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Measures and Dynamics on Pascal–Bratteli Diagrams

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We introduce and study dynamical systems and measures on stationary generalized Bratteli diagrams B that are represented as the union of countably many classical Pascal–Bratteli diagrams. We describe all ergodic tail invariant measures on B. For every probability tail invariant measure ν_p on the classical Pascal–Bratteli diagram, we approximate the support of ν_p by the path space of a subdiagram. By considering various orders on the edges of B, we define dynamical systems with various properties. We show that there exist orders such that the sets of infinite maximal and infinite minimal paths are empty. This implies that the corresponding Vershik map is a homeomorphism. We also describe orders on both B and the classical Pascal–Bratteli diagram that generate either uncountably many minimal infinite and uncountably many maximal infinite paths, or uncountably many minimal infinite paths alongside countably infinitely many maximal infinite paths.

Key words: Borel dynamical systems, Bratteli-Vershik model, tail invariant measures, Pascal–Bratteli diagram

Mathematical Subject Classification 2020: 37A05, 37B05, 37A40, 54H05, 28D05

1. Introduction

We have initiated the study of generalized Bratteli diagrams in a series of recent works (see, e.g., [2, 5, 6, 10, 11]). In this paper, we focus on investigating invariant measures and dynamics for both the standard Pascal–Bratteli diagram and a broader class of generalized Bratteli diagrams, which can be represented as countable unions of classical Pascal graphs.

Generalized Bratteli diagrams serve as models for Borel automorphisms of standard Borel spaces. The tail equivalence relation on the path space of a generalized Bratteli diagram describes the dynamical properties of the corresponding automorphisms. These diagrams feature countably infinite vertices at each level, and their path spaces are non-compact Polish spaces. The corresponding incidence matrices are also infinite. These characteristics introduce new challenges in studying invariant measures and dynamics for generalized Bratteli diagrams compared to the standard Bratteli diagrams typically used in Cantor dynamics.

The set of various classes of generalized Bratteli diagrams is large and significantly different from that of standard Bratteli diagrams. For example, there are no simple generalized Bratteli diagrams, although the tail equivalence relation can be minimal for some diagrams.

The class of stationary standard Bratteli diagrams plays an important role in this area, as they provide models for substitution dynamical systems on finite alphabets [12, 16, 19]. The set of invariant measures for stationary standard Bratteli diagrams is fully described in [13]. However, for stationary generalized Bratteli diagrams, the situation is much more complex. The results in [13] no longer apply because generalized Bratteli diagrams can admit infinite σ-finite measures that assign finite values to all cylinder sets (see [10]). It was shown in [6] that stationary generalized Bratteli diagrams serve as models for a class of substitution dynamical systems on infinite alphabets. Substitutions on infinite alphabets were also studied in [17, 18, 25]

Pascal–Bratteli diagrams (also called Pascal diagrams, Pascal graphs, Pascal adic systems) were studied in numerous papers, see, for example, [14, 20–22, 27–31]. In [11], $\mathbb{N}$-infinite and $\mathbb{Z}$-infinite generalized Pascal–Bratteli diagrams were studied, and the set of probability ergodic invariant measures for these diagrams was completely described.

In this paper, we study ergodic invariant probability measures on standard Pascal–Bratteli diagrams using vertex subdiagrams. We describe subdiagrams of the standard Pascal–Bratteli diagram that are pairwise disjoint for different ergodic invariant probability measures and can be chosen so that the corresponding ergodic measure of their path space is arbitrarily close to 1.

We introduce stationary generalized Bratteli diagrams B, which are represented as the union of countably many standard Pascal–Bratteli diagrams, and describe all ergodic invariant probability measures on B. Additionally, we study various orders on both the standard Pascal–Bratteli diagram and the generalized Bratteli diagram B, which contains infinitely many classical Pascal–Bratteli diagrams. We introduce an order on the standard Pascal–Bratteli diagram that has uncountably many minimal infinite and uncountably many maximal infinite paths, and an order that has uncountably many minimal infinite and countably infinitely many maximal infinite paths. Conversely, we show that B can be endowed with an order that has no minimal infinite or maximal infinite paths (in which case the corresponding Vershik map is a homeomorphism), as well as with an order that has uncountably many minimal infinite and uncountably many maximal infinite paths and an order that has uncountably many minimal infinite and countably infinitely many maximal infinite paths.

The outline of the paper is as follows. In Section 2, we provide all necessary definitions and recall the procedure for measure extension from a subdiagram. In Section 3, we describe subdiagrams X_p of a standard Pascal–Bratteli diagram such that $\{X_p\}_{p\in(0,1)}$ are pairwise disjoint for different ergodic probability invariant measures ν_p, and can be chosen so that the measure $\nu_p(X_p)$ is arbitrarily close to 1. Section 4 is devoted to various orders on the standard Pascal–Bratteli diagram. We construct an order on the standard Pascal–Bratteli diagram that has a continuum of minimal infinite paths and a continuum of maximal infinite paths. We also answer the question posed in [22] and construct an order on the standard Pascal–Bratteli diagram that has a continuum of minimal infinite paths and countably infinitely many maximal infinite paths. In Section 5, we consider a stationary generalized Bratteli diagram B which contains countably many standard Pascal–Bratteli diagrams as vertex subdiagrams. We describe all probability ergodic invariant measures on B. Using the results from Section 4, we show that B can be ordered so that it has a continuum of minimal infinite paths and a continuum of maximal infinite paths, thus providing an example of a stationary generalized Bratteli diagram which has such a property. We also construct an order on B that has uncountably many minimal infinite and countably infinitely many maximal infinite paths. We also demonstrate that there is an order on the two-sided version of B that has no maximal infinite paths and no minimal infinite paths; thus, the corresponding Vershik map is a homeomorphism.

Our main results are contained in Theorems 3.1, 4.1, 4.2 and 5.3.

2. Preliminaries

In this section, we briefly remind the reader of the definitions of main objects considered in the paper.

2.1. Standard and generalized Bratteli diagrams Standard Bratteli diagrams and Vershik maps on them are models for homeomorphisms of a Cantor set [15, 24, 26].

Definition 2.1. A *standard Bratteli diagram* is an infinite graded graph $B = (V, E)$ such that the vertex set $V = \bigsqcup_{i\geq 0} V_i$ and the edge set $E = \bigsqcup_{i\geq 0} E_i$ are partitioned into disjoint subsets V_i and E_i, where

- (i) $V_0 = \{v_0\}$ is a single point;
- (ii) V_i and E_i are finite sets for all i;
- (iii) there exists a range map $r\colon E \to V$ and a source map $s\colon E \to V$ such that $r(E_i) = V_{i+1}$ and $s(E_i) = V_i$ for all $i \geq 1$.

A generalized Bratteli diagram is a natural extension of the notion of a standard Bratteli diagram. Generalized Bratteli diagrams have countably many vertices on each level and provide models for Borel automorphisms of standard Borel spaces [2, 6].

Definition 2.2. A *generalized Bratteli diagram* is an infinite graded graph $B = (V, E)$ such that the vertex set V and the edge set E can be partitioned $V = \bigsqcup_{i=0}^{\infty} V_i$ and $E = \bigsqcup_{i=0}^{\infty} E_i$ so that the following properties hold (there is no need to assume that V_0 is a singleton):

- (i) For every $i \in \mathbb{Z}_+$, the number of vertices at each level V_i is countably infinite, and the set E_i of all edges between V_i and V_{i+1} is countable.
- (ii) For every edge $e \in E$, we define the range and the source maps r and s such that $r(E_i) = V_{i+1}$ and $s(E_i) = V_i$ for $i \in \mathbb{Z}_+$.
- (iii) For every vertex $v \in V \setminus V_0$, we have $|r^{-1}(v)| < \infty$.

A (finite or infinite) *path* in the diagram is a (finite or infinite) sequence of edges $(e_i : e_i \in E_i)$ such that $s(e_i) = r(e_{i-1})$. Denote by X_B the set of all infinite paths that start at V_0. The set

$$[\overline{e}] := \{x = (x_i) \in X_B : x_0 = e_0, \dots, x_n = e_n\}$$

is called the *cylinder set* associated with a finite path $\overline{e} = (e_0, \dots, e_n)$. Cylinder sets generate a topology on X_B such that X_B becomes a 0-dimensional Polish space. Recall that for a standard Bratteli diagram, X_B is compact, while the path space of a generalized Bratteli diagram is just a Polish space (it can be locally compact for some special classes of generalized Bratteli diagrams).

For $n = 0, 1, \dots$, let the element $f^{(n)}_{v,w}$ of the n-th incidence matrix $F_n = (f^{(n)}_{v,w})$ be the number of all edges between $v \in V_{n+1}$ and $w \in V_n$. If $F_n = F$ for all n, we call the corresponding generalized Bratteli diagram stationary. If all incidence matrices F_n are $\mathbb{N} \times \mathbb{N}$ matrices, we call the corresponding generalized Bratteli diagram B one-sided. If all F_n are $\mathbb{Z} \times \mathbb{Z}$ matrices, we call B two-sided.

2.2. Vershik map and tail equivalence relation To define dynamics on Bratteli diagrams, we need the notion of an ordered Bratteli diagram. An *ordered* (standard or generalized) Bratteli diagram $B = (B, V, >)$ is a (standard or generalized) Bratteli diagram $B = (V, E)$ together with a partial order $>$ on E such that edges e, e' are comparable if and only if $r(e) = r(e')$ (see [5, 7, 24] for more details). A (finite or infinite) path $e = (e_0, e_1, \dots, e_i, \dots)$ is called *maximal* (respectively *minimal*) if every e_i is maximal (respectively minimal) among all elements of $r^{-1}(r(e_i))$. We denote the sets of all maximal infinite and all minimal infinite paths by $X_{\max}$ and $X_{\min}$, respectively. It is not hard to see that these sets are closed. For brevity, we call the elements of $X_{\max}$ and $X_{\min}$ extreme paths. Note that for a standard Bratteli diagram, the sets $X_{\max}$ and $X_{\min}$ are necessarily nonempty, while for generalized Bratteli diagrams, there are orders for which $X_{\min} = X_{\max} = \varnothing$ (see Section 4 in [5] or [2]).

To introduce the Vershik map, first define a Borel transformation $\varphi_B : X_B \setminus X_{\max} \to X_B \setminus X_{\min}$ as follows: given $x = (x_0, x_1, \dots) \in X_B \setminus X_{\max}$, let k be the smallest number such that x_k is not maximal. Let y_k be the successor of x_k in the finite set $r^{-1}(r(x_k))$. Define $\varphi_B(x) := (y_0, y_1, \dots, y_{k-1}, y_k, x_{k+1}, \dots)$, where $(y_0, \dots, y_{k-1})$ is the unique minimal path from $s(y_k)$ to V_0. Note that such defined φ_B is, in fact, a homeomorphism. To extend the definition of φ_B to X_B, we need to determine a bijection from $X_{\max} \to X_{\min}$. It can be always done in a Borel manner if $X_{\max}$ and $X_{\min}$ have the same cardinality. Then (X_B, φ_B) is called a generalized Bratteli-Vershik dynamical system associated with an ordered Bratteli diagram $(B, >)$. In some cases (for example, when $X_{\max}$ and $X_{\min}$ are empty sets), this extension can be made continuous, and φ_B will be a homeomorphism of X_B. These questions have been discussed in [2, 5, 10]).

Two paths $x = (x_i)$ and $y = (y_i)$ in X_B are called *tail equivalent* if there exists an $n \in \mathbb{Z}_+$ such that $x_i = y_i$ for all $i \geq n$. This notion defines a countable Borel equivalence relation $\mathcal{R}$ on the path space X_B which is called the *tail equivalence relation*.

Throughout the paper, by the term *measure* we will mean a non-atomic positive Borel measure on a Polish space. We will use the fact that any such measure is completely determined by its values on cylinder sets.

Definition 2.3. A measure μ on X_B is called *tail invariant* if, for any cylinder sets $[\overline{e}]$ and $[\overline{e}']$ such that $r(\overline{e}) = r(\overline{e}')$, we have $\mu([\overline{e}]) = \mu([\overline{e}'])$.

2.3. Measure extension from subdiagrams In this subsection, we briefly describe the procedure of measure extension from a vertex subdiagram of a Bratteli diagram. This procedure works in the same way for both standard and generalized Bratteli diagrams. For more details, see [1, 8, 10].

Let B be a standard or generalized Bratteli diagram. A vertex subdiagram $\overline{B}$ of B is a (standard or generalized) Bratteli diagram such that the set of vertices W is formed by nonempty proper subsets $W_n \subset V_n$ and the set of edges consists of all edges of B whose source and range are in W_n and W_{n+1}, respectively. In other words, the incidence matrix $\overline{F}_n$ of $\overline{B}$ has the size $|W_{n+1}| \times |W_n|$ (it may be $\infty \times \infty$), and it is represented by a block of F_n corresponding to the vertices from W_{n+1} and W_n. Note that for a generalized Bratteli diagram B, we consider both standard and generalized Bratteli subdiagrams $\overline{B}$, i.e., $|W_n|$ can be finite or infinite.

Let B be a standard or generalized Bratteli diagram and $\overline{B}$ be its vertex subdiagram. Let $\overline{\mu}$ be an ergodic tail invariant probability measure on $X_{\overline{B}}$. Then $\overline{\mu}$ can be canonically extended to the ergodic measure $\widehat{\overline{\mu}}$ on the space X_B by tail invariance: let $p_w^{(n)}$ be a measure $\overline{\mu}([\overline{e}])$ of any cylinder set $[\overline{e}] \subset X_{\overline{B}}$ such that $r(\overline{e}) = w \in W_n$. Then for any cylinder set $[\overline{e}'] \subset X_B$ such that $r(\overline{e}') = w \in W_n$, we set $\widehat{\overline{\mu}}([\overline{e}']) := \overline{\mu}([\overline{e}]) = \overline{p}_w^{(n)}$. Denote by $\widehat{X}_{\overline{B}} := \mathcal{R}(X_{\overline{B}})$ the subset of all paths in X_B that are tail equivalent to paths from $X_{\overline{B}}$. Then $\widehat{X}_{\overline{B}}$ is the smallest $\mathcal{R}$-invariant subset of X_B containing $X_{\overline{B}}$. After setting $\widehat{\overline{\mu}}(X_B \setminus \widehat{X}_{\overline{B}}) = 0$, we obtain an ergodic tail invariant measure $\widehat{\overline{\mu}}$ on the whole path space X_B.

One can compute the measure $\widehat{\overline{\mu}}(X_B)$ as follows. Let $\widehat{X}_{\overline{B}}^{(n)}$ be the set of all paths $x = (x_i)_{i=0}^{\infty}$ from X_B such that the finite path $(x_0, \ldots, x_{n-1})$ ends at a vertex $w \in W_n$ of $\overline{B}$, and the tail $(x_n, x_{n+1}, \ldots)$ belongs to $\overline{B}$, i.e.,

$$\widehat{X}_{\overline{B}}^{(n)} = \{x = (x_i) \in \widehat{X}_{\overline{B}} : r(x_{i-1}) \in W_i,\ \forall i \geq n\}. \tag{2.1}$$

It is clear that $\widehat{X}_{\overline{B}}^{(n)} \subset \widehat{X}_{\overline{B}}^{(n+1)}$ and $\widehat{X}_{\overline{B}} = \bigcup_n \widehat{X}_{\overline{B}}^{(n)}$. Moreover, we have

$$\begin{aligned}\widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}) = \lim_{n\to\infty} \widehat{\overline{\mu}}(\widehat{X}_{\overline{B}}^{(n)}) &= \lim_{n\to\infty} \sum_{w \in W_n} H_w^{(n)} \overline{p}_w^{(n)} \\ &= 1 + \sum_{n=0}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W_n'} f_{vw}^{(n)} H_w^{(n)} \overline{p}_v^{(n+1)},\end{aligned} \tag{2.2}$$

where $W_n' = V_n \setminus W_n,\ n = 0, 1, 2, \ldots$, and $H_w^{(n)}$ is the number of paths between the vertex $w \in V_n$ and vertices of V_0 (see Theorem 3.1 in [10]). The value $\widehat{\overline{\mu}}(X_B)$ can be either finite or infinite. If it is finite, then we say that $\overline{\mu}$ admits a finite measure extension.

3. Disjoint subdiagrams for ergodic invariant measures

In [9, Subection 5.2], a class of (standard) Bratteli diagrams was considered such that every ergodic invariant probability measure can be obtained as the measure extension from a uniquely ergodic vertex subdiagram. It was shown in [9, Subsection 6.2] that the Pascal–Bratteli diagram does not belong to this class. In this section, we prove that for every ergodic invariant probability measure ν_p on the Pascal–Bratteli diagram, one can find a subdiagram B_p such that the measure $\nu_p(X_p)$ is arbitrarily close to 1, and the sets X_p are pairwise disjoint, where X_p is the path space of B_p, $0 < p < 1$.

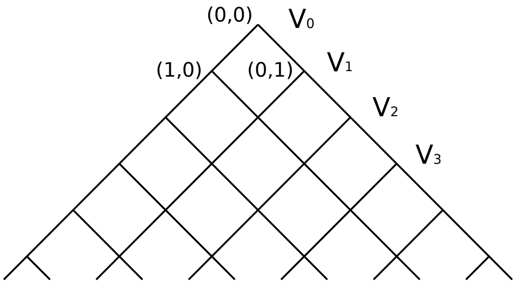

Fig. 3.1: Pascal diagram

The vertices of the *Pascal diagram* can be labeled by pairs of non-negative integers (i, j). Each vertex (i, j) is connected by an edge to two vertices $(i+1, j)$ and $(i, j+1)$. Typically, this diagram is drawn expanding downwards with the vertex $O = (0, 0)$ at the top (see Fig. 3.1). The vertices are subdivided into levels $V_n = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i+j = n\}$. Observe that each vertex $v \in V_n$ is connected by an edge with one or two vertices from V_{n-1} for $n \geqslant 1$. Geometrically, it will be useful for us to view the Pascal diagram (triangle) as a subset of $\mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R} \times \mathbb{R}$. By a path in the Pascal diagram, we will mean a (finite or infinite) sequence of edges $\{e_0, e_1, e_2, \ldots\}$ of the diagram such that e_j connects a vertex of V_j with a vertex of V_{j+1} and is connected to e_{j+1} for each j.

Ergodic tail invariant measures on the Pascal diagram are labeled by a real number (probability) $0 < p < 1$ such that each edge of the form $(v, v + (1, 0))$ is given weight p and each edge of the form $(v, v + (0, 1))$ is given weight $1 - p$ (see, e.g., [28]). Denote the corresponding measure on the path space of the Pascal diagram by ν_p. We can formally include the values $p = 0$ and $p = 1$ in the consideration, but this case gives atomic measures ν_p and is not interesting.

Theorem 3.1. *For each $0 < p < 1$ and each $\varepsilon > 0$, there exists a subdiagram B_p of the Pascal diagram such that*

1) $\nu_p(X_p) > 1 - \varepsilon$ *for each p, where X_p is the path space of B_p;*
2) *the subspaces $X_p, 0 \leqslant p \leqslant 1$, are pairwise disjoint.*

Proof. Fix $0 < p < 1$. We will construct by induction a sequence $N_i = N_i(p)$ such that for the sets

$$A_i = A_i(p) = \left\{\{(x_n, y_n)\} \in X_B : \left|\frac{x_k}{k} - p\right| < 2^{1-i} \ \text{ for all } \ k > N_i\right\},$$

one has $\nu_p(\bigcap_{i\leqslant j} A_i) > 1-\varepsilon$ for every $j \in \mathbb{N}$. Let $N_0 = 0$.

Base of induction. Since $0 \leqslant x_n \leqslant n$ for all n, we set $A_0 = X_B$. One has $\nu_p(A_0) = 1$.

Step of induction. Let $j \in \mathbb{N}$. Assume that N_i are constructed for $0 \leqslant i \leqslant j$ such that

$$\nu_p\left(\bigcap_{i\leqslant j} A_i\right) > 1-\varepsilon.$$

By the law of large numbers, for ν_p almost all $\{(x_n, y_n)\} \in X_B$, one has

$$\lim_{n\to\infty} \frac{x_n}{n} = p.$$

It follows that for N_{j+1} large enough, one can make $\nu_p(A_{j+1})$ be arbitrarily close to 1. This implies that for N_{j+1} large enough, we have

$$\nu_p\left(\bigcap_{i\leqslant j+1} A_i\right) > 1-\varepsilon,$$

and that proves the induction step.

Now, let $A(p) = \bigcap_{i\in\mathbb{Z}_+} A_i(p)$. Introduce a subdiagram $B_p = (V_p, E_p)$ of the Pascal diagram as follows. Let V_p be the set of all vertices $v \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that there exists a path in $A(p)$ containing v. Let E_p be the set of all edges of the Pascal diagram having both ends in V_p.

Condition 1) of Theorem 3.1 is satisfied by construction. Let $0 < p \neq q < 1$. Then there exists i such that $|p-q| > 2^{2-i}$. By definition, $A_i(p)$ and $A_i(q)$ are disjoint. This implies that condition 2) is satisfied as well. □

Remark 3.2. The subdiagrams B_p are not pairwise disjoint as graphs, but their path spaces X_p are disjoint.

4. Orders on the Pascal–Bratteli diagram

The following theorem shows that there is an order on the Pascal–Bratteli diagram for which the sets of minimal infinite and maximal infinite paths are of the cardinality continuum. This result can also be found in [22, Example 7.2], where the authors provide a sketch of its proof. Here we provide a detailed proof, which has an idea similar to the proof in [22], but the construction has principal differences with the construction from [22]. The main idea of our proof is constructing minimal paths along every direction. This is motivated in part by Theorem 3.1, where every subdiagram is constructed along a particular direction of the Pascal–Bratteli diagram. We hope that this second geometrical approach will contribute to the further development of the theory.

Theorem 4.1. *There exists an ordering of the edges of the Pascal diagram into* $0, 1$ *such that both the set of minimal paths* $X_{\min}$ *and the set of maximal paths* $X_{\max}$ *have the cardinality of the continuum.*

Proof. Let $\mathbb{D}$ be the subset of dyadic numbers of $[0, 1)$, i.e., $\mathbb{D} = \{\frac{p}{2^n} : 0 \leqslant p \leqslant 2^n - 1,\ p, n \in \mathbb{Z}_+\}$. We will construct inductively a countable collection of infinite paths C_r, $r \in \mathbb{D}$, which start at vertices v_r. We set C_0 to be the "left side of the Pascal triangle", i.e., the sequence of edges $((j, 0), (j + 1, 0)), j \in \mathbb{Z}_+$. For $r \in [0, 1]$, we denote by L_r the ray inside $\mathbb{R}_+ \times \mathbb{R}_+$ starting at $(0, 0)$ and making the angle $r\frac{\pi}{2}$ with the real positive half-line. Thus, L_0 and L_1 are the sides of the Pascal triangle, and, as a set of points, C_0 coincides with L_0.

On the n-th step, $n \in \mathbb{Z}_+$, given that C_r are constructed for all r of the form $\frac{p}{2^n}$, where $0 \leqslant p \leqslant 2^n - 1$, we construct the paths C_r for all $r \in \mathbb{D}$ of the form $\frac{p}{2^{n+1}}$, where p is odd. Moreover, we will do it such that the following inductive conditions hold for each of the constructed paths C_r:

1) any two paths C_{r_1}, C_{r_2}, $r_1 \neq r_2$, intersect at most at one point, which coincides with the beginning of one of them;
2) for any $n, p \in \mathbb{Z}_+, 0 \leqslant p \leqslant 2^n - 1$ and $r = p/2^n$, the path $C_{r+1/2^{n+1}}$ starts at a vertex $v_{r+1/2^{n+1}}$ of the path C_r;
3) there exists $N = N(r)$ such that for all $(x, y) \in C_r$ with $x + y \geqslant N$, one has $d((x, y), L_r) < 2$.

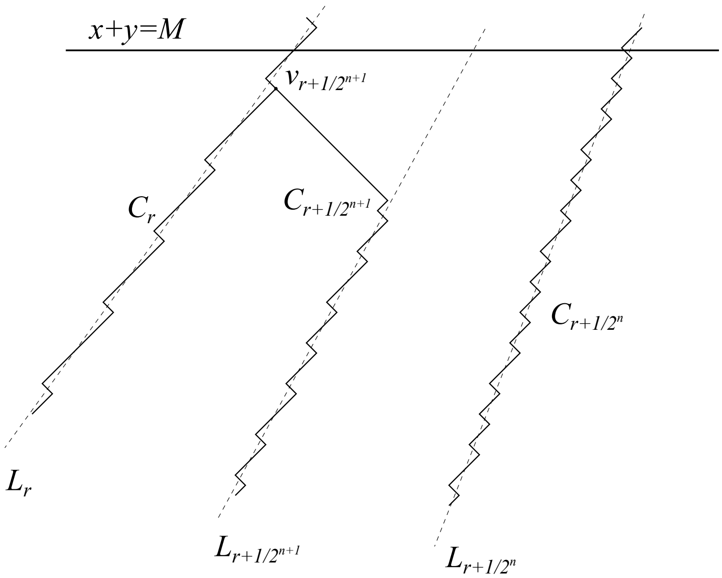

Fig. 4.1: Illustration to the proof of Theorem 4.1

For $n \in \mathbb{N}$, set $\mathbb{D}_n = \{\frac{p}{2^n} : p \in \mathbb{Z}_+, 0 \leqslant p \leqslant 2^n - 1\}$. Assume that C_r are constructed for all $r \in \mathbb{D}_n$ such that the inductive conditions $1) - 3)$ hold. Set

$$M = \max\{\max\{N(r) : r \in \mathbb{D}_n\}, 2^{n+5}\}.$$

Let H_M be the half-plane $\{(x, y) : x + y \geqslant M\}$.

Now, fix $r \in \mathbb{D}_n$. We will show how to construct the path $C_{r+1/2^{n+1}}$. The inductive condition implies that

$$d(C_t \cap H_M, L_{r+1/2^{n+1}} \cap H_M) \geqslant 5 \text{ for all } t \in \mathbb{D}_n. \tag{4.1}$$

Indeed, by 3), $C_t \cap H_M$ lies in the neighborhood of radius 2 of L_t. Since $M \geqslant 2^{n+5}$, for any two points $A \in L_t \cap H_M, B \in L_{r+1/2^{n+1}} \cap H_M$, one has

$$AB \geqslant OA \cdot \sin(1/2^{n+1}) > 2^{n+4}/2^{n+1} = 8.$$

Thus, $d(L_t \cap H_M, L_{r+1/2^{n+1}}) > 8$. From the above, equation 4.1 follows.

Let (a, b) be a vertex of the part of the path C_r belonging to H_M minimizing the distance from $C_r \cap H_M$ to $L_{r+1/2^{n+1}} \cap H_M$. Starting with $w_0 = (a, b)$, we construct a sequence $\{w_n\}$ of vertices of consecutive levels (i.e., $w_n \in V_{a+b+n}$) such that w_{n+1} is the closest to $L_{r+1/2^{n+1}}$ vertex among two downward neighbors of w_n (belonging to $V_{a+b+n+1}$). We claim that the infinite path $C_{r+1/2^{n+1}}$ starting with w_0 and passing through the sequence of vertices $\{w_n\}$ satisfies the inductive assumptions.

Indeed, by the inductive assumptions 1) and 2), the graph Γ_n consisting of the union of the paths C_t, $t \in \mathbb{D}_n$, is a connected planar graph forming an infinite tree. By (4.1), C_t does not intersect $L_{r+1/2^{n+1}} \cap H_M$ for all $t \in \mathbb{D}_n$. Observe that C_t intersects ∂H_M exactly at one point A_t for each $t \in \mathbb{D}_n$. Using condition 3) and the choice of M, we obtain that $C_r \cap H_M$, the segment $A_r A_{r+1/2^n} \subset \partial H_M$, and $C_{r+1/2^n} \cap H_M$ bound a connected infinite domain U_r which contains $H_M \cap L_{r+1/2^{n+1}}$.

Consider the ray $w_0 + \mathbb{R}_+ = \{w_0 + (0, s) : s \in \mathbb{R}_+\}$. Simple geometric considerations show that $w_0 + \mathbb{R}_+$ intersects $L_{r+1/2^{n+1}}$ at some point P. By the choice of w_0, the ray $w_0 + \mathbb{R}_+$ intersects C_r only at the point w_0. Let $l = [|w_0 P|]$ (the integer part of the length of the segment joining w_0 and P). By construction, the points $w_1, w_2, \ldots, w_l$ are consecutive integer points on $w_0 + \mathbb{R}_+$. One has $d(w_l, C_{r+1/2^{n+1}}) < 1$. Recall that for every $n \geqslant l$, the point w_{n+1} is the closest to $L_{r+1/2^{n+1}}$ among the two neighbors of w_n in $V_{a+b+n+1}$. Using induction, we obtain that $d(w_n, C_{r+1/2^{n+1}}) < 1$ for every $n \geqslant l$. Using (4.1), we derive that the path $C_{r+1/2^{n+1}}$ lies inside the domain U_r, except for the point w_0 belonging to C_r. From the above considerations, the conditions $1) - 3)$ are satisfied for the path $C_{r+1/2^{n+1}}$.

Furthermore, the union

$$\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n = \bigcup_{r \in \mathbb{D}} C_r,$$

as a graph, is a planar infinite tree such that there are no two vertices $v \neq w$ of the same level V_n which are connected by an edge of Γ to the same vertex $u \in V_{n+1}$. Notice that Γ is a subgraph of the Pascal diagram. Set $\Gamma' = \bigcup_{r \in \mathbb{D}/4} C_r$ and let Γ'' be the graph symmetric to Γ' with respect to $y = x$. Since $C_{1/4}$ asymptotically lies in 2-neighborhood of the line $y = \sin(\pi/8)x$, there exists N such that $(N, 0) + C_{1/4}$ does not intersect the line $y = x$. Then the shifted graphs $(N, 0) + \Gamma'$ and $(0, N) + \Gamma''$ do not intersect. Assign 0 to each edge of $(N, 0) + \Gamma'$ and 1 to each edge of $(0, N) + \Gamma''$ (except the edges on the y-axis, which are labeled 0 automatically). The rest of the edges of the Pascal diagram are numbered by 0's and 1's consistently in an arbitrary way.

Finally, in addition to already constructed paths $C_r, r \in \mathbb{D}$, for each $r \in [0, 1/4) \setminus \mathbb{D}/4$, we construct a path C_r in Γ (thus, giving a minimal path and a symmetric maximal path in the ordered Pascal diagram) as follows. Write the dyadic expansion of r and the corresponding partial sums:

$$r = \sum_{i=1}^{\infty} 2^{-n_i}, \quad r_k = \sum_{i=1}^{k} 2^{-n_i}, \quad \text{where } n_i \in \mathbb{N}, \quad n_{i+1} > n_i \text{ for each } i.$$

Set $r_0 = n_0 = 0$. For $k \in \mathbb{Z}_+$, denote by $T_k = T_k(r)$ the part of the path $C_{r_{n_k}}$ joining the vertices $v_{r_{n_k}}$ and $v_{r_{n_{k+1}}}$. Set $C_r = \bigcup_{k \in \mathbb{Z}_+} T_k$.

By construction, $((0,0),(N,0)) \cup ((N,0)+C_r), r \in [0,1/4)$ are pairwise distinct minimal paths of the ordered Pascal diagram. The maximal paths in the diagram are taken symmetrically to the minimal ones with respect to $y = x$. This finishes the proof. □

The following result answers Question 7.3 from [22].

Theorem 4.2. *There exists an ordering of the edges of the Pascal diagram into* $0, 1$ *such that the set of minimal paths* $X_{\min}$ *has the cardinality of the continuum and the set of maximal paths* $X_{\max}$ *is countably infinite.*

Proof. We will construct in a certain way a tree of minimal paths. For each $k \in \mathbb{N}$, let $S_k \subset V_{4^k}$ be the set of vertices given by

$$S_k = \{(2j, 4^k - 2j) : 1 \leqslant j \leqslant 2^k\}.$$

For each k, we connect the vertices from S_k by a collection of paths Υ_k to the vertices from S_{k+1} as follows. For each $1 \leqslant j \leqslant 2^k$, connect $(2j, 4^k - 2j) \in S_k$ by a straight path to $(2j, 4^{k+1} - 4j)$, the next one connect by a straight path to $(4j - 2, 4^{k+1} - 4j)$, and the last one connect by segments of length 2 to $(4j, 4^{k+1} - 4j) \in S_{k+1}$ and $(4j-2, 4^{k+1} - 4j + 2) \in S_{k+1}$. Also, let Υ_0 be the union of the segment $((0,0),(4,0))$ and the segment $((2,0),(2,2))$. Let $\Upsilon = \bigcup_{k \in \mathbb{Z}_+} \Upsilon_k$. Notice that Υ is an infinite tree such that for any vertex $(i,j) \in \mathbb{N} \times \mathbb{N}$ it contains at most one of the edges $((i-1,j),(i,j))$ and $((i,j-1),(i,j))$. Mark all the edges belonging to Υ with 0. Regardless of how we mark the rest of the edges, Υ contains a continuum of minimal paths.

Next, let Ψ be the set of edges of the form $((i,j),(i,j+1))$, not belonging to Υ, where $(i,j) \in \mathbb{N} \times \mathbb{Z}_+$ is such that

$$((i-1, j+1), (i, j+1))$$

does not belong to Υ either. Mark all edges from Ψ by 0 as well. In addition, all the edges belonging to the coordinate axis (the sides of the Pascal graph) are marked with 0 automatically. Mark all other edges by 1. We claim that such numbering of the edges of the Pascal graph has only countably many maximal paths.

Indeed, observe that by construction, for each $n \in \mathbb{Z}_+$, the line $y = n$ (containing all vertices of the form (i, n), $i \in \mathbb{Z}_+$) intersects the union of edges marked by 0 by at most one segment. Namely, when $n = 4^k - 4j$ for some $k \in \mathbb{N}$ and $1 \leqslant j \leqslant 2^{k-1}$. Since such n is necessarily even (in fact, divisible by four), we obtain by definition of Ψ that every edge of the form $((i, 2j), (i, 2j + 1))$ is marked with 0 for $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$.

Let $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $(i, j) \neq (0, 0)$ be a branching point for maximal paths, i.e., both edges $((i, j), (i + 1, j))$ and $((i, j), (i, j + 1))$ are marked by 1. Since $((i, j), (i, j+1))$ does not belong to Ψ, we conclude that either $i = 0$ or $((i-1, j+1), (i, j + 1))$ belongs to Υ. In the latter case, $j + 1$ is even. By construction, all the edges of the form $((l, j + 1), (l, j + 2))$, $l \in \mathbb{Z}_+$, are marked with 0. Thus, any maximal path passing through $(i, j + 1)$ may continue only along the line $y = j + 1$ to infinity without any further branching possible. We obtain that any maximal infinite path may contain at most two branching points. Since for any two branching points, only two maximal infinite paths might pass through both of them, there are only countably many maximal infinite paths. □

The following proposition about the measure of minimal and maximal infinite paths is true from general observations for other diagrams (see [13, Lemma 2.7]), and was proved in [22, Proposition 2.1]. To illustrate the ideas, we present here direct calculations for the Pascal–Bratteli diagram.

Proposition 4.3. *Let B be a Pascal–Bratteli diagram. Then for any ordering on B and any non-atomic probability ergodic invariant measure μ, the set of minimal infinite paths $X_{\min}$ and the set of maximal infinite paths $X_{\max}$ have μ-measure zero.*

Proof. Let $\mu = \nu_p$, where $p \in (0, 1)$ (see Section 3), be any non-atomic ergodic probability invariant measure on B, and ω be any order on B. We say that a cylinder set is a minimal cylinder set of length n if it corresponds to a minimal finite path of length n. Let $X_{\min}^{(n)}$ be a union of minimal cylinder sets of length n. Then

$$X_{\min}^{(n)} \supset X_{\min}^{(n+1)} \text{ and } X_{\min} = \bigcap_{n=1}^{\infty} X_{\min}^{(n)}.$$

Since for every $n \in \mathbb{N}$ and every vertex $w \in V_n$ there is a unique path which joins w with V_0, for $p < \frac{1}{2}$, we have

$$\begin{aligned} \mu(X_{\min}^{(n)}) = \sum_{k=0}^{n} p^k (1-p)^{n-k} &= (1-p)^n \sum_{k=0}^{n} \left(\frac{p}{1-p}\right)^k \\ &\leq (1-p)^n \frac{1}{1 - \frac{p}{1-p}} = (1-p)^n \frac{1-p}{1-2p} \to 0 \end{aligned}$$

as n tends to infinity. By switching p and $1 - p$, we obtain that $\mu(X_{\min}^{(n)})$ tends to 0 as n tends to infinity for $p > \frac{1}{2}$. We also have

$$\mu\left(X_{\min}^{(n)}\right) = \sum_{k=0}^{n} \frac{1}{2^n} = \frac{n+1}{2^n} \quad \text{for } p = \frac{1}{2}.$$

Thus, for all $p \in (0,1)$, we have

$$\mu(X_{\min}) = \lim_{n\to\infty} \mu\left(X_{\min}^{(n)}\right) = 0.$$

Similarly, $\mu(X_{\max}) = 0$. □

Remark 4.4. In [23], the authors study the so-called polynomial shape Bratteli diagrams, which are generalizations of a Pascal–Bratteli diagram. They show that for such diagrams, the sets of minimal infinite and maximal infinite paths have measure zero for a fully supported ergodic invariant probability measure (see [23, Proposition 3.6]). It is also shown that under some mild conditions, the orbits of the sets of minimal infinite and maximal infinite paths are meager (see [23, Proposition 2.3, Remark 2.4]).

5. Stationary generalized Pascal–Bratteli diagrams

In this section, we consider generalized Bratteli diagrams that are formed by countably many overlapping Pascal triangles. To the best of our knowledge, these diagrams have not been considered before.

5.1. One-sided stationary generalized Pascal–Bratteli diagram. First, we focus on a one-sided stationary generalized Bratteli diagram B that has infinitely many Pascal–Bratteli diagrams as vertex subdiagrams and the $\mathbb{N} \times \mathbb{N}$ incidence matrix

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

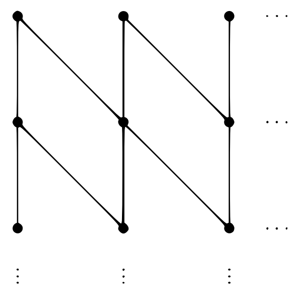

Fig. 5.1: A stationary generalized Bratteli diagram with infinitely many Pascal subdiagrams

To be consistent with other papers on generalized Bratteli diagrams, we draw it so that the vertices of consecutive levels are aligned one below another, see Fig. 5.1. Diagram B is stationary, and this property allows us to use the techniques from the papers [5], [10]. For any $i \in \mathbb{N}$, denote by X_i the set of all paths in X_B that start at vertex i on level V_0. The sets $\{X_i\}_{i\in\mathbb{N}}$ form a clopen partition of X_B. Note that X_i is also the path space of a vertex subdiagram B_i of B that is supported by the set of vertices $W = (W_n)_{n=0}^{\infty}$, where $W_n = \{i, \dots, i+n\}$. Obviously, each diagram B_i is the classical Pascal–Bratteli diagram. It is well known that there are uncountably many ergodic probability tail invariant measures $\nu_p^{(i)}$, $0 < p < 1$, on B_i (see Section 3). In the proposition below, we describe the invariant measures on B given by eigenpairs for the incidence matrix (see, e.g., [3, Theorem 2.3.2]) or [5, Theorem 6.6]).

Proposition 5.1. *Let $\lambda > 1$ and $\overline{\xi}_\lambda = (\xi_i)$ be the vector such that*

$$\xi_i = (\lambda - 1)^{i-1}, \quad i \in \mathbb{N}.$$

Then $F^T \overline{\xi}_\lambda = \lambda \overline{\xi}_\lambda$ and the eigenpair $(\overline{\xi}_\lambda, \lambda)$ defines a tail invariant measure μ_λ. If $1 < \lambda < 2$, then μ_λ is finite. The measure μ_λ is infinite for $\lambda \geq 2$.

Proof. First, we find a non-negative vector $\overline{\xi} = (\xi_i)$ satisfying $F^T \overline{\xi} = \lambda \overline{\xi}$. It is easy to see that if $\xi_1 = 0$, then $\xi_i = 0$ for all $i \in \mathbb{N}$. Set $\xi_1 = 1$. Then $\xi_1 + \xi_2 = \lambda \xi_1$ and $\xi_2 = \lambda - 1$. Similarly, $\xi_3 = \lambda(\lambda - 1) - (\lambda - 1) = (\lambda - 1)^2$ and one can prove by induction that for all $i \in \mathbb{N}$,

$$\xi_i = (\lambda - 1)^{i-1}.$$

Hence $\overline{\xi} = \overline{\xi}_\lambda$ is the eigenvector corresponding to λ. Set

$$\mu_\lambda[\overline{e}^{(n)}(i)] = \frac{\xi_i}{\lambda^n} = \frac{(\lambda - 1)^{i-1}}{\lambda^n},$$

where $[\overline{e}^{(n)}(i)]$ is a cylinder set corresponding to a finite path $\overline{e}^{(n)}(i)$ which ends at a vertex $i \in V_n$. Clearly, if $1 < \lambda < 2$, then μ_λ is finite and

$$\mu_\lambda(X_B) = \sum_{i=0}^{\infty} (\lambda - 1)^i = \frac{1}{2 - \lambda}.$$

If $\lambda \geq 2$, then μ_λ is infinite. □

Lemma 5.2. *Let $i \in \mathbb{N}$ and $\lambda > 1$. Then the measure $\mu_\lambda|_{X_{B_i}}$ (after normalization) coincides with $\nu_p^{(i)}$.*

Proof. Let $m_\lambda^{(i)}$ be the normalized measure $\mu_\lambda|_{X_{B_i}}$, that is,

$$m_\lambda^{(i)} = \frac{1}{(\lambda - 1)^i} \mu_\lambda|_{X_{B_i}}.$$

We claim that $m_\lambda^{(i)} = \nu_p^{(i)}$, where $p = \frac{1}{\lambda}$. Indeed, let $\overline{e}^{(n)}(j)$ be a finite path from i to $j \in V_n$. Then

$$m_\lambda^{(i)}([\overline{e}^{(n)}(j)]) = \frac{(\lambda - 1)^{j-1}}{(\lambda - 1)^{i-1} \lambda^n} = \frac{1}{\lambda^n} (\lambda - 1)^{j-i} = \frac{1}{\lambda^{n-j+i}} \left(\frac{\lambda - 1}{\lambda} \right)^{j-i}.$$

Let $j - i = k$. Then

$$m_\lambda^{(i)}([\overline{e}^{(n)}(j)]) = p^{n-k}q^k,$$

where

$$p = \frac{1}{\lambda}, \quad q = \frac{\lambda - 1}{\lambda}.$$

This means that $m_\lambda^{(i)} = \nu_p^{(i)}$ for $p = \frac{1}{\lambda}$. □

In the following theorem, we show that all ergodic invariant probability measures on B can be obtained as extensions of the measures $\nu_p^{(i)}$ (see Section 2).

Theorem 5.3. *For any $i \in \mathbb{N}$, the set of measures $\{\widehat{\nu}_p^{(i)}, p \in (\frac{1}{2}, 1)\}$ is (after normalization) the set of all ergodic probability tail invariant measures on B.*

Proof. Let μ be any ergodic invariant measure on B. Without loss of generality, assume that μ takes the value 1 on a cylinder set $X_u = [e^{(0)}(u)]$ formed by infinite paths that start at a vertex $u \in V_0$. Notice that each cylinder set of B is a standard Pascal–Bratteli diagram. Hence, the restriction of μ onto X_u must coincide with one of the measures ν_p. Because the measure $\mu|_{X_u}$ coincides with ν_p on a set of positive measures, and both measures are tail invariant, the extensions of these measures must also coincide on the saturation of X_u with respect to the tail equivalence relation. Thus, $\mu = \widehat{\nu}_p$. □

The following result is a corollary of Theorem 4.1.

Corollary 5.4. *There exists a stationary generalized Bratteli diagram together with a (non-stationary) order such that both the set of minimal paths $X_{\min}$ and the set of maximal paths $X_{\max}$ have the cardinality of the continuum.*

Proof. By Theorem 4.1, there exists an ordering of a (standard) Pascal–Bratteli diagram such that it has continuum minimal infinite paths and continuum maximal infinite paths. Let B be a stationary generalized Bratteli diagram with infinitely many Pascal subdiagrams. Pick any of its Pascal subdiagrams of level 0 and enumerate the edges of the subdiagram using Theorem 4.1. All other edges of B can be enumerated in an arbitrary way such that the order is well defined. Then the obtained order on B has continuum infinite number of both minimal paths and maximal paths. □

The following result is a corollary of Theorem 4.2.

Corollary 5.5. *There exists a stationary generalized Bratteli diagram together with a (non-stationary) order such that the set of minimal paths $X_{\min}$ has the cardinality of the continuum and the set of maximal paths $X_{\max}$ is countably infinite.*

Proof. By Theorem 4.2, there exists an ordering of a (standard) Pascal–Bratteli diagram such that it has continuum minimal infinite paths and countably infinitely many maximal infinite paths. Let B be the one-sided stationary generalized Bratteli diagram with infinitely many Pascal subdiagrams. Pick its leftmost Pascal subdiagram, i.e., the Pascal subdiagram which starts at the vertex 1 of level 0, and enumerate the edges of the subdiagram using Theorem 4.2. Enumerate all other edges of B from left to right. Then the obtained order on B has continuum minimal infinite paths and infinitely countably many maximal infinite paths. $\square$

Note that for every order on B, the leftmost vertical path is simultaneously a minimal infinite path and a maximal infinite path, hence there is no order on B such that the sets $X_{\min}$ and $X_{\max}$ are empty. Even if a generalized Bratteli diagram has a unique minimal and a unique maximal path, it does not necessarily follow that the corresponding Vershik map is a homeomorphism [5, Theorem 4.7]. In the following, we consider a two-sided stationary generalized Pascal–Bratteli diagram $\widetilde{B}$ and show in Proposition 5.6 that there is an order on $\widetilde{B}$ such that $X_{\min} = X_{\max} = \varnothing$, and hence the corresponding Vershik map is a homeomorphism.

5.2. Two-sided infinite stationary generalized Pascal–Bratteli diagram. We can also consider a two-sided generalized Bratteli diagram $\widetilde{B}$ which has infinitely many Pascal–Bratteli diagrams as subdiagrams. Diagram $\widetilde{B}$ has a $\mathbb{Z} \times \mathbb{Z}$ incidence matrix (we use the boldface to indicate the entries on the main diagonal):

$$\widetilde{F} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbf{1} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & \mathbf{1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \mathbf{1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & \mathbf{1} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & \mathbf{1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Diagram $\widetilde{B}$ is both vertically and horizontally stationary (see [4]). Corollaries 5.4 and 5.5 also hold for the two-sided generalized Bratteli diagram $\widetilde{B}$. Indeed, to prove Corollary 5.5, it is enough to pick any Pascal subdiagram B' of level 0 and enumerate the edges of the subdiagram using Theorem 4.2. All other edges of $\widetilde{B}$ which are to the right of B' should be enumerated from left to right, and all edges to the left of B' should be enumerated from right to left.

In the following proposition, we show that there is an order on $\widetilde{B}$ that does not admit any minimal or maximal infinite paths.

Proposition 5.6. *There exists an ordering on $\widetilde{B}$ that does not have minimal infinite or maximal infinite paths.*

Proof. Fix an injective map G from $\mathbb{Z} \times \{0, 1\}$ to $\mathbb{N}$. For each $(n, i) \in \mathbb{Z} \times \{0, 1\}$, set $K = K(n, i) = 3^{G(n,i)}$. For $n \in \mathbb{Z}$, $k \in \mathbb{Z}_+$, let $n^{(k)}$ be the n-th vertex of the k-th level of the generalized Bratteli diagram B. Let $\mathcal{S}(n, i)$ be the set of all the edges of the form $((l + n)^{(K+l)}, (l + n)^{(K+l+1)})$ and of the form $((K + n)^{(K+l)}, (K + n + 1)^{(K+l+1)})$, where $0 \leqslant l \leqslant K = K(n, i)$. Mark all the edges from $\mathcal{S}(n,i)$ with i. It is not hard to see that the sets $\mathcal{S}(n,i)$, $(n,i) \in \mathbb{Z} \times \{0,1\}$, are pairwise disjoint. Therefore, the above operation is well defined. Number all the other edges in an arbitrary consistent way to obtain an ordering on $\widetilde{B}$.

For any top vertex $n^{(0)}$, $n \in \mathbb{N}$, of the diagram $\widetilde{B}$ and any $i \in \{0,1\}$, any infinite path starting at $n^{(0)}$ passes through an edge of $\mathcal{S}(n,i)$. It follows that there are no minimal or maximal paths. $\square$

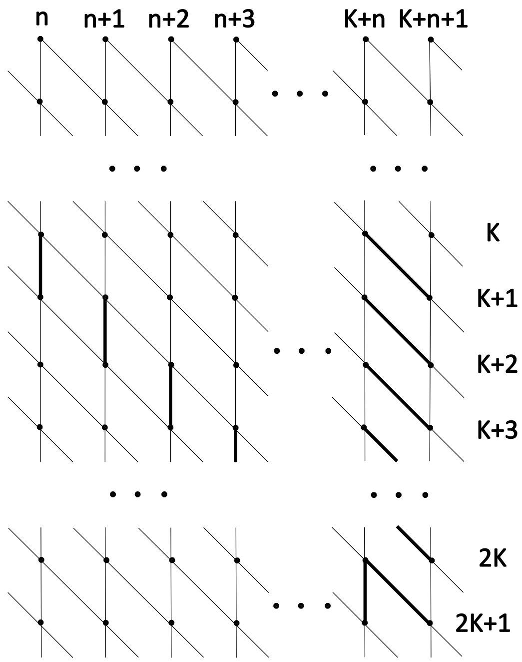

Fig. 5.2: Illustration to the proof of Proposition 5.6: edges from $\mathcal{S}(n,i)$ are in bold

For the two-sided generalized Bratteli diagram $\widetilde{B}$, for every $p \in (0,1)$, there is a positive right eigenvector (the boldface indicates the zero coordinate of the eigenvector):

$$\mathbf{x}^T = \left(\ldots, \frac{(1-p)^2}{p^2}, \frac{1-p}{p}, \mathbf{1}, \frac{p}{1-p}, \frac{p^2}{(1-p)^2}, \ldots\right)$$

of $\widetilde{A} = \widetilde{F}^T$ which corresponds to the eigenvalue $\lambda = \frac{1}{p}$. In other words, we have $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$ and $x_i = \frac{p^i}{(1-p)^i}$ for $i \in \mathbb{Z}$. Indeed, for every $i \in \mathbb{Z}$, we have

$$(\widetilde{A}\mathbf{x})_i = \left(\frac{p}{1-p}\right)^{i-1} + \left(\frac{p}{1-p}\right)^{i} = \frac{1}{p}\left(\frac{p}{1-p}\right)^{i} = \lambda x_i.$$

In particular, for $p = \frac{1}{2}$, we obtain $\lambda = 2$ and $\mathbf{x}^T = (\ldots, 1, 1, 1, \ldots)$. Note that for every $p \in (0,1)$, the corresponding eigenvector $\mathbf{x}$ has an infinite sum of coordinates and hence generates an infinite σ-finite measure.

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Міри та динаміка на діаграмах Паскаля–Браттелі

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Ми вводимо та досліджуємо динамічні системи та міри на стаціонарних узагальнених діаграмах Браттелі B, які представлені як об'єднання зліченної кількості класичних діаграм Паскаля–Браттелі. Ми описуємо всі ергодичні міри на B, інваріантні відносно хвостового відношення еквівалентності. Для кожної ймовірнісної інваріантної міри ν_p на класичній діаграмі Паскаля–Браттелі, ми апроксимуємо носій ν_p простором шляхів піддіаграми. Розглядаючи різні порядки на ребрах B, ми визначаємо динамічні системи з різними властивостями. Ми показуємо, що існують порядки, для яких множини нескінченних максимальних та нескінченних мінімальних шляхів порожні. Це означає, що відповідне відображення Вершика є гомеоморфізмом. Ми також описуємо порядки як на B, так і на класичній діаграмі Паскаля–Браттелі, які генерують або незліченну кількість мінімальних нескінченних шляхів та незліченну кількість максимальних нескінченних шляхів, або незліченну кількість мінімальних нескінченних шляхів та зліченну нескінченну кількість максимальних нескінченних шляхів.

Ключові слова: борелівські динамічні системи, моделі Браттелі–Вершика, інваріантні міри, хвостове відношення еквівалентності, діаграма Паскаля–Браттелі