* f*-harmonic vector field if and only if the following conditions are satisfied:*

$$c\nabla_\zeta\zeta - c\,\text{Tr}_g\, g(\nabla_*\zeta, *)\zeta = \text{grad}_g(\ln f), \tag{3.3}$$

$$\text{Tr}_g\, \nabla^2\zeta = -\nabla_{\text{grad}_g(\ln f)}\zeta. \tag{3.4}$$

Furthermore, if the vector field $\zeta : M^m \to TM$ *is a Killing vector field, then the relations* (3.3) *and* (3.4) *can be simplified as follows:*

$$c\nabla_\zeta\zeta = \text{grad}_g(\ln f), \tag{3.5}$$

$$c(m-1)\zeta = -\nabla_{\text{grad}_g(\ln f)}\zeta, \tag{3.6}$$

and ζ *is an* f*-harmonic vector field if and only if* $f = \text{const}$, *and* (M^m, g) *is flat or* $m = 1$, *and* ζ *is itself parallel.*

Proof. Let $\{E_i\}_{i=1,\ldots,m}$ be an orthonormal basis on (M^m, g). By employing

$$R(X, Y)Z = c[g(Y, Z)X - g(X, Z)Y]$$

for all vector fields X, Y and Z on M^m, we can derive the following:

$$\begin{aligned}\text{Tr}_g\, R(\nabla_*\zeta, \zeta)* &= \sum_i R(\nabla_{E_i}\zeta, \zeta)E_i \\ &= c\sum_i [g(\zeta, E_i)\nabla_{E_i}\zeta - g(\nabla_{E_i}\zeta, E_i)\zeta \\ &= c\nabla_\zeta\zeta - c\,\text{Tr}_g\, g(\nabla_*\zeta, *)\zeta.\end{aligned}$$

Referring to the Killing vector fields properties, we can observe that the relations (3.3) and (3.4) transform into the relations (3.5) and (3.6), respectively. Additionally, we can establish the following relationship:

$$c(m-1)|\zeta|^2 = g(\nabla_{\text{grad}_g(\ln f)}\zeta, \zeta).$$

By manipulating this equation, we can further deduce:

$$c(m-1)|\zeta|^2 = -c|\nabla_\zeta \zeta|^2.$$

This implies that $c = 0$, or $m = 1$, and $\nabla_\zeta \zeta = 0$. Consequently, from (3.5), we can conclude that f is a constant. □

4. The tangent bundle with a deformed Sasaki metric

In this section, our focus shifts towards the investigation of f-harmonic vector fields when they serve as mappings from the Riemannian manifold (M^m, g) to its tangent bundle. The tangent bundle is equipped with a particular class of Riemannian natural metrics, which are derived through the vertical deformation of the Sasaki metric. We refer to this specific metric as to a deformed Sasaki metric. It is worth noting that this class of metrics encompasses the Cheeger-Gromoll metric as a special case providing a broader framework for our study (for more comprehensive details, see [7]).

Definition 4.1. We define a deformed Sasaki metric, denoted as g_{DS}, on the tangent bundle TM of a Riemannian manifold (M^m, g) using smooth functions α and β, both defined on the positive real numbers $\mathbb{R}^+$. This metric is defined as follows:

$$\begin{aligned} g_{DS}({}^HX, {}^HY)_p &= g_x(X,Y), \\ g_{DS}({}^HX, {}^VY)_p &= 0, \\ g_{DS}({}^VX, {}^VY)_p &= \alpha(r) g_x(X,Y) + \beta(r) g_x(X,u) g_x(Y,u), \end{aligned}$$

where X and Y belong to the space of smooth vector fields on M^m, $p = (x,u) \in TM$, $r = g(u,u)$, $\alpha > 0$ and $\alpha + \beta r > 0$ (see [7]).

Remark 4.2.

1) If $\alpha = 1$ and $\beta = 0$, then g_{DS} is the Sasaki metric [20].
2) If $\beta = 0$, then g_{DS} is one case of the Mus–Sasaki metric [23].
3) If $\alpha = \beta = \dfrac{1}{r+1}$, then g_{DS} is the Cheeger–Gromoll metric [2].

Let us establish our notations:

$$\lambda = \alpha + \beta r, \qquad \overline{\alpha} = \frac{\alpha'}{\alpha}, \qquad \overline{\beta} = \frac{\beta - \alpha'}{\alpha + r\beta}, \qquad \overline{\delta} = \frac{\alpha\beta' - 2\alpha'\beta}{\alpha(\alpha + r\beta)} \tag{4.1}$$

and

$$\eta = \frac{1}{\lambda\alpha}[\lambda\alpha' + \alpha(\beta - \alpha') + (\alpha\beta' - 2\beta\alpha')\, r] = [\overline{\alpha} + \overline{\beta} + \overline{\delta}\, r].$$

Applying Definition 4.1 and utilizing the Koszul formula, we derive the following lemma.

Lemma 4.3 ([7]). *When considering the Levi-Civita connection ∇ (respectively, $\widetilde{\nabla}$) of the Riemannian manifold (M^m, g) (respectively, the tangent bundle (TM, g_{DS})) and denoting R as the Riemannian curvature tensor of (M^m, g), we can express the following relationships:*

$$\begin{aligned}
(\widetilde{\nabla}_{^HX}{}^HY)_p &= (^H\nabla_X Y)_p - \frac{1}{2}{}^V(R_x(X,Y)u),\\
((\widetilde{\nabla}_{^HX}{}^VY)_p &= {}^V(\nabla_X Y)_p + \frac{\alpha}{2}{}^H(R_x(u,Y)X),\\
(\widetilde{\nabla}_{^VX}{}^HY)_p &= \frac{\alpha}{2}{}^H(R_x(u,X)Y),\\
(\widetilde{\nabla}_{^VX}{}^VY)_p &= \overline{\alpha}\big[g_x(X,u)(^VY)_p + g_x(Y,u)(^VX)_p\big]\\
&\quad + \big[\overline{\beta}g_x(X,Y) + \overline{\delta}g_x(X,u)g_x(Y,u)\big]U_p,
\end{aligned}$$

where X, Y are vector fields on M^m, $p = (x,u) \in TM$ and U_p is the canonical vertical vector at p, defined as $U_p = u^i \frac{\partial}{\partial u^i} \in T_p(TM)$.

By utilizing Theorem 4.3 and (1.1), we obtain the following lemma.

Lemma 4.4. *Consider a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the deformed Sasaki metric g_{DS}. For a smooth vector field $\zeta : M^m \to TM$, we have the following relationship:*

$$\begin{aligned}
\tau(\zeta) = {}^V\mathrm{Tr}_g[\nabla^2_*\zeta + 2\overline{\alpha}g(\nabla_*\zeta,\zeta)\nabla_*\zeta + \overline{\beta}|\nabla_*\zeta|^2\zeta\\
+ \overline{\delta}g(\nabla_*\zeta,\zeta)^2\zeta] + {}^H[\alpha\,\mathrm{Tr}_g\, R(\zeta,\nabla_*\zeta)*],
\end{aligned}$$

where $r = g(\zeta,\zeta)$ and $\overline{\alpha}, \overline{\beta}$ and $\overline{\delta}$ are smooth functions defined by (4.1).

Proof. Let $(x,u) \in TM$, ζ be a vector field on M^m with the property that $\zeta_x = u$ and let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M^m such that $(\nabla^M_{E_i}E_i)_x = 0$. Then

$$\begin{aligned}
\tau(\zeta)_x &= \sum_{i=1}^m \{(\nabla^\zeta_{E_i} d\zeta(E_i))_x - d\zeta(\nabla^M_{E_i}E_i)_x\}\\
&= \sum_{i=1}^m \{\widetilde{\nabla}_{d\zeta(E_i)} d\zeta(E_i)\}_{(x,u)} = \sum_{i=1}^m \{\widetilde{\nabla}_{(^HE_i + {}^V(\nabla_{E_i}\zeta))}(^HE_i + {}^V(\nabla_{E_i}\zeta))\}_{(x,u)}\\
&= \sum_{i=1}^m \{\widetilde{\nabla}_{^HE_i}{}^HE_i + \widetilde{\nabla}_{^HE_i}{}^V(\nabla_{E_i}\zeta) + \widetilde{\nabla}_{^V(\nabla_{E_i}\zeta)}{}^HE_i + \widetilde{\nabla}_{^V(\nabla_{E_i}\zeta)}{}^V(\nabla_{E_i}\zeta)\}_{(x,u)}.
\end{aligned}$$

By using Lemma 4.3, we obtain

$$\begin{aligned}
\tau(\zeta) = \sum_{i=1}^m \Big\{&{}^V(\nabla_{E_i}\nabla_{E_i}\zeta) + \alpha{}^HR(\zeta,\nabla_{E_i}\zeta)E_i + 2\bar{\alpha}g(\nabla_{E_i}\zeta,\zeta){}^V(\nabla_{E_i}\zeta)\\
&+ (\bar{\beta}|\nabla_{E_i}\zeta|^2 + \bar{\delta}g(\nabla_{E_i}\zeta,\zeta)^2){}^V\zeta\Big\}
\end{aligned}$$

$$= \sum_{i=1}^{m} {}^{V}\Big[\nabla_{E_i}\nabla_{E_i}\zeta + 2\bar{\alpha}g(\nabla_{E_i}\zeta,\zeta)\nabla_{E_i}\zeta + \bar{\beta}|\nabla_{E_i}\zeta|^2\zeta$$
$$+ \bar{\delta}g(\nabla_{E_i}\zeta,\zeta)^2\zeta\Big] + {}^{H}\Big[\alpha R(\zeta,\nabla_{E_i}\zeta)E_i\Big].$$

Hence, Lemma 4.4 follows. □

Theorem 4.5. *Consider a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the deformed Sasaki metric g_{DS}. In this context, a vector field $\zeta : M^m \to TM$ is an f-harmonic vector field if and only if the following conditions are satisfied:*

$$\alpha \operatorname{Tr}_g R(\nabla_*\zeta,\zeta)* = -\operatorname{grad}_g(\ln f),$$
$$\operatorname{Tr}_g[\overline{\beta}\,|\nabla\zeta|^2 + \overline{\delta}g(\nabla\zeta,\zeta)^2]\zeta = -\operatorname{Tr}_g[\nabla^2\zeta + 2\overline{\alpha}g(\nabla\zeta,\zeta)\nabla\zeta] - \nabla_{\operatorname{grad}_g(\ln f)}\zeta. \quad (4.2)$$

When $|\zeta| = \text{const}$, the relation (4.2) becomes

$$\operatorname{Tr}_g[\overline{\beta}\,|\nabla\zeta|^2\zeta + \nabla^2\zeta] = -\nabla_{\operatorname{grad}_g(\ln f)}\zeta.$$

From Theorem 4.5, we can state the following proposition.

Proposition 4.6. *Under the hypotheses of Theorem 4.5 and if $\zeta : M^m \to TM$ is a Killing vector field, then ζ is considered as an f-harmonic vector field if and only if the following conditions hold:*

$$\alpha \operatorname{Tr}_g R(\nabla_*\zeta,\zeta)* = -\operatorname{grad}_g(\ln f), \quad (4.3)$$
$$\overline{\beta}\,(\operatorname{Tr}_g|\nabla\zeta|^2)\zeta = Q(\zeta) - \overline{\delta}\,|\nabla_\zeta\zeta|^2\zeta - 2\overline{\alpha}\,\nabla_{\nabla_\zeta\zeta}\zeta - \nabla_{\operatorname{grad}_g(\ln f)}\zeta. \quad (4.4)$$

Moreover, if (M^m, g) is a real space form manifold $M^m(c)$, then equations (4.3) and (4.4) become

$$c\alpha\nabla_\zeta\zeta = -\operatorname{grad}_g(\ln f),$$
$$B_\zeta(\zeta) = c(m-1)\zeta,$$

where B_ζ is a real operator given for any vector field X on M^m by

$$B_\zeta X = \overline{\beta}\,(\operatorname{Tr}_g|\nabla\zeta|^2)X + \overline{\delta}\,|\nabla_\zeta\zeta|^2 X + 2\overline{\alpha}\,\nabla_{\nabla_\zeta\zeta}X + \nabla_{\operatorname{grad}_g(\ln f)}X.$$

Therefore, ζ is an eigenvector of the operator B_ζ corresponding to the eigenvalue $c(m-1)$.

Proof. We deduce the proof from $Q(\zeta) = -\operatorname{Tr}_g[\nabla^2\zeta]$. Additionally, by the Killing vector fields properties, we obtain

$$\operatorname{Tr}_g g(\nabla\zeta,\zeta)^2 = \sum_i g(\nabla_{E_i}\zeta,\zeta)g(\nabla_{E_i}\zeta,\zeta) = g(\nabla_\zeta\zeta,\nabla_\zeta\zeta)$$

and

$$\operatorname{Tr}_g g(\nabla\zeta,\zeta)\nabla\zeta = \sum_i g(\nabla_{E_i}\zeta,\zeta)\nabla_{E_i}\zeta = \nabla_{\nabla_\zeta\zeta}\zeta.$$

Moreover, if $M^m(c)$ is a real space form manifold, we can deduce that $Q(\zeta) = c(m-1)\zeta$. Additionally, we get $\operatorname{Tr}_g R(\nabla_*\zeta,\zeta)* = c\nabla_\zeta\zeta$. □

5. The tangent bundle with a Mus–Sasaki metric

This section is dedicated to the study of the vector field ζ, which represents the mapping between (M^m, g) and the tangent bundle TM equipped with an alternative class of natural metric known as the Mus–Sasaki metric g_{SF}.

Consider a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the Mus–Sasaki metric denoted as g_{SF}. This metric is defined as follows:

$$\begin{aligned} g_{SF}({}^HX, {}^HY)_p &= g_x(X,Y), \\ g_{SF}({}^VX, {}^HY)_p &= g_{SF}({}^HX, {}^VY)_p = 0, \\ g_{SF}({}^VX, {}^VY)_p &= F(\alpha(x), \beta(r))g_x(X,Y) \end{aligned}$$

for all vector fields X, Y on M^m, where $F : (s,t) \in \mathbb{R}^2 \to F(s,t) \in]0, +\infty[$ is a smooth function, $\alpha \in C^\infty(M^m)$, $\beta \in C^\infty(\mathbb{R})$, $r = g(u,u)$ and $p = (x,u) \in TM$ (see [16, 23]).

Theorem 5.1 ([16, 23]). *Consider a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the Mus–Sasaki metric g_{SF}. If we denote ∇ (respectively, $\widehat{\nabla}$) as the Levi-Civita connection of (M^m, g) (respectively, (TM, g_{SF})), we can express the following relationships:*

$$\begin{aligned} (\widehat{\nabla}_{{}^HX}{}^HY)_p &= {}^H(\nabla_X Y)_p - \frac{1}{2}{}^V(R_x(X,Y)u), \\ (\widehat{\nabla}_{{}^HX}{}^VY)_p &= {}^V(\nabla_X Y)_p + \frac{F(\alpha(x),\beta(r))}{2}{}^H(R_x(u,Y)X) \\ &\quad + \frac{1}{2F(\alpha(x),\beta(r))}g_x(\operatorname{grad}_M \alpha, X)\frac{\partial F}{\partial s}(\alpha(x),\beta(r)){}^V(Y)_p, \\ (\widehat{\nabla}_{{}^VX}{}^HY)_p &= \frac{F(\alpha(x),\beta(r))}{2}{}^H(R_x(u,X)Y)) \\ &\quad + \frac{1}{2F(\alpha(x),\beta(r))}g_x(\operatorname{grad}_M \alpha, Y)\frac{\partial F}{\partial s}(\alpha(x),\beta(r)){}^V(X)_p, \\ (\widehat{\nabla}_{{}^VX}{}^VY)_p &= \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\Big[g_x(Y,U){}^V(X)_p) \\ &\quad + g_x(X,U){}^V(Y)_p) - g_x(X,Y){}^VU_p\Big] \\ &\quad - \frac{1}{2}g_x(X,Y)\frac{\partial F}{\partial t}(\alpha(x),\beta(r)){}^H(\operatorname{grad}_M \alpha)_p \end{aligned}$$

for all vector fields X, Y on M^m and $p = (x,u) \in TM$, where R denotes the Riemannian curvature tensor of (M^m, g).

Lemma 5.2 ([16]). *Consider a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the Mus–Sasaki metric g_{SF}. If X, Y are vector fields on M^m and (x,u) is a point on TM, then we have the following relationship:*

$$\tau(\zeta) = \left(\frac{1}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial s}{}^V(\alpha(x),\beta(r))\nabla_{\operatorname{grad}(\alpha)}\zeta\right) + {}^V(\operatorname{Tr}_g A(\zeta)) + {}^H(\operatorname{Tr}_g B(\zeta)),$$

where $A(\zeta)$ and $B(\zeta)$ are bilinear maps defined by

$$A(\zeta) = \nabla^2\zeta + \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\Big[2g(\nabla\zeta,\zeta)\nabla\zeta - g(\nabla\zeta,\nabla\zeta)\zeta\Big],$$
$$B(\zeta) = F(\alpha(x),\beta(r))R(\zeta,\nabla\zeta) * -\frac{1}{2}g(\nabla\zeta,\nabla\zeta)\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{grad}_M(\alpha).$$

Theorem 5.3. *In the context of a Riemannian manifold (M^m, g) and its tangent bundle TM equipped with the Mus–Sasaki metric g_{SF}, a vector field $\zeta : M^m \to TM$ is considered as an f-harmonic vector field if and only if the following conditions are satisfied:*

$$-\nabla_{\mathrm{grad}_g(\ln f)}\zeta = \mathrm{Tr}_g\Big\{\nabla^2\zeta + \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\Big[2g_x(\nabla\zeta,\zeta)\nabla\zeta - g_x(\nabla\zeta,\nabla\zeta)\zeta\Big]\Big\} + \frac{\frac{\partial F}{\partial s}(\alpha(x),\beta(r))}{F(\alpha(x),\beta(r))}\nabla_{\mathrm{grad}(\alpha)}\zeta \tag{5.1}$$

$$-\mathrm{grad}_g(\ln f) = F(\alpha(x),\beta(r))\,\mathrm{Tr}_g\, R(\zeta,\nabla\zeta)* - \frac{1}{2}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{Tr}_g\, g_x(\nabla\zeta,\nabla\zeta)\,\mathrm{grad}_M(\alpha), \tag{5.2}$$

where $r = g(\zeta_x,\zeta_x) = \|\zeta_x\|^2$.

In the case where $\zeta : M^m \to TM$ is a vector field with a constant norm, ζ is an f-harmonic vector field if and only if ζ is parallel or

$$1 + \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))|\zeta|^2 = 0. \tag{5.3}$$

Proof. Equations (5.1) and (5.2) immediately follow from (1.1) and Lemma 5.2.

Given that $|g(\zeta,\zeta)| = \text{const}$, we can infer the relationships

$$g(\nabla^2\zeta,\zeta) = -|\nabla\zeta|^2$$

and

$$g(\nabla_Y\zeta,\zeta) = 0$$

for any vector field Y on M^m. Hence, from (5.1), we arrive at

$$0 = \mathrm{Tr}_g\Big\{g_x(\nabla\zeta,\nabla\zeta)(1 + \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))|\zeta|^2)\Big\}. \qquad \square$$

Theorem 5.4. *In the context of a real space form manifold $M^m(c)$ represented as (M^m, g) and its tangent bundle TM equipped with the Mus–Sasaki metric g_{SF}, if $\zeta : M^m \to TM$ is a Killing vector field, then ζ is an f-harmonic vector field if and only if the following conditions are met:*

$$L_\zeta(\zeta) = c(m-1)\zeta,$$
$$\mathrm{grad}_g(\ln f) = -cF(\alpha(x),\beta(r))\nabla_\zeta\zeta + \frac{1}{2}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{Tr}_g\,|\nabla\zeta|^2\,\mathrm{grad}_M(\alpha),$$

where L_ζ *is a real operator given for any vector field* Y *on* M^m *by*

$$L_\zeta Y = \frac{\beta'(r)}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{Tr}_g\left\{\left[2\nabla_{\nabla_\zeta\zeta}Y - |\nabla\zeta|^2 Y\right]\right\}$$
$$+\nabla_{\mathrm{grad}_g(\ln f)}Y + \frac{1}{F(\alpha(x),\beta(r))}\frac{\partial F}{\partial s}(\alpha(x),\beta(r))\nabla_{grad(\alpha)}Y.$$

Therefore, ζ *is an eigenvector of the operator* L_ζ *corresponding to the eigenvalue* $c(m-1)$*.*

Proof. The proof comes directly from Theorem 5.3, taking into account the properties of a real space form manifold $M^m(c)$ and the Killing vector field ζ. □

6. Applications and examples

In this section, our focus turns towards studying the necessary and sufficient conditions for certain vector fields (such as Killing vector fields, vector fields with constant norm and gradient vector fields) on the 3-dimensional Heisenberg group H_3 to be f-harmonic maps (for more comprehensive details, see [4, 17]).

First, we recall some basic results on the geometry of the 3-dimensional Heisenberg group H_3. Let H_3 be the 3-dimensional Heisenberg group. H_3 is realized as a Lie group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,z\in\mathbb{R} \right\}$$

equipped with a left-invariant metric

$$g = dx^2 + dy^2 + (dz - xdy)^2. \tag{6.1}$$

An orthonormal basis on (H_3, g) is given by

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

The non-zero coefficients of the Levi-Civita connection are given by

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = -\frac{1}{2}e_3, \quad \nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}e_2, \quad \nabla_{e_2}e_3 = -\nabla_{e_3}e_2 = \frac{1}{2}e_1.$$

The Riemannian curvature tensor is expressed as

$$R(e_1,e_2)e_1 = -\frac{1}{4}e_2, \qquad R(e_1,e_2)e_2 = \frac{1}{4}e_1,$$
$$R(e_1,e_3)e_1 = -\frac{1}{4}e_3, \qquad R(e_1,e_3)e_3 = \frac{1}{4}e_1,$$
$$R(e_2,e_3)e_3 = -\frac{3}{4}e_2, \qquad R(e_2,e_3)e_2 = \frac{3}{4}e_3.$$

The left-invariant Killing vector fields on (H_3, g) are given by

$$\zeta = a(\frac{1}{2}(x^2+y^2)e_1 + ye_2 - xe_3) + b(ye_1 + e_2) + c(-xe_1 + e_3) + d(e_1),$$

where a, b, c and d are constants, which means that the Killing vector fields are provided by

$$\zeta_1 = e_1, \quad \zeta_2 = ye_1 + e_2, \quad \zeta_3 = -xe_1 + e_3,$$
$$\zeta_4 = \frac{1}{2}(x^2+y^2)e_1 + ye_2 - xe_3.$$

Consider a left-invariant vector field $V = ae_1 + be_2 + ce_3$, where a, b and c are constants. V is a vector field with a constant norm as follows:

$$\nabla_{e_1}V = \frac{1}{2}(ce_2 - be_3), \quad \nabla_{e_2}V = \frac{1}{2}(ce_1 - ae_3), \quad \nabla_{e_3}V = \frac{1}{2}(ae_2 - be_1).$$

After a straightforward computation, we obtain

$$\mathrm{Tr}_g\, R(\nabla_* V, V)* = \frac{a}{4}(be_3 - ce_2),\ \mathrm{Tr}_g\, |\nabla V|^2 = \frac{1}{2}|V|^2,\ \mathrm{Tr}_g\, \nabla^2 V = -\frac{1}{2}V. \quad (6.2)$$

Let X be a left-invariant vector field defined by $X = \mathrm{grad}_g(\gamma)$, where γ is a smooth function on H_3 that depends on x and y. The vector X can be described as

$$X = \frac{\partial\gamma}{\partial x}(x,y)e_2 + \frac{\partial\gamma}{\partial y}(x,y)e_3. \quad (6.3)$$

6.1. With the Sasaki metric

Theorem 6.1. *Let (H_3, g) be the 3-dimensional Heisenberg group with the left-invariant metric g given by (6.1) and let (TH_3, g_S) be its tangent bundle equipped with the Sasaki metric. In this context, we can state the followings:*

1. *The vector field V with a constant norm cannot be an f-harmonic vector field from (H_3, g) into (TH_3, g_S).*
2. *The vector field X given in (6.3) is an f-harmonic vector field from (H_3, g) into (TH_3, g_S) if and only if the following system holds:*

$$\begin{cases} \dfrac{\partial\gamma}{\partial y}\dfrac{\partial\gamma}{\partial x}\left(\dfrac{\partial^2\gamma}{\partial y^2} - \dfrac{\partial^2\gamma}{\partial x^2}\right) + \dfrac{\partial^2\gamma}{\partial x\partial y}\left(\left(\dfrac{\partial\gamma}{\partial x}\right)^2 - \left(\dfrac{\partial\gamma}{\partial y}\right)^2\right) = 0, \\ \dfrac{3}{4}\dfrac{\partial\gamma}{\partial x}\left(\dfrac{\partial^2\gamma}{\partial x^2}\dfrac{\partial^2\gamma}{\partial y^2} - \left(\dfrac{\partial^2\gamma}{\partial x\partial y}\right)^2\right) = -\left(\dfrac{\partial^3\gamma}{\partial x^3} + \dfrac{\partial^3\gamma}{\partial x\partial^2 y}\right) + \dfrac{1}{2}\dfrac{\partial\gamma}{\partial x}, \\ \dfrac{3}{4}\dfrac{\partial\gamma}{\partial y}\left(\dfrac{\partial^2\gamma}{\partial x^2}\dfrac{\partial^2\gamma}{\partial y^2} - \left(\dfrac{\partial^2\gamma}{\partial x\partial y}\right)^2\right) = -\left(\dfrac{\partial^3\gamma}{\partial y^3} + \dfrac{\partial^3\gamma}{\partial x^2\partial y}\right) + \dfrac{1}{2}\dfrac{\partial\gamma}{\partial y} \end{cases} \quad (6.4)$$

and f satisfies the equation

$$\mathrm{grad}_g(\ln f) = \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial y^2}\frac{\partial\gamma}{\partial x} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial y}\right)e_2 + \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial x^2}\frac{\partial\gamma}{\partial y} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial x}\right)e_3.$$

Proof. 1. One can easily notice that the non-zero vector field V is not parallel. Thus, by virtue of Corollary 3.3, V is not an f-harmonic vector field with respect to the Sasaki metric.

2. Let X be the vector field given in (6.3). By developing the left sides of equations (3.1) and (3.2) provided in Theorem 3.2, we obtain

$$\operatorname{Tr}_g R(\nabla_* X, X)* = \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial y^2}\frac{\partial\gamma}{\partial x} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial y}\right)e_2 + \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial x^2}\frac{\partial\gamma}{\partial y} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial x}\right)e_3, \tag{6.5}$$

$$\operatorname{Tr}_g \nabla^2 X = \left(\frac{\partial^3\gamma}{\partial x^3} + \frac{\partial^3\gamma}{\partial x\partial^2 y} - \frac{1}{2}\frac{\partial\gamma}{\partial x}\right)e_2 + \left(\frac{\partial^3\gamma}{\partial y^3} + \frac{\partial^3\gamma}{\partial y\partial^2 x} - \frac{1}{2}\frac{\partial\gamma}{\partial y}\right)e_3. \tag{6.6}$$

From (3.1) and (6.5), we find

$$\operatorname{grad}_g(\ln f) = \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial y^2}\frac{\partial\gamma}{\partial x} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial y}\right)e_2 + \frac{3}{4}\left(\frac{\partial^2\gamma}{\partial x^2}\frac{\partial\gamma}{\partial y} - \frac{\partial^2\gamma}{\partial x\partial y}\frac{\partial\gamma}{\partial x}\right)e_3.$$

Then we compute

$$\begin{aligned}\nabla_{\operatorname{grad}_g(\ln f)} X = {} & \frac{3}{8}\left(\frac{\partial\gamma}{\partial y}\frac{\partial\gamma}{\partial x}\left(\frac{\partial^2\gamma}{\partial y^2} - \frac{\partial^2\gamma}{\partial x^2}\right) + \frac{\partial^2\gamma}{\partial x\partial y}\left(\left(\frac{\partial\gamma}{\partial x}\right)^2 - \left(\frac{\partial\gamma}{\partial y}\right)^2\right)\right)e_1 \\ & + \frac{3}{4}\frac{\partial\gamma}{\partial x}\left(\frac{\partial^2\gamma}{\partial x^2}\frac{\partial^2\gamma}{\partial y^2} - \left(\frac{\partial^2\gamma}{\partial x\partial y}\right)^2\right)e_2 \\ & + \frac{3}{4}\frac{\partial\gamma}{\partial y}\left(\frac{\partial^2\gamma}{\partial x^2}\frac{\partial^2\gamma}{\partial y^2} - \left(\frac{\partial^2\gamma}{\partial x\partial y}\right)^2\right)e_3. \end{aligned} \tag{6.7}$$

Upon comparing equations (3.2), (6.6) and (6.7), we can derive the system (6.4). □

Numerous examples can be generated based on Theorem 6.1. As an illustration, we give an example.

Example 6.2. Consider the Heisenberg group (H_3, g) with the left-invariant metric g defined by (6.1). Let (TH_3, g_S) be its tangent bundle equipped with the Sasaki metric. Suppose $X = \operatorname{grad}_g(\gamma)$, where $\gamma(x,y,z) = \frac{\sqrt{2}}{2\sqrt{3}}(x^2+y^2) + c_1x + c_2y + k$ and c_1, c_2, k are constants. Then the vector field $X : (H_3, g) \to (TH_3, g_S)$ is an f-harmonic vector field with $f(x,y,z) = \exp\left(\frac{\sqrt{3}}{2\sqrt{2}}\gamma(x,y,z)\right)$.

6.2. With the deformed Sasaki metric

Proposition 6.3. *Consider the* 3*-dimensional Heisenberg group* (H_3, g) *with the left-invariant metric* g *given by* (6.1)*. Let* (TH_3, g_{DS}) *be its tangent bundle equipped with the deformed Sasaki metric. Then the vector field* $V = ae_1 + be_2 + ce_3$*, where* a, b, c *are constants, is an* f*-harmonic vector field if and only if* $\overline{\beta} = \frac{\beta-\alpha'}{\alpha+r\beta} = \frac{1}{|V|^2}$*, and either* $(a = 0)$ *or* $(b = c = 0)$*, with* f *being a constant.*

Proof. From Theorem 4.5, a vector field V is an f-harmonic vector field if and only if the following conditions hold:

$$\alpha \operatorname{Tr}_g R(\nabla_* V, V)* = -\operatorname{grad}_g(\ln f), \tag{6.8}$$
$$\operatorname{Tr}_g[\overline{\beta}\,|\nabla V|^2 V + \nabla^2 V] = -\nabla_{\operatorname{grad}_g(\ln f)} V.$$

Substituting the above results into (6.8), we get the system

$$\begin{cases} \operatorname{grad}_g(\ln f) = \dfrac{a\alpha}{4}(ce_2 - be_3), \\ \nabla_{\operatorname{grad}_g(\ln f)} V = \dfrac{1}{2}(1 - \overline{\beta}|V|^2)V. \end{cases} \tag{6.9}$$

On the other hand, we have

$$\nabla_{\operatorname{grad}_g(\ln f)} V = \frac{a\alpha}{8}[(c^2 + b^2)e_1 - abe_2 - ace_3]. \tag{6.10}$$

By using (6.10), the second equation of the system (6.9) yields

$$\begin{cases} \dfrac{a\alpha}{4}(c^2 + b^2) = (1 - \overline{\beta}\,|V|^2)a, \\ -\dfrac{a^2\alpha}{4}b = (1 - \overline{\beta}\,|V|^2)b, \\ -\dfrac{a^2\alpha}{4}c = (1 - \overline{\beta}\,|V|^2)c. \end{cases} \tag{6.11}$$

The solution of the system (6.11) is $\overline{\beta} = \dfrac{\beta - \alpha'}{\alpha + r\beta} = \frac{1}{|V|^2}$. Again, by substituting $\overline{\beta}$ into the system (6.11), we find $a = 0$ or $b = c = 0$. □

Remark 6.4. The Killing vector field $\zeta_1 = e_1$ is an f-harmonic vector field with respect to g_{DS} if and only if $\overline{\beta} = 1$ and $f = \text{const}$.

6.3. With the Mus–Sasaki metric

Theorem 6.5. *Consider the* 3*-dimensional Heisenberg group* (H_3, g) *with the left-invariant metric* g *given by* (6.1)*. Let* (TH_3, g_{SF}) *be its tangent bundle equipped with the Mus–Sasaki metric. If* V *is a vector field on* H_3 *with a constant norm, then* V *is an* f*-harmonic vector field if and only if the system* (6.12) *holds*

$$\begin{cases} \dfrac{a(c^2 + b^2)}{4}F(\alpha, \beta) \\ \quad + \left(\dfrac{1}{4}|V|^2 \dfrac{\partial F}{\partial t}(\alpha, \beta) + \dfrac{1}{F(\alpha, \beta)}\dfrac{\partial F}{\partial s}(\alpha, \beta)\right)(be_3(\alpha) - ce_2(\alpha)) = 0, \\ \dfrac{a^2 b}{4}F(\alpha, \beta) \\ \quad + \left(\dfrac{1}{4}|V|^2 \dfrac{\partial F}{\partial t}(\alpha, \beta) + \dfrac{1}{F(\alpha, \beta)}\dfrac{\partial F}{\partial s}(\alpha, \beta)\right)(ce_1(\alpha) + ae_3(\alpha)) = 0, \\ -\dfrac{a^2 c}{4}F(\alpha, \beta) \\ \quad + \left(\dfrac{1}{4}|V|^2 \dfrac{\partial F}{\partial t}(\alpha, \beta) + \dfrac{1}{F(\alpha, \beta)}\dfrac{\partial F}{\partial s}(\alpha, \beta)\right)(be_1(\alpha) + ae_2(\alpha)) = 0 \end{cases} \tag{6.12}$$

and f satisfies the equation

$$\mathrm{grad}_g(\ln f) = F(\alpha(x),\beta(r))\frac{a}{4}(be_3 - ce_2) + \frac{1}{4}|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{grad}_g(\alpha).$$

Proof. Initially, our task is to derive equations (5.1) and (5.2) presented in Theorem 5.3. With direct standard calculations and by using (5.3) and (6.2), we find

$$\begin{aligned}\mathrm{grad}_g(\ln f) &= F(\alpha(x),\beta(r))\frac{a}{4}(be_3 - ce_2) + \frac{1}{4}|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{grad}_g(\alpha)\\ &= \left(\frac{1}{4}|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))e_1(\alpha)\right)e_1\\ &\quad+\left(-F(\alpha(x),\beta(r))\frac{ac}{4}+\frac{1}{4}|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))e_2(\alpha)\right)e_2\\ &\quad+\left(F(\alpha(x),\beta(r))\frac{ab}{4}+\frac{1}{4}|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))e_3(\alpha)\right)e_3\end{aligned}$$

and

$$\begin{aligned}\nabla_{\mathrm{grad}_g(\ln f)}V = {}&\frac{1}{2F(\alpha(x),\beta(r))}\frac{\partial F}{\partial s}(\alpha(x),\beta(r))\big((be_3(\alpha)-ce_2(\alpha))e_1\\ &-(ce_1(\alpha)+ae_3(\alpha))e_2+(be_1(\alpha)+ae_2(\alpha))e_3\big).\end{aligned}\tag{6.13}$$

On the other hand, from (6.13), we have

$$\begin{aligned}&\nabla_{\mathrm{grad}_g(\ln f)}V\\ &=\frac{1}{8}\Bigg[\left(|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))(ce_2(\alpha)-be_3(\alpha))-aF(\alpha(x),\beta(r))(c^2+b^2)\right)e_1\\ &+\left(a^2bF(\alpha(x),\beta(r))+|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))(ce_1(\alpha)+ae_3(\alpha))\right)e_2\\ &+\left(a^2cF(\alpha(x),\beta(r))+|V|^2\frac{\partial F}{\partial t}(\alpha(x),\beta(r))(be_1(\alpha)+ae_2(\alpha))\right)e_3\Bigg].\end{aligned}\tag{6.14}$$

Hence, from (6.13) and (6.14), we obtain the system (6.12). □

From Theorem 6.5, we can state the following.

Corollary 6.6. *Let (H_3, g) be the 3-dimensional Heisenberg group with the left-invariant metric g given by (6.1) and let (TH_3, g_{SF}) be its tangent bundle equipped with the Mus–Sasaki metric. The Killing vector field $\zeta_1 = e_1$ is an f-harmonic vector field if and only if*

$$\begin{cases}(\frac{1}{4}\frac{\partial F}{\partial t}(\alpha,\beta)+\frac{1}{F(\alpha,\beta)}\frac{\partial F}{\partial s}(\alpha,\beta))e_3(\alpha)=0,\\ (\frac{1}{4}\frac{\partial F}{\partial t}(\alpha,\beta)+\frac{1}{F(\alpha,\beta)}\frac{\partial F}{\partial s}(\alpha,\beta))e_2(\alpha)=0\end{cases}$$

and f satisfies the following equation:

$$\mathrm{grad}_g(\ln f) = \frac{1}{4}\frac{\partial F}{\partial t}(\alpha(x),\beta(r))\,\mathrm{grad}_g(\alpha).$$

Example 6.7. Consider the Heisenberg group (H_3, g) with the left-invariant metric g given by (6.1). Let (TH_3, g_{SF}) be its tangent bundle equipped with the Mus–Sasaki metric, where $F(s,t) = 4\frac{t}{s}$. In this context, the Killing vector field $\zeta_1 = e_1$ is an f-harmonic vector field with $f(x,y,z) = \alpha(x,y,z)$.

Corollary 6.8. *Consider the* 3*-dimensional Heisenberg group* (H_3, g) *with the left-invariant metric* g *given by* (6.1)*. Let* (TH_3, g_{SF}) *be its tangent bundle equipped with the Mus–Sasaki metric, with* $\alpha(x,y,z) = h(bx+cy)$*, where* $h(t)$ *is an arbitrary smooth function and* b, c *are real constants. In this context, the vector field* $V = be_2 + ce_3$ *is an* f*-harmonic vector field from* (H_3, g) *into* (TH_3, g_{SF})*, where* f *satisfies the following equation:*

$$\operatorname{grad}_g(\ln f) = \frac{1}{4}|V|^2 \frac{\partial F}{\partial t}(\alpha(x), \beta(r))\big(e_2(\alpha)e_2 + e_3(\alpha)e_3\big).$$

Proof. The proof directly comes from Theorem 6.5 taking into account $a = 0$ and $\alpha(x,y,z) = h(bx+cy)$, where $h(t)$ is an arbitrary smooth function and b, c are real constants. □

The aforementioned corollary enables the construction of numerous examples.

Example 6.9. Consider the Heisenberg group (H_3, g) with the left-invariant metric g given by (6.1). Let (TH_3, g_{SF}) be its tangent bundle equipped with the Mus–Sasaki metric, where $F(s,t) = st$, $\beta = e^{-\frac{r}{|V|^2}}$, and $\alpha(x,y,z) = e^{h(bx+cy)}$, where $h(t)$ is an arbitrary smooth function. Then, for the vector field $V = be_2 + ce_3$, it is an f-harmonic vector field with $\ln f = \frac{b^2+c^2}{8}\exp(2h(bx+cy)) + k$, where k is a constant.

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Дослідження властивостей f-гармонічних векторних полів

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У цій статті ми ставимо за мету дослідити специфічні характеристики f-гармонічних векторних полів. По-перше, ми досліджуємо властивості f-гармонічного векторного поля Кіллінга, коли воно діє як f-гармонічне відображення між рімановим многовидом, позначеним як (M, g), та його дотичним пучком (TM, g_S), який має метрику Сасакі. Ми наголошуємо на тому, що (M, g) має вигляд або айнштайнівського многовиду, або просторової форми. По-друге, ми досліджуємо властивості f-гармонічного векторного поля між рімановим многовидом (M, g) та його дотичним пучком TM, який має або деформовану метрику Сасакі g_{DS}, або метрику Муса–Сасакі g_{SF}. Насамкінець ми завершуємо статтю розглядом прикладів f-гармонічних векторних полів у контексті групи Гайзенберґа.

Ключові слова: f-гармонічне векторне поле Кілінґа, многовид Айнштайна, дотичний пучок