

On Shimurian Generalizations of the Stack

$$\mathrm{BT}_1 \otimes \mathbb{F}_p$$

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Let G be a smooth group scheme over $\mathbb{F}_p$ equipped with a $\mathbb{G}_m$ -action such that all weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1 . Let Disp_n^G be Eike Lau’s stack of n -truncated G -displays (this is an algebraic $\mathbb{F}_p$ -stack). In the case $n = 1$ we introduce an algebraic stack equipped with a morphism to Disp_1^G . We conjecture that if $G = \mathrm{GL}(d)$ then the new stack is canonically isomorphic to the reduction modulo p of the stack of 1-truncated Barsotti–Tate groups of height d and dimension d' , where d' depends on the action of $\mathbb{G}_m$ on $\mathrm{GL}(d)$.

We also discuss how to define an analog of the new stack for $n > 1$ and how to replace $\mathbb{F}_p$ by $\mathbb{Z}/p^m\mathbb{Z}$.

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1. Introduction

Once and for all, we fix a prime p . The words “algebraic stack” are understood in the sense of [20] unless stated otherwise.

1.1. The goal. The notion of n -truncated Barsotti–Tate group was introduced by Grothendieck [13]. It is reviewed in [14, 15, 22]. n -truncated Barsotti–Tate groups of height d and dimension $d' \leq d$ form an algebraic stack over $\mathbb{Z}$, denoted by $\mathrm{BT}_n^{d,d'}$. This stack is smooth by a deep theorem of Grothendieck, whose proof is given in Illusie’s article [14].

The stack $\mathrm{BT}_n^{d,d'}$ is related to the group $G = \mathrm{GL}(d)$; e.g., $\mathrm{BT}_n^{d,d'} \otimes \mathbb{Z}[p^{-1}]$ is the classifying stack of $\mathrm{GL}(d, \mathbb{Z}/p^n\mathbb{Z})$. Since there exist Shimura varieties of type E_7 , it is natural to expect that the stacks $\mathrm{BT}_n^{d,d'} \otimes \mathbb{Z}/p^m\mathbb{Z}$ (and maybe the stacks $\mathrm{BT}_n^{d,d'}$ themselves) have interesting E_7 -analogs.

In the main body of this article we focus on the case $n = m = 1$. In this case we give a rather elementary definition of the stack $\mathrm{BT}_{1,\mathbb{F}_p}^G = \mathrm{BT}_1^G \otimes \mathbb{F}_p$, where G is any smooth affine group scheme over $\mathbb{F}_p$ equipped with a 1-bounded $\mathbb{G}_m$ -action (1-boundedness means that the weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1). Note that a semisimple group of type E_7 can be equipped with a nontrivial 1-bounded $\mathbb{G}_m$ -action.

In Appendix C we suggest a definition of $\mathrm{BT}_{n,\mathbb{Z}/p^m\mathbb{Z}}^G$ for arbitrary n and m , which uses the prismatic theory.¹

¹In the case $m = 1$ one only needs the crystalline theory, and in the case $n = m = 1$ the de Rham theory is enough. This is why in the case $m = n = 1$ an elementary approach is possible.

We prove that the stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$ defined in §4.2 is algebraic (this is not an immediate consequence of the definition). We conjecture that in the case $G = \mathrm{GL}(d)$ one has $\mathrm{BT}_{1,\mathbb{F}_p}^G = \mathrm{BT}_1^{d,d'} \otimes \mathbb{F}_p$, where d' depends on the action of $\mathbb{G}_m$ on $\mathrm{GL}(d)$. The precise formulations of this result and conjecture are given in §4.4-4.5.

1.2. Structure of the article. In §2 we recall some facts about group schemes equipped with a $\mathbb{G}_m$ -action. We also introduce a (2,1)-category $\mathbf{Shim}_n$, whose objects are smooth affine group schemes over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a 1-bounded $\mathbb{G}_m$ -action.

In §3 we recall a certain $\mathbb{F}_p$ -stack Disp_1^G , $G \in \mathbf{Shim}_1$, which is called the stack of 1-truncated displays (or the stack of F -zips). We also introduce in §3.3 a commutative group scheme over Disp_1^G denoted by Lau_1^G .

In §4 we define an $\mathbb{F}_p$ -stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which depends functorially on $G \in \mathbf{Shim}_1$. We also formulate the main theorems and a conjecture, see §4.4-4.5.

The proof of the main theorems is given in §5-6.

In Appendices A-B we discuss some material on algebraic stacks.

In Appendix C we suggest a definition of $\mathrm{BT}_{n,\mathbb{Z}/p^m\mathbb{Z}}^G$ for arbitrary n and m , which uses the prismatic theory. We also formulate Conjecture C.3.1; it is motivated by Corollary 4.4.3 and by Grothendieck's smoothness theorem mentioned at the beginning of §1.1.

In Appendix D we explain why the definition of $\mathrm{BT}_{1,\mathbb{F}_p}^G$ given in Appendix C is equivalent to the one from §4.

1.3. Status of the conjectures. As far as I understand, both conjectures formulated in this article are proved in [12].

2. The (2,1)-category $\mathbf{Shim}_n$

2.1. 1-bounded $\mathbb{G}_m$ -actions. Let G be a smooth group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathfrak{g} := \mathrm{Lie}(G)$. An action of $\mathbb{G}_m$ on G induces its action on $\mathfrak{g}$ and therefore a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i. \quad (2.1)$$

Following E. Lau [19], we say that an action of $\mathbb{G}_m$ on G is *1-bounded* if $\mathfrak{g}_i = 0$ for $i > 1$.

Example 2.1.1. If G is reductive and connected then an action of $\mathbb{G}_m$ on G is given by a cocharacter $\mu : \mathbb{G}_m \rightarrow G_{\mathrm{ad}}$. In this case 1-boundedness implies that all weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ belong to $\{-1, 0, 1\}$. Cocharacters μ with this property are called *minuscule*².

²This notion of minuscule cocharacter is slightly more general than that of Bourbaki (e.g., Bourbaki requires a minuscule cocharacter to be nonzero).

2.2. The (2,1)-category $\mathbf{Shim}_n$. Let $\mathbf{Shim}_n$ be the following (2,1)-category: its objects are smooth affine group schemes over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a 1-bounded $\mathbb{G}_m$ -action, and for $G_1, G_2 \in \text{Ob } \mathbf{Shim}_n$, the groupoid of morphisms $G_1 \rightarrow G_2$ is the quotient groupoid³ of $\text{Hom}^{\mathbb{G}_m}(G_1, G_2)$ by the conjugation action of $G_2^{\mathbb{G}_m}(\mathbb{Z}/p^n\mathbb{Z})$. Here $\text{Hom}^{\mathbb{G}_m}$ is the set of $\mathbb{G}_m$ -equivariant homomorphisms, and $G_2^{\mathbb{G}_m} \subset G_2$ is the group subscheme of $\mathbb{G}_m$ -fixed points.

The above (2,1)-category $\mathbf{Shim}_n$ is not quite good from a certain viewpoint⁴, but we work with it in this article.

2.3. The subgroups $M, P^\pm, U^\pm$

2.3.1. Recollections from [8]. Let k be a ring and G a smooth affine group k -scheme equipped with a $\mathbb{G}_m$ -action, i.e., a homomorphism $\mu : \mathbb{G}_m \rightarrow \text{Aut } G$. Let $M := G^{\mathbb{G}_m}$, i.e., $M \subset G$ is the subgroup of $\mathbb{G}_m$ -fixed points. Let P^+ be the attractor for μ , i.e., $P^+ \subset G$ is the maximal closed subscheme such that the action map $\mathbb{G}_m \times P^+ \rightarrow G$ extends to a morphism of schemes $\mathbb{A}^1 \times P^+ \rightarrow G$. Restricting the latter to $\{0\} \times P^+ \subset \mathbb{A}^1 \times P^+$, we get a retraction $P^+ \rightarrow M \subset P^+$. Let $P^- \subset G$ be the attractor for μ^{-1} ; we have a retraction $P^- \rightarrow M \subset P^-$. Let $U^\pm$ be the preimage of 1 with respect to the map $P^\pm \rightarrow M$. The next lemma is well known if G is reductive and connected (in this case P^+ and P^- are parabolics, and M is a Levi).

Lemma 2.3.2.

- (i) $P^\pm$ and $U^\pm$ are group subschemes of G . The maps $P^\pm \rightarrow M$ are group homomorphisms. One has $P^\pm = M \ltimes U^\pm$.
- (ii) $P^+ \cap P^- = M$.
- (iii) $P^\pm, U^\pm$, and M are smooth. In terms of the grading (2.1), one has

$$\begin{aligned} \text{Lie}(M) &= \mathfrak{g}_0, & \text{Lie}(P^+) &= \mathfrak{g}_{\geq 0}, & \text{Lie}(P^-) &= \mathfrak{g}_{\leq 0}, \\ \text{Lie}(U^+) &= \mathfrak{g}_{> 0}, & \text{Lie}(U^-) &= \mathfrak{g}_{< 0}, \end{aligned}$$

where $\mathfrak{g}_{\geq 0} := \bigoplus_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{g}_{\leq 0}, \mathfrak{g}_{> 0}, \mathfrak{g}_{< 0}$ are defined similarly.

- (iv) The multiplication map $U^- \times M \times U^+ \rightarrow G$ is an open immersion.
- (v) The fibers of $U^\pm$ over $\text{Spec } k$ are connected. If the fibers of G over $\text{Spec } k$ are connected then the same holds for $P^\pm$ and M .

Proof. Apply the results from [8, p. 48-56] (especially [8, Prop. 2.1.8]) to the semidirect product $\mathbb{G}_m \ltimes G$. $\square$

2.3.3. The 1-bounded case. Now assume that $\mathfrak{g}_i = 0$ for $i > 1$. Then $\mathfrak{g}_1$ is abelian. Moreover, by [19, Lemma 6.3.2], there is a unique $\mathbb{G}_m$ -equivariant isomorphism of group schemes

$$f : U^+ \xrightarrow{\sim} \mathfrak{g}_1 \tag{2.2}$$

³If a group Γ acts on a set X then the *quotient groupoid* is defined as follows: the set of objects is X , a morphism $x \rightarrow x'$ is an element $\gamma \in \Gamma$ such that $\gamma x = x'$, and the composition of morphisms is given by multiplication in G .

⁴See §9.1.4 of version 1 of [12].

such that $\mathrm{Lie}(f) = \mathrm{id}_{\mathfrak{g}_1}$. If the ring k is local then the projective k -module $\mathfrak{g}_1$ is free, so the isomorphism (2.2) implies that U^+ is isomorphic to a product of several copies of $\mathbb{G}_a$.

If k is an $\mathbb{F}_p$ -algebra then $\mathfrak{g}$ is equipped with a p -operation $x \mapsto x^{(p)}$. By (2.2), if $\mathfrak{g}$ is 1-bounded then

$$x^{(p)} = 0 \text{ for } x \in \mathfrak{g}_1. \quad (2.3)$$

Another way to prove (2.3) is to note that more generally, the p -operation takes $\mathfrak{g}_i$ to $\mathfrak{g}_{pi}$. This follows from the fact that for any k -algebra $\tilde{k}$ and any $a \in \tilde{k}^\times$ the action of a on $\mathfrak{g} \otimes_k \tilde{k}$ preserves the structure of restricted Lie algebra on $\mathfrak{g} \otimes_k \tilde{k}$.

3. Recollections on the stacks Disp_1^G

3.1. Plan of this section. For any $n \in \mathbb{N}$ and a smooth group scheme G over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a $\mathbb{G}_m$ -action, one has the $\mathbb{F}_p$ -stack of n -truncated displays Disp_n^G defined⁵ in [19].

In §3.2 we recall the stack Disp_1^G . Assuming that $G \in \mathbf{Shim}_1$, we define in §3.3 a commutative group scheme on Disp_1^G ; we denote it by Lau_1^G (because it is conjecturally related to E. Lau's work [18], see §4.5 below).

The stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which will be defined in §4 assuming that $G \in \mathbf{Shim}_1$, will be equipped with a map to Disp_1^G , which makes it into a gerbe over Disp_1^G banded by Lau_1^G .

Let us make some historical remarks related to Disp_1^G . First, as noted in Example 3.7.5 of [19], Disp_1^G is the same as the stack of F -zips in the sense of [23, 25]. Second, the projective limit of Disp_n^G is the reduction modulo p of a stack over $\mathrm{Spf} \mathbb{Z}_p$, which was defined in [5] (at least, assuming that G is reductive and the cocharacter $\mu : \mathbb{G}_m \rightarrow G_{\mathrm{ad}}$ is minuscule) following the ideas of Thomas Zink. The articles [5, 23, 25] preceded [19].

3.2. The $\mathbb{F}_p$ -stack Disp_1^G .

3.2.1. Definition of the stack Disp_1^G . Let G be a smooth group scheme over $\mathbb{F}_p$ equipped with a $\mathbb{G}_m$ -action. Let M and $P^\pm$ be as in §2.3. Let S be an $\mathbb{F}_p$ -scheme. Then $\mathrm{Disp}_1^G(S)$ is the groupoid of the following data:

- (i) a $P^\pm$ -torsor $\mathcal{F}^\pm$ on S ;
- (ii) an isomorphism between the G -torsors corresponding to $\mathcal{F}^+$ and $\mathcal{F}^-$;
- (iii) an isomorphism $\mathcal{F}_M^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}_M^-$, where $\mathcal{F}_M^\pm$ is the M -torsor corresponding to $\mathcal{F}^\pm$.

3.2.2. Disp_1^G as a quotient stack. As noted in [25], the stack Disp_1^G has an explicit realization as a quotient, which shows that Disp_1^G is a quasi-compact smooth algebraic stack over $\mathbb{F}_p$ of pure dimension 0 with affine diagonal. To get this realization, note that the combination of data (i) and (iii) is the same as a principal K -bundle $\mathcal{E} \rightarrow S$, where $K \subset G \times G$ is the following subgroup:

$$K := \{(g, h) \in P_+ \times P_- \mid g_M = \mathrm{Fr}(h_M)\}, \quad (3.1)$$

⁵We are talking about the particular case of a general construction of [19] corresponding to the “ n -truncated Witt frame” in the sense of [19, Example 2.1.6].

(here g_M, h_M are the images of g, h in M). Moreover, data (ii) is the same as a K -equivariant morphism $\mathcal{E} \rightarrow G$, where $(g, h) \in K$ acts on G by

$$x \mapsto hxg^{-1}. \quad (3.2)$$

Thus Disp_1^G identifies with the quotient of the scheme G by the action of K given by (3.2).

3.2.3. The generic locus of Disp_1^G . By (3.1), the stabilizer of $1 \in G$ under the K -action given by (3.2) equals the finite group $M(\mathbb{F}_p)$. Moreover, the K -orbit of 1 is open.

By §3.2.2, Disp_1^G is a quotient of the scheme G . The composite morphism

$$\mathrm{Spec} \mathbb{F}_p \xrightarrow{1} G \twoheadrightarrow \mathrm{Disp}_1^G \quad (3.3)$$

will be called the *generic point* of the stack Disp_1^G . The group scheme of automorphisms of this point equals $M(\mathbb{F}_p)$, and the image of (3.3) is an open substack of Disp_1^G , which identifies with the classifying stack of $M(\mathbb{F}_p)$. This open substack will be called the *generic locus* of Disp_1^G and denoted by $\mathrm{Disp}_{1,\mathrm{gen}}^G$. The word “generic” is justified at least if G is connected (which implies that Disp_1^G is irreducible).

3.2.4. Motivation. A pair $(\mathcal{F}^+, \mathcal{F}^-)$ equipped with data (ii) from §3.2.1 is the same as a G -torsor $\mathcal{F}$ equipped with a P^+ -structure and a P^- -structure. If $G = GL(d)$ this means that the vector bundle ξ corresponding to $\mathcal{F}$ is equipped with two filtrations. As explained at the beginning of [23], the typical example is when ξ is the m -th relative de Rham cohomology of a smooth proper scheme X over S satisfying certain conditions⁶. In this situation the P^- -structure corresponds to the Hodge filtration, and the P^+ -structure corresponds to the conjugate one.

Note that the Gauss–Manin connection on ξ is not included into the definition of Disp_1^G . However, it is included into the definition of $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which will be given in §4.2.

3.2.5. Reductive case. If G is reductive it is known that the set of points of Disp_1^G is finite; moreover, Theorem 1.6 of [25] gives a complete description of this set (together with the topology on it), and Theorem 1.7 of [25] describes the automorphism group scheme of each point.

3.3. The group scheme Lau_1^G

3.3.1. Definition of Lau_1^G . From now on, we assume⁷ that $G \in \mathbf{Shim}_1$ (i.e., the action of $\mathbb{G}_m$ on G is 1-bounded). Given an $\mathbb{F}_p$ -scheme S and an S -point of Disp_1^G , we will construct a group scheme over S . Let us use the notation

⁶The conditions for the smooth proper morphism $f : X \rightarrow S$ are as follows: the sheaves $R^b f_* \Omega_{X/S}^a$ should be locally free, and the Hodge–de Rham spectral sequence for f should degenerate at the E_1 page.

⁷Without this assumption, the definition given below formally makes sense (but is probably useless) if one replaces $\mathfrak{g}_1$ by $\mathfrak{g}_{>0}$.

of §3.2.1. Note that both P^+ and P^- act on $\mathfrak{g}_1$ via M ; we have a P^+ -equivariant monomorphism $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ and a P^- -equivariant epimorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{g}_{\leq 0} = \mathfrak{g}_1$. Let $(\mathfrak{g}_1)_{\mathcal{F}^\pm}$ be the vector bundle on S corresponding to $\mathcal{F}^\pm$ and the $P^\pm$ -module $\mathfrak{g}_1$. We have the diagram

$$\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+} \hookrightarrow \mathfrak{g}_{\mathcal{F}^+} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{F}^-} \twoheadrightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \quad (3.4)$$

in which the first isomorphism comes from data (iii) of §3.2.1 and the second one from (ii). Consider $(\mathfrak{g}_1)_{\mathcal{F}^-}$ as a commutative restricted Lie $\mathcal{O}_S$ -algebra with p -operation⁸ (3.4). This restricted Lie $\mathcal{O}_S$ -algebra yields a finite locally free group S -scheme of height 1 in the usual way (see [26, Exp.VIIA] or [15, §2]). We denote it by $\mathrm{Lau}_{1,S}^G$. The formation of $\mathrm{Lau}_{1,S}^G$ commutes with base change $S' \rightarrow S$. As S varies, the group schemes $\mathrm{Lau}_{1,S}^G$ define a commutative finite flat group scheme over Disp_1^G of height 1, which we denote by Lau_1^G .

The stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which will be defined in §4, will turn out to be a gerbe over Disp_1^G banded by Lau_1^G , see Theorem 4.4.2.

3.3.2. Restriction of Lau_1^G to the generic locus of Disp_1^G . Let $\mathrm{Lau}_{1,\mathrm{gen}}^G$ be the fiber of Lau_1^G over the generic point of Disp_1^G in the sense of §3.2.3, so $\mathrm{Lau}_{1,\mathrm{gen}}^G$ is a finite group scheme over $\mathbb{F}_p$ equipped with an action of $M(\mathbb{F}_p)$. Then one has a canonical $M(\mathbb{F}_p)$ -equivariant isomorphism

$$\mathrm{Lau}_{1,\mathrm{gen}}^G \xrightarrow{\sim} \mathfrak{g}_1 \otimes_{\mathbb{F}_p} \mu_p \quad (3.5)$$

(indeed, in the case of the generic point of Disp_1^G the torsors $\mathcal{F}^\pm$ are canonically trivial, and the composite map (3.4) is $\mathrm{id}_{\mathfrak{g}_1}$). Less canonically, $\mathrm{Lau}_{1,\mathrm{gen}}^G$ is a direct sum of $\dim \mathfrak{g}_1$ copies of μ_p .

4. Definition of $\mathrm{BT}_{1,\mathbb{F}_p}^G$

4.1. General remarks on smooth algebraic stacks

4.1.1. An easy lemma. Let $\mathrm{Sch}_{\mathbb{F}_p}$ be the category of $\mathbb{F}_p$ -schemes and $\mathrm{Sm}_{\mathbb{F}_p}$ the full subcategory of smooth $\mathbb{F}_p$ -schemes; unless specified otherwise, we will specify each of these categories with the étale topology. Let Grpds be the (2,1)-category of groupoids.

Let $\mathcal{X}$ be an $\mathbb{F}_p$ -stack, i.e., a functor $\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \mathrm{Grpds}$ satisfying the sheaf property. Then the restriction of $\mathcal{X} : \mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \mathrm{Grpds}$ to $\mathrm{Sm}_{\mathbb{F}_p}^{\mathrm{op}}$ will be denoted by $\mathcal{X}_{\mathrm{Sm}}$. The following variant of Yoneda's lemma is easy and well known.

Lemma 4.1.2. *The functor $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{Sm}}$ becomes fully faithful when restricted to the (2,1)-category of **smooth algebraic stacks** over $\mathbb{F}_p$.*

This variant of Yoneda's lemma is probably well known. For completeness, we give a proof of a more general statement in Proposition A.0.1 of Appendix A.

⁸This p -operation is *different* from the one induced by the p -operation on $\mathfrak{g}_1$; the latter is zero by formula (2.3).

Remark 4.1.3. One can ask how to reconstruct a smooth algebraic stack $\mathcal{X}$ over $\mathbb{F}_p$ in terms of $\mathcal{X}_{\text{Sm}}$. Abstract nonsense gives the following “answer” (see Proposition A.0.2 and the sentence after it): $\mathcal{X}$ is the sheafified left Kan extension of $\mathcal{X}_{\text{Sm}} : \text{Sm}_{\mathbb{F}_p}^{\text{op}} \rightarrow \text{Grpds}$ to $\text{Sch}_{\mathbb{F}_p}^{\text{op}}$.

4.1.4. Example. In this article we will be describing smooth algebraic $\mathbb{F}_p$ -stacks by specifying their restriction to $\text{Sm}_{\mathbb{F}_p}^{\text{op}}$. Here is a baby example.

Let $\mathcal{X}$ be the classifying stack of the group scheme

$$\alpha_p := \text{Ker}(\text{Fr} : \mathbb{G}_a \rightarrow \mathbb{G}_a), \text{ where } \mathbb{G}_a := (\mathbb{G}_a)_{\mathbb{F}_p}; \quad (4.1)$$

in other words, $\mathcal{X}(S)$ is the groupoid of α_p -torsors on S for the fppf topology. The stack $\mathcal{X}$ is smooth: indeed, the map

$$\mathbb{G}_a \times \mathbb{A}^1 \rightarrow \mathbb{A}^1, \quad (a, x) \mapsto x + a^p.$$

defines an action of $\mathbb{G}_a$ on $\mathbb{A}^1$ such that the quotient stack is $\mathcal{X}$. We claim that $\mathcal{X}_{\text{Sm}}$ is the following sheaf of sets (rather than groupoids):

$$S \mapsto H^0(S, \Omega_{S, \text{exact}}^1), \text{ where } S \in \text{Sm}_{\mathbb{F}_p} \text{ and } \Omega_{S, \text{exact}}^1 := \text{Im}(\mathcal{O}_{S, \text{et}} \xrightarrow{d} \Omega_{S, \text{et}}^1).$$

Indeed, by (4.1), the derived direct image of α_p under the morphism $S_{\text{fppf}} \rightarrow S_{\text{et}}$ equals $\mathcal{A}[-1]$, where $\mathcal{A} := \text{Cone}(\text{Fr} : \mathcal{O}_{S, \text{et}} \rightarrow \mathcal{O}_{S, \text{et}})$. Finally, smoothness of S implies that $\mathcal{A} = \Omega_{S, \text{exact}}^1$.

4.2. Definition of $\text{BT}_1^G(S)$, where $S \in \text{Sm}_{\mathbb{F}_p}$. We will use the notion of p -curvature of an integrable connection (see [16, §5]). We say that a connection is p -integrable if it is integrable and its p -curvature is zero.

4.2.1. Definition. Let $G \in \mathbf{Shim}_1$. Let M and $P^\pm$ be as in §2.3. Let $S \in \text{Sm}_{\mathbb{F}_p}$. Then $\text{BT}_1^G(S)$ is the groupoid of the following data:

- (i) a $P^\pm$ -torsor $\mathcal{F}^\pm$ on S ;
- (ii) an isomorphism $\mathcal{F}_G^+ \xrightarrow{\sim} \mathcal{F}_G^-$, where $\mathcal{F}_G^\pm$ is the G -torsor corresponding to $\mathcal{F}^\pm$;
- (iii) an isomorphism

$$\mathcal{F}_M^+ \xrightarrow{\sim} \text{Fr}_S^* \mathcal{F}_M^-, \quad (4.2)$$

where $\mathcal{F}_M^\pm$ is the M -torsor corresponding to $\mathcal{F}^\pm$;

- (iv) an integrable connection ∇ on $\mathcal{F}^+$ satisfying the following conditions: first, the corresponding connection on $\mathcal{F}_M^+$ should equal the one that comes from (4.2) and the usual connection on a Fr_S -pullback; second, the following *Katz condition* should hold:

$$\text{p-Curv}_\nabla = -\text{KS}_\nabla, \quad (4.3)$$

where p-Curv_∇ is the p -curvature of ∇ and $\text{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1)$ is the *Kodaira–Spencer*⁹ 1-form defined in §4.2.2 below.

⁹The terminology is motivated by the picture from §3.2.4. One can think of data (i)–(ii) as a G -bundle on S equipped with a P^+ -structure and a P^- -structure. Informally, these are the conjugate filtration and Hodge filtration, respectively. Also informally, we think of ∇ as a Gauss–Manin connection.

Let us explain why (4.3) makes sense. The right-hand side of (4.3) is a section of the sheaf $(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1$. By the first condition from (iv), the connection on $\mathcal{F}_M^+$ induced by ∇ is p -integrable, so $\mathrm{p}\text{-Curv}_{\nabla}$ is a section of $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^{\nabla} \otimes \Omega_S^1$, where $\mathrm{Fr} := \mathrm{Fr}_S$ and $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^{\nabla}$ is the horizontal part of $\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+}$. But (4.2) induces an isomorphism $\mathrm{Fr}^*(\mathfrak{g}_1)_{\mathcal{F}^-} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+}$, so $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^{\nabla} = (\mathfrak{g}_1)_{\mathcal{F}^-}$ and (4.3) makes sense.

4.2.2. The Kodaira–Spencer 1-form. By §4.2.1(ii), $\mathcal{F}^+$ and $\mathcal{F}^-$ induce the same G -torsor, which we denote by $\mathcal{F}$. The connection ∇ on $\mathcal{F}^+$ induces a connection ∇_G on $\mathcal{F}$, and KS_{∇} “measures” the failure of ∇_G to preserve the P^- -structure on $\mathcal{F}$. More precisely, ∇_G is a section of the Atiyah extension¹⁰

$$0 \rightarrow \mathfrak{g}_{\mathcal{F}} \rightarrow \mathcal{A} \rightarrow \Theta_S \rightarrow 0, \quad \Theta_S := (\Omega_S^1)^*, \quad (4.4)$$

and $\mathrm{KS}_{\nabla} : \Theta_S \rightarrow (\mathfrak{g}/\mathfrak{g}_{\leq 0})_{\mathcal{F}^-}$ is the composition of $\nabla_G : \Theta_S \rightarrow \mathcal{A}$ and the map

$$\mathcal{A} \rightarrow (\mathfrak{g}/\mathrm{Lie}(P^-))_{\mathcal{F}^-} = (\mathfrak{g}/\mathfrak{g}_{\leq 0})_{\mathcal{F}^-}$$

that comes from the P^- -structure on $\mathcal{F}$. Note that $\mathfrak{g}/\mathfrak{g}_{\leq 0} = \mathfrak{g}_1$ by the 1-boundedness assumption.

4.2.3. Remark. The above definition of $\mathrm{BT}_1^G(S)$ makes sense without the 1-boundedness assumption if (4.3) is understood as an equality in $H^0(S, (\mathfrak{g}/\mathfrak{g}_{\leq 0})_{\mathcal{F}^-} \otimes \Omega_S^1)$; this equality implies the Griffiths transversality condition $\mathrm{KS}_{\nabla} \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1)$ because $\mathrm{p}\text{-Curv}_{\nabla}$ is in $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1)$. However, the 1-boundedness assumption is used in the proof of the theorems formulated in §4.4 below.

4.2.4. On the Katz condition. Condition (4.3) is inspired by Theorem 3.2 of [17]. As far as I understand, the minus sign in (4.3) does not agree¹¹ with [17, Thm. 3.2], but it agrees with Remark 3.20 of [24], which explains a modern point of view on Theorem 3.2 of [17]. (In [24, Remark 3.20] the sign appears when one computes explicitly the functor $C_{Y/S}^{\bullet}$, which occurs in the l.h.s of (3.19.2) and (3.19.3).)

4.2.5. Remark. Forgetting the connection ∇ from §4.2.1(iv), one gets for each $S \in \mathrm{Sm}_{\mathbb{F}_p}$ a functor $\mathrm{BT}_1^G(S) \rightarrow \mathrm{Disp}_1^G(S)$, where Disp_1^G is as in §3.2.1.

4.3. Functoriality in $G \in \mathbf{Shim}_1$. It is clear that the functor

$$\mathrm{Sm}_{\mathbb{F}_p}^{\mathrm{op}} \rightarrow \mathrm{Grpds}, \quad S \mapsto \mathrm{BT}_1^G(S). \quad (4.5)$$

from §4.2.1 depends functorially on $G \in \mathbf{Shim}_1$, where $\mathbf{Shim}_1$ is the (2,1)-category from §2.2.

¹⁰Recall that in terms of the principal G -bundle $E \rightarrow S$ corresponding to $\mathcal{F}$, the sheaf $\mathcal{A}$ from (4.4) is the sheaf of G -equivariant vector fields on E .

¹¹Theorem 3.2 of [17] involves $(-1)^{b+1}$, where b is the number of the cohomology group. The case relevant for us is $b = 1$.

4.4. Formulation of the main theorems

Theorem 4.4.1. *There exists a smooth algebraic stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$ over $\mathbb{F}_p$ whose restriction to $\mathrm{Sm}_{\mathbb{F}_p}^{\mathrm{op}}$ is the functor (4.5).*

By Lemma 4.1.2, $\mathrm{BT}_{1,\mathbb{F}_p}^G$ is unique. Combining §4.2.5 and Lemma 4.1.2, we get a morphism

$$\mathrm{BT}_{1,\mathbb{F}_p}^G \rightarrow \mathrm{Disp}_1^G \quad (4.6)$$

Theorem 4.4.2. *The morphism (4.6) is an fppf gerbe banded by the group scheme Lau_1^G from §3.3.1.*

Theorems 4.4.1 and 4.4.2 will be proved in §6.3.

Corollary 4.4.3. *$\mathrm{BT}_{1,\mathbb{F}_p}^G$ is a quasi-compact smooth algebraic stack over $\mathbb{F}_p$ of pure dimension 0 with affine diagonal.*

Proof. By Theorems 4.4.1–4.4.2, this follows from similar properties of Disp_1^G (see §3.2.2). $\square$

4.4.4. Remark. As explained in §B.0.2 of Appendix B, Theorem 4.4.2 implies that the morphism (4.6) is smooth.

4.4.5. A simple example. Let G be $\mathbb{G}_a$ equipped with the usual action of $\mathbb{G}_m$. In this case $\mathrm{Disp}_1^G = \mathrm{Spec} \mathbb{F}_p$ and $\mathrm{Lau}_1^G = \mu_p$. Since $H_{\mathrm{fppf}}^i(\mathrm{Spec} \mathbb{F}_p, \mu_p) = 0$ for all i , Theorem 4.4.2 implies that $\mathrm{BT}_{1,\mathbb{F}_p}^G$ is canonically isomorphic to the classifying stack of μ_p over $\mathbb{F}_p$.

4.5. A conjecture about $\mathrm{BT}_{1,\mathbb{F}_p}^G$ in the case $G = GL(d)$

4.5.1. Some results of E. Lau. Let d, d' be integers such that $0 \leq d' \leq d$. Let $n \in \mathbb{N}$. Let G be the group scheme $GL(d)$ equipped with the $\mathbb{G}_m$ -action corresponding to the composite map

$$\mathbb{G}_m \rightarrow GL(d) \rightarrow PGL(d), \quad (4.7)$$

where the first map takes t to a diagonal matrix with d' diagonal entries equal to t and $d - d'$ diagonal entries equal to 1.

Let $\mathrm{BT}_n^{d,d'}$ be the stack of n -truncated Barsotti-Tate groups of height d and dimension d' . Using a covariant version of Dieudonné theory¹², E. Lau defined in [18] a canonical morphism $\mathrm{BT}_n^{d,d'} \otimes_{\mathbb{F}_p} \rightarrow \mathrm{Disp}_n^G$. Moreover, according to Theorem B of [18], this morphism is a gerbe banded by a commutative locally free finite group scheme over Disp_n^G . We denote this group scheme by $\mathrm{Lau}_n^{G,\mathrm{true}}$.

According to [18], the group scheme $\mathrm{Lau}_n^{G,\mathrm{true}}$ is infinitesimal and has order $p^{nd'(d-d')}$ (see Theorem B of [18] and the paragraph after it). Moreover, Remark 4.8 of [18] describes the restriction of $\mathrm{Lau}_n^{G,\mathrm{true}}$ to the generic locus of Disp_n^G ; in the case $n = 1$, it is canonically isomorphic to the restriction of Lau_1^G (which was described in §3.3.2).

¹²Those who prefer *contravariant* Dieudonné theory should replace d' by $d - d'$ in the definition of (4.7).

Proposition 4.5.2. *The isomorphism between the restrictions of $\mathrm{Lau}_1^{G, \text{true}}$ and Lau_1^G to the generic locus of Disp_1^G extends¹³ to an isomorphism over the whole Disp_1^G .*

Proof. Follows from [10, Thm. 4.4.2(ii)] and [10, § 9.2.2]. $\square$

Combining Proposition 4.5.2 with Theorem 4.4.1, we see that $\mathrm{BT}_{1, \mathbb{F}_p}^G$ and $\mathrm{BT}_1^{d, d'} \otimes \mathbb{F}_p$ are gerbes over Disp_1^G banded by the same group scheme Lau_1^G .

Conjecture 4.5.3. *These two gerbes are isomorphic.*

As far as I understand, Conjecture 4.5.3 has already been proved in [12].

5. Analyzing the conditions for the connection ∇

The definition of $\mathrm{BT}_1^G(S)$ from §4.2.1 involves a connection ∇ , which has to satisfy certain conditions. In this section we show that these conditions are affine- $\mathbb{F}_p$ -linear; for a precise statement, see Theorem 5.2.6 below. In §6 we will deduce Theorems 4.4.1 and 4.4.2 from Theorem 5.2.6.

5.1. The sheaves $\mathcal{T}$ and $\mathcal{T}'$

5.1.1. The category $\mathrm{Sm} / \mathrm{Disp}_1^G$. Let $\mathrm{Sm} / \mathrm{Disp}_1^G$ be the category of pairs (S, f) , where $S \in \mathrm{Sm}_{\mathbb{F}_p}$ and $f \in \mathrm{Disp}_1^G(S)$. We equip $\mathrm{Sm} / \mathrm{Disp}_1^G$ with the étale topology.

5.1.2. The sheaf $\mathcal{T}$. Let $(S, f) \in \mathrm{Sm} / \mathrm{Disp}_1^G$, so $f \in \mathrm{Disp}_1^G(S)$ is given by data (i)–(iii) from §4.2.1. We will use the notation of §4.2.1 for these data (i.e., $\mathcal{F}^\pm$, $\mathcal{F}_M^\pm$, etc.). Let $\mathcal{T}(S, f)$ be the fiber of the functor $\mathrm{BT}_1^G(S) \rightarrow \mathrm{Disp}_1^G(S)$ over $f \in \mathrm{Disp}_1^G(S)$. This fiber is a set rather than a groupoid; namely, $\mathcal{T}(S, f)$ is the set of connections ∇ on the P^+ -bundle $\mathcal{F}^+$ satisfying certain conditions. The conditions are as follows:

- (a) the connection on $\mathcal{F}_M^+$ induced by ∇ is equal to the one that comes from the isomorphism $\mathcal{F}_M^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}_M^-$;
- (b) ∇ is integrable;
- (c) ∇ satisfies the Katz condition, i.e.,

$$p\text{-Curv}_\nabla = -\mathrm{KS}_\nabla, \quad (5.1)$$

where $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1)$ is the Kodaira–Spencer 1-form defined in §4.2.2 and $p\text{-Curv}_\nabla \in H^0(S, (\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^\nabla \otimes \Omega_S^1) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1)$ is the p -curvature of ∇ (see §4.2.1 for details).

The assignment $(S, f) \mapsto \mathcal{T}(S, f)$ is a sheaf on $\mathrm{Sm} / \mathrm{Disp}_1^G$ (with respect to the étale topology).

¹³Such an extension is unique. Indeed, by §3.2.2, Disp_1^G is a quotient of the reduced irreducible scheme $G = \mathrm{GL}(n)$.

5.1.3. The sheaf $\mathcal{T}' \supset \mathcal{T}$. We keep the notation of §5.1.2. Let $\mathcal{T}'(S, f)$ be the set of connections ∇ on $\mathcal{F}^+$ satisfying condition (a) from §5.1.2. Then $\mathcal{T}'$ is a sheaf of sets on $\mathrm{Sm} / \mathrm{Disp}_1^G$, and $\mathcal{T} \subset \mathcal{T}'$.

5.2. $\mathcal{T}$ and $\mathcal{T}'$ as torsors

5.2.1. The sheaf $\mathcal{A}'$. Define a sheaf of $\mathbb{F}_p$ -vector spaces $\mathcal{A}'$ on $\mathrm{Sm} / \mathrm{Disp}_1^G$ as follows:

$$\mathcal{A}'(S, f) := H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1). \quad (5.2)$$

Then $\mathcal{A}'$ acts on $\mathcal{T}'$ in the usual way (adding a 1-form to a connection). Moreover, $\mathcal{T}'$ is an $\mathcal{A}'$ -torsor.

Using the isomorphism $\mathcal{F}^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}^-$ (which is a part of the data), we can rewrite (5.2) as

$$\mathcal{A}'(S, f) = H^0(S, \mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1). \quad (5.3)$$

Note that $\mathrm{Fr}_S : S \rightarrow S$ induces the identity on the underlying set of S , so $(\mathrm{Fr}_S)_* \Omega_S^1$ is just the sheaf Ω_S^1 equipped with the Frobenius-twisted $\mathcal{O}_S$ -action.

5.2.2. The next goals. In §5.2.4 we will define a subsheaf of $\mathbb{F}_p$ -vector spaces $\mathcal{A} \subset \mathcal{A}'$. Then we will formulate Theorem 5.2.6, which says that $\mathcal{T}$ is an $\mathcal{A}$ -torsor.

5.2.3. The maps φ , $\tilde{\varphi}$, C , and $\tilde{C}$. Let $\varphi : (\mathfrak{g}_1)_{\mathcal{F}^+} \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-}$ be the p -linear map corresponding to the $\mathcal{O}_S$ -linear map $\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^+} \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-}$ from formula (3.4). The latter is the composition

$$\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^+} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+} \hookrightarrow \mathfrak{g}_{\mathcal{F}^+} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{F}^-} \twoheadrightarrow (\mathfrak{g}_1)_{\mathcal{F}^-}.$$

The map φ induces a p -linear map

$$\tilde{\varphi} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1 \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1;$$

namely, $\tilde{\varphi}$ is the tensor product of $\varphi : (\mathfrak{g}_1)_{\mathcal{F}^+} \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-}$ and the identity¹⁴ map

$$(\mathrm{Fr}_S)_* \Omega_S^1 \rightarrow \Omega_S^1$$

(the latter is p -linear).

We have the $\mathcal{O}_S$ -submodule $((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \subset (\mathrm{Fr}_S)_* \Omega_S^1$ and the Cartier operator¹⁵

$$C : ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \rightarrow \Omega_S^1,$$

which is $\mathcal{O}_S$ -linear and surjective. C induces a surjective $\mathcal{O}_S$ -linear map

$$\tilde{C} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1, \quad \tilde{C} := \mathrm{id} \otimes C$$

¹⁴See the end of §5.2.1.

¹⁵In Lemma 5.4.2 we will recall the explicit description of the Cartier operator assuming that S is equipped with a coordinate system.

5.2.4. The sheaf $\mathcal{A}$. Let us use formula (5.3) for $\mathcal{A}'$. Define a sheaf of $\mathbb{F}_p$ -vector spaces $\mathcal{A} \subset \mathcal{A}'$ as follows:

$$\mathcal{A}(S, f) = \{\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \mid \tilde{C}(\omega) = \tilde{\varphi}(\omega)\}, \quad (5.4)$$

where $\tilde{C}$ and $\tilde{\varphi}$ were defined in §5.2.3.

5.2.5. Remarks about $\mathcal{A}$

- (i) By definition, $\mathcal{A}$ is the kernel of a certain morphism of sheaves. Later we will see that this morphism is surjective, see Lemma 5.3.4.
- (ii) The “true nature” of $\mathcal{A}$ will be explained in Proposition 6.2.3.

Theorem 5.2.6. *The subsheaf $\mathcal{T} \subset \mathcal{T}'$ is stable under the action of $\mathcal{A} \subset \mathcal{A}'$. Moreover, $\mathcal{T}$ is an $\mathcal{A}$ -torsor.*

The proof will be given in §5.3, 5.4.

Remark 5.2.7. In particular, Theorem 5.2.6 says that for every $S \in \mathrm{Sm}_{\mathbb{F}_p}$, every object of $\mathrm{Disp}_1^G(S)$ admits a lift to $\mathrm{BT}_1^G(S)$ étale-locally on S . As explained in §4.4.4, this is a part of Theorem 4.4.2, which we want to prove.

5.3. Proof of Theorem 5.2.6. Given $(S, f) \in \mathrm{Sm} / \mathrm{Disp}_1^G$, let $\mathcal{T}_S$ be the restriction of $\mathcal{T}$ to S_{et} ; similarly, we have $\mathcal{T}'_S, \mathcal{A}_S, \mathcal{A}'_S$. The problem is to prove that the subsheaf $\mathcal{T}_S \subset \mathcal{T}'_S$ is an $\mathcal{A}_S$ -torsor.

Lemma 5.3.1. *Zariski-locally on S , there exists a connection ∇ on $\mathcal{F}^+$ satisfying conditions (a) and (b) from §5.1.2.*

Proof. Zariski-locally, there exists an isomorphism between $\mathcal{F}^+$ and the P^+ -torsor induced from the M -torsor $\mathcal{F}_M^+$ via the inclusion $M \hookrightarrow P^+$. Choose such an isomorphism and take ∇ to be induced by the canonical connection on $\mathcal{F}_M^+ = \mathrm{Fr}_S^* \mathcal{F}_M^-$. $\square$

Lemma 5.3.2. *Let ∇ be as in Lemma 5.3.1. A connection $\tilde{\nabla}$ on $\mathcal{F}^+$ satisfies conditions (a) and (b) from §5.1.2 if and only if $\tilde{\nabla} = \nabla + \omega$, where $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)$ and $d\omega = 0$.*

Proof. Write $\tilde{\nabla} = \nabla + \omega$, where $\omega \in H^0(S, (\mathfrak{g}_{\geq 0})_{\mathcal{F}^+} \otimes \Omega_S^1)$. Condition (a) means that $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)$. Condition (b) is equivalent to the Maurer-Cartan equation for ω . Since $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$, this is just the equation $d\omega = 0$. $\square$

As noted in §5.2.1, $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1)$. So the ω from Lemma 5.3.2 is in $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}})$.

Lemma 5.3.3. *In the situation of Lemma 5.3.2, we have*

$$\mathrm{KS}_{\tilde{\nabla}} - \mathrm{KS}_{\nabla} = \tilde{\varphi}(\omega), \quad (5.5)$$

$$\mathrm{p-Curv}_{\tilde{\nabla}} - \mathrm{p-Curv}_{\nabla} = -\tilde{C}(\omega), \quad (5.6)$$

where $\tilde{\varphi} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1 \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1$ and $\tilde{C} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \rightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1$ are as in §5.2.3.

Formula (5.5) immediately follows from the definition of $\tilde{\varphi}$. A proof of (5.6) will be given in §5.4.

Theorem 5.3 follows from (5.5) and (5.6) and the next lemma.

Lemma 5.3.4. *The map $(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \xrightarrow{\tilde{C}-\tilde{\varphi}} (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1$ is a surjective morphism of étale sheaves of $\mathbb{F}_p$ -vector spaces.*

Proof. Let $\mathcal{E}_1 := (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}}$, $\mathcal{E}_2 := (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1$; these are finitely generated locally free $\mathcal{O}_S$ -modules. Let E_1, E_2 be the corresponding vector bundles; these are schemes over S (namely, E_i is the spectrum of the symmetric algebra of $\mathcal{E}_i^*$). Recall that $\tilde{C} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a surjective $\mathcal{O}_S$ -linear map and $\tilde{\varphi} : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is p -linear. So the morphism of S -schemes $E_1 \xrightarrow{\tilde{C}-\tilde{\varphi}} E_2$ is smooth. It is also surjective: indeed, the image is an open subgroup scheme of E_2 , so it has to be equal to E_2 . Therefore every section of E_2 étale-locally admits a lift to a section of E_1 . $\square$

5.4. Proof of formula (5.6). A proof will be given in §5.4.4; it is parallel to that of Proposition 7.1.2 of [17]. Let us first recall some facts used in the proof.

5.4.1. Recollections. Let $\mathbb{F}_p\langle x, y \rangle$ be the free associative $\mathbb{F}_p$ -algebra on x, y . Let us define $J \in \mathbb{F}_p\langle x, y \rangle$ by $J(x, y) := (x + y)^p - x^p - y^p$. We have $J = J_1 + \dots + J_{p-1}$, where J_i is homogeneous of degree i in y . Jacobson proved that each J_i belongs to the free Lie algebra on x, y (which is a Lie subalgebra of $\mathbb{F}_p\langle x, y \rangle$) and that

$$J_1(x, y) = \mathrm{ad}_x^{p-1}(y). \quad (5.7)$$

Hochschild's proof of these results can be found in [6, p. 199–200]. Since J_i is a Lie polynomial in x, y , which is homogeneous of degree i in y , we see that

$$J_i \in [\mathfrak{a}, \mathfrak{a}] \quad \text{for } i > 1, \quad (5.8)$$

where $\mathfrak{a} \subset \mathbb{F}_p\langle x, y \rangle$ is the Lie subalgebra generated by the elements $\mathrm{ad}_x^j(y)$, $j \geq 0$.

We will also need the following coordinate description of the Cartier operator

$$C : H^0(S, \Omega_S^1)_{\mathrm{closed}} \rightarrow H^0(S, \Omega_S^1).$$

Lemma 5.4.2. *Let S be a scheme étale over $\mathrm{Spec} \mathbb{F}_p[x_1, \dots, x_n]$ and $\omega \in H^0(S, \Omega_S^1)_{\mathrm{closed}}$. Write $\omega = \sum_i f_i \cdot dx_i$, $C(\omega) = \sum_i h_i \cdot dx_i$, where $f_i, h_i \in H^0(S, \mathcal{O}_S)$. Then*

$$h_i^p = -\partial_i^{p-1}(f_i), \quad \text{where } \partial_i := \frac{\partial}{\partial x_i}. \quad (5.9)$$

This lemma is well known, see formula (7.1.2.6) of [17]; moreover, P. Cartier used (5.9) to define C , see p. 200 and p. 202–203 of [6]. We prove the lemma for completeness.

Proof of Lemma 5.4.2. By Cartier's isomorphism, the space of closed 1-forms is generated by locally exact 1-forms and by

$$g^p \cdot x_i^{p-1} dx_i, \quad g \in H^0(S, \mathcal{O}_S), \quad 1 \leq i \leq n. \quad (5.10)$$

If ω is given by (5.10) then $C(\omega) = g \cdot dx_i$, and (5.9) holds because

$$\partial_i^{p-1}(g^p \cdot x_i^{p-1}) = (p-1)! \cdot g^p = -g^p$$

by Wilson's theorem. On the other hand, if $\omega = \sum_i f_i \cdot dx_i$ is exact then $C(\omega) = 0$, and the problem is to show that $\partial_i^{p-1}(f_i) = 0$. Indeed, if $\omega = du$ then $f_i = \partial_i(u)$ and we have $\partial_i^{p-1}(f_i) = \partial_i^p(u) = 0$. $\square$

Remark 5.4.3. Formula 5.10 tells us that in the situation of Lemma 5.4.2, the function $\partial_i^{p-1}(f_i)$ is a p -th power. Here is a direct proof of this. It suffices to show that $\partial_j \partial_i^{p-1}(f_i) = 0$ for all j . But the equality $d\omega = 0$ means that $\partial_j(f_i) = \partial_i(f_j)$, so $\partial_j \partial_i^{p-1}(f_i) = \partial_i^p(f_j) = 0$.

5.4.4. Proof of formula (5.6). We can assume that S is equipped with an étale morphism to $\mathrm{Spec} \mathbb{F}_p[x_1, \dots, x_n]$. Write $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)_{\mathrm{closed}}$ as

$$\omega = \sum_i f_i \otimes dx_i, \quad \text{where } f_i \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+}).$$

We have $(\mathfrak{g}_1)_{\mathcal{F}^+} = \mathrm{Fr}^* \mathcal{E}$, where $\mathcal{E} = (\mathfrak{g}_1)_{\mathcal{F}^-}$; so $\mathcal{E}$ is the horizontal part of $\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+}$. The problem is to prove that

$$\mathrm{p}\text{-Curv}_{\nabla+\omega} - \mathrm{p}\text{-Curv}_{\nabla} = -\tilde{C}(\omega), \quad (5.11)$$

where $\tilde{C} : H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)_{\mathrm{closed}} = H^0(S, \mathcal{E} \otimes (\mathrm{Fr}_* \Omega_S^1)_{\mathrm{closed}}) \rightarrow H^0(S, \mathcal{E} \otimes \Omega_S^1)$ is induced by $C : (\mathrm{Fr}_* \Omega_S^1)_{\mathrm{closed}} \rightarrow \Omega_S^1$. Write $\tilde{C}(\omega) = \sum_i h_i \otimes dx_i$, where $h_i \in H^0(S, \mathcal{E})$, then Lemma 5.4.2 implies that

$$\mathrm{Fr}^*(h_i) = -\nabla_i^{p-1}(f_i), \quad \text{where } \nabla_i := \nabla_{\frac{\partial}{\partial x_i}}. \quad (5.12)$$

Let us now compute the l.h.s. of (5.11). By §2.3.3, $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ and the p -operation on $\mathfrak{g}_1$ is zero. So by (5.7), (5.8), we get $(\nabla_i + f_i)^p - \nabla_i^p = \nabla_i^{p-1}(f_i)$. This means that the l.h.s. of (5.11) equals $-\sum_i h_i \otimes dx_i$, where the h_i 's are as in (5.12). $\square$

6. Proof of Theorems 4.4.1 and 4.4.2

In §6.3 we will see that Theorems 4.4.1 and 4.4.2 easily follow from Theorem 5.2.6 and a result of Artin–Milne [1], which we are going to recall now.

6.1. A result of Artin–Milne

6.1.1. The setting. Let S be a smooth $\mathbb{F}_p$ -scheme and H a finite flat commutative group scheme over S . Assume that H has height 1 (i.e., is killed by Frobenius). We have a morphism of sites $\pi : S_{\text{fppf}} \rightarrow S_{\text{et}}$. Artin and Milne described the sheaves $R^q\pi_*H$.

To formulate their result, we need some notation. Let $\mathfrak{h} := \text{Lie}(H)$; this is a finitely generated locally free $\mathcal{O}_S$ -module. The p -operation on $\mathfrak{h}$ is a p -linear map $\varphi : \mathfrak{h} \rightarrow \mathfrak{h}$. Similarly to §5.2.3, one defines a p -linear map $\tilde{\varphi} : \mathfrak{h} \otimes (\text{Fr}_S)_*\Omega_S^1 \rightarrow \mathfrak{h} \otimes \Omega_S^1$ and a surjective $\mathcal{O}_S$ -linear map $\tilde{C} : \mathfrak{h} \otimes ((\text{Fr}_S)_*\Omega_S^1)_{\text{closed}} \rightarrow \mathfrak{h} \otimes \Omega_S^1$ (the definition of $\tilde{\varphi}$ uses φ , and the definition of $\tilde{C}$ uses the Cartier operator C).

Proposition 6.1.2. *In the situation of §6.1.1 one has a canonical exact sequence*

$$0 \rightarrow R^1\pi_*H \xrightarrow{f} \mathfrak{h} \otimes ((\text{Fr}_S)_*\Omega_S^1)_{\text{closed}} \xrightarrow{\tilde{C}-\tilde{\varphi}} \mathfrak{h} \otimes \Omega_S^1 \rightarrow 0 \quad (6.1)$$

*of sheaves on S_{et} , which is functorial with respect to H and with respect to base changes $S' \rightarrow S$; moreover, $R^q\pi_*H = 0$ if $q > 1$.*

This is Proposition 2.4 of [1]. The construction of the map f from (6.1) will be recalled in §6.1.5.

6.1.3. Example. Let $H = (\alpha_p)_S$. In this case Proposition 6.1.2 says that

$$R\pi_*H = ((\text{Fr}_S)_*\Omega_S^1)_{\text{exact}}[-1].$$

Since Fr_S induces the identity functor $S_{\text{et}} \rightarrow S_{\text{et}}$, we can rewrite this as $R\pi_*H = \Omega_{S,\text{exact}}^1[-1]$. This is well known (and explained at the end of §4.1.4).

6.1.4. Remarks

- (i) The equality $R^0\pi_*H = 0$ is clear because S is reduced and the reduced part of the fiber of H over each point of S is zero.
- (ii) The proof of the equality $R^q\pi_*H = 0$ for $q > 1$ given in [1] works for any scheme S and any finite locally free commutative group scheme over S .
- (iii) In a particular situation, surjectivity of the map $\tilde{C} - \tilde{\varphi}$ from (6.1) was proved in Lemma 5.3.4. The same argument works in general.

6.1.5. Interpretation and noncommutative generalization of (6.1).

An element of $H_{\text{fppf}}^1(S, H)$ is an isomorphism class of a principal H -bundle $E \rightarrow S$. Since H is killed by Frobenius, the geometric Frobenius $F : E \rightarrow \text{Fr}_S^*E$ factors as $E \rightarrow S \xrightarrow{\sigma} \text{Fr}_S^*E$. The section σ trivializes the bundle Fr_S^*E . On the other hand, any Fr_S -pullback (e.g., Fr_S^*E) is equipped with a p -integrable connection. A connection ∇ on the trivialized Fr_S^*H -bundle Fr_S^*E is given by an element $\omega \in H^0(S, \text{Fr}_S^*\mathfrak{h} \otimes \Omega_S^1) = H^0(S, \mathfrak{h} \otimes (\text{Fr}_S)_*\Omega_S^1)$. Moreover, integrability of ∇ is equivalent to the Maurer–Cartan equation for ω ; since H is commutative, this equation just means that $\omega \in H^0(S, \mathfrak{h} \otimes ((\text{Fr}_S)_*\Omega_S^1)_{\text{closed}})$. Thus we get a map

$$H_{\text{fppf}}^1(S, H) \rightarrow H^0(S, \mathfrak{h} \otimes ((\text{Fr}_S)_*\Omega_S^1)_{\text{closed}}).$$

This map and similar maps for all schemes étale over S give rise to the map f from (6.1). (This is essentially the description of f from [1, §2.7] although the word “connection” is not used there).

It is easy to check¹⁶ that the inclusion $\mathrm{Im} f \subset \mathrm{Ker}(\tilde{C} - \tilde{\varphi})$ (which is a part of Proposition 6.1.2) just means that the p -curvature of ∇ is zero. So the exact sequence (6.1) essentially says that $H_{\mathrm{fppf}}^1(S, H)$ canonically identifies (via Fr_S -pullback) with the set of p -integrable connections on the trivial H -bundle on S . One can prove that this statement remains valid without assuming H commutative; we will not need this fact.

6.2. The classifying stack of Lau_1^G

6.2.1. The sheaf $\mathcal{B}$. Let $\mathrm{Sch} / \mathrm{Disp}_1^G$ be the category of pairs (S, f) , where S is a scheme and $f : S \rightarrow \mathrm{Disp}_1^G$ is a morphism. Let $\mathrm{Sm} / \mathrm{Disp}_1^G$ be the full subcategory of $\mathrm{Sch} / \mathrm{Disp}_1^G$ formed by pairs (S, f) such that $S \in \mathrm{Sm}_{\mathbb{F}_p}$. We equip $\mathrm{Sch} / \mathrm{Disp}_1^G$ and $\mathrm{Sm} / \mathrm{Disp}_1^G$ with the étale topology.

In §3.3.1 we defined a commutative finite flat group scheme Lau_1^G over Disp_1^G ; this group scheme has height 1. By definition, the classifying stack of Lau_1^G is the stack of Picard groupoids on $\mathrm{Sch} / \mathrm{Disp}_1^G$ whose sections over $(S, f) \in \mathrm{Sch} / \mathrm{Disp}_1^G$ are $(f^* \mathrm{Lau}_1^G)$ -torsors. Let $\mathcal{B}$ be the restriction of this stack to $\mathrm{Sm} / \mathrm{Disp}_1^G$. For any $(S, f) \in \mathrm{Sm} / \mathrm{Disp}_1^G$ one has $H^0(S, f^* \mathrm{Lau}_1^G) = 0$ (because S is reduced), so the groupoid of sections of $\mathcal{B}$ over (S, f) is discrete. Therefore $\mathcal{B}$ is just a sheaf of abelian groups.

6.2.2. Remark. Restricting from $\mathrm{Sch} / \mathrm{Disp}_1^G$ to $\mathrm{Sm} / \mathrm{Disp}_1^G$ does not lead to loss of information: this follows from Lemma 4.1.2 and the fact that the classifying stack of Lau_1^G is smooth over $\mathbb{F}_p$. The latter follows from §B.0.1 of Appendix B because by §3.2.2, Disp_1^G is smooth over $\mathbb{F}_p$.

Proposition 6.2.3. *The sheaf $\mathcal{B}$ from §6.2.1 is canonically isomorphic to the sheaf $\mathcal{A}$ from §5.2.4.*

Proof. Follows from Proposition 6.1.2 and the definitions of $\mathcal{A}$ and Lau_1^G . $\square$

6.3. Proof of Theorems 4.4.1 and 4.4.2. In §5.1.2 we introduced the notation $\mathcal{T}$ for $\mathrm{BT}_{1, \mathbb{F}_p}^G$ viewed as a sheaf of sets on $\mathrm{Sm} / \mathrm{Disp}_1^G$. The problem is to show that $\mathcal{T}$ is locally isomorphic to the sheaf $\mathcal{B}$ from §6.2.1. By Proposition 6.2.3, $\mathcal{B} = \mathcal{A}$. By Theorem 5.2.6, $\mathcal{T}$ is locally isomorphic to $\mathcal{A}$. $\square$

A. Generalities on stacks

Let P be a smooth-local property of schemes. Let Sch be the category of all schemes and Sch_P the category of schemes satisfying P ; we equip Sch and Sch_P with the étale topology. Let Stacks_P be the (2,1)-category of algebraic stacks satisfying P (this makes sense because P is smooth-local). Let Grpds be the (2,1)-category of groupoids.

¹⁶The verification is parallel to §5.4.4; the only difference is that the p -operation on $\mathfrak{h}$ is not assumed to be zero, while the restricted Lie algebra $(\mathfrak{g}_1)_{\mathcal{F}^+}$ from §5.4.4 has zero p -operation.

Given a stack $\mathcal{Y} : \text{Sch}^{\text{op}} \rightarrow \text{Grpds}$, let $\mathcal{Y}_P$ denote the restriction of $\mathcal{Y}$ to Sch_P^{op} .

Proposition A.0.1. *Let $\mathcal{X} \in \text{Stacks}_P$. Then the functor $\text{Mor}(\mathcal{X}, \mathcal{Y}) \rightarrow \text{Mor}(\mathcal{X}_P, \mathcal{Y}_P)$ is an equivalence for any stack $\mathcal{Y} : \text{Sch}^{\text{op}} \rightarrow \text{Grpds}$.*

Proof. Let $\pi : X \rightarrow \mathcal{X}$ be a smooth surjective morphism with X being a scheme. Let $X_\bullet$ be the Čech nerve of π . Note that each X_n is an algebraic space with property P . We have

$$\text{Mor}(\mathcal{X}, \mathcal{Y}) = \text{Tot}(\text{Mor}(X_\bullet, \mathcal{Y})), \quad \text{Mor}(\mathcal{X}_P, \mathcal{Y}_P) = \text{Tot}(\text{Mor}((X_\bullet)_P, \mathcal{Y}_P)).$$

Thus we have reduced the problem to the case where $\mathcal{X}$ is an algebraic space.

Running the same argument again, we reduce to the case where $\mathcal{X} \in \text{Sch}_P$, which is covered by Yoneda's lemma. $\square$

Our next goal is to reformulate Proposition A.0.1 in terms of a certain morphism of toposes. Our approach is influenced by §4.10 of Exposé IV of [27] and by the more sophisticated Proposition A.0.4 of [11] (which goes back to A. Mathew and is about *derived stacks*).

Let $\widetilde{\text{Sch}}$ (respectively $\widehat{\text{Sch}}$) be the category of presheaves (respectively sheaves) of groupoids on Sch . Similarly, we have $\widehat{\text{Sch}}_P$ and $\widetilde{\text{Sch}}_P$. Let $\tilde{g}_* : \widetilde{\text{Sch}} \rightarrow \widetilde{\text{Sch}}_P$, $\hat{g}_* : \widehat{\text{Sch}} \rightarrow \widehat{\text{Sch}}_P$ be the restriction functors. They have left adjoints $\tilde{g}^*$ and $\hat{g}^*$. The functor $\tilde{g}^* : \widetilde{\text{Sch}}_P \rightarrow \widetilde{\text{Sch}}$ is the *left Kan extension*. The functor $\hat{g}^* : \widehat{\text{Sch}}_P \rightarrow \widehat{\text{Sch}}$ is the *sheafified left Kan extension*, i.e., the composition

$$\widehat{\text{Sch}}_P \hookrightarrow \widetilde{\text{Sch}}_P \xrightarrow{\tilde{g}^*} \widetilde{\text{Sch}} \rightarrow \widehat{\text{Sch}}.$$

Each of the adjoint pairs $(\tilde{g}^*, \tilde{g}_*)$ and $(\hat{g}^*, \hat{g}_*)$ defines a morphism of toposes.

Since the functor $\text{Sch}_P \rightarrow \text{Sch}$ is fully faithful, so is $\tilde{g}^*$; equivalently, the unit of the adjunction $\text{id} \rightarrow \tilde{g}_* \tilde{g}^*$ is an isomorphism. The same is true for $\hat{g}^*$. For sheaves of sets, this is [29, Tag 00XT]. For sheaves of groupoids, one can argue as follows: if $\mathcal{F} \in \widehat{\text{Sch}}_P$ then $\hat{g}_* \hat{g}^*(\mathcal{F})$ is the restriction of the sheafification of $\tilde{g}^* \mathcal{F}$, which equals¹⁷ the sheafification of the restriction of $\tilde{g}^* \mathcal{F}$, i.e., the sheafification of $\mathcal{F}$, i.e., $\mathcal{F}$ itself.

Now we can reformulate Proposition A.0.1 as follows.

Proposition A.0.2. *If $\mathcal{X} \in \text{Stacks}_P$ then the canonical morphism*

$$\hat{g}^* \mathcal{X}_P = \hat{g}^* \hat{g}_* \mathcal{X} \rightarrow \mathcal{X}$$

is an isomorphism. Equivalently, $\mathcal{X}$ belongs to the essential image of the fully faithful functor $\hat{g}^$.* $\square$

Proposition A.0.2 tells us how to reconstruct $\mathcal{X} \in \text{Stacks}_P$ from $\mathcal{X}_P$: namely, $\mathcal{X} = \hat{g}_* \mathcal{X}_P$, i.e., $\mathcal{X}$ is the sheafified left Kan extension of $\mathcal{X}_P$.

¹⁷Sheafification commutes with restriction by part 1 of [7, Prop. 7.1]. Probably this can also be proved by interpreting $\widehat{\text{Sch}}$ and $\widehat{\text{Sch}}_P$ as explained in part 2 of Exercise 4.10.6 of Exposé IV of [27] (i.e., a sheaf on Sch is just a collection of sheaves $\mathcal{F}_S$ on S_{et} for all $S \in \text{Sch}$ plus certain morphisms relating the sheaves $\mathcal{F}_S$ with each other).

B. Recollections on gerbes

B.0.1. Classifying stacks. Let H be a flat affine group scheme of finite presentation over an algebraic stack $\mathcal{X}$. Then the classifying stack BH is known to be an algebraic stack smooth over $\mathcal{X}$: indeed, we can assume that $\mathcal{X}$ is a scheme, in which case the statement is proved in [29, Tag 0DLS] using [29, Tag 05B5] and a deep theorem of M. Artin, see [29, Tag 06FI]. (On the other hand, if H is finite over $\mathcal{X}$, one can give an elementary constructive proof similar to the one given in §4.1.4 in the case $H = \alpha_p$.)

B.0.2. Gerbes. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be a morphism of finite presentation between algebraic stacks. If f is an fppf gerbe then f is smooth. Indeed, smoothness can be checked fppf-locally on $\mathcal{X}$, so we can assume that the gerbe is trivial. Then we can apply §B.0.1.

B.0.3. Remark. Let $f : \mathcal{Y} \rightarrow \mathcal{X}$ be as in §B.0.2. Since f is smooth and surjective, it admits a section locally for the *smooth* topology of $\mathcal{X}$. This implies that if $\mathcal{X}$ is a scheme then f admits a section étale-locally on $\mathcal{X}$.

C. A definition of BT_n^G via syntomification

Let $n \in \mathbb{N}$ and $G \in \mathbf{Shim}_n$, where $\mathbf{Shim}_n$ is as in §2.2. For every derived p -adic formal scheme S , we define in §C.2 an ∞ -groupoid $\mathrm{BT}_n^G(S)$. The assignment $S \mapsto \mathrm{BT}_n^G(S)$ is an étale sheaf. Conjecture C.3.1 says that for each $m \in \mathbb{N}$ the restriction of BT_n^G to the category of derived schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a smooth algebraic stack over $\mathbb{Z}/p^m\mathbb{Z}$.

C.1. Recollections on syntomification

C.1.1. The stack S^Δ . Bhatt and Lurie [4] define a *prismatization* functor $S \mapsto S^\Delta$ from the category of derived p -adic formal schemes to the category of fpqc-stacks of ∞ -groupoids on the category of p -nilpotent¹⁸ derived schemes. The stacks S^Δ are not very far from being algebraic in the sense of Definition 2.3.5 of [9] (whose essential point is that the quotient of a scheme over a ring R by an action of a flat affine group scheme H over R is considered to be an algebraic stack even if H has infinite type).

The above words “not very far” are necessary for two reasons. First, the derived setting is not considered in [9]. Second, already the stacks $(\mathrm{Spf} \mathbb{Z}_p)^\Delta$ and $(\mathrm{Spec} \mathbb{Z}/p^n\mathbb{Z})^\Delta$ are *formal* (in the sense of [9, §2.9.1]) rather than algebraic; e.g., $(\mathrm{Spec} \mathbb{F}_p)^\Delta = \mathrm{Spf} \mathbb{Z}_p$.

C.1.2. The stacks $S^\mathcal{N}$ and S^{Syn} . Bhatt defines in his lecture notes [2] the (Nygaard-)filtered prismatization functor $S \mapsto S^\mathcal{N}$. As before, this is a functor from the category of derived p -adic formal schemes to the category of fpqc-stacks of ∞ -groupoids on the category of p -nilpotent derived schemes. However, for pedagogical reasons, Bhatt assumes in [2] that S is a classical¹⁹ scheme, and he

¹⁸This means that p is Zariski-locally nilpotent.

¹⁹“Classical” means “not really derived”.

defines in [2] only the restriction of the stack $S^{\mathcal{N}}$ to the category of classical p -nilpotent schemes. As before, the stacks $S^{\mathcal{N}}$ are not very far from being algebraic in the sense of [9].

The stack $S^{\mathcal{N}}$ has two open substacks canonically isomorphic to S^{Δ} . Gluing them together, Bhatt gets a stack which is called the *syntomification* of S and denoted by S^{Syn} . The assignment $S \mapsto S^{\text{Syn}}$ is a functor. By definition, one has a natural morphism

$$S^{\mathcal{N}} \rightarrow S^{\text{Syn}}.$$

There is a canonical line bundle on $(\text{Spf } \mathbb{Z}_p)^{\text{Syn}}$ called the Breuil-Kisin twist. It defines a morphism $(\text{Spf } \mathbb{Z}_p)^{\text{Syn}} \rightarrow B\mathbb{G}_m$, where $B\mathbb{G}_m$ is the classifying stack of $\mathbb{G}_m$. By functoriality, it induces a morphism $S^{\text{Syn}} \rightarrow B\mathbb{G}_m$ for any S . The corresponding line bundle on S^{Syn} is denoted by $\mathcal{O}_{S^{\text{Syn}}}\{1\}$ or $\mathcal{O}\{1\}$.

C.1.3. The canonical morphism $S \times B\mathbb{G}_m \rightarrow S^{\mathcal{N}}$. There is a canonical morphism

$$S \times B\mathbb{G}_m \rightarrow S^{\mathcal{N}} \tag{C.1}$$

over $B\mathbb{G}_m$; namely, (C.1) is the composite map

$$S \times \{0\}/\mathbb{G}_m \hookrightarrow S \times \mathbb{A}^1/\mathbb{G}_m \rightarrow S^{dR,+} \rightarrow S^{\mathcal{N}},$$

where $S^{dR,+}$ and the maps $S \times \mathbb{A}^1/\mathbb{G}_m \rightarrow S^{dR,+} \rightarrow S^{\mathcal{N}}$ are defined in [2, §5.3.13].

We need the morphism (C.1) only if $S = \text{Spec } k$, where k is a perfect field. In this case it can be described as follows. It is known²⁰ that

$$(\text{Spec } k)^{\mathcal{N}} \otimes \mathbb{F}_p = (\text{Spec } k[u, t]/(ut))/\mathbb{G}_m,$$

where the $\mathbb{G}_m$ -action is such that $\deg t = 1$, $\deg u = -1$; so the closed substack of the stack $(\text{Spec } k)^{\mathcal{N}} \otimes \mathbb{F}_p$ given by $t = u = 0$ equals $\text{Spec } k \times B\mathbb{G}_m$. (Let us note that this closed substack is called the *Hodge locus*.)

C.1.4. On notation. The definition of $S^{\mathcal{N}}$ and S^{Syn} is sketched in [9, §1.7], but the notation is different there ($S^{\Delta'}$ instead of $S^{\mathcal{N}}$ and $S^{\Delta''}$ instead of S^{Syn}). In [3, 4] the stack S^{Δ} was denoted by WCart_S .

C.2. Definition of $\text{BT}_n^G(S)$. Let S be a derived p -adic formal scheme. Let $G \in \mathbf{Shim}_n$.

C.2.1. The group scheme $G_{\mathcal{O}\{1\}}$. Our G is a group scheme over $\mathbb{Z}/p^n\mathbb{Z}$. Twisting G by the canonical $\mathbb{G}_m$ -torsor on $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, one gets a group scheme over $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, which we denote by $G_{\mathcal{O}\{1\}}$. One has

$$\text{Lie}(G_{\mathcal{O}\{1\}}) = \bigoplus_i \mathfrak{g}_i \otimes \mathcal{O}\{i\},$$

where $\mathfrak{g}_i$ is the i -th graded component of $\text{Lie}(G)$ and $\mathcal{O}\{i\}$ is the i -th tensor power of the canonical line bundle on $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$.

Since $S^{\text{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is equipped with a morphism to $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, we can consider $G_{\mathcal{O}\{1\}}$ -torsors on $S^{\text{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

²⁰See [2, §3.3 and §5.4].

C.2.2. Definition. For any derived p -adic formal scheme S of finite type²¹ over $\mathrm{Spf} \mathbb{Z}_p$, let $\mathrm{BT}_n^G(S)$ be the ∞ -groupoid of $G_{\mathcal{O}\{1\}}$ -torsors $\mathcal{E}$ on $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ such that for every geometric point $\alpha : \mathrm{Spec} k \rightarrow S$, the pullback of $\mathcal{E}$ via the composite morphism

$$\mathrm{Spec} k \times B\mathbb{G}_m \rightarrow (\mathrm{Spec} k)^{\mathcal{N}} \otimes \mathbb{F}_p \xrightarrow{\alpha_*} S^{\mathcal{N}} \otimes \mathbb{F}_p \hookrightarrow S^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z} \rightarrow S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z} \quad (\mathrm{C}.2)$$

is trivial. The first arrow in (C.2) was described in §C.1.3.

C.2.3. Remarks

- (i) The isomorphism class of the above-mentioned pullback is an element $\nu_\alpha \in H^1((\mathbb{G}_m)_k, G_k)$, where G_k is the base change of G to k , and we want ν_α to be zero for all geometric points α . One can show that if S is connected then it suffices to check the condition $\nu_\alpha = 0$ for a single α .
- (ii) $H^1((\mathbb{G}_m)_k, G_k)$ is the set of $G(k)$ -conjugacy classes of splittings for the canonical epimorphism $(\mathbb{G}_m)_k \ltimes G_k \twoheadrightarrow (\mathbb{G}_m)_k$. The latter set depends only on the quotient of G_k by its unipotent radical (together with the action of $\mathbb{G}_m$ on this quotient).
- (iii) The definition of $\mathrm{BT}_n^G(S)$ from §C.2.2 makes sense without assuming the action of $\mathbb{G}_m$ on G to be 1-bounded. But the conjecture formulated below is *unlikely to hold without this assumption*.

C.3. The conjecture. For each $n \in \mathbb{N}$ and $G \in \mathbf{Shim}_n$ we have defined a contravariant functor $S \mapsto \mathrm{BT}_n^G(S)$, where S is a derived p -adic formal scheme of finite type over $\mathrm{Spf} \mathbb{Z}_p$.

Conjecture C.3.1. *For each $m \in \mathbb{N}$ the restriction of this functor to the category of derived schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a quasicompact smooth algebraic stack²² over $\mathbb{Z}/p^m\mathbb{Z}$ with affine diagonal.*

In particular, the conjecture would imply that the restriction of the functor BT_n^G to the category of classical schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a smooth algebraic stack in the sense of [20].

As far as I understand, Conjecture C.3.1 has already been proved in [12].

C.3.2. Remarks

- (i) If S is a smooth $\mathbb{F}_p$ -scheme then the definitions of $\mathrm{BT}_1^G(S)$ given in §4.2.1 and §C.2.2 are equivalent, see [28, Thm. 3.8] or Appendix D below.
- (ii) Let G be the group $\mathbb{G}_a \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with the usual $\mathbb{G}_m$ -action. Then the triviality condition from §C.2.2 is automatic by §C.2.3, so $\mathrm{BT}_n^G(S)$ is the ∞ -groupoid of all $\mathcal{O}\{1\}$ -torsors on $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Bhatt and Lurie proved²³

²¹“Finite type” means that the classical truncation of $S \otimes \mathbb{F}_p$ has finite type in the usual sense. For the motivation of the finite type precaution, see §C.3.2(ii) below.

²²Here the words “algebraic stack” are understood in the derived sense (since S is allowed to be derived); see the definition of 1-stack in [21, Def. 5.1.3] (which goes back to [30]).

²³Theorem 7.5.6 of [3] identifies the p -adic completion of $R\Gamma(S_{\mathrm{et}}, \mathcal{O}_S^\times)[-1]$ with a certain complex $R\Gamma_{\mathrm{syn}}(S, \mathbb{Z}_p(1))$, whose definition does not involve S^{Syn} . If S is quasisyntomic then it is known that $R\Gamma_{\mathrm{syn}}(S, \mathbb{Z}_p(1)) = R\Gamma(S^{\mathrm{Syn}}, \mathcal{O}\{1\})$.

that

$$R\Gamma(S^{\text{Syn}} \otimes \mathbb{Z}/p^n \mathbb{Z}, \mathcal{O}\{1\}) = R\Gamma(S_{\text{et}}, \text{Cone}(\mathcal{O}_S^\times \xrightarrow{p^n} \mathcal{O}_S^\times)[-1]). \quad (\text{C.3})$$

if S is quasisyntomic. Moreover, (C.3) is expected to hold²⁴ for any p -nilpotent derived scheme of finite type over $\text{Spf } \mathbb{Z}_p$ (similarly to [4, Prop. 8.16]). If so then BT_n^G identifies with the classifying stack of μ_{p^n} . This agrees with Conjecture C.3.1; in the case $n = m = 1$ this also agrees with §4.4.5.

D. Equivalence between the two definitions of $\text{BT}_1^G(S)$

Let G be a smooth affine group scheme over $\mathbb{F}_p$ equipped with an action of $\mathbb{G}_m$. Given a smooth $\mathbb{F}_p$ -scheme S , we have two definitions of the groupoid $\text{BT}_1^G(S)$: an elementary one (see §4.2.1) and a definition via the stack S^{Syn} (see §C.2.2); both definitions make sense without assuming that the $\mathbb{G}_m$ -action is 1-bounded, see §4.2.3 and §C.2.3(iii). In this Appendix we sketch a proof of the equivalence between the two definitions.

D.1. Recollections on $S^{\text{Syn}} \otimes \mathbb{F}_p$. We will follow [2, Ch. 2].

D.1.1. As mentioned in §C.1.2, S^{Syn} is obtained from $S^\mathcal{N}$ by gluing together the two open substacks canonically isomorphic S^Δ . One has $S^\Delta \otimes \mathbb{F}_p = S^{\text{dR}} := S^{\text{dR}}/\mathbb{F}_p$. Following §2.8 of [2], we use the notation $S^C := S^\mathcal{N} \otimes \mathbb{F}_p$. The stack S^C contains two open substacks canonically isomorphic to S^{dR} , and after gluing them together, one gets $S^{\text{Syn}} \otimes \mathbb{F}_p$.

Let us now recall the material on S^C from [2, Ch. 2].

D.1.2. Just as in Construction 2.8.2 from [2], let C denote the quotient of the coordinate cross $\text{Spec } \mathbb{F}_p[u, t]/(ut)$ by the hyperbolic action of $\mathbb{G}_m$. One has $(\text{Spec } \mathbb{F}_p)^C = C$. The two irreducible components of C are denoted by $C_{u=0}$ and $C_{t=0}$.

D.1.3. The preimage of $C_{t=0}$ (respectively $C_{u=0}$) in S^C is denoted in [2] by $S^{\text{dR},c}$ (respectively $S^{\text{dR},+}$); the superscript c stands for “conjugate filtration”. Let $S^{\text{Hodge}} := S^{\text{dR},c} \cap S^{\text{dR},+}$.

In §D.1.1 we mentioned two open substacks of S^C canonically isomorphic S^{dR} ; these substacks are $S^{\text{dR},c} \setminus S^{\text{Hodge}}$ and $S^{\text{dR},+} \setminus S^{\text{Hodge}}$.

D.1.4. By §D.1.1 and §D.1.3, we have a pushout diagram of stacks

$$\begin{array}{ccc} S^{\text{Syn}} \otimes \mathbb{F}_p & \longleftarrow & S^{\text{dR},+} \\ \uparrow & & \uparrow \\ S^{\text{dR},c} & \longleftarrow & S^{\text{dR}} \sqcup S^{\text{Hodge}} \end{array} \quad (\text{D.1})$$

Chapter 2 of [2] contains a rather explicit description of the whole diagram (D.1) via the procedure of *transmutation* (see [2, Rem. 2.3.8]). The key point is that

²⁴Again, the problem is to show that $R\Gamma_{\text{syn}}(S, \mathbb{Z}_p(1)) = R\Gamma(S^{\text{Syn}}, \mathcal{O}\{1\})$.

it suffices to describe (D.1) in the particular case $S = \mathbb{G}_a$ as a *diagram of ring stacks*. For such a description, see [2, 2.8.3] and references therein, as well as [2, 2.5.1], [2, 2.7.8], and [2, Prop. 2.7.12]. We paraphrase this description in §D.4.

D.2. $\mathrm{BT}_1^G(S)$ as a fiber product. Let $\mathrm{BT}_1^G(S)$ denote the groupoid defined in §C.2.2. Thus $\mathrm{BT}_1^G(S)$ is the groupoid of $G_{\mathcal{O}_{\{1\}}}$ -torsors $\mathcal{E}$ on $S^{\mathrm{Syn}} \otimes \mathbb{F}_p$ such that for every geometric point $\alpha : \mathrm{Spec} k \rightarrow S$, the pullback of $\mathcal{E}$ via a certain morphism $f_\alpha : \mathrm{Spec} k \times B\mathbb{G}_m \rightarrow S^{\mathrm{Syn}} \otimes \mathbb{F}_p$ is trivial. Let us note that f_α factors as

$$\mathrm{Spec} k \times B\mathbb{G}_m \rightarrow S^{\mathrm{Hodge}} \rightarrow S^{\mathrm{Syn}} \otimes \mathbb{F}_p, \quad (\mathrm{D}.2)$$

see the last paragraph of §C.1.3

Diagram (D.1.4) yields a pullback diagram of groupoids

$$\begin{array}{ccc} \mathrm{BT}_1^G(S) & \longrightarrow & \mathcal{X}^{\mathrm{dR},+}(S) \\ \downarrow & & \downarrow \\ \mathcal{X}^{\mathrm{dR},c}(S) & \longrightarrow & \mathcal{X}^{\mathrm{dR}}(S) \times \mathcal{X}^{\mathrm{Hodge}}(S) \end{array} \quad (\mathrm{D}.3)$$

where $\mathcal{X}^{\mathrm{dR}}(S)$ is the groupoid of G -torsors²⁵ on S^{dR} and $\mathcal{X}^{\mathrm{dR},+}(S)$ (respectively $\mathcal{X}^{\mathrm{dR},c}(S)$ or $\mathcal{X}^{\mathrm{Hodge}}(S)$) is the groupoid of $G_{\mathcal{O}_{\{1\}}}$ -torsors on $S^{\mathrm{dR},+}$ (respectively $S^{\mathrm{dR},c}$ or S^{Hodge}) satisfying the triviality condition²⁶ similar to the one from the definition of $\mathrm{BT}_1^G(S)$.

In the next subsection we will translate the diagram

$$\mathcal{X}^{\mathrm{dR},c}(S) \rightarrow \mathcal{X}^{\mathrm{dR}}(S) \times \mathcal{X}^{\mathrm{Hodge}}(S) \leftarrow \mathcal{X}^{\mathrm{dR},+}(S) \quad (\mathrm{D}.4)$$

into an elementary language. The isomorphism between $\mathrm{BT}_1^G(S)$ and the groupoid from §4.2.1 will immediately follow from this description because diagram (D.3) is Cartesian.

D.3. Diagram (D.4) in elementary terms. Let $\mathfrak{g} := \mathrm{Lie}(G)$. We will use the subgroups $M, P^\pm \subset G$ defined in §2.3 and the grading $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ corresponding to the $\mathbb{G}_m$ -action.

D.3.1. $\mathcal{X}(S)$ is the groupoid of G -torsors on S equipped with a nilpotent integrable connection.

D.3.2. $\mathcal{X}^{\mathrm{Hodge}}(S)$ is the groupoid of pairs $(\mathcal{F}_M, \sigma)$, where $\mathcal{F}_M$ is an M -torsor on S and $\sigma \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}_M} \otimes \Omega_S^1)$, see [2, Rem. 2.5.9].

D.3.3. $\mathcal{X}^{\mathrm{dR},+}(S)$ is the groupoid of P^- -torsors $\mathcal{F}^-$ on S equipped with a nilpotent integrable connection ∇ on the corresponding G -bundle $\mathcal{F}_G^-$ satisfying the Griffiths transversality condition $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1) \subset H^0(S, (\mathfrak{g}/\mathfrak{g}_{\leq 0})_{\mathcal{F}^-} \otimes \Omega_S^1)$ (we are using the notation from §4.2.2). See [2, Rem. 2.5.8].

²⁵A $G_{\mathcal{O}_{\{1\}}}$ -torsor on S^{dR} is the same as a G -torsor because $(\mathrm{Spec} \mathbb{F}_p)^{\mathrm{dR}} = \mathrm{Spec} \mathbb{F}_p$.

²⁶The formulation of this condition uses the factorization (D.2).

D.3.4. $\mathcal{X}^{\text{dR},c}(S)$ is the groupoid of P^+ -torsors $\mathcal{F}^+$ on S equipped with an integrable connection ∇ inducing a p -integrable connection on the M -torsor $\mathcal{F}_M^+$ corresponding to $\mathcal{F}^+$. See [2, Rem. 2.7.10].

D.3.5. The functor $\mathcal{X}^{\text{dR},c}(S) \rightarrow \mathcal{X}^{\text{Hodge}}(S)$ from diagram (D.4) is as follows:

(i) the M -torsor $\mathcal{F}_M$ from §D.3.2 is the Fr_S -descent of the M -torsor $\mathcal{F}_M^+$ from §D.3.4 via the p -integrable connection;

(ii) $\sigma \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}_M} \otimes \Omega_S^1)$ is the image of $-\text{p-Curv}_{\nabla} \in H^0(S, (\mathfrak{g}_{\geq 1})_{\mathcal{F}_M} \otimes \Omega_S^1)$.

The other functors from diagram (D.4) are self-explanatory.

D.3.6. Origin of the sign. The “minus” sign in §D.3.5(ii) comes from the “minus” sign in formula (D.7) below.

D.4. The ring stack $\mathbb{G}_a^{\text{Syn}} \otimes \mathbb{F}_p$. Let $\mathbb{G}_a$ denote the additive group over $\mathbb{F}_p$. In this subsection we recall the description of $\mathbb{G}_a^{\text{Syn}} \otimes \mathbb{F}_p$ given in [2, Ch. 2] and explain the origin of the “minus” sign in §D.3.5(ii).

D.4.1. Plan. $\mathbb{G}_a$ is a ring scheme, so $\mathbb{G}_a^{\text{Syn}} \otimes \mathbb{F}_p$ is a ring stack over $(\text{Spec } \mathbb{F}_p)^{\text{Syn}} \otimes \mathbb{F}_p$. One gets $(\text{Spec } \mathbb{F}_p)^{\text{Syn}} \otimes \mathbb{F}_p$ from the stack C considered in §D.1.2 by gluing together the two open points of C . Equivalently, $(\text{Spec } \mathbb{F}_p)^{\text{Syn}} \otimes \mathbb{F}_p$ is obtained from $\mathbb{P}^1/\mathbb{G}_m$ by gluing $\{0\}/\mathbb{G}_m$ with $\{\infty\}/\mathbb{G}_m$.

Let $\mathcal{R}$ be the pullback of $\mathbb{G}_a^{\text{Syn}} \otimes \mathbb{F}_p$ via the map $\mathbb{P}^1 \rightarrow (\text{Spec } \mathbb{F}_p)^{\text{Syn}} \otimes \mathbb{F}_p$. To describe $\mathbb{G}_a^{\text{Syn}} \otimes \mathbb{F}_p$, we will describe in §D.4.2–D.4.3 the $\mathbb{G}_m$ -equivariant ring stack $\mathcal{R}$ over $\mathbb{P}^1$ and the isomorphism $\mathcal{R}_0 \xrightarrow{\sim} \mathcal{R}_{\infty}$, where $\mathcal{R}_0, \mathcal{R}_{\infty}$ are the fibers of $\mathcal{R}$ over $0, \infty \in \mathbb{P}^1$.

D.4.2. The ring stack $\mathcal{R}$. Let $(\mathbb{G}_a)_{\mathbb{P}^1} = \mathbb{G}_a \times \mathbb{P}^1$; this is a ring scheme over $\mathbb{P}^1$. One has

$$\mathcal{R} = \text{Cone}(G \xrightarrow{d} (\mathbb{G}_a)_{\mathbb{P}^1}),$$

where (G, d) is a quasi-ideal²⁷ in $(\mathbb{G}_a)_{\mathbb{P}^1}$, which we are going to define.

Let $\mathbb{G}_a^{\sharp}$ denote the PD-hull of zero in $\mathbb{G}_a$. Then $\mathbb{G}_a^{\sharp}$ is a $\mathbb{G}_a$ -module, and the natural map $f : \mathbb{G}_a^{\sharp} \rightarrow \mathbb{G}_a$ is a $\mathbb{G}_a$ -module homomorphism such that

$$\text{Im } f = \alpha_p := \text{Ker}(\text{Fr} : \mathbb{G}_a \rightarrow \mathbb{G}_a).$$

One defines G to be a certain $\mathbb{G}_a$ -submodule of $\mathbb{G}_a^{\sharp} \times \mathbb{G}_a \times \mathbb{P}^1$ and $d : G \rightarrow \mathbb{G}_a \times \mathbb{P}^1$ to be the projection. The equations defining $G \subset \mathbb{G}_a^{\sharp} \times \mathbb{G}_a \times \mathbb{P}^1$ are as follows:

$$uy = f(x), \tag{D.5}$$

$$y^p = 0, \tag{D.6}$$

where $x \in \mathbb{G}_a^{\sharp}$, $y \in \mathbb{G}_a$, $u \in \mathbb{P}^1$. Strictly speaking, (D.5) means that

$$uy = f(x) \text{ if } u \neq \infty, \quad y = u^{-1}f(x) \text{ if } u \neq 0.$$

Note that if $u \neq 0$ then (D.6) follows from (D.5).

Finally, the action of $\lambda \in \mathbb{G}_m$ on G and $\mathbb{G}_a \times \mathbb{P}^1$ is given by $\tilde{x} = \lambda^{-1}x$, $\tilde{u} = \lambda^{-1}u$, $\tilde{y} = y$.

²⁷For the language of quasi-ideals and cones, see [9, §1.3.3–1.3.4].

D.4.3. The isomorphism $\mathcal{R}_0 \xrightarrow{\sim} \mathcal{R}_\infty$. Let $K := \mathrm{Ker}(\mathbb{G}_a^\# \xrightarrow{f} \mathbb{G}_a)$. Then

$$\mathcal{R}_0 = \mathrm{Cone}(K \oplus \alpha_p \xrightarrow{(0,1)} \mathbb{G}_a) = \mathrm{Cone}(K \xrightarrow{0} \mathbb{G}_a/\alpha_p) = \mathrm{Cone}(K \xrightarrow{0} \mathbb{G}_a)$$

(we have used the ring isomorphism $\mathbb{G}_a/\alpha_p \xrightarrow{\sim} \mathbb{G}_a$ induced by $\mathrm{Fr} : \mathbb{G}_a \rightarrow \mathbb{G}_a$). On the other hand,

$$\mathcal{R}_\infty = \mathrm{Cone}(\mathbb{G}_a^\# \xrightarrow{0} \mathbb{G}_a).$$

The isomorphism $\mathcal{R}_0 \xrightarrow{\sim} \mathcal{R}_\infty$ comes from the isomorphism

$$-V : \mathbb{G}_a^\# \xrightarrow{\sim} K, \tag{D.7}$$

where V is the Verschiebung of $\mathbb{G}_a^\#$.

D.4.4. Origin of formula (D.7). One gets (D.7) by comparing [2, §2.8.3] with [2, §2.8.1] and [2, Prop. 2.7.12]. The “minus” sign in (D.7) comes from diagram (2.7.5) of [2].

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Шимурові узагальнення стеку $\mathrm{BT}_1 \otimes \mathbb{F}_p$

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Нехай G є гладкою груповою схемою над $\mathbb{F}_p$, оснащеною дією $\mathbb{G}_m$ так, що всі ваги $\mathbb{G}_m$ на $\mathrm{Lie}(G)$ не перевищують 1. Нехай Disp_n^G є стеком n -зрізаних G -дисплеїв Айке Лау (це алгебраїчний $\mathbb{F}_p$ -стек). У випадку $n = 1$ ми вводимо алгебраїчний стек, оснащений морфізмом до Disp_1^G . Ми припускаємо, що якщо $G = GL(d)$, то новий стек канонічно ізоморфний редукції за модулем p стеку 1-зрізаних груп Барсотті–Тейта висоти d та розмірності d' , де d' залежить від дії $\mathbb{G}_m$ на $GL(d)$.

Ми також обговорюємо, як визначити аналог нового стеку для $n > 1$ та як замінити $\mathbb{F}_p$ на $\mathbb{Z}/p^m\mathbb{Z}$.

Ключові слова: група Барсотті–Тейта, многовид Шимури, дисплей, F -зіп, зв'язність, p -кривина, синтоміфікація