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Finite-Time Blow-Up for the Fractional Superdiffusion Equation with Time ψ-Caputo Derivative

Yacine Arioua and Brahim Benaissi

This article investigates a time-space fractional nonlinear superdiffusion equation defined on the entire Euclidean space. The time-fractional derivative is the ψ-Caputo derivative of order in $(1, 2)$, while the spatial derivative is expressed using the fractional Laplacian of order in $(0, 1)$. The associated Cauchy problem is analyzed by constructing both mild and weak solutions. It is demonstrated that the mild solution also satisfies the weak formulation. The finite-time blow-up behavior of solutions to the superdiffusion equation is also examined.

Key words: fractional superdiffusion equation, ψ-Caputo derivative, fractional Laplacian, blow-up, mild solution, weak solution

Mathematical Subject Classification 2020: 26A33, 35R11, 35B44, 35D30

1. Introduction

Fractional calculus operators—non-integer-order integro-differential operators—have recently received growing attention. These operators are widely used to model memory and hereditary properties of dynamical systems across various fields, including continuum mechanics [17], anomalous diffusion [20], atmospheric dust dynamics [25], plasma physics [3], open quantum systems [24], electrode–electrolyte interfaces [21], and cellular biochemistry [23]. A recent review [5] summarizes current research directions, emerging developments, and open problems in fractional calculus.

The most commonly used fractional differential operators are the Riemann–Liouville and Caputo operators [11,22], both defined using integral kernels involving monomial power functions. Another class of operators, such as the Hadamard and Caputo–Hadamard types, employs kernels based on powers of logarithmic functions [9]. More generally, fractional integro-differential operators can be defined using arbitrary kernel functions. These generalized forms—often referred to as ψ-fractional integrals and derivatives—extend the classical definitions by allowing the relation of one function to another through a kernel function ψ. This broader framework has been investigated in works such as [1,14].

This study focuses on the phenomenon of solution blow-up for the Cauchy problem associated with a time-space fractional superdiffusion equation, expressed as follows:

$$
\begin{cases}
{}^{C}\mathrm{D}^{\alpha,\psi}_{a,t}\, u(\mathrm{x},t) + (-\Delta)^{\delta} u(\mathrm{x},t) = |u(\mathrm{x},t)|^{p}, & \mathrm{x}\in\mathbb{R}^d,\ t>a,\\
u(\mathrm{x},a) = u_a(\mathrm{x}), & \mathrm{x}\in\mathbb{R}^d,\\
\gamma_{\psi}\, u(\mathrm{x},a) = \tilde{u}_a(\mathrm{x}), & \mathrm{x}\in\mathbb{R}^d.
\end{cases} \tag{1.1}
$$

In this formulation, the parameters satisfy $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, $d \in \mathbb{N}$, and $a \in \mathbb{R}$. The initial values u_a and $\tilde{u}_a$ are elements of the space $C_0(\mathbb{R}^d) = \{\upsilon \in C(\mathbb{R}^d) \mid \lim_{|\mathrm{x}|\to\infty} \upsilon(\mathrm{x}) = 0\}$. The operators ${}^{C}\mathrm{D}^{\alpha,\psi}_{a,t}$, $(-\Delta)^{\delta}$, and γ_{ψ} represent the ψ-Caputo derivative with respect to the initial time $t = a$, the fractional Laplacian defined over the entire space $\mathbb{R}^d$, and the Gamma derivative operator, respectively; these will be formally introduced in the subsequent section.

Numerous studies have investigated exceptional cases and variants of Eq. (1.1). For instance, the earliest research on solution blow-up for the associated Cauchy problem was reported by Fujita [7] in 1966. In Fujita's formulation, the temporal derivative is the classical first-order derivative, and the spatial operator is the standard Laplacian. Since Fujita's seminal work, this area's research volume has significantly expanded, as comprehensively reviewed by Hu [8]. Notably, much of this research has focused on models involving integer-order calculus.

In 2008, a semilinear heat equation incorporating nonlinear memory effects was studied by Cazenave et al. [2], who established conditions for both finite-time blow-up and global existence of solutions. Later, in 2012, Fino and Kirane [6] demonstrated finite-time blow-up and provided conditions for global existence within a generalized framework that extended beyond that of Cazenave et al. [2]. Their analysis employed a spatial derivative defined via the fractional Laplacian. In 2015, Zhang and Sun [27] examined the finite-time blow-up and global existence properties of a time-fractional semilinear heat equation. Their study replaced the ψ-Caputo derivative with the standard Caputo derivative, and substituted the fractional Laplacian with the conventional Laplacian, simplifying Eq. (1.1) to a more tractable form aligned with their model.

In [15], the Cauchy problem for a semilinear time-space fractional diffusion equation is formulated as:

$$
\begin{cases}
{}^{C}\mathrm{D}^{\alpha,\psi}_{a,t} u(\mathrm{x},t) + (-\Delta)^{\delta} u(\mathrm{x},t) = |u(\mathrm{x},t)|^{p-1} u(\mathrm{x},t), & \mathrm{x}\in\mathbb{R}^d,\ t>a,\\
u(\mathrm{x},a) = u_a(\mathrm{x}), & \mathrm{x}\in\mathbb{R}^d,
\end{cases} \tag{1.2}
$$

where $\alpha \in (0,1)$, $\delta \in (0,1)$, $p > 1$, $d \in \mathbb{N}$, $a \in \mathbb{R}$, and the initial value $u_a \in C_0(\mathbb{R}^d)$. It was shown that if $0 \le u_a \not\equiv 0$ and $1 < p < 1 + \frac{2\delta}{d}$, then the mild solution of Equation (1.2) blows up in finite time.

When $\psi(t) = t$, $\delta = 1$, and $a = 0$ in Eq. (1.1), it was shown in [10] that for $u_0, \tilde{u}_0 \in L^q(\mathbb{R}^d)$, with $q > \max\left\{\frac{dp\alpha}{2}, 1\right\}$, the following blow-up results hold:

(i) If $1 < p \le 1 + \frac{2\alpha}{\alpha d + 2 - 2\alpha}$, $\tilde{u}_0 \equiv 0$, and $0 \le u_0 \not\equiv 0$, then any mild solution of blows up in finite time.

(ii) If $d > 1$, $1 < p < 1+\frac{2\alpha}{\alpha d-2}$, $0 \leq \tilde{u}_0 \not\equiv 0$, and $u_0 \geq 0$, then any mild solution of Eq. (1.1) blows up in finite time. If $d = 1$ and $u_0, \tilde{u}_0 \geq 0$, then any nontrivial solution of Eq. (1.1) blows up in finite time for every $p > 1$.

In this paper, we study a generalized form of Eq. (1.1), with a focus on the finite-time blow-up behavior of its solutions. The structure of the paper is as follows. In Section 2, we present essential definitions and preliminary results that form the basis of our analysis. Section 3 introduces the concept of a mild solution to Eq. (1.1) and defines weak solutions within a variational framework. We prove that every mild solution also satisfies the definition of a weak solution. In addition, we derive sufficient conditions that lead to finite-time blow-up. The main results are summarized in the final section.

2. Notations and lemmas

This section presents several fundamental concepts and essential lemmas that form the basis for the theorems established in the following two sections.

Definition 2.1 ([16, 18]). Let (a, b), where $-\infty \leq a < b \leq \infty$, be a finite or infinite interval on the real line $\mathbb{R}$. Consider a strictly increasing function $\psi(t)$ with $\psi(t) \to +\infty$ as $t \to +\infty$, and assume that ψ has a continuous derivative $\psi'(t)$ on (a, b). Let $f(t) \in L^1(a, b)$. The left-sided ψ-fractional integral of order $\alpha > 0$ is defined by

$$\mathrm{I}^{\alpha,\psi}_{a,t} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(\tau)\, (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t > a.$$

Similarly, the right-sided ψ-fractional integral is defined as

$$\mathrm{I}^{\alpha,\psi}_{t,b} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \psi'(\tau)\, (\psi(\tau) - \psi(t))^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t < b.$$

Special cases include:

- $\psi(t) = t$: yields the classical left- and right-sided Riemann–Liouville fractional integrals [11, 22].
- $\psi(t) = \log t$ with $a > 0$: gives the Hadamard fractional integrals [11, 22].
- $\psi(t) = t^\sigma$ with $a > 0$ and $\sigma \geq 0$: corresponds to the Erdélyi–Kober fractional integrals [11, 22].

We define the function space $AC^n_{\gamma_\psi}[a, b]$ as

$$AC^n_{\gamma_\psi}[a, b] = \left\{ f : [a, b] \to \mathbb{R} \,\middle|\, \gamma^{n-1}_\psi f(t) \in AC[a, b] \right\},$$

where the operator γ_ψ is defined recursively by

$$\gamma_\psi f(t) = \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \right) f(t), \quad \gamma^{n-1}_\psi f(t) = \gamma_\psi \left(\gamma^{n-2}_\psi f(t) \right) \quad \text{with} \quad \gamma^0_\psi f(t) = f(t).$$

Here, $AC[a, b]$ denotes the space of absolutely continuous functions on $[a, b]$. For unbounded intervals, where a and/or b are infinite, the definition of $AC[a, b]$ extends in the usual way; see Definition 6.1, p. 130 in Samko et al. [22].

Definition 2.2. Let (a,b), where $-\infty \le a < b \le \infty$, be a finite or infinite interval on the real line $\mathbb{R}$. Let $\psi(t) \in C^n(a,b)$ be a strictly increasing function satisfying $\psi'(t) \neq 0$ and $\psi(t) \to +\infty$ as $t \to +\infty$. Suppose that $f(t) \in AC^n_{\gamma_\psi}[a,b]$. The left-sided ψ-Caputo fractional derivative of order α, with $n-1<\alpha<n$ and $n \in \mathbb{N}$, is defined by

$$^{C}\mathrm{D}^{\alpha,\psi}_{a,t} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(\tau)\,(\psi(t)-\psi(\tau))^{n-\alpha-1}\,\gamma^n_\psi f(\tau)\,\mathrm{d}\tau, \quad t>a.$$

Similarly, the right-sided ψ-Caputo fractional derivative is defined as

$$^{C}\mathrm{D}^{\alpha,\psi}_{t,b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \psi'(\tau)\,(\psi(\tau)-\psi(t))^{n-\alpha-1}\,\gamma^n_\psi f(\tau)\,\mathrm{d}\tau, \quad t<b.$$

The left-sided ψ-Caputo fractional derivative was originally introduced by Kilbas et al. [11] (see Eq. (2.7.35), p. 115), although it was not explicitly named or analyzed under the condition $\psi(t) \to +\infty$ as $t \to +\infty$. For specific choices of ψ, the operators reduce to known fractional derivatives:

- $\psi(t) = t$: yields the classical Caputo fractional derivatives [11,22];
- $\psi(t) = \log t$: gives the Caputo–Hadamard fractional derivatives [11,13];
- $\psi(t) = e^t$: corresponds to the exponential Caputo derivatives as defined in [14].

The following fractional integration by parts formula holds under appropriate conditions:

$$\int_a^T \psi'(t)\,g(t)\,{}^{C}\mathrm{D}^{\alpha,\psi}_{a,t} f(t)\,\mathrm{d}t = \int_a^T \psi'(t)\,(f(t)-(\psi(t)-\psi(a))\,\gamma_\psi f(a) - f(a))\,{}^{C}\mathrm{D}^{\alpha,\psi}_{t,T} g(t)\,\mathrm{d}t \tag{2.1}$$

provided that the operators $\gamma_\psi f(t)$ and $\gamma_\psi g(t)$ are defined almost everywhere, the fractional order satisfies $1<\alpha<2$, and the boundary conditions $g(T) = \gamma_\psi g(T) = 0$ hold for any $T>a>0$.

For a given $T>a>0$ and $k>0$, define the function $g(t)$ by

$$g(t) = \begin{cases} \left(1-\dfrac{\psi(t)-\psi(a)}{\psi(T)-\psi(a)}\right)^k & \text{for } t \le T, \\ 0 & \text{for } t > T. \end{cases}$$

A direct computation yields, for $1<\alpha<2$,

$$^{C}\mathrm{D}^{\alpha,\psi}_{t,T} g(t) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\,(\psi(T)-\psi(a))^{-\alpha}\left(1-\frac{\psi(t)-\psi(a)}{\psi(T)-\psi(a)}\right)^{k-\alpha}, \quad t \le T. \tag{2.2}$$

Next, let $v \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space. The fractional Laplacian $(-\Delta)^\delta$ is then defined as follows [4,22]:

$$\begin{aligned}(-\Delta)^\delta v(\mathrm{x}) &= C(d,\delta)\,\mathrm{P.V.}\int_{\mathbb{R}^d} \frac{v(\mathrm{x})-v(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{d+2\delta}}\,\mathrm{dy} \\ &= -\frac{1}{2}C(d,\delta)\int_{\mathbb{R}^d} \frac{v(\mathrm{x}+\mathrm{y})-2v(\mathrm{x})+v(\mathrm{x}-\mathrm{y})}{|\mathrm{y}|^{d+2\delta}}\,\mathrm{dy}, \quad \mathrm{x} \in \mathbb{R}^d,\end{aligned}$$

where P.V. denotes the Cauchy principal value, and the normalization constant $C(d,\delta)$ is given by

$$C(d,\delta) = \left(\int_{\mathbb{R}^d} \frac{1-\cos y_1}{|\mathrm{y}|^{d+2\delta}} \,\mathrm{dy} \right)^{-1}, \quad \mathrm{y} = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d.$$

For $v \in \mathcal{S}(\mathbb{R}^d)$, we define the norm:

$$\|v\|^2_{H^{2\delta}} = \|v\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^\delta v\|^2_{L^2(\mathbb{R}^d)}.$$

The fractional Sobolev space $H^{2\delta}(\mathbb{R}^d)$ is defined as the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{H^{2\delta}}$. By density, the fractional Laplacian $(-\Delta)^\delta$ extends uniquely to a bounded operator on $H^{2\delta}(\mathbb{R}^d)$.

Since $(-\Delta)^\delta$ is a self-adjoint operator [6], it satisfies the identity

$$\int_{\mathbb{R}^d} \phi(\mathrm{x})\,(-\Delta)^\delta \psi(\mathrm{x})\,\mathrm{dx} = \int_{\mathbb{R}^d} \psi(\mathrm{x})\,(-\Delta)^\delta \phi(\mathrm{x})\,\mathrm{dx}, \tag{2.3}$$

for any functions $\phi, \psi \in H^{2\delta}(\mathbb{R}^d)$. Moreover, it is known from [6] that for any $q \geq 1$ and for any $\chi(\mathrm{x}) \in \mathcal{S}(\mathbb{R}^d)$ with $\chi(\mathrm{x}) \geq 0$, the following inequality holds:

$$(-\Delta)^\delta \chi^q(\mathrm{x}) \leq q\,\chi^{q-1}(\mathrm{x})\,(-\Delta)^\delta \chi(\mathrm{x}). \tag{2.4}$$

Let $0 \leq m \leq l$ and $0 \leq n \leq k$, with $m, n, k, l \in \mathbb{Z}$. For complex parameters $a_\nu, b_j \in \mathbb{C}$ and positive real parameters $\alpha_\nu, \beta_j \in \mathbb{R}^+$, where $\nu = 1, \ldots, k$ and $j = 1, \ldots, l$, the *Fox H-function* $H^{mn}_{kl}(z)$ is defined by [10, 11] as

$$\begin{aligned} H^{mn}_{kl}(z) &\equiv H^{mn}_{kl}\left(z \,\middle|\, \begin{matrix} (a_1,\alpha_1),\ldots,(a_n,\alpha_n);\,(a_{n+1},\alpha_{n+1}),\ldots,(a_k,\alpha_k) \\ (b_1,\beta_1),\ldots,(b_m,\beta_m);\,(b_{m+1},\beta_{m+1}),\ldots,(b_l,\beta_l) \end{matrix} \right) \\ &:= \frac{1}{2\pi i} \int_{\mathbf{C}} \mathbf{H}^{mn}_{kl}(\tau)\, z^{-\tau}\,\mathrm{d}\tau, \end{aligned} \tag{2.5}$$

where the integrand $\mathbf{H}^{mn}_{kl}(\tau)$ is given by

$$\mathbf{H}^{mn}_{kl}(\tau) := \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j\tau)\,\prod_{\nu=1}^n \Gamma(1 - a_\nu - \alpha_\nu\tau)}{\prod_{\nu=n+1}^k \Gamma(a_\nu + \alpha_\nu\tau)\,\prod_{j=m+1}^l \Gamma(1 - b_j - \beta_j\tau)}. \tag{2.6}$$

The function is well-defined under the condition that the poles of the gamma functions in the numerator and denominator do not overlap. Specifically, the poles of the form:

$$b_{j\sigma} = -\frac{b_j + \sigma}{\beta_j}, \quad j = 1, \ldots, m, \ \sigma = 0, 1, 2, \ldots, \tag{2.7}$$

and those of the form:

$$a_{\nu\mu} = \frac{1 - a_\nu + \mu}{\alpha_\nu}, \quad \nu = 1, \ldots, n, \ \mu = 0, 1, 2, \ldots, \tag{2.8}$$

must not coincide; that is,

$$\alpha_\nu(b_j+\sigma)\neq\beta_j(a_\nu-\mu-1),\quad j=1,\dots,m;\ \nu=1,\dots,n;\ \sigma,\mu\in\mathbb{N}_0. \tag{2.9}$$

The contour $\mathbf{C}$ in (2.5) denotes a Mellin–Barnes integral path that separates all poles $\tau=b_{j\sigma}$ of the gamma functions in the numerator (to the left of $\mathbf{C}$) from all poles $\tau=a_{\nu\mu}$ of the gamma functions in the denominator (to the right of $\mathbf{C}$).

We first need to construct the solution to the linear and nonhomogeneous Eq. (2.10):

$$\begin{cases} {}^C\mathrm{D}^{\alpha,\psi}_{a,t}\,u(\mathrm{x},t)+(-\Delta)^\delta u(\mathrm{x},t)=f(\mathrm{x},t), & \mathrm{x}\in\mathbb{R}^d,\ t>a,\\ u(\mathrm{x},a)=u_a(\mathrm{x}), & \mathrm{x}\in\mathbb{R}^d,\\ \gamma_\psi\,u(\mathrm{x},a)=\tilde{u}_a(\mathrm{x}), & \mathrm{x}\in\mathbb{R}^d. \end{cases} \tag{2.10}$$

Theorem 2.3. *The solution of Eq. (2.10) is given by*

$$\begin{aligned} u(\mathrm{x},t)&=(G^\psi_a(t)*u_a)(\mathrm{x})+(\tilde{G}^\psi_a(t)*\tilde{u}_a)(\mathrm{x})\\ &\quad+\int_a^t\psi'(\tau)G^\psi_f\left(\mathrm{x},\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)*f(\mathrm{x},\tau)\mathrm{d}\tau, \end{aligned} \tag{2.11}$$

where $*$ is the convolution symbol and the solutions are as follows:

$$\begin{aligned} G^\psi_a(\mathrm{x},t)&=\frac{1}{|\mathrm{x}|^d\pi^{d/2}}\,H^{21}_{23}\left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^\alpha}\,\middle|\,\begin{matrix}(1,1);\,(1,\alpha)\\(1,1),\,(\frac{d}{2},\delta);\,(1,\delta)\end{matrix}\right),\\ \tilde{G}^\psi_a(\mathrm{x},t)&=\frac{\psi(t)-\psi(a)}{|\mathrm{x}|^d\pi^{d/2}}\,H^{21}_{23}\left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^\alpha}\,\middle|\,\begin{matrix}(1,1);\,(2,\alpha)\\(1,1),\,(\frac{d}{2},\delta);\,(1,\delta)\end{matrix}\right),\\ G^\psi_f(\mathrm{x},t)&=\frac{(\psi(t)-\psi(a))^{\alpha-1}}{|\mathrm{x}|^d\pi^{d/2}}\,H^{21}_{23}\left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^\alpha}\,\middle|\,\begin{matrix}(1,1);\,(\alpha,\alpha)\\(1,1),\,(\frac{d}{2},\delta);\,(1,\delta)\end{matrix}\right). \end{aligned}$$

Remark 2.4. The proof of Theorem 2.3 is carried out in a manner similar to that employed in [12, 15].

Lemma 2.5. *For $\alpha\in(1,2)$ and $\delta\in(0,1)$, we have*

1. $G^\psi_a(\mathrm{x},t)>0$, $\tilde{G}^\psi_a(\mathrm{x},t)>0$ *and* $G^\psi_f(\mathrm{x},t)>0$.
2. $\mathrm{I}^{2-\alpha,\psi}_{a,t}G^\psi_f(\mathrm{x},t)=\tilde{G}^\psi_a(\mathrm{x},t)$.

Proof. 1. The proof of positivity of the functions $G^\psi_a(\mathrm{x},t)$, $\tilde{G}^\psi_a(\mathrm{x},t)$ and $G^\psi_f(\mathrm{x},t)$ is similar to that of Theorem 1 in [19].

2. We have

$$\begin{aligned} &\mathrm{I}^{2-\alpha,\psi}_{a,t}G^\psi_f(\mathrm{x},t)\\ &\quad=\mathrm{I}^{2-\alpha,\psi}_{a,t}\left[\frac{(\psi(t)-\psi(a))^{\alpha-1}}{|\mathrm{x}|^d\pi^{\frac{d}{2}}}H^{21}_{23}\left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^\alpha}\,\middle|\,\begin{matrix}(1,1);(\alpha,\alpha)\\(1,1),\left(\frac{d}{2},\delta\right);(1,\delta)\end{matrix}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\mathrm{x}|^d \pi^{\frac{d}{2}}} \cdot \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\Gamma(1+\tau)\,\Gamma\left(\frac{d}{2}+\delta\tau\right)\,\Gamma(-\tau)}{\Gamma(\alpha+\alpha\tau)\,\Gamma(-\delta\tau)} \left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}}\right)^{-\tau} \\
&\quad \times \mathrm{I}_{a,t}^{2-\alpha,\psi} \left(\psi(t)-\psi(a)\right)^{\alpha\tau+\alpha-1} \,\mathrm{d}\tau \\
&= \frac{\psi(t)-\psi(a)}{|\mathrm{x}|^d \pi^{\frac{d}{2}}} \cdot \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{\Gamma(1+\tau)\,\Gamma\left(\frac{d}{2}+\delta\tau\right)\,\Gamma(-\tau)}{\Gamma(2+\alpha\tau)\,\Gamma(-\delta\tau)} \\
&\quad \times \left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^{\alpha}}\right)^{-\tau} \mathrm{d}\tau \\
&= \frac{\psi(t)-\psi(a)}{|\mathrm{x}|^d \pi^{\frac{d}{2}}} H_{23}^{21} \left(\frac{|\mathrm{x}|^{2\delta}}{2^{2\delta}(\psi(t)-\psi(a))^{\alpha}} \,\middle|\, \begin{matrix} (1,1);(2,\alpha) \\ (1,1),\left(\frac{d}{2},\delta\right);(1,\delta) \end{matrix} \right) \\
&= \tilde{G}_a^{\psi}(\mathrm{x},t).
\end{aligned}
$$

Therefore, the proof is concluded. □

From this point onward, for the sake of brevity and where no confusion arises, the notations $G_a^{\psi}(\mathrm{x},t)$, $\tilde{G}_a^{\psi}(\mathrm{x},t)$, $G_f^{\psi}(\mathrm{x},t)$, and $u(\mathrm{x},t)$ will be abbreviated as $G_a^{\psi}(t)$, $\tilde{G}_a^{\psi}(t)$, $G_f^{\psi}(t)$, and $u(t)$, respectively.

Lemma 2.6. *Let $\alpha \in (1,2), \delta \in (0,1)$, and $u_a, \tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$. Then, for any $t > a$, $G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$, and*

$$
{}^{C}\mathrm{D}_{a,t}^{\alpha,\psi} \left(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\right) = -(-\Delta)^{\delta} \left(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\right), \quad t > a.
$$

Proof. For any $t > a$, it is clear that $G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$ by the continuity of $G_a^{\psi}(\mathrm{x},t), \tilde{G}_a^{\psi}(\mathrm{x},t)$ and $u_a, \tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$. Moreover, from (2.11) one finds that $G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a$ is the solution of Eq. (2.10) when $f \equiv 0$, and thus it satisfies

$$
{}^{C}\mathrm{D}_{a,t}^{\alpha,\psi} \left(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\right) = -(-\Delta)^{\delta} \left(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\right), t > a.
$$

Therefore, the proof is complete. □

Lemma 2.7. *For $\alpha \in (1,2), \delta \in (0,1)$ and $k > 1$, if $f \in L^k\left((a,T), C_0\left(\mathbb{R}^d\right)\right)$ for $T > a$ and*

$$
\hbar(t) = \int_a^t \psi'(\tau) G_f^{\psi}\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right) * f(\tau)\,\mathrm{d}\tau
$$

then

$$
\mathrm{I}_{a,t}^{2-\alpha,\psi} \hbar(t) = \int_a^t \psi'(\tau) \tilde{G}_a^{\psi}\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right) * f(\tau)\,\mathrm{d}\tau.
$$

Proof. In view of Definition 2.1 and Lemma 2.5, we obtain

$$
\mathrm{I}_{a,t}^{2-\alpha,\psi} \hbar(t) = \frac{1}{\Gamma(2-\alpha)} \int_a^t (\psi(t)-\psi(\tau))^{1-\alpha} \int_a^{\tau} \psi'(\tau)\psi'(w)
$$

$$
\begin{aligned}
&\times G_f^\psi\left(\psi^{-1}[\psi(\tau)+\psi(a)-\psi(w)]\right) * f(w)\,\mathrm{d}w\,\mathrm{d}\tau \\
&= \frac{1}{\Gamma(2-\alpha)} \int_a^t \int_w^t (\psi(t)-\psi(\tau))^{1-\alpha} \psi'(w)\,\psi'(\tau) \\
&\times G_f^\psi\left(\psi^{-1}[\psi(\tau)+\psi(a)-\psi(w)]\right) * f(w)\,\mathrm{d}\tau\,\mathrm{d}w \\
&= \frac{1}{\Gamma(2-\alpha)} \int_a^t \int_a^{\psi^{-1}[\psi(t)+\psi(a)-\psi(w)]} (\psi(t)-\psi(\tau))^{1-\alpha} \psi'(w)\,\psi'(z) \\
&\times G_f^\psi(z) * f(w)\,\mathrm{d}z\,\mathrm{d}w \\
&= \int_a^t \psi'(w)\,\tilde{G}_a^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(w)]\right) * f(w)\,\mathrm{d}w.
\end{aligned}
$$

Therefore, the proof is complete. □

3. Blow-up of Solution to Eq. (1.1)

This section is devoted to the study of finite-time blow-up of the solution to Eq. (1.1). We begin by introducing the definition of a mild solution to Eq. (1.1) as follows.

Definition 3.1. Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and suppose u_a, $\tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$. A function $u \in C\left([a,T], C_0\left(\mathbb{R}^d\right)\right)$, with $T > a$, is said to be a *mild solution* of Eq. (1.1) if the following integral equation holds:

$$
u(t) = G_a^\psi(t) * u_a + \tilde{G}_a^\psi(t) * \tilde{u}_a \tag{3.1}
$$

$$
+ \int_a^t \psi'(\tau) G_f^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right) * |u(\tau)|^p\,\mathrm{d}\tau. \tag{3.2}
$$

To study the finite-time blow-up of solutions to Eq. (1.1), we first introduce the following definition of a weak solution.

Definition 3.2. Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and assume $u_a, \tilde{u}_a \in L^2\left(\mathbb{R}^d\right)$. A function $u \in L^p\left((a,T), L^{2p}\left(\mathbb{R}^d\right)\right) \cap L^1\left((a,T), L^2\left(\mathbb{R}^d\right)\right)$, with $T > a$, is said to be a *weak solution* of Eq. (1.1) if the following integral identity holds:

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_a^T \psi'(t) \left[|u|^p \varphi + (u_a + (\psi(t)-\psi(a))\,\tilde{u}_a)\ {}^C\mathrm{D}_{t,T}^{\alpha,\psi} \varphi \right] \mathrm{d}t\,\mathrm{dx} \\
&= \int_{\mathbb{R}^d} \int_a^T \psi'(t)\, u\, (-\Delta)^\delta \varphi\,\mathrm{d}t\,\mathrm{dx} + \int_{\mathbb{R}^d} \int_a^T \psi'(t)\, u\, {}^C\mathrm{D}_{t,T}^{\alpha,\psi} \varphi\,\mathrm{d}t\,\mathrm{dx}
\end{aligned}
$$

for all compactly supported functions $\varphi \in C^2\left([a,T], H^{2\delta}(\mathbb{R}^d)\right)$ such that $\gamma_\psi \varphi \in C^0\left([a,T], H^{2\delta}(\mathbb{R}^d)\right)$, and $\varphi(T) \equiv 0, \quad \gamma_\psi \varphi(T) \equiv 0$.

The relationship between the mild and weak solutions of Eq. (1.1) is established in the following theorem.

Theorem 3.3. *Let $\alpha \in (1,2)$, $\delta \in (0,1)$, $p > 1$, and suppose u_a, $\tilde{u}_a \in C_0\left(\mathbb{R}^d\right)$.*

If $u \in C\left([a,T], C_0\left(\mathbb{R}^d\right)\right)$ is a mild solution of Eq. (1.1), then u is also a weak solution of Eq. (1.1) for any $T > a$.

Proof. In view of the assumptions and Definition 3.1, it follows that

$$u - u_a - (\psi(t) - \psi(a))\tilde{u}_a = G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a + \int_a^t \psi'(\tau) G_f^{\psi}\left(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)]\right) * |u(\tau)|^p \mathrm{d}\tau.$$

Applying the ψ-fractional integral to both sides of the above equality and invoking Lemma 2.7 yields

$$\begin{aligned} &\mathrm{I}_{a,t}^{2-\alpha,\psi}\left(u - u_a - (\psi(t) - \psi(a))\tilde{u}_a\right) \\ &\quad = \mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big) \\ &\quad\quad + \int_a^t \psi'(\tau)\tilde{G}_a^{\psi}\left(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)]\right) * |u(\tau)|^p \mathrm{d}\tau. \end{aligned} \tag{3.3}$$

Let $\varphi \in C^2\left([a,T], H^{2\delta}(\mathbb{R}^d)\right)$ be compactly supported and such that $\gamma_\psi \varphi \in C^0\left([a,T], H^{2\delta}(\mathbb{R}^d)\right)$, $\varphi(T) \equiv 0$ and $\gamma_\psi \varphi(T) \equiv 0$. Then multiplying both sides of (3.3) by $\gamma_\psi \varphi(t)$ and integrating over $\mathbb{R}^d$ yields

$$\int_{\mathbb{R}^d} \mathrm{I}_{a,t}^{2-\alpha,\psi}\left(u - u_a - (\psi(t) - \psi(a))\tilde{u}_a\right)\gamma_\psi \varphi \,\mathrm{dx} = I + II,$$

where

$$I = \int_{\mathbb{R}^d} \mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big)\gamma_\psi \varphi \mathrm{dx},$$

$$II = \int_{\mathbb{R}^d}\int_a^t \psi'(\tau)\tilde{G}_a\left(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)]\right) * |u(\tau)|^p \mathrm{d}\tau \gamma_\psi \varphi \mathrm{dx}.$$

We now begin by considering the term I. Applying Lemma 2.6 to I yields

$$\begin{aligned} \gamma_\psi I &= \int_{\mathbb{R}^d} \gamma_\psi\left[\mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big)\right]\gamma_\psi \varphi \,\mathrm{dx} \\ &\quad + \int_{\mathbb{R}^d}\left[\mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big)\right]\gamma_\psi^2 \varphi \,\mathrm{dx} \\ &= -\int_{\mathbb{R}^d} {}^{C}\mathrm{D}_{a,t}^{\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\Big)\varphi \,\mathrm{dx} \\ &\quad + \int_{\mathbb{R}^d}\left[\mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big)\right]\gamma_\psi^2 \varphi \,\mathrm{dx} \\ &= \int_{\mathbb{R}^d} (-\Delta)^{\delta}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a\Big)\varphi \,\mathrm{dx} \\ &\quad + \int_{\mathbb{R}^d}\left[\mathrm{I}_{a,t}^{2-\alpha,\psi}\Big(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a - u_a - (\psi(t) - \psi(a))\tilde{u}_a\Big)\right]\gamma_\psi^2 \varphi \,\mathrm{dx}. \end{aligned} \tag{3.4}$$

The next step is to consider the term II. Then, one attains

$$\gamma_\psi II = \int_{\mathbb{R}^d}\int_a^t \psi'(\tau)\tilde{G}_a^{\psi}\left(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)]\right) * |u(\tau)|^p \mathrm{d}\tau \gamma_\psi^2 \varphi \,\mathrm{dx}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^d}\int_a^t \psi'(\tau)\gamma_\psi \tilde{G}_a^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)*|u(\tau)|^p \mathrm{d}\tau \gamma_\psi \varphi \,\mathrm{dx}\\
&=\int_{\mathbb{R}^d}\int_a^t \psi'(\tau)\tilde{G}_a^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)*|u(\tau)|^p \mathrm{d}\tau \gamma_\psi^2 \varphi \,\mathrm{dx}\\
&+\int_{\mathbb{R}^d}\int_a^t \psi'(\tau)G_f^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)*|u(\tau)|^p \mathrm{d}\tau \gamma_\psi \varphi \,\mathrm{dx}\\
&=II_1+II_2. \qquad (3.5)
\end{aligned}
$$

Integrating (3.4) from a to T , one obtains

$$
\begin{aligned}
&\int_a^T \psi'(t)\gamma_\psi I \mathrm{d}t=\int_a^T\int_{\mathbb{R}^d}\psi'(t)(-\Delta)^\delta\left(G_a^\psi(t)*u_a+\tilde{G}_a^\psi(t)*\tilde{u}_a\right)\varphi \,\mathrm{dx}\,\mathrm{d}t\\
&+\int_a^T\int_{\mathbb{R}^d}\psi'(t)\left(G_a^\psi(t)*u_a+\tilde{G}_a^\psi(t)*\tilde{u}_a-u_a-(\psi(t)-\psi(a))\tilde{u}_a\right){}^C\mathrm{D}_{t,T}^{\alpha,\psi}\varphi \,\mathrm{dx}\,\mathrm{d}t.
\end{aligned} \tag{3.6}
$$

Next, we evaluate the integral $\int_a^T \psi'(t)\gamma_\psi II \mathrm{d}t$. It follows from (3.5) and Lemma 2.7 that

$$
\begin{aligned}
\int_a^T \psi'(t)II_1\mathrm{d}t&=\int_a^T\int_{\mathbb{R}^d}\int_a^t \psi'(t)\psi'(\tau)\tilde{G}_a\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)\\
&\quad *|u(\tau)|^p\mathrm{d}\tau\gamma_\psi^2\varphi \mathrm{dxd}t\\
&=\int_a^T\int_{\mathbb{R}^d}\psi'(t)\mathrm{I}_{a,t}^{2-\alpha,\psi}\hbar(t)\gamma_\psi^2\varphi\,\mathrm{dx}\,\mathrm{d}t\\
&=\int_a^T\int_{\mathbb{R}^d}\psi'(t)\hbar(t){}^C\mathrm{D}_{t,T}^{\alpha,\psi}\varphi\,\mathrm{dx}\,\mathrm{d}t. \qquad (3.7)
\end{aligned}
$$

Using the mean value theorem, one obtains that

$$
\begin{aligned}
\int_a^T \psi'(t)II_2\mathrm{d}t&=\int_a^T\int_{\mathbb{R}^d}\int_a^t \psi'(t)\psi'(\tau)G_f^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right) \qquad (3.8)\\
&\quad *|u(\tau)|^p\mathrm{d}\tau\gamma_\psi\varphi\,\mathrm{dx}\,\mathrm{d}t\\
&=-\int_a^T\int_{\mathbb{R}^d}\psi'(t)|u(t)|^p\varphi\,\mathrm{dx}\,\mathrm{d}t\\
&\quad+\int_a^T\int_{\mathbb{R}^d}\int_a^t \psi'(t)\psi'(\tau)G_f^\psi\left(\psi^{-1}[\psi(t)+\psi(a)-\psi(\tau)]\right)\\
&\quad *|u(\tau)|^p\mathrm{d}\tau(-\Delta)^\delta\varphi\,\mathrm{dx}\,\mathrm{d}t. \qquad (3.9)
\end{aligned}
$$

As a result, combining (3.6), (3.7), and (3.8), we obtain

$$
\begin{aligned}
0&=\int_a^T \psi'(t)\gamma_\psi\int_{\mathbb{R}^d}\mathrm{I}_{a,t}^{2-\alpha,\psi}\left(u-u_a-(\psi(t)-\psi(a))\tilde{u}_a\right)\gamma_\psi\varphi\,\mathrm{dx}\,\mathrm{d}t\\
&=\int_a^T \psi'(t)(\gamma_\psi I+\gamma_\psi II)\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^T \int_{\mathbb{R}^d} \psi'(t) \left(G_a^{\psi}(t) * u_a + \tilde{G}_a^{\psi}(t) * \tilde{u}_a \right) (-\Delta)^{\delta} \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&\quad + \int_a^T \int_{\mathbb{R}^d} \psi'(t) \left(u - u_a - (\psi(t) - \psi(a)) \tilde{u}_a \right) {}^C\mathrm{D}_{t,T}^{\alpha,\psi} \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&\quad - \int_a^T \int_{\mathbb{R}^d} \psi'(t) |u(t)|^p \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&\quad + \int_a^T \int_{\mathbb{R}^d} \int_a^t \psi'(t) G_f^{\psi} \left(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)] \right) \\
&\quad * \psi'(\tau) |u(\tau)|^p \, \mathrm{d}\tau (-\Delta)^{\delta} \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&= \int_a^T \int_{\mathbb{R}^d} \psi'(t) u (-\Delta)^{\delta} \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&\quad + \int_a^T \int_{\mathbb{R}^d} \psi'(t) \left(u - u_a - (\psi(t) - \psi(a)) \tilde{u}_a \right) {}^C\mathrm{D}_{t,T}^{\alpha,\psi} \varphi \, \mathrm{dx} \, \mathrm{d}t \\
&\quad - \int_a^T \int_{\mathbb{R}^d} \psi'(t) |u(t)|^p \varphi \, \mathrm{dx} \, \mathrm{d}t.
\end{aligned}
$$

In other words, u is a weak solution of Eq. (1.1). This completes the proof. $\square$

Next, we investigate the blow-up property of the solution to Eq. (1.1). As usual, the solution u to Eq. (1.1) is said to blow up in finite time T if

$$\lim_{t \to T} \|u(t)\|_{L^\infty(\mathbb{R}^d)} = +\infty.$$

The following theorem asserts that the mild solution of Eq. (1.1) blows up in finite time, provided the initial value satisfy certain conditions.

Theorem 3.4. *Let $d \in \mathbb{N}$, $\alpha \in (1,2)$, $\delta \in (0,1)$, and $p > 1$. Suppose the initial value u_a, $\tilde{u}_a \in C_0(\mathbb{R}^d)$ satisfy u_a, $\tilde{u}_a \geq 0$, and*

$$\int_{\mathbb{R}^d} u_a(\mathrm{x}) \, \eta(\mathrm{x}) \, \mathrm{dx} > 3^{\frac{1}{p-1}}, \qquad \int_{\mathbb{R}^d} \tilde{u}_a(\mathrm{x}) \, \eta(\mathrm{x}) \, \mathrm{dx} \geq 0,$$

where

$$\eta(\mathrm{x}) = \left(\int_{\mathbb{R}^d} e^{-\sqrt{|\mathrm{x}|^2 + d^2}} \, \mathrm{dx} \right)^{-1} e^{-\sqrt{|\mathrm{x}|^2 + d^2}}.$$

Then the mild solution of Eq. (1.1) blows up in finite time.

Proof. Let the function ϕ satisfy

$$\phi \in C_0^\infty(\mathbb{R}), \quad \phi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases} \quad \text{and} \quad 0 \leq \phi \leq 1. \tag{3.10}$$

Define $\varphi_n(\mathrm{x}, t) = \eta(\mathrm{x}) \, \phi_n(\mathrm{x}) \, \widetilde{\varphi}(t)$, where $\phi_n(\mathrm{x}) = \phi\left(\frac{|\mathrm{x}|}{n}\right)$ for $n = 1, 2, \ldots$, and

$$\widetilde{\varphi}(t) = \left(1 - \frac{\psi(t) - \psi(a)}{\psi(T) - \psi(a)} \right)^k,$$

with $k \geq \max\left\{2, \frac{p\alpha}{p-1}\right\}$ and $t \in [a, T]$.

Since the mild solution of Eq. (1.1) is also a weak solution, it follows from Definition 3.2 that

$$\int_{\mathbb{R}^d}\int_a^T \psi'(t)\left[u^p\,\varphi_n + (u_a + (\psi(t)-\psi(a))\,\tilde{u}_a)\;{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\varphi_n\right]\mathrm{d}t\,\mathrm{dx}$$
$$= \int_{\mathbb{R}^d}\int_a^T \psi'(t)\,u\,(-\Delta)^{\delta}\varphi_n\,\mathrm{d}t\,\mathrm{dx} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)\,u\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\varphi_n\,\mathrm{d}t\,\mathrm{dx}. \tag{3.11}$$

Owing to the useful inequality $(-\Delta)^{\delta}\eta(\mathrm{x}) \leq \eta(\mathrm{x})$ (see Theorem 4.3 in [13]) and the Lebesgue dominated convergence theorem, letting $n \to \infty$ in (3.11) gives

$$\int_{\mathbb{R}^d}\int_a^T \psi'(t)\,u^p\,\eta\,\widetilde{\varphi}\,\mathrm{d}t\,\mathrm{dx} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)\;(u_a + (\psi(t)-\psi(a))\,\tilde{u}_a)\;\eta\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}\,\mathrm{d}t\,\mathrm{dx}$$
$$\leq \int_{\mathbb{R}^d}\int_a^T \psi'(t)\;\left(3u\,\eta\,\widetilde{\varphi} + u\,\eta\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}\right)\,\mathrm{d}t\,\mathrm{dx}. \tag{3.12}$$

As $n \to \infty$, employing Jensen's inequality in (3.12) leads to

$$\int_a^T \psi'(t)\left(\int_{\mathbb{R}^d} u\,\eta\,\mathrm{dx}\right)^p \widetilde{\varphi}(t)\,\mathrm{d}t$$
$$+ \int_{\mathbb{R}^d}\int_a^T \psi'(t)\;(u_a + (\psi(t)-\psi(a))\,\tilde{u}_a)\;\eta\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\,\mathrm{d}t\,\mathrm{dx}$$
$$\leq \int_{\mathbb{R}^d}\int_a^T \psi'(t)\;\left(3u\,\eta\,\widetilde{\varphi}(t) + u\,\eta\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\right)\,\mathrm{d}t\,\mathrm{dx}. \tag{3.13}$$

Now let $f(t) = \int_{\mathbb{R}^d} u\,\eta\,\mathrm{d}\mathbf{x}$. Then, (3.13) becomes

$$\int_a^T \psi'(t)\,(f^p(t) - 3f(t))\;\widetilde{\varphi}(t)\,\mathrm{d}t$$
$$+ \int_a^T \psi'(t)\left(f(a) + \tilde{f}(a)(\psi(t)-\psi(a))\right)\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\,\mathrm{d}t$$
$$\leq \int_a^T \psi'(t)\,f(t)\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\,\mathrm{d}t. \tag{3.14}$$

Direct computation gives

$$\int_a^T \psi'(t)\,f(a)\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\,\mathrm{d}t = \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)}\,f(a)\,(\psi(T)-\psi(a))^{1-\alpha}. \tag{3.15}$$

Using Young's inequality yields

$$\int_a^T \psi'(t)\,f(t)\,{}^{C}\mathrm{D}_{t,T}^{\alpha,\psi}\widetilde{\varphi}(t)\,\mathrm{d}t \leq \epsilon\int_a^T \psi'(t)\,f^p(t)\,\widetilde{\varphi}(t)\,\mathrm{d}t + C(\epsilon)\,(\psi(T)-\psi(a))^{1-\frac{\alpha p}{p-1}}. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) gives

$$\begin{aligned}\int_a^T \psi'(t)\left(f^p(t)-3f(t)\right)\widetilde{\varphi}(t)\,\mathrm{d}t + \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)}\, f(a)\,(\psi(T)-\psi(a))^{1-\alpha} \\ + \frac{\Gamma(k+1)}{\Gamma(k+2-\alpha)}\, \tilde{f}(a)\,(\psi(T)-\psi(a))^{2-\alpha} \\ \le \epsilon \int_a^T \psi'(t)\, f^p(t)\,\widetilde{\varphi}(t)\,\mathrm{d}t + C(\epsilon)\,(\psi(T)-\psi(a))^{1-\frac{\alpha p}{p-1}}. \qquad (3.17)\end{aligned}$$

Choose $\epsilon > 0$ sufficiently small such that

$$f(a) > \left(\frac{3}{1-\epsilon}\right)^{\frac{1}{p-1}}.$$

Then, from (3.17), it follows that

$$f(a) \le C\,(\psi(T)-\psi(a))^{\alpha-\frac{\alpha p}{p-1}}, \qquad (3.18)$$

for some constant $C > 0$.

We now proceed by contradiction. Assume that Eq. (1.1) admits a global solution. Then letting $T \to \infty$ in (3.18) yields $f(a) = 0$. However, this contradicts the initial condition

$$f(a) = \int_{\mathbb{R}^d} u_a(\mathrm{x})\,\eta(\mathrm{x})\,\mathrm{dx} > 3^{\frac{1}{p-1}}.$$

Hence, the mild solution of Eq. (1.1) blows up in finite time. This completes the proof. □

We now establish some further finite-time blow-up conditions. This time, we will consider a general initial data, while the aforementioned conditions will be expressed in terms of the semilinearity exponent. We need however to distinguish among two cases: let us first consider $\tilde{u}_a \equiv 0$.

Theorem 3.5. *Let $d \in \mathbb{N}$, $\alpha \in (1,2)$, $\delta \in (0,1)$, and suppose $0 \le u_a \in C_0(\mathbb{R}^d)$ with $u_a \not\equiv 0$. Then the mild solution of Eq. (1.1) blows up in finite time, provided that $1 < p < 1 + \frac{2\delta}{d}$.*

Proof. Assume that the function ϕ is given as in (3.10). Define

$$\chi(\mathrm{x}) = \left(\phi\left((\psi(T)-\psi(a))^{-\frac{\alpha}{2\delta}}|\mathrm{x}|\right)\right)^{\frac{2p}{p-1}}, \quad \varpi(t) = \left(1-\frac{\psi(t)-\psi(a)}{\psi(T)-\psi(a)}\right)^k,$$

where $k \ge \max\left\{2, \frac{p\alpha}{p-1}\right\}$ and $t \in [a,T]$.

In view of Definition 3.1 and Theorem 3.3, we have

$$\int_{\mathbb{R}^d}\int_a^T \psi'(t)\left(u^p\,\chi\,\varpi + u_a\,\chi\,{}^{C}\mathrm{D}^{\alpha,\psi}_{t,T}\varpi\right)\mathrm{d}t\,\mathrm{dx}$$

$$= \int_{\mathbb{R}^d}\int_a^T \psi'(t)\left(u\,(-\Delta)^{\delta}\chi\,\varpi + u\,\chi\,{}^C\mathrm{D}^{\alpha,\psi}_{t,T}\varpi\right)\,\mathrm{d}t\,\mathrm{d}\mathrm{x}. \tag{3.19}$$

Note that

$$(-\Delta)^{\delta}\chi(\mathrm{x})\varpi(t) \le \frac{2p}{p-1}\phi\left((\psi(T)-\psi(a))^{-\frac{\alpha}{2\delta}}|\mathrm{x}|\right)\chi^{\frac{1}{p}}(\psi(T)-\psi(a))^{-\alpha}$$
$$\times(-\Delta)^{\delta}\phi(|\mathrm{x}|)[\varpi(t)]^{\frac{1}{p}}[\varpi(t)]^{1-\frac{1}{p}} \le C_1(\psi(T)-\psi(a))^{-\alpha}\chi^{\frac{1}{p}}\varpi^{\frac{1}{p}} \tag{3.20}$$

and

$$\chi\,{}^C\mathrm{D}^{\alpha,\psi}_{t,T}\varpi(t) \le \phi\left((\psi(T)-\psi(a))^{-\frac{\alpha}{2\delta}}|\mathrm{x}|\right)\chi^{\frac{1}{p}}\frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(\psi(T)-\psi(a))^{-\alpha}$$
$$\times\left(1-\frac{\psi(t)-\psi(a)}{\psi(T)-\psi(a)}\right)^{k-\alpha} \le C_2(\psi(T)-\psi(a))^{-\alpha}\chi^{\frac{1}{p}}\varpi^{\frac{1}{p}}, \tag{3.21}$$

where the constants C_1 and C_2 are independent of T. Substituting (3.20) and (3.21) into (3.19), and taking Hölder inequality and Young's inequality into account, one has

$$\int_{\mathbb{R}^d}\int_a^T \psi'(t)u^p\chi\varpi\,\mathrm{d}t\,\mathrm{d}\mathrm{x} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)u_a\chi\,{}^C\mathrm{D}^{\alpha,\psi}_{t,T}\varpi\,\mathrm{d}t\,\mathrm{d}\mathrm{x}$$
$$\le C(p)(\psi(T)-\psi(a))^{1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)u^p\chi\varpi\mathrm{d}t\,\mathrm{d}\mathrm{x}.$$

As a result,

$$\int_{\mathbb{R}^d} u_a\chi\,\mathrm{d}\mathrm{x} \le C(\alpha,p)(\psi(T)-\psi(a))^{\alpha+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}. \tag{3.22}$$

Since $1<p<1+\frac{2\delta}{d}$, $\alpha+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}<0$. Thus, if a global solution of Eq. (1.1) exists, then one can arrive at $u_a\equiv 0$ as $T\to\infty$ in (3.22). However, this contradicts with the assumption $u_a\not\equiv 0$.Hence, the mild solution of Eq. (1.1) blows up in a finite time. $\square$

For the case $\tilde{u}_a\not\equiv 0$, we have the following result.

Theorem 3.6. *Let $d\ge 2$. Assume that $0\le u_a,\tilde{u}_a\in C_0(\mathbb{R}^d)$ and $\tilde{u}_a\not\equiv 0$. If $1<p<1+\frac{2\alpha\delta}{\alpha d-2\delta}$, then any mild solution of Eq. (1.1) blows up in a finite time.*

Proof. By an argument similar to that used in the proof of Theorem 3.5, we obtain

$$\int_{\mathbb{R}^d}\int_a^T \psi'(t)u^p\chi\varpi\,\mathrm{d}t\,\mathrm{d}\mathrm{x} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)(u_a+(\psi(t)-\psi(a))\tilde{u}_a)\chi\,{}^C\mathrm{D}^{\alpha,\psi}_{t,T}\varpi\,\mathrm{d}t\,\mathrm{d}\mathrm{x}$$
$$\le C(p)(\psi(T)-\psi(a))^{1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}} + \int_{\mathbb{R}^d}\int_a^T \psi'(t)u^p\chi\varpi\,\mathrm{d}t\,\mathrm{d}\mathrm{x}.$$

As a result, we have

$$\int_{\mathbb{R}^d}\tilde{u}_a\chi\mathrm{d}\mathrm{x} \le C(\alpha,p)(\psi(T)-\psi(a))^{\alpha-1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}}. \tag{3.23}$$

Since $1<p<1+\frac{2\alpha\delta}{\alpha d-2\delta}$, $\alpha-1+\frac{\alpha d}{2\delta}-\frac{\alpha p}{p-1}<0$ Thus, if the solution of Eq. (1.1) exists globally, then taking $T\to\infty$, we obtain $\tilde{u}_a\equiv 0$ by (3.23), which contradicts $\tilde{u}_a\not\equiv 0$. $\square$

4. Conclusion

In this study, we have investigated the blow-up phenomena of solutions to the time-space fractional superdiffusion Eq. (1.1), which involves the ψ-Caputo fractional derivative and the fractional Laplacian. We have rigorously established the conditions under which solutions to Eq. (1.1) exhibit finite-time blow-up.

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Yacine Arioua,

Laboratory for Pure and Applied Mathematics, University of M'sila University Pole, Road Bourdj Bou Arreiridj, M'sila, 28000, Algeria,
E-mail: yacine.arioua@univ-msila.dz

Brahim Benaissi,

Laboratory for Pure and Applied Mathematics, University of M'sila University Pole, Road Bourdj Bou Arreiridj, M'sila, 28000, Algeria,
E-mail: brahim.benaissi@univ-msila.dz

Скінченночасовий вибух для дробового рівняння супердифузії з часовою похідною ψ-Капуто

Yacine Arioua and Brahim Benaissi

У цій статті досліджується часово-просторове дробове нелінійне рівняння супердифузії, розглянуте на всьому евклідовому просторі. Дробова похідна за часом є похідною ψ-Капуто порядку з інтервалу $(1, 2)$, тоді як просторова похідна виражена за допомогою дробового лапласіана порядку з інтервалу $(0, 1)$. Пов'язана задача Коші аналізується шляхом побудови як м'яких (mild), так і слабких (weak) розв'язків. Доведено, що м'який розв'язок також задовольняє слабке формулювання. Також досліджено поведінку розв'язків рівняння супердифузії в умовах скінченночасового вибуху (blow-up).

Ключові слова: дробове рівняння супердифузії, похідна ψ-Капуто, дробовий лапласіан, вибух розв'язку, м'який розв'язок, слабкий розв'язок