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On 2-Convex Non-Orientable Surfaces in Four-Dimensional Euclidean Space

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We prove that a 2-convex closed surface $S \subset E^4$ in the four-dimensional Euclidean space E^4, which is either C^2-smooth or polyhedral, provided that each vertex is incident to at most five edges, admits a mapping of degree one to a two-dimensional torus, where the degree is assumed to be $\mod 2$ if S is non-orientable. As a corollary, we show that the projective plane and the Klein bottle do not admit such a 2-convex embedding in E^4.

Key words: k-convex set, Euclidean space, non-orientable surface

Mathematical Subject Classification 2020: 53A05, 57R19

1. Introduction

Let us recall the following definition (see [8, §16]).

Definition 1.1. A subset C of the Euclidean space E^n is called k-convex if through each point $x \in E^n \setminus C$ there passes a k-dimensional plane π_x that does not intersect C.

Notice that for the case $k = n-1$, we obtain one of the equivalent definitions of usual convexity. If we, moreover, replace E^n by the complex affine space $\mathbb{C}^n$ and take the hyperplane π_x to be complex, we obtain the definition of linear convexity of $C \subset \mathbb{C}^n$ (see [8, Definition 1.6]). Recall that the concept of linear convexity in two-dimensional complex plane $\mathbb{C}^2$ was first introduced in [2] and has gained great importance in complex analysis in several variables (see [1, 5]). Topological properties of linearly convex sets and their real analogues are represented in [8].

Clearly, linearly convex sets in $\mathbb{C}^n$ are $(n-2)$-convex in $\mathbb{R}^{2n} \simeq \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n$. It is not hard to see that the Clifford torus ($S^1 \times S^1 \subset \mathbb{C}^2$) is a linearly convex subset of the complex affine plane $\mathbb{C}^2$ (see [8, Example 1.3]), in particular, this gives an example of a 2-convex subset of E^4.

Yu. Zelinsky asked (see [8, Question 30.5a]): *Is there a 2-convex embedding of a compact K, which is a homological sphere S^2, in the Euclidean space E^4?*

In papers [3, 4], we gave a partial answer to this question and showed that there is no 2-convex embedding $f : S^2 \to E^4$ which is C^2 or PL-embedding such that the valence of vertices does not exceed five. For simplicity, denote such PL-embeddings by $PL(5)$.

In the present paper, we show that our result obtained for the sphere is actually a corollary of the following topological property of 2-convex closed surfaces, which are either C^2-smooth or $PL(5)$ embedded in E^4. Namely, we prove that such a surface admits a mapping of degree one onto a two-dimensional torus (Theorem 3.1). As a corollary, we show that the projective plane and the Klein bottle do not admit C^2- smooth or $PL(5)$-embedding in E^4 (Theorem 3.4, Corollary 3.5). Notice that the Euler characteristic formula shows that the Klein bottle, unlike the projective plane, does not admit triangulations in which the valence of vertices does not exceed five.

2. Preliminaries

2.1. Degree mod 2 of a mapping. Let $f : M \to N$ be a differentiable mapping between two n-dimensional differentiable closed manifolds.

Definition 2.1 (see [7])**.** Let $y \in N$ be a regular value, then we define the degree mod 2 of f (or $\deg_2 f$) by

$$\deg_2 f := \# f^{-1} y \pmod 2. \tag{2.1}$$

It can be shown that the degree mod 2 does not depend on the regular value y.

As in the orientable case, using Poincaré duality with coefficients $\mathbb{Z}_2$, we have

$$\deg_2 f := f^* \mathbf{u}, \tag{2.2}$$

where $\mathbf{u}$ is the generator of $H^n(N; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and $f^* : H^n(N; \mathbb{Z}_2) \to H^n(M; \mathbb{Z}_2)$ is the homomorphism of the cohomology groups induced by f.

Let us recall the cohomology ring structure of the torus T^2, the projective plane P^2 and the Klein bottle K^2. Let $\mathbb{Z}_2[x_1, \dots, x_n]$ denote the polynomial ring over $\mathbb{Z}_2$ in n variables.

A surface S	T^2	K^2	P^2
Cohomology group $H^1(S; \mathbb{Z}_2)$: generators	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ **a**, **b**	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ **c**, **d**	$\mathbb{Z}_2$ **e**
Cohomology ring $H^*(S; \mathbb{Z}_2)$:	$\mathbb{Z}_2[\mathbf{a}, \mathbf{b}]/(\mathbf{a}^2, \mathbf{b}^2)$	$\mathbb{Z}_2[\mathbf{c}, \mathbf{d}]/(\mathbf{c}^2, \mathbf{d}^3, \mathbf{d}^2 - \mathbf{cd})$	$\mathbb{Z}_2[\mathbf{e}]/\mathbf{e}^3$

Table 2.1: Ring structure

2.2. Construction. Recall the construction that underlies the proof of the main result in the case of the sphere (see [3, 4]). Denote by S a closed surface which is C^2-smooth or $PL(5)$ embedded into the Euclidean space E^4. Since S is compact, there exists a ball $B^4 \subset E^4$ that contains S. We will decrease the radius of B^4 until its boundary $S^3 := \partial B^4$ touches S at some point p. Let $T_p S^3$ denote a 3-dimensional plane in E^4 tangent to S^3 at the point p. As was shown in [3, 4], there exists another 3-dimensional plane Π parallel and sufficiently close to $T_p S^3$, which divides E^4 into closed half-spaces $E^4_+ \supset p$ and E^4_- such that the intersection $\gamma := \Pi \cap S = S_+ \cap S_-$, where $S_\pm = E^4_\pm \cap S$, satisfies the properties represented in Table 2.2 below.

C^2-case	PL(5)-case
γ is a connected C^2-smooth curve, $\gamma \subset \partial L_\gamma$, where L_γ denotes the convex hull of γ	γ is a closed k-segmented polygonal chain, $3 \leq k \leq 5$
$p \in S_+$ and S_+ is homeomorphic to the disk D^2	$S_+ = C\gamma$ is the cone with vertex p and base γ

Table 2.2: Properties of γ.

These properties are obvious in $PL(5)$ - case. In C^2 - case, we can represent the surface S in a small neighborhood of the point p as follows:

$$\begin{cases} x^3 = f(x^1, x^2), \\ x^4 = g(x^1, x^2), \end{cases}$$

where $\{x_i : i = 1, \ldots, 4\}$ are the Euclidean coordinates in E^4 such that the point p is the origin and the coordinate frame $\{e_i : i = 1, \ldots, 4\}$ has the property that $\{e_1, e_2\}$ is a basis of the tangent plane T_pS and e_3 is orthogonal to the tangent plane T_pS^3, and f is a convex function. Thus, the desired 3-dimensional plane Π can be taken as $\{x^3 = \varepsilon\}$, where ε is chosen such that the curve $f(x^1, x^2) = \varepsilon$ is a convex closed curve in the 3-dimensional plane $\Pi_0 := \{x^4 = 0\}$.

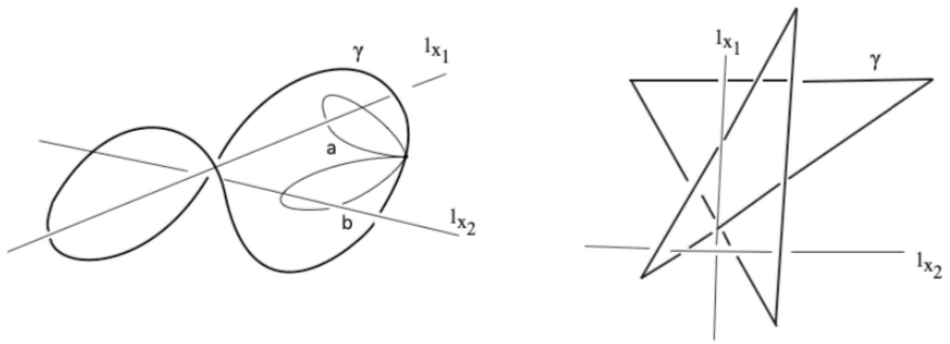
Observe that $l_x := \pi_x \cap \Pi$ is a line if $x \in L_\gamma \setminus \gamma$ (see Definition 1.1). The basic observation we made for the case of sphere in [3, 4] is transferred one-to-one to the case of an arbitrary closed surface of the following theorem.

Theorem 2.2. *Let $S \subset E^4$ be a 2-convex closed surface which is C^2 or $PL(5)$-embedded in E^4. Let Π be a 3-dimensional plane in E^4 satisfying the conditions above (see Table 2.2). Then one of the following possibilities occurs:*

1. *There exist $x \in L_\gamma \setminus \gamma$ and π_x from Definition 1.1 such that $[\gamma]$ is a generator of $\pi_1(\Pi \setminus l_x) \cong \mathbb{Z}$.*
2. *There exist $x_1, x_2 \in \operatorname{int} L_\gamma$ and π_{x_1}, π_{x_2} from Definition 1.1 such that:*
   (a) *π_{x_1} and π_{x_2} are in a general position;*
   (b) *$l_{x_1} \cap l_{x_2} = \varnothing$;*
   (c) *$[\gamma] = [a][b][a]^{-1}[b]^{-1}$, where a, b are the circles of the bouquet $S^1 \vee S^1$ representing the generators of the group $\pi_1(\Pi \setminus (l_{x_1} \cup l_{x_2})) \cong \mathbb{Z} * \mathbb{Z}$ (see Fig. 2.1).*

Sketch of the proof for C^2-case. If γ is flat, then item 1 holds obviously. Suppose, γ is not flat, then γ separates ∂L_γ into two connected parts A_i, $i = 1, 2$, homeomorphic to open discs. If there exists a point $x \in \operatorname{int} L_\gamma$ and π_x such that $l_x \cap A_i$, $i \in \{1, 2\}$ is one point, then item 1 holds obviously. Otherwise, $\operatorname{int} L_\gamma = \bigcup_i C_i$, $i = 1, 2$, where C_i is characterized as follows:
$x_i \in C_i$ iff there exists a plane π_{x_i} through x_i from Definition 1.1 such that $l_{x_i} \cap A_i \neq \varnothing$.

In this case, $l_{x_i} \cap A_i$ consists of two points. Observe that $C_i \neq \varnothing$. Indeed, let $y_i \in A_i$ be a smooth boundary point, i.e., there exists only one supporting



a) C^2-case b) PL(5) - case

Fig. 2.1: Link

plane T_{y_i} through y_i. Such points are everywhere dense in A_j [6]. If $l_{y_i} \subset T_{y_i}$, we can move the plane π_{y_i} a little bit to get $l_{y_i} \cap \operatorname{int} L_\gamma \neq \varnothing$. On the other hand, obviously, C_i are open subsets in $\operatorname{int} L_\gamma$. From the connectivity of $\operatorname{int} L_\gamma$, it follows that $C_1 \cap C_2 \neq \varnothing$. This means that there exists $x \in \operatorname{int} L_\gamma$ and planes π_i, $i = 1, 2$, through x from Definition 1.1 such that $l_1 \cap l_2 = x$ and $l_i \cap A_i \neq \varnothing$, where $l_i := \pi_i \cap \Pi$. Now, by a small perturbation of the planes π_i to the planes π_{x_i} through some points $x_1, x_2 \in \operatorname{int} L_\gamma$, we can get 2(a) and 2(b) of Theorem 2.2 with the mutual arrangement of the straight lines l_{x_1}, l_{x_2} and the curve γ as a nontrivial link as shown in Fig. 2.1a). The diagram of this link can be obtained by the orthogonal projection $p : \Pi \to \Pi \cap \Pi_0$. Item 2(c) is checked directly. □

Remark 2.3. As will be shown in the proof of Theorem 3.1, the situation described in item 1 of Theorem 2.2 is impossible.

3. Main result

Theorem 3.1. *If $S \subset E^4$ is a 2-convex closed surface which is either C^2 or $PL(5)$-embedded into E^4, then it admits a mapping of degree one to a two-dimensional torus. If S is non-orientable, we assume that the degree is taken modulo 2.*

Proof. Suppose $f : S \to E^4$ is a 2-convex embedding satisfying the condition of the theorem. Then Theorem 2.2 is satisfied. If item 1 of Theorem 2.2 holds, then we immediately come to a contradiction since γ bounds $S_+ \simeq D^2$, which contradicts to $[\gamma] \neq 0$ in $\pi_1(\Pi \setminus l_x)$ and therefore $[\gamma] \neq 0$ in $\pi_1(E^4 \setminus \pi_x)$ since $\Pi \setminus l_x$ is the deformation retract of $E^4 \setminus \pi_x$.

Let us suppose that item 2 of Theorem 2.2 is satisfied. Then, from 2(a), it follows that $\pi_{x_1} \cap \pi_{x_2}$ is a point, which we denote by O. Thus, there are two possible cases.

Case 1. $O \in E^4_-$. In this case, we have the deformation retraction

$$r_+ : E^4_+ \setminus (\pi_{x_1} \cup \pi_{x_2}) \to \Pi \setminus (l_{x_1} \cup l_{x_2}),$$

which is defined as follows:

$$r_+(x) = l_{Ox} \cap \Pi,$$

where l_{Ox} denotes the straight line passing through O and $x \in E^4_+$.

Recalling that deformation retraction induces an isomorphism of fundamental groups, we immediately come to a contradiction since γ bounds the disk

$$S_+ \subset E^4_+ \setminus (\pi_{x_1} \cup \pi_{x_2})$$

and $[\gamma]$ must be equal to zero in

$$\pi_1(E^4_+ \setminus (\pi_{x_1} \cup \pi_{x_2})) \simeq \pi_1(\Pi \setminus (l_{x_1} \cup l_{x_2})),$$

which contradicts to 2(c) of Theorem 2.2.

Case 2. $O \in E^4_+$. In this case, as above, we have the deformation retraction

$$r_- : E^4_- \setminus (\pi_{x_1} \cup \pi_{x_2}) \to \Pi \setminus (l_{x_1} \cup l_{x_2})$$

and the homotopically inverse to r_- embedding

$$i_- : \Pi \setminus (l_{x_1} \cup l_{x_2}) \to E^4_- \setminus (\pi_{x_1} \cup \pi_{x_2}).$$

Moreover, we have the following homotopy equivalences:

$h_1 :\ \Pi \setminus (l_{x_1} \cup l_{x_2}) \overset{r_1}{\to} S^1 \vee S^1 \overset{i_1}{\hookrightarrow} \Pi \setminus (l_{x_1} \cup l_{x_2}),$

$h_2 :\ S_- \overset{r_2}{\to} \vee_p S^1_p \overset{i_2}{\hookrightarrow} S_-$ (see Remark 3.2),

where $\vee_{p=1}^n S^1_p$ is a bouquet of circles generating $\pi_1(S_-)$, and r_k, i_k $(k = 1, 2)$ are the deformation retractions and the homotopically inverse embeddings to them.

Remark 3.2. Notice that the choice of the bouquet $\vee_{p=1}^n S^1_p$ generating $\pi_1(S_-)$ and the deformation retraction $r_2 : S_- \to \vee_{p=1}^n S^1_p$ are ambiguous. Moreover, we can always choose circles S^1_p such that $\cap_{p=1}^n S^1_p \cap \gamma$ is a single point (the basepoint of S_-), which we denote by o. Fixing the orientations of the circles S^1_p and cutting along them, we obtain a representation of S_- by a 2n-gon P with a hole bounded by γ and oriented sides that are pairwise identified (see Fig. 3.1 for the case where S is either P^2 or K^2). The deformation retraction

$$r_2 : P \setminus \operatorname{int} S_+ \to \partial P$$

is shown in Fig. 3.1 by thin arrows. This determines the equality $[\gamma] = [w] \in \pi_1(S_-)$, where $w = r_2(\gamma)$.

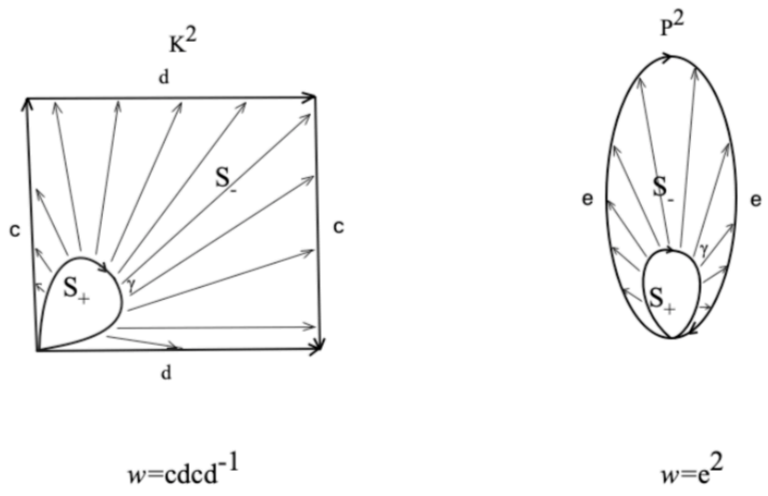


Fig. 3.1: Deformation retraction.

Remark 3.3. The marking of the sides of polygons in Fig. 3.1 is not accidental since the sides represent homological classes of the group $H_1(S;\mathbb{Z}_2)$ of the surface S obtained by the gluing of the corresponding sides of the polygons, which are Poincaré dual to the classes of cohomologies in $H^1(S;\mathbb{Z}_2)$ denoted by the same letters in Table 2.1. Note that the modulo 2 indices of intersection of homological classes correspond to the multiplication of cohomology classes with coefficients in $\mathbb{Z}_2$.

Let us consider the following composition of the mappings:

$$\psi := r_1 \circ r_- \circ \bar{f} \circ i_2 : \vee_p S^1_p \to S^1 \vee S^1,$$

where $\bar{f}$ is defined from the composition

$$f|_{S_-} : S_- \xrightarrow{\bar{f}} E^4_- \setminus (\pi_{x_1} \cup \pi_{x_2}) \hookrightarrow E^4.$$

Taking into account the definition of w and 2(c) of Theorem 2.2, we have

$$\psi_*[w] = \psi_* r_{2*}[\gamma] = (r_1 \circ r_-)_* \bar{f}_* i_{2*} r_{2*}[\gamma] = (r_1 \circ r_-)_* \bar{f}_*[\gamma] = [a][b][a]^{-1}[b]^{-1},$$

where $*$ indicates the induced homomorphisms of fundamental groups.

Let us complete the natural cellular decomposition of $S^1 \vee S^1$ to the cellular decomposition of the torus T^2 as shown in Fig. 3.2. Observe that S has a natural cellular decomposition generated by o, $\vee_p S^1_p \setminus o$, $\gamma \setminus o$, $\operatorname{int} S_-$, $\operatorname{int} S_+$. We claim that one can extend ψ to the continuous map $\Psi : S \to T^2$ of degree one.

One can extend ψ to the map of 1-skeletons $\Psi^{(1)} : S^{(1)} \to T^{2(1)}$ by means of some diffeomorphism $\Psi^{(1)}|_\gamma : \gamma \to u$ preserving orientations. By the construction, we have

$$(\Psi^{(1)})_*[\gamma]^{-1}[w] = [u]^{-1}[a][b][a]^{-1}[b]^{-1}.$$

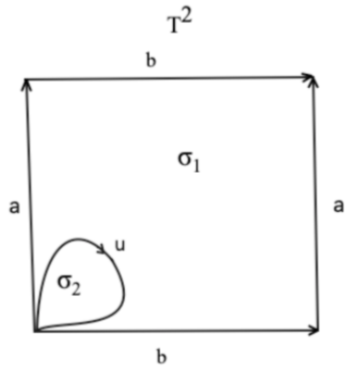


Fig. 3.2: Cellular decomposition of T^2.

Thus, we can extend $\Psi^{(1)}$ to S_-, whose images are in $\bar{\sigma}_1$ (see Fig. 3.2), and then to S_+ by some diffeomorphism $\Psi|_{S_+} : S_+ \to \bar{\sigma}_2$. Smoothing Ψ and leaving them unchanged in a small disk $B \subset \operatorname{int} S_+$, show that the degree of Ψ is equal to one (see (2.1)). □

Theorem 3.4. *Neither the projective plane nor the Klein bottle admits a continuous mapping onto the torus T^2 of degree one* mod 2.

Proof. Let us suppose that $g : S \to T^2$ is a mapping of a closed surface S to the torus T^2 and $\deg_2 g = 1$. Then, using the ring structure of $H^*(T^2; \mathbb{Z}_2)$ (see Table 2.1) and (2.2), we have

$$g^*(\mathbf{a} \cup \mathbf{b}) = g^*\mathbf{a} \cup g^*\mathbf{b} \neq 0. \tag{3.1}$$

It should be noticed that

$$\mathbf{a} \cup \mathbf{a} = \mathbf{b} \cup \mathbf{b} = 0. \tag{3.2}$$

Case 1. $S = P^2$. In this case, it follows from the ring structure of $H^*(P^2; \mathbb{Z}_2)$ (see Table 2.1) and (3.1) that $g^*\mathbf{a} = g^*\mathbf{b} = \mathbf{e}$. But this leads to a contradiction with (3.2):

$$0 \neq \mathbf{e}^2 = g^*\mathbf{a} \cup g^*\mathbf{a} = g^*(\mathbf{a} \cup \mathbf{a}) = 0.$$

Case 2. $S = K^2$. Let $\mathbf{c}, \mathbf{d}$ be generators of the group $H^1(K^2; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (see Table 2.1). If $g^*\mathbf{a} = g^*\mathbf{b} = \mathbf{c}$, then we obtain the contradiction:

$$0 = \mathbf{c}^2 = g^*\mathbf{a} \cup g^*\mathbf{b} \neq 0.$$

The last inequality is caused by (3.1). Thus, one of two things happens:

(a) $g^*\mathbf{a} \in \{\mathbf{d}, \mathbf{d} + \mathbf{c}\}$ or
(b) $g^*\mathbf{b} \in \{\mathbf{d}, \mathbf{d} + \mathbf{c}\}$.

Without loss of generality, we can assume that (a) is satisfied. Using the ring structure of $H^*(K^2; \mathbb{Z}_2)$, we have

- $\mathbf{d}^2 \neq 0$,
- $(\mathbf{c} + \mathbf{d}) \cup (\mathbf{c} + \mathbf{d}) = \mathbf{c}^2 + 2\mathbf{c} \cup \mathbf{d} + \mathbf{d}^2 = \mathbf{d}^2 \neq 0$.

But this yields to a contradiction: $0 \neq g^*\mathbf{a} \cup g^*\mathbf{a} = g^*(\mathbf{a} \cup \mathbf{a}) = 0$ (the last equality follows from (3.2)). □

Theorems 3.1 and 3.4 imply the corollary.

Corollary 3.5. *Neither the projective plane nor the Klein bottle admits a C^2-smooth or $PL(5)$-embedding in a four-dimensional Euclidean space as a 2-convex surface.*

4. Final remarks

Finally, we note that the question under consideration for non-orientable surfaces M^2_μ, $\mu \geq 3$, where μ denotes the number of Möbius bands glued into the sphere S^2, remains open. Indeed, any such surface is homeomorphic to the torus T^2 with $\mu - 2$ Möbius bands glued into it. The mapping $g : M^2_\mu \to T^2$, contracting each such Mobius band to a point, has $\deg_2 g = 1$ (see Fig. 4.1) and Theorem 3.4 is wrong for $S = M^2_\mu$, $\mu \geq 3$.

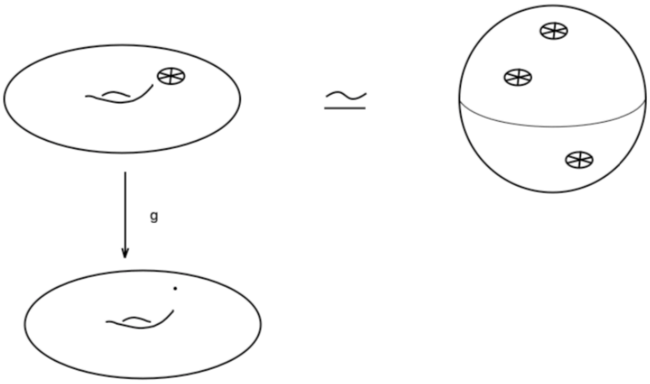


Fig. 4.1: Degree one mod 2 map of M^2_3 to the torus T^2.

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Про 2-опуклі неорієнтовні поверхні в чотиривимірному евклідовому просторі

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Доведено, що 2-опукла замкнута поверхня $S \subset E^4$ у чотиривимірному евклідовому просторі E^4, яка є або C^2-гладкою, або поліедральною, за умови, що кожна вершина інцидентна не більше ніж п'яти ребрам, допускає відображення степеня один у двовимірний тор, де степінь розглядається за $\mod 2$, якщо S є неорієнтовною. Як наслідок, ми показуємо, що проєктивна площина і пляшка Кляйна не допускають такого 2-опуклого вкладення в E^4.

Ключові слова: k-опукла множина, евклідів простір, неорієнтовна поверхня