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On the Blow-up of Coupled Wave Equations with Time-Varying Delay and Logarithmic Nonlinearity

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This paper investigates a system of nonlinear wave equations exhibiting time-varying delay and logarithmic nonlinearity. We first prove the local existence and uniqueness of solutions using semigroup theory. Our main result establishes that solutions with negative initial energy undergo finite-time blow-up, generalizing the scalar case [5, 9]. The proof relies on a carefully constructed Lyapunov functional and nonlinear differential inequalities, adapting methods from [4, 14] to handle the coupled structure. This work extends the blow-up theory to coupled systems with delay.

Key words: coupled wave system, time-varying delay, logarithmic nonlinearity, local existence, finite-time blow-up, Lyapunov functional, semigroup theory

Mathematical Subject Classification 2020: 35L51, 35L71, 35B44, 35R10

1. Introduction

The study of nonlinear wave equations constitutes a central theme in modern mathematical physics, offering a rich framework to model a vast array of phenomena, from the propagation of sound and light to the vibrations of elastic structures and cosmological models. Among the diverse classes of nonlinearities, those of logarithmic type have garnered significant and sustained interest due to their unique mathematical properties and their fundamental appearance in various physical theories. Logarithmic nonlinearities arise naturally in quantum mechanics, particularly in the context of nonlinear wave mechanics as proposed by Bialynicki-Birula and Mycielski [3], where they introduced a Schrödinger equation with a logarithmic term to describe a physically plausible nonlinear wave equation that satisfies the separability property for noninteracting systems. This model has found applications in nuclear physics, optics, geophysics, and even in the theory of superfluids. The mathematical appeal of such nonlinearities lies in their specific growth properties: they lie on the borderline between polynomial and exponential growth, being sign-changing and non-differentiable at the origin. This behavior introduces profound challenges and subtleties in the analysis of well-posedness, stability, and the long-time behavior of solutions, distinguishing them sharply from the more commonly studied power-type nonlinearities.

The foundational work on evolution equations with logarithmic nonlinearities was established by Cazenave and Haraux [4], who provided a comprehensive analysis of the Cauchy problem, exploring questions of global existence, uniqueness, and the asymptotic behavior of solutions. Following this pioneering effort, a substantial body of literature has been developed. For instance, Zeng et al. [16] studied a class of fractional p-Laplacian Kirchhoff diffusion equation with logarithmic nonlinearity, for both subcritical and critical states, by means of the Galerkin approximations, the potential well theory and the Nehari manifold, they proved the global existence and finite time blow-up of the weak solutions. Al-Gharabli et al. [1] concerned with a viscoelastic wave equation with variable exponent and logarithmic nonlinearities, they proved a global existence result using the well-depth method and established explicit and general decay results under a wide class of relaxation functions and some specific conditions on the variable exponent function. These studies highlight the delicate balance that logarithmic terms create: they can, under certain conditions, have a regularizing effect, yet under others, particularly when acting as source terms, they can drive solutions to blow up in finite time.

In parallel to the development of theories for logarithmic nonlinearities, the past two decades have witnessed a surge of interest in the analysis of evolution equations incorporating time-delay effects. Delays are ubiquitous in real-world applications, arising inevitably from feedback mechanisms, signal processing times, hydraulic actuation, and biological processes. In control theory, delays are often intentionally introduced for stabilization purposes. However, it is well-known that even small delays can destabilize a system that is otherwise stable, a phenomenon famously illustrated in the context of boundary damping for the wave equation. The seminal work of Nicaise, Pignotti, and Valein [14] rigorously addressed this issue, establishing exponential stability results for the wave equation with a boundary time-varying delay, provided the delay term is sufficiently small relative to the damping. This line of research has been extensively developed for various hyperbolic, parabolic, and coupled systems, underscoring the critical need to carefully account for delay terms in stability analysis.

The intersection of these two active research areas “logarithmic nonlinearities and time delay” presents a fascinating and highly non-trivial problem. The combined effects can lead to a complex interplay between the destabilizing potential of the delay and the unique forcing nature of the logarithmic source. Recent years have seen several important contributions that begin to explore this confluence. Kafini and Messaoudi [9] made significant strides by analyzing a single logarithmic nonlinear wave equation with a constant delay term. They established a local existence theorem and, crucially, provided a blow-up result for solutions with negative initial energy, demonstrating that the presence of delay does not necessarily preclude finite-time singularity formation. Building upon this, Choucha and Ouchenane [5] extended the analysis to the more complex and technically demanding case of a *time-varying delay*. Their work provided refined local existence results and blow-up criteria, highlighting the additional challenges posed by the non-constant nature of the delay function $\tau(t)$, such as the need for more intricate energy functionals and the assumption $\tau'(t) \leq d < 1$ to manage the derivative of the delay.

The research landscape has further expanded to include even more sophisticated models. Very recently, a study [2] investigated a logarithmic wave equation incorporating both *infinite memory* and a *strong time-varying delay*, proving conditions for exponential energy decay and the occurrence of blow-up phenomena, thus showcasing the rich dynamics possible in such systems. Furthermore, the analysis has been extended to complex boundary conditions. For example, [6] examined a nonlinear wave equation coupled with acoustic and fractional boundary conditions through logarithmic source and delay terms, establishing results on global existence and the asymptotic behavior of solutions. On the numerical and phenomenological side, [10] demonstrated through analysis and simulation how boundary delay effects can precipitate finite-time blow-up in nonlinear logarithmic wave dynamics. Conversely, it is essential to note that the introduction of delay is not always benign; as discussed by Racke [7], delay terms can indeed lead to instability and even ill-posedness in certain coupled systems like those in thermoelasticity and plate equations, emphasizing the need for a cautious and rigorous approach.

Despite these advanced studies on scalar equations, the theory for *coupled systems* of wave equations with both logarithmic nonlinearities and time-varying delays remains largely unexplored. Coupled systems are paramount for modeling interactive fields, such as in thermoelasticity, piezoelectricity, or the description of complex elastic materials like porous-elastic solids. The interaction between components can lead to new and unexpected phenomena not present in the scalar case. The logarithmic coupling term in our system, $u|u|^{p-2}|v|^p\ln(|uv|^k)$ and its symmetric counterpart, introduces a strong and non-standard nonlinear interaction that makes the analysis of energy dissipation and potential blow-up considerably more challenging.

Logarithmic nonlinearities appear in quantum mechanics, nuclear physics, and geophysics, while coupled wave systems model interactive fields in thermoelasticity, piezoelectricity, and porous media. The specific coupling form mentioned above arises naturally in models where the interaction energy between components depends on both amplitudes and their product, with a logarithmic scaling law. Such forms are studied in the context of nonlinear Schrödinger and Klein–Gordon systems with logarithmic potentials, and they represent a natural extension of power-law couplings to the logarithmic case.

In this paper, we bridge this gap in the literature by conducting a comprehensive analysis of the following coupled wave equations with time-varying delay and logarithmic nonlinearity:

$$\begin{cases} u_{tt} - \Delta u + \alpha_1 u_t + \alpha_2 u_t(t-\tau(t)) = u|u|^{p-2}|v|^p \ln(|uv|^k), \\ v_{tt} - \Delta v + \beta_1 v_t + \beta_2 v_t(t-\tau(t)) = v|v|^{p-2}|u|^p \ln(|uv|^k), \end{cases} \tag{1.1}$$

where $x \in \Omega$, $t > 0$, k and $p \geq 2$ are positive constants, with the initial data and boundary conditions

$$\begin{aligned} &u(x,t) = v(x,t) = 0, && x \in \partial\Omega,\ t > 0, \\ &u(x,0) = u_0,\ u_t(x,0) = u_1, && x \in \Omega, \end{aligned}$$

$$\begin{aligned}
&v(x,0) = v_0,\ v_t(x,0) = v_1, && x \in \Omega,\\
&u_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), && (x,t) \in \Omega \times (0,\tau(0)),\\
&v_t(x,t-\tau(0)) = g_0(x,t-\tau(0)), && (x,t) \in \Omega \times (0,\tau(0)).
\end{aligned}$$

The time-varying delay $\tau(t)$ is assumed to be a positive, bounded function with a derivative bounded away from one ($\tau'(t) \le d < 1$).

Our main contributions are twofold:

Local Well-Posedness: We reformulate the problem as an abstract Cauchy problem on an extended energy space designed to incorporate the delay terms as additional variables satisfying transport equations. Utilizing the theory of monotone operators and the Lax–Milgram theorem, we prove the local existence and uniqueness of weak solutions.

Finite-Time Blow-Up: For the case of negative initial energy, we prove that solutions blow up in finite time. This result generalizes the blow-up phenomena known for scalar equations to the coupled system setting.

This work significantly extends recent results on scalar equations to a coupled system, and it provides a unified framework that encompasses well-posedness, stability, and instability analysis for a class of problems with combined logarithmic and delay effects.

The paper is structured as follows: in Section 2, we introduce the functional setting, assumptions, and prove the local existence and uniqueness theorem; Section 3 is devoted to the proof of the finite-time blow-up result.

2. Preliminaries and local existence

In this section, we shall study the well-posedness of solutions to problem (1.1). Throughout this paper, C or c represents a generic positive constant and is different in various occurrences and C_p is used to denote the Poincaré-type constant. First, we adopt the following assumptions:

The delay function $\tau(t)$ satisfies

$$0 < \tau_0 \le \tau(t) \le \bar{\tau}, \quad \tau(t) \in W^{2,\infty}([0,T]),\ T > 0. \tag{2.1}$$

The damping and delay coefficients satisfy

$$|\alpha_2| \le \sqrt{1-d}\alpha_1, \quad |\beta_2| \le \sqrt{1-d}\beta_1, \tag{2.2}$$

where d is a constant satisfies

$$\tau'(t) \le d < 1, \quad t > 0. \tag{2.3}$$

To establish local existence, we introduce new variables to handle the delay terms (see [8]):

$$\begin{aligned}
&z_1(x,\rho,t) = u_t(x,t-\tau(t)\rho),\\
&z_2(x,\rho,t) = v_t(x,t-\tau(t)\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).
\end{aligned} \tag{2.4}$$

These satisfy the transport equations:

$$\begin{aligned}
&\tau(t)\partial_t z_1 + (1-\tau'(t)\rho)\partial_\rho z_1 = 0,\\
&\tau(t)\partial_t z_2 + (1-\tau'(t)\rho)\partial_\rho z_2 = 0,\\
&z_1(x,0,t) = u_t(x,t),\ z_1(x,1,t) = u_t(t-\tau(t)),\\
&z_2(x,0,t) = v_t(x,t),\ z_2(x,1,t) = v_t(t-\tau(t)).
\end{aligned}$$

The use of the auxiliary variables z_1, z_2 is essential for handling time-varying delays in a way that preserves the dissipative structure of the system. Unlike constant delays, time-varying delays prevent a direct application of semigroup theory due to the non-autonomous nature of the delay terms. The transformation defined in (2.4) converts the delay term into a transport equation, which can be incorporated into an extended phase space. This technique, introduced by Nicaise, Pignotti, and Valein [14] for scalar equations, allows us to treat the system as an autonomous Cauchy problem and apply semigroup theory rigorously. The auxiliary variable technique is known for scalar equations, its application to coupled systems with logarithmic nonlinearity introduces significant and non-trivial challenges not present in the scalar case.

Next, let set $u_t = w$ and $v_t = z$, we define the energy space as follows:

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times (0,1)) \times L^2(\Omega \times (0,1)),$$

with the inner product

$$\langle U, \overline{U}\rangle_{\mathcal{H}} = \int_\Omega (\nabla u \cdot \nabla \overline{u} + w\overline{w} + \nabla v \cdot \nabla \overline{v} + z\overline{z})\, dx + \xi \int_\Omega \int_0^1 (z_1 \overline{z}_1 + z_2 \overline{z}_2)\, d\rho\, dx,$$

where $U = (u, w, v, z, z_1, z_2)^T$, $\overline{U} = (\overline{u}, \overline{w}, \overline{v}, \overline{z}, \overline{z}_1, \overline{z}_2)^T$, and $\xi > 0$ is a constant chosen such that

$$\frac{|\alpha_2|}{\sqrt{1-d}} \le \xi \le 2\alpha_1 - \frac{|\alpha_2|}{\sqrt{1-d}}, \quad \frac{|\beta_2|}{\sqrt{1-d}} \le \xi \le 2\beta_1 - \frac{|\beta_2|}{\sqrt{1-d}}.$$

The original system (1.1) is thus equivalent to

$$\begin{cases}
u_{tt} - \Delta u + \alpha_1 w + \alpha_2 z_1(x,1,t) = u|u|^{p-2}|v|^p \ln(|uv|^k),\ x \in \Omega,\ t>0,\\
v_{tt} - \Delta v + \beta_1 z + \beta_2 z_2(x,1,t) = v|v|^{p-2}|u|^p \ln(|uv|^k),\ x \in \Omega,\ t>0,\\
\tau(t)\partial_t z_1 + (1-\tau'(t)\rho)\partial_\rho z_1 = 0,\ (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty),\\
\tau(t)\partial_t z_2 + (1-\tau'(t)\rho)\partial_\rho z_2 = 0,\ (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty)
\end{cases} \tag{2.5}$$

with the initial data and boundary conditions

$$\begin{cases}
u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t>0,\\
u(x,0) = u_0,\ w(x,0) = u_1,\ v(x,0) = v_0,\ z(x,0) = v_1, & x \in \Omega,\\
z_1(x,\rho,0) = f_0(x,-\tau(0)\rho), z_2(x,\rho,0) = g_0(x,-\tau(0)\rho), & (x,\rho) \in \Omega \times (0,1),
\end{cases} \tag{2.6}$$

where $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$, and $f_0, g_0 \in L^2(\Omega \times (0,1))$.

The functions f_0 and g_0 represent the history of the velocity terms on the delay interval. Under the transformation (2.4), the initial functions f_0 and g_0 are mapped to

$$z_1(x,\rho,0) = f_0(x,-\tau(0)\rho), \quad z_2(x,\rho,0) = g_0(x,-\tau(0)\rho).$$

This ensures that $z_1, z_2 \in L^2(\Omega \times (0,1))$ at $t = 0$, which is consistent with the energy space $\mathcal{H}$. We define the operator $\mathcal{A}$ and the nonlinearity $\mathcal{J}$ as follows

$$\mathcal{A}(U) = \begin{pmatrix} -w \\ -\Delta u + \alpha_1 w + \alpha_2 z_1(\cdot,1) \\ -z \\ -\Delta v + \beta_1 z + \beta_2 z_2(\cdot,1) \\ \frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho z_1 \\ \frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho z_2 \end{pmatrix}, \tag{2.7}$$

$$\mathcal{J}(U) = \begin{pmatrix} 0 \\ u|u|^{p-2}|v|^p \ln(|uv|^k) \\ 0 \\ v|v|^{p-2}|u|^p \ln(|uv|^k) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ F(u,v) \\ 0 \\ G(u,v) \\ 0 \\ 0 \end{pmatrix}.$$

The domain of the operator $\mathcal{A}$ is defined as

$$\mathcal{D}(\mathcal{A}) = \{(u,w,v,z,z_1,z_2)^T \in \mathcal{H} : u,v \in H^2(\Omega) \cap H_0^1(\Omega),\ z_2(\cdot,0,\cdot) = z$$
$$w,z \in H_0^1(\Omega),\ z_1,z_2,\partial_\rho z_1,\partial_\rho z_2 \in L^2(\Omega \times (0,1)),\ z_1(\cdot,0,\cdot) = w\}.$$

Thus, the problem can be written as the abstract Cauchy problem:

$$\begin{cases} U_t + \mathcal{A}(t)(U) = \mathcal{J}(U), \\ U(0) = U_0 = (u_0, u_1, v_0, v_1, f_0(\cdot, -\cdot\,\tau(0)), g_0(\cdot, -\cdot\,\tau(0)))^T. \end{cases} \tag{2.8}$$

Lemma 2.1 (Growth at Infinity and near Zero)**.** *For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that*
for all $s \geq 1$,

$$\ln s \leq C_\epsilon s^\epsilon, \tag{2.9}$$

for all $0 < s \leq 1$,

$$|\ln s| \leq C_\epsilon s^{-\epsilon}. \tag{2.10}$$

Proof. The proof for (2.9) involves applying l'Hôpital's rule to show that $\lim_{s\to\infty} \frac{\ln s}{s^\epsilon} = 0$, proving the function $h(s) = \frac{\ln s}{s^\epsilon}$ is bounded for $s \geq 1$. Using the substitution $t = 1/s$, which maps $(0,1]$ to $[1,\infty)$, and applying (2.9) yields (2.10). ☐

Definition 2.2 (Weak Solution)**.** A weak solution of system (2.5) on $[0,T]$ is a quadruple (u,v,z_1,z_2) satisfying the following:

Regularity: We have

$$u, v \in C([0,T]; H_0^1(\Omega)),\ \ w, z \in C([0,T]; L^2(\Omega)), z_1, z_2 \in L^2(\Omega \times (0,1) \times (0,T)).$$

Weak Formulation: For all test functions $\phi, \psi \in C_0^\infty(\Omega \times [0,T])$:

$$\int_0^T \int_\Omega [w\phi_t - \nabla u \cdot \nabla \phi + (\alpha_1 w + \alpha_2 z_1(\cdot,1,t))\phi]\, dxdt = \int_0^T \int_\Omega F(u,v)\phi dxdt,$$

and similarly for v, with $F(u,v) = u|u|^{p-2}|v|^p \ln(|uv|^k)$.

Delay Compatibility: The transport equations hold weakly, i.e.,

$$\tau(t)z_{1t} + (1-\tau'(t)\rho)z_{1\rho} = 0,\ \ \tau(t)z_{2t} + (1-\tau'(t)\rho)z_{2\rho} = 0,$$

with $z_1(x,0,t) = w(x,t)$, $z_2(x,0,t) = z(x,t)$.

Initial Data: We have $u(0) = u_0,\ w(0) = u_1,\ v(0) = v_0,\ z(0) = v_1$.

We now give and prove existence and uniqueness results for our system using the semigroup theory.

Theorem 2.3 (Local Existence)**.** *Under the assumptions* (2.1)–(2.3)*, for any* $U_0 \in \mathcal{H}$*, there exists* $T > 0$ *such that the* (2.5) *has a unique weak solution* $U \in C([0,T];\mathcal{H})$*. If* $U_0 \in \mathcal{D}(\mathcal{A})$*, then*

$$U \in C\left(\mathbb{R}_+, \mathcal{D}(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}\right).$$

In order to prove Theorem 2.3, we will use the variable norm technique developed by Kato in [11]. Following Theorem 2.2 mentioned in [12] and proved in [11].

Proof of Theorem 2.3. As a consequence of Theorem 1.4 pp. 185–186, Pazy [15] (see also Komornik [13], pp. 117–118), we prove that the operator $\mathcal{A}$ defined by (2.7) is the infinitesimal generator of C^0-semigroup in $\mathcal{H}$, and that the nonlinear operator $\mathcal{F}$ is locally Lipschitz in $\mathcal{H}$. We will follow the method used in [2,5,12,14] with the necessary modification imposed by the nature of our problem.

As in [12], one can show easly that $\mathcal{D}(\mathcal{A}(0))$ is dense in $\mathcal{H}$. With our choice, $\mathcal{D}(\mathcal{A}(t))$ is independent of t, consequently, $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0))$, $t > 0$.

Now, we define the time-dependent inner-product on $\mathcal{H}$, which is the classical inner product for every $U = (u, w, v, z, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$. We have

$$\langle U, \overline{U}\rangle_t = \int_\Omega (\nabla u \cdot \nabla \overline{u} + w\overline{w} + \nabla v \cdot \nabla \overline{v} + z\overline{z})\, dx + \xi\tau(t) \int_\Omega \int_0^1 (z_1\overline{z}_1 + z_2\overline{z}_2)\, d\rho\, dx, \tag{2.11}$$

Let us set $\overline{\mathcal{A}}(t) = \mathcal{A}(t) + k(t)I$, where $k(t) = \frac{\sqrt{(1+\tau'(t)^2)}}{2\tau(t)}$.

We show that the operator $\overline{\mathcal{A}}(t)$ is monotonous in the following step. For fixed t and $(u, w, v, z, z_1, z_2)^T \in \mathcal{D}(\mathcal{A}(t))$, we get

$$\langle \mathcal{A}(t)U, U\rangle_t = \int_\Omega (-w)w\, dx + \int_\Omega (-\Delta u + \alpha_1 w + \alpha_2 z_1(\cdot,1))u_t\, dx$$

$$
\begin{aligned}
&+\int_\Omega (-z)z\,dx+\int_\Omega(-\Delta v+\beta_1 z+\beta_2 z_2(\cdot,1))v_t\,dx\\
&+\xi\int_\Omega\int_0^1\left(1-\tau'(t)\rho)\partial_\rho z_2\right)z_2 d\rho\,dx\\
&+\xi\int_\Omega\int_0^1\left(1-\tau'(t)\rho)\partial_\rho z_1\right)z_1 d\rho\,dx.
\end{aligned}
$$

Using integration by parts and the fact that $u,v\in H^2(\Omega)\cap H_0^1(\Omega)$, we have

$$
-\int_\Omega \Delta uw\,dx=\int_\Omega \nabla u\cdot\nabla w\,dx=\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2.
$$

However, for the purpose of proving monotonicity at a fixed time, we treat this term as part of the energy identity and focus on the sign of the remaining expression. The critical terms are those involving the delays and the transport equations. Let us analyze the z_1-terms:

$$
I_{z_1}=\alpha_2\int_\Omega z_1(x,1)w(x)\,dx+\xi\int_\Omega\int_0^1\left(1-\tau'(t)\rho\right)\partial_\rho z_1 z_1 d\rho\,dx.
$$

Note that

$$
\left(1-\tau'(t)\rho\right)\partial_\rho z_1 z_1=\frac{1}{2}(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z_1^2).
$$

Therefore,

$$
\begin{aligned}
\xi\int_0^1\left(1-\tau'(t)\rho\right)\partial_\rho z_1 z_1\,d\rho&=\frac{\xi}{2}\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z_1^2)\,d\rho\\
&=\frac{\xi}{2}\left[(1-\tau'(t))z_1^2(1)-z_1^2(0)+\tau'(t)\int_0^1 z_1^2(x,\rho)\,d\rho\right].
\end{aligned}
$$

Since $z_1(x,0)=w(x)$, we get

$$
\begin{aligned}
I_{z_1}=\;&\alpha_2\int_\Omega z_1(x,1)w(x)\,dx\\
&+\frac{\xi}{2}\int_\Omega\left[(1-\tau'(t))z_1^2(x,1)-w^2(x)+\tau'(t)\int_0^1 z_1^2(x,\rho)\,d\rho\right]dx.
\end{aligned}
$$

Now, applying Young's inequality to the first term gives

$$
\alpha_2\int_\Omega z_1(x,1)w(x)\,dx\geq-\frac{|\alpha_2|}{2\sqrt{1-d}}\int_\Omega w^2\,dx-\frac{|\alpha_2|\sqrt{1-d}}{2}\int_\Omega z_1^2(x,1)\,dx.
$$

Using the assumption (2.3), $\tau'(t)\leq d<1$, we have $(1-\tau'(t))\geq(1-d)>0$. Thus,

$$
\begin{aligned}
I_{z_1}\geq\;&-\left(\frac{|\alpha_2|}{2\sqrt{1-d}}+\frac{\xi}{2}\right)\int_\Omega w^2\,dx\\
&+\left(\frac{\xi(1-d)}{2}-\frac{|\alpha_2|\sqrt{1-d}}{2}\right)\int_\Omega z_1^2(x,1)\,dx+\frac{\xi\tau'(t)}{2}\int_\Omega\int_0^1 z_1^2\,d\rho\,dx.
\end{aligned}
$$

By choosing ξ such that

$$\frac{|\alpha_2|}{\sqrt{1-d}} \le \xi \le 2\alpha_1 - \frac{|\alpha_2|}{\sqrt{1-d}},$$

the coefficients become non-negative:

$$\alpha_1 - \frac{|\alpha_2|}{2\sqrt{1-d}} - \frac{\xi}{2} \ge 0, \quad \frac{\xi(1-d)}{2} - \frac{|\alpha_2|\sqrt{1-d}}{2} \ge 0.$$

An identical analysis holds for the terms involving z and z_2. That is

$$\begin{aligned} I_{z_1} \ge &-\left(\frac{|\beta_2|}{2\sqrt{1-d}} + \frac{\xi}{2}\right)\int_\Omega z^2\,dx \\ &+\left(\frac{\xi(1-d)}{2} - \frac{|\beta_2|\sqrt{1-d}}{2}\right)\int_\Omega z_2^2(x,1)dx + \frac{\xi\tau'(t)}{2}\int_\Omega\int_0^1 z_2^2\,d\rho\,dx. \end{aligned}$$

By choosing ξ such that

$$\frac{|\beta_2|}{\sqrt{1-d}} \le \xi \le 2\beta_1 - \frac{|\beta_2|}{\sqrt{1-d}},$$

the coefficients become non-negative:

$$\beta_1 - \frac{|\beta_2|}{2\sqrt{1-d}} - \frac{\xi}{2} \ge 0, \quad \frac{\xi(1-d)}{2} - \frac{|\beta_2|\sqrt{1-d}}{2} \ge 0.$$

By using Cauchy–Schwartz inequality, we get

$$\begin{aligned} \langle \mathcal{A}(t)U, U\rangle_t = &\left(\frac{\xi(1-d)}{2} - \frac{|\alpha_2|\sqrt{1-d}}{2}\right)\int_\Omega z_1^2(x,1)dx \\ &+\left(\alpha_1 - \frac{|\alpha_2|}{2\sqrt{1-d}} - \frac{\xi}{2}\right)\int_\Omega w^2dx \\ &+\left(\beta_1 - \frac{|\beta_2|}{2\sqrt{1-d}} - \frac{\xi}{2}\right)\int_\Omega z^2dx \\ &+\left(\frac{\xi(1-d)}{2} - \frac{|\beta_2|\sqrt{1-d}}{2}\right)\int_\Omega z_2^2(x,1)dx - k(t)\langle U, U\rangle_t \ge 0. \end{aligned}$$

Consequently, the operator $\overline{\mathcal{A}}(t)$ is monotone. To show that $\mathcal{A}(t)$ is maximal, we prove that for any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ (fixed), there exists $U \in \mathcal{D}(\mathcal{A})$ such that $(I + \mathcal{A})U = F$. This yields the system

$$u - w = f_1 \tag{2.12}$$

$$w - \Delta u + \alpha_1 w + \alpha_2 z_1(\cdot, 1) = f_2, \tag{2.13}$$

$$v - z = f_3, \tag{2.14}$$

$$z - \Delta v + \beta_1 z + \beta_2 z_2(\cdot, 1) = f_4, \tag{2.15}$$

$$z_1 + \frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho z_1 = f_5, \tag{2.16}$$

$$z_2 + \frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho z_2 = f_6. \tag{2.17}$$

From (2.12) and (2.14), we get

$$w = u - f_1, \tag{2.18}$$
$$z = v - f_3. \tag{2.19}$$

The equations (2.16) and (2.17) are linear first-order ODEs in ρ. Let's solve (2.16). This is an ODE for $z_1(x,\rho)$ for a.a. fixed x,t:

$$\partial_\rho z_1 + \frac{\tau(t)}{1-\tau'(t)\rho}z_1 = \frac{\tau(t)}{1-\tau'(t)\rho}f_5.$$

The integrating factor is $\mu(\rho) = \exp\left(\int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}ds\right)$. This integral evaluates to

$$\eta(\rho) = \int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}\,ds = \begin{cases} -\frac{\tau(t)}{\tau'(t)}\ln(1-\tau'(t)\rho) & \text{if } \tau'(t)\neq 0,\\ \tau(t)\rho & \text{if } \tau'(t)=0. \end{cases}$$

Thus, the integrating factor is $\mu(\rho) = e^{\eta(\rho)}$. The general solution is

$$z_1(x,\rho) = e^{-\int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}\,ds}\left[z_1(x,0) + \int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}f_5(x,s)e^{\int_0^s \frac{\tau(t)}{1-\tau'(t)r}\,dr}\,ds\right].$$

From the domain condition, $z_1(x,0) = w(x) = u(x) - f_1(x)$. Thus,

$$z_1(x,\rho) = [u(x) - f_1(x)]e^{-\eta(\rho)} + e^{-\eta(\rho)}\int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}f_5(x,s)e^{\eta(s)}\,ds,$$

where $\eta(\rho) = \int_0^\rho \frac{\tau(t)}{1-\tau'(t)s}ds$. In particular, at $\rho = 1$, we have

$$z_1(x,1) = u(x)e^{-\eta(1)} + \tilde{f}_5(x), \tag{2.20}$$

where

$$\tilde{f}_5(x) = -f_1(x)e^{-\eta(1)} + e^{-\eta(1)}\int_0^1 \frac{\tau(t)}{1-\tau'(t)s}f_5(x,s)e^{\eta(s)}\,ds \in L^2(\Omega).$$

A similar expression holds for $z_2(x,1)$:

$$z_2(x,1) = v(x)e^{-\delta(1)} + \tilde{f}_6(x), \tag{2.21}$$

with $\tilde{f}_6 \in L^2(\Omega)$. Now substituting (2.18) and (2.20) into (2.13), we obtain

$$(u-f_1) - \Delta u + \alpha_1(u-f_1) + \alpha_2(ue^{-\eta(1)} + \tilde{f}_5) = f_2.$$

Rearranging terms gives

$$-\Delta u + (1+\alpha_1+\alpha_2 e^{-\eta(1)})u = f_2 + f_1 + \alpha_1 f_1 - \alpha_2\tilde{f}_5 = G_1.$$

Similarly, from (2.15), we get

$$-\Delta v + (1 + \beta_1 + \beta_2 e^{-\delta(1)})v = f_4 + f_3 + \beta_1 f_3 - \beta_2 \tilde{f}_6 = G_2.$$

So, we arrive at a coupled system of elliptic boundary value problems:

$$-\Delta u + \gamma_{11} u + \gamma_{12} v = G_1 \qquad \text{in } \Omega, \tag{2.22}$$
$$-\Delta v + \gamma_{21} u + \gamma_{22} v = G_2 \qquad \text{in } \Omega, \tag{2.23}$$
$$u = v = 0 \qquad \text{on } \partial\Omega. \tag{2.24}$$

In our specific case, this system is decoupled because the equations for u and v were derived independently). However, for the sake of generality and a truly rigorous approach, we will demonstrate the method for a potentially coupled system. If the system is decoupled ($\gamma_{12} = \gamma_{21} = 0$), the method simplifies to applying Lax–Milgram separately. Next, the Hilbert space for the weak formulation of the system (2.22)–(2.24) is $\mathcal{V} = H_0^1(\Omega) \times H_0^1(\Omega)$.

This space is a Hilbert space when equipped with the inner product

$$\langle (u, v), (\phi, \psi) \rangle_{\mathcal{V}} = (\nabla u, \nabla \phi)_{L^2} + (\nabla v, \nabla \psi)_{L^2},$$

and the induced norm $\|(u, v)\|_{\mathcal{V}} = \left(\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right)^{1/2}$.

To obtain the weak formulation, we multiply (2.22) by a test function $\phi \in H_0^1(\Omega)$ and (2.23) by a test function $\psi \in H_0^1(\Omega)$, integrate over Ω, and add the resulting equations

$$\begin{aligned} &\int_\Omega (-\Delta u)\phi dx + \gamma_{11} \int_\Omega u\phi \, dx + \gamma_{12} \int_\Omega v\phi \, dx \\ &\quad + \int_\Omega (-\Delta v)\psi \, dx + \gamma_{21} \int_\Omega u\psi \, dx + \gamma_{22} \int_\Omega v\psi \, dx = \int_\Omega G_1 \phi dx + \int_\Omega G_2 \psi \, dx. \end{aligned}$$

Applying Green's identity to the Laplacian terms (and noting the boundary terms vanish due to $\phi, \psi \in H_0^1(\Omega)$), we get

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla \phi \, dx + \gamma_{11} \int_\Omega u\phi \, dx + \gamma_{12} \int_\Omega v\phi \, dx \\ &\quad + \int_\Omega \nabla v \cdot \nabla \psi \, dx + \gamma_{21} \int_\Omega u\psi \, dx + \gamma_{22} \int_\Omega v\psi \, dx = \int_\Omega G_1 \phi \, dx + \int_\Omega G_2 \psi \, dx. \end{aligned}$$

This must hold for all test functions $(\phi, \psi) \in \mathcal{V}$.

We now define the necessary forms on the product space $\mathcal{V}$.

The Bilinear Form $\mathbf{a} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$:

$$\begin{aligned} \mathbf{a}((u, v), (\phi, \psi)) = &\int_\Omega \nabla u \cdot \nabla \phi \, dx + \int_\Omega \nabla v \cdot \nabla \psi \, dx + \gamma_{12} \int_\Omega v\phi \, dx \\ &+ \gamma_{11} \int_\Omega u\phi \, dx + \gamma_{22} \int_\Omega v\psi \, dx + \gamma_{21} \int_\Omega u\psi \, dx. \end{aligned}$$

This form combines the energy from both equations and their coupling.

The Linear Functional $\mathbf{L} : \mathcal{V} \to \mathbb{R}$:

$$\mathbf{L}((\phi, \psi)) = \int_\Omega G_1 \phi dx + \int_\Omega G_2 \psi dx.$$

The weak formulation is equivalent to

Find $(u, v) \in \mathcal{V}$ such that $\mathbf{a}((u, v), (\phi, \psi)) = \mathbf{L}((\phi, \psi))$ for all $(\phi, \psi) \in \mathcal{V}$.

We now verify that $\mathbf{a}(\cdot, \cdot)$ and $\mathbf{L}(\cdot)$ satisfy the hypotheses of the Lax–Milgram theorem on the Hilbert space $\mathcal{V}$.

Boundedness of $a(\cdot, \cdot)$: We need to show there exists $M > 0$ such that

$$|\mathbf{a}((u, v), (\phi, \psi))| \leq M \|(u, v)\|_{\mathbf{V}} \|(\phi, \psi)\|_{\mathbf{V}}$$

for all $(u, v), (\phi, \psi) \in \mathcal{V}$.
We estimate each term using the Cauchy–Schwarz inequality and the Poincaré inequality ($\|w\|_{L^2} \leq C_P \|\nabla w\|_{L^2}$ for $w \in H_0^1(\Omega)$).

$$\begin{aligned}
|\mathbf{a}(\cdot)| \leq\ & |(\nabla u, \nabla \phi)| + |(\nabla v, \nabla \psi)| + |\gamma_{11}||(u, \phi)| + |\gamma_{12}||(v, \phi)| \\
& + |\gamma_{21}||(u, \psi)| + |\gamma_{22}||(v, \psi)| \\
\leq\ & \|\nabla u\|\|\nabla \phi\| + \|\nabla v\|\|\nabla \psi\| + |\gamma_{11}|C_P^2 \|\nabla u\|\|\nabla \phi\| + |\gamma_{12}|C_P^2 \|\nabla v\|\|\nabla \phi\| \\
& + |\gamma_{21}|C_P^2 \|\nabla u\|\|\nabla \psi\| + |\gamma_{22}|C_P^2 \|\nabla v\|\|\nabla \psi\|.
\end{aligned}$$

Let $A = \max\{|\gamma_{11}|, |\gamma_{12}|, |\gamma_{21}|, |\gamma_{22}|\}$. Then,

$$\begin{aligned}
|\mathbf{a}(\cdot)| \leq\ & \|\nabla u\|\|\nabla \phi\| + \|\nabla v\|\|\nabla \psi\| \\
& + AC_P^2 \left(\|\nabla u\|\|\nabla \phi\| + \|\nabla v\|\|\nabla \phi\| + \|\nabla u\|\|\nabla \psi\| + \|\nabla v\|\|\nabla \psi\|\right) \\
\leq\ & (1 + AC_P^2) \left(\|\nabla u\|\|\nabla \phi\| + \|\nabla v\|\|\nabla \psi\| + \|\nabla v\|\|\nabla \phi\| + \|\nabla u\|\|\nabla \psi\|\right).
\end{aligned}$$

Using the inequality $ab + cd + bc + ad \leq 2(a^2 + c^2)^{1/2}(b^2 + d^2)^{1/2}$ for $a, b, c, d \geq 0$, we get

$$\begin{aligned}
& \|\nabla u\|\|\nabla \phi\| + \|\nabla v\|\|\nabla \psi\| + \|\nabla v\|\|\nabla \phi\| + \|\nabla u\|\|\nabla \psi\| \\
& \quad \leq 2 \left(\|\nabla u\|^2 + \|\nabla v\|^2\right)^{1/2} \left(\|\nabla \phi\|^2 + \|\nabla \psi\|^2\right)^{1/2} = 2\|(u, v)\|_{\mathcal{V}} \|(\phi, \psi)\|_{\mathcal{V}}.
\end{aligned}$$

Therefore,

$$|\mathbf{a}((u, v), (\phi, \psi))| \leq 2(1 + AC_P^2) \|(u, v)\|_{\mathcal{V}} \|(\phi, \psi)\|_{\mathcal{V}}.$$

Thus, $\mathbf{a}(\cdot, \cdot)$ is bounded with $M = 2(1 + AC_P^2)$.

Coercivity of $a(\cdot, \cdot)$: We need to show there exists $\alpha > 0$ such that $\mathbf{a}((u, v), (u, v)) \geq \alpha \|(u, v)\|_{\mathcal{V}}^2$ for all $(u, v) \in \mathcal{V}$.

$$\begin{aligned}
\mathbf{a}((u, v), (u, v)) =\ & \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla v|^2 \, dx + \gamma_{11} \int_\Omega |u|^2 \, dx \\
& + \gamma_{12} \int_\Omega uv \, dx + \gamma_{21} \int_\Omega uv \, dx + \gamma_{22} \int_\Omega |v|^2 dx
\end{aligned}$$

$$= \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \gamma_{11}\|u\|_{L^2}^2 + \gamma_{22}\|v\|_{L^2}^2 + (\gamma_{12}+\gamma_{21})(u,v)_{L^2}.$$

In our specific case, the system is decoupled ($\gamma_{12} = \gamma_{21} = 0$), and $\gamma_{11}, \gamma_{22} > 0$. Therefore,

$$\begin{aligned}\mathbf{a}((u,v),(u,v)) &= \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \gamma_{11}\|u\|_{L^2}^2 + \gamma_{22}\|v\|_{L^2}^2 \\ &\geq \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = \|(u,v)\|_{\mathcal{V}}^2.\end{aligned}$$

Thus, $\mathbf{a}(\cdot,\cdot)$ is coercive with $\alpha = 1$.
Boundedness of $L(\cdot)$: We need to show that $|\mathbf{L}((\phi,\psi))| \leq C\|(\phi,\psi)\|_{\mathcal{V}}$.

$$|\mathbf{L}((\phi,\psi))| = \left|\int G_1\phi\, dx + \int G_2\psi\, dx\right| \leq \|G_1\|_{L^2}\|\phi\|_{L^2} + \|G_2\|_{L^2}\|\psi\|_{L^2}.$$

By the Poincaré inequality, $\|\phi\|_{L^2} \leq C_P\|\nabla\phi\|_{L^2}$ and $\|\psi\|_{L^2} \leq C_P\|\nabla\psi\|_{L^2}$. Thus,

$$|\mathbf{L}((\phi,\psi))| \leq C_P\left(\|G_1\|_{L^2}\|\nabla\phi\|_{L^2} + \|G_2\|_{L^2}\|\nabla\psi\|_{L^2}\right).$$

Using the Cauchy–Schwarz inequality for sums, we get

$$\begin{aligned}|\mathbf{L}((\phi,\psi))| &\leq C_P\left(\|G_1\|_{L^2}^2 + \|G_2\|_{L^2}^2\right)^{1/2}\left(\|\nabla\phi\|_{L^2}^2 + \|\nabla\psi\|_{L^2}^2\right)^{1/2} \\ &= C_P\|(G_1,G_2)\|_{L^2}\|(\phi,\psi)\|_{\mathcal{V}}.\end{aligned}$$

Therefore, $\mathbf{L}$ is bounded.

Since all hypotheses are satisfied, the Lax–Milgram theorem guarantees the existence of a unique vector $(u,v) \in \mathcal{V} = H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\mathbf{a}((u,v),(\phi,\psi)) = \mathbf{L}((\phi,\psi)) \quad \text{for all } (\phi,\psi) \in \mathcal{V}.$$

This (u,v) is a weak solution of the coupled system (2.22)–(2.24). The weak solution satisfies

$$\begin{cases}-\Delta u = G_1 - \gamma_{11}u - \gamma_{12}v \\ -\Delta v = G_2 - \gamma_{21}u - \gamma_{22}v\end{cases} \quad \text{in the sense of distributions.}$$

The right-hand sides belong to $L^2(\Omega)$ because $G_1, G_2 \in L^2$ and $u, v \in H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. By elliptic regularity for the system (or component-wise, since the Laplacian is diagonal and the domain is smooth), we conclude that

$$(u,v) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)].$$

From (2.18) and (2.19), we have $w = u - f_1 \in H_0^1(\Omega)$ and $z = v - f_3 \in H_0^1(\Omega)$. The formulas for z_1 and z_2 show that $z_1, z_2, \partial_\rho z_1, \partial_\rho z_2 \in L^2(\Omega \times (0,1))$ and satisfy $z_1(x,0) = w(x)$, $z_2(x,0) = z(x)$. Therefore, $U = (u,w,v,z,z_1,z_2)^T \in \mathcal{D}(\mathcal{A})$ and $(I + \mathcal{A})U = F$. So, $\mathcal{A}$ is maximal monotone. Therefore, the operator $I + \mathcal{A}$ is surjective for any fixed $t > 0$. Since $k(t) > 0$ and

$$\overline{\mathcal{A}}(t) + I = \mathcal{A}(t) + (1 + k(t))\, I.$$

From this, we establish that for all $t > 0$, the operator $\overline{\mathcal{A}}(t) + I$ is likewise surjective and that $\overline{\mathcal{A}}(t)$ is maximal. Thus, based on the analysis above, we conclude that the problem

$$\begin{cases} \overline{U}_t + \overline{\mathcal{A}}(t)\overline{U} = 0, \\ \overline{U}(0) = U_0 \end{cases} \tag{2.25}$$

possesses a unique solution $\overline{U} \in C([0,\infty),\mathcal{H})$ and if $U_0 \in \mathcal{D}(\mathcal{A}(0))$, hence,

$$\overline{U} \in C([0,+\infty), \mathcal{D}(\mathcal{A}(0))) \cap C^1([0,+\infty),\mathcal{H}).$$

Now, let us suppose that $U(t) = \exp(\theta(t))\overline{U}(t)$. Using (2.25), we have with $\theta(t) = \int_0^1 k(s)ds$ that

$$\begin{aligned} U_t(t) &= \exp(\theta(t))[\overline{U}_t(t) + \tau(t)\overline{U}(t)] = -\exp(\theta(t))[\overline{\mathcal{A}}(t) - \tau(t)I]\overline{U}(t) \\ &= -\exp(\theta(t))\mathcal{A}(t)\overline{U}(t) = \mathcal{A}(t)\left(-\exp(\theta(t))\overline{U}(t)\right) = -\mathcal{A}(t)U(t). \end{aligned}$$

Thus, $U(t)$ is the unique solution of problem.

Now, we prove that the operator $\mathcal{F}$ is locally Lipschitz in $\mathcal{H}$. Let $U = (u_1, u_{1t}, v_1, v_{1t}, z_1, z_2)^T$and $\overline{U} = (u_2, u_{2t}, v_2, v_{2t}, \overline{z}_1, \overline{z}_2)^T \in \mathcal{H}$, we want to show that bounded set $I \subset \mathcal{H}$, there exists $C_I > 0$ such that

$$\|F(U_1) - F(U_2)\|_{\mathcal{H}} \leq C_I\|U_1 - U_2\|_{\mathcal{H}}.$$

First, applying Lemma 2.1 with $s = |uv|$, we get

$$|\ln(|uv|^k)| = |k| \cdot |\ln|uv|| \leq |k|C_\epsilon(|uv|^\epsilon + |uv|^{-\epsilon}).$$

A central challenge in analyzing the system is controlling the logarithmic terms within the nonlinearities F and G:

$$F(u,v) = u|u|^{p-2}|v|^p \ln(|uv|^k), \quad G(u,v) = v|v|^{p-2}|u|^p \ln(|uv|^k).$$

For $(u_1, v_1), (u_2, v_2)$ in a bounded set $I \subset \mathcal{H}$, we analyze $|F(u_1, v_1) - F(u_2, v_2)|$ and $|G(u_1, v_1) - G(u_2, v_2)|$ similarly by symmetry of F and G. We set

$$F(u,v) = u|u|^{p-2}|v|^p \ln(|uv|^k) = u\Psi(u,v),$$

where

$$\Psi(u,v) = |u|^{p-2}|v|^p \ln(|uv|^k)$$

Next, we estimate the difference and it decomposed as

$$\begin{aligned} \Psi(u_1,v_1) - \Psi(u_2,v_2) = k &\left[(|u_1|^{p-2}|v_1|^p - |u_2|^{p-2}|v_2|^p)\ln|u_1v_1|\right] \\ &+ k[|u_2|^{p-2}|v_2|^p(\ln|u_1v_1| - \ln|u_2v_2|)]. \end{aligned}$$

Term 1. Apply the mean value theorem to $f(u,v) = |u|^{p-2}|v|^p$:

$$|f(u_1,v_1) - f(u_2,v_2)| \leq (p-2)|\tilde{u}|^{p-3}|\tilde{v}|^p|u_1 - u_2| + p|\tilde{u}|^{p-2}|\tilde{v}|^{p-1}|v_1 - v_2|, \tag{2.26}$$

where the point $(\tilde{u}, \tilde{v})$ is an intermediate point on the line segment connecting (u_1, v_1) and (u_2, v_2). The logarithmic factor is bounded by

$$|\ln|u_1v_1|| \leq \begin{cases} C_\epsilon|u_1v_1|^\epsilon & \text{if } |u_1v_1| \geq 1, \\ C_\epsilon|u_1v_1|^{-\epsilon} & \text{if } |u_1v_1| < 1. \end{cases}$$

Term 2. For the logarithmic difference, we have

$$|\ln|u_1v_1| - \ln|u_2v_2|| \leq \frac{1}{|u^*v^*|}(|v^*||u_1 - u_2| + |u^*||v_1 - v_2|). \tag{2.27}$$

The point (u^*, v^*) is an intermediate point on the line segment connecting (u_1, v_1) and (u_2, v_2). That is,

$$(u^*, v^*) = (u_1, v_1) + t((u_2, v_2) - (u_1, v_1)) \quad \text{for some } t \in [0, 1].$$

Combining estimates (2.26) and (2.27) and using the boundedness of u_i, v_i in $H_0^1(\Omega)$ gives

$$|\Psi(u_1, v_1) - \Psi(u_2, v_2)| \leq C_I(|u_1 - u_2| + |v_1 - v_2|).$$

Where C_I depends on the H^1-norms of u_i, v_i. Integrating over Ω and using Hölder's inequality, we obtain

$$\|\Psi(u_1, v_1) - \Psi(u_2, v_2)\|_{L^2} \leq C_I(\|u_1 - u_2\|_{L^2} + \|v_1 - v_2\|_{L^2}).$$

Using $F(u, v) = u\Psi_1(u, v)$, we get

$$|F(u_1, v_1) - F(u_2, v_2)| \leq |u_1 - u_2||\Psi(u_1, v_1)| + |u_2||\Psi(u_1, v_1) - \Psi(u_2, v_2)|. \tag{2.28}$$

The first term in the right side of (2.28) is bounded by $C|u_1 - u_2|$ (since Ψ is bounded on I). While the second term is bounded by $C_I|u_2|(|u_1 - u_2| + |v_1 - v_2|)$, we have

$$|F(u_1, v_1) - F(u_2, v_2)| \leq C_I(|u_1 - u_2| + |v_1 - v_2|).$$

Square and integrate the pointwise estimate, gives us

$$\|F(u_1, v_1) - F(u_2, v_2)\|_{L^2} \leq C_I(\|u_1 - u_2\|_{L^2} + \|v_1 - v_2\|_{L^2}).$$

Since $\|u\|_{L^2} \leq \|u\|_{H^1}$, we have

$$\|F(u_1, v_1) - F(u_2, v_2)\|_{L^2} \leq C_I \|U_1 - U_2\|_{\mathcal{H}}.$$

Next, the operator $\mathcal{J}$ satisfies

$$\|\mathcal{J}(U_1) - \mathcal{J}(U_2)\|_{\mathcal{H}} = \|F(u_1, v_1) - F(u_2, v_2)\|_{L^2} + \|G(u_1, v_1) - G(u_2, v_2)\|_{L^2}.$$

From the symmetry of F and G, the same estimate holds for G.

Thus,

$$\|\mathcal{J}(U_1) - \mathcal{J}(U_2)\|_{\mathcal{H}} \leq 2C_I\|U_1 - U_2\|_{\mathcal{H}}.$$

Therefore, the operator $\mathcal{J}$ is locally Lipschitz in $\mathcal{H}$. Thanks to [13,15], the proof is completed. $\square$

3. Blow-up result

In this section, we prove that solutions to the system (2.5) with negative initial energy blow up in finite time. The proof relies on constructing a Lyapunov functional whose derivative satisfies a differential inequality that forces it to reach infinity in finite time.

We begin by defining the natural energy functional for the system, which will be a cornerstone of the blow-up argument.

The energy functional $E(t)$ for the system (2.5)–(2.6) is defined as

$$E(t) = \frac{1}{2}\|w\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|z\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 + \frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 \left[z_1^2 + z_2^2\right]\,d\rho\,dx - \frac{1}{p}\int_\Omega |u|^p|v|^p\ln(|uv|^k)\,dx + \frac{k}{p^2}\int_\Omega |u|^p|v|^p\,dx. \tag{3.1}$$

where $\xi > 0$ is a constant chosen to satisfy the conditions in the subsequent lemma.

The terms $\frac{1}{2}\|w\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|z\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2$ represent the sum of the kinetic and potential (strain) energies for the two wave equations. The terms involving the integrals of $|u|^p|v|^p$ represent the potential energy associated with the logarithmic source. The final term represents the energy stored in the delay mechanisms.

Lemma 3.1 (Energy Dissipation)**.** *Assume conditions* (2.1)–(2.3) *hold. Let* (u, v, z_1, z_2) *be a solution of system* (2.5)—(2.6)*. Then the energy functional* $E(t)$ *defined in* (3.1) *is non-increasing and satisfies the dissipation inequality*:

$$E'(t) \le -C\left(\|w\|_2^2 + \|z\|_2^2 + \|z_1(\cdot,1,t)\|_2^2 + \|z_2(\cdot,1,t)\|_2^2\right) \le 0, \tag{3.2}$$

for some constant $C > 0$*, provided* ξ *is chosen such that*

$$\frac{|\alpha_2|}{\sqrt{1-d}} \le \xi \le 2\alpha_1 - \frac{|\alpha_2|}{\sqrt{1-d}} \quad \text{and} \quad \frac{|\beta_2|}{\sqrt{1-d}} \le \xi \le 2\beta_1 - \frac{|\beta_2|}{\sqrt{1-d}}. \tag{3.3}$$

Proof. We differentiate $E(t)$ with respect to time. For the first line of (3.1), we have the following (by integration by parts):

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|w\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2\right) &= \int_\Omega w u_{tt}\,dx + \int_\Omega \nabla u\cdot\nabla w\,dx \\ &= \int_\Omega w u_{tt}\,dx - \int_\Omega \Delta u w\,dx = \int_\Omega w\left(u_{tt} - \Delta u\right)\,dx.\end{aligned}$$

Using the first equation in (2.5):

$$u_{tt} - \Delta u = -\alpha_1 w - \alpha_2 z_1(x,1,t) + u|u|^{p-2}|v|^p\ln(|uv|^k),$$

we get

$$\frac{d}{dt}\left(\frac{1}{2}\|w\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2\right) = -\alpha_1\|w\|_2^2 - \alpha_2\int_\Omega w z_1(x,1,t)\,dx$$

$$+\int_\Omega w\left(u|u|^{p-2}|v|^p\ln(|uv|^k)\right)dx. \tag{3.4}$$

Similarly, for the v-terms, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|z\|_2^2+\frac{1}{2}\|\nabla v\|_2^2\right)=-\beta_1\|z\|_2^2-\beta_2\int_\Omega zz_2(x,1,t)\,dx$$
$$+\int_\Omega z\left(v|v|^{p-2}|u|^p\ln(|uv|^k)\right)dx. \tag{3.5}$$

Now, we differentiate the potential energy terms from the source. Let us define

$$\Phi(u,v)=-\frac{1}{p}|u|^p|v|^p\ln(|uv|^k)+\frac{k}{p^2}|u|^p|v|^p.$$

One can verify by direct computation that the derivative of Φ is

$$\frac{d}{dt}\Phi(u,v)=-w\left(u|u|^{p-2}|v|^p\ln(|uv|^k)\right)-z\left(v|v|^{p-2}|u|^p\ln(|uv|^k)\right).$$

Therefore, the time derivative of the integral term is

$$\frac{d}{dt}\int_\Omega\Phi(u,v)\,dx=-\int_\Omega\left[w\left(u|u|^{p-2}|v|^p\ln(|uv|^k)\right)+z\left(v|v|^{p-2}|u|^p\ln(|uv|^k)\right)\right]dx. \tag{3.6}$$

Adding (3.4), (3.5), and (3.6), the source terms cancel out yielding the following:

$$\frac{d}{dt}\,(\text{Kinetic+Strain+Potential})$$
$$=-\alpha_1\|w\|_2^2-\beta_1\|z\|_2^2-\alpha_2\int_\Omega wz_1(x,1,t)\,dx-\beta_2\int_\Omega zz_2(x,1,t)\,dx. \tag{3.7}$$

Finally, we differentiate the delay energy term:

$$\frac{d}{dt}\left(\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1\left[z_1^2+z_2^2\right]d\rho\,dx\right)$$
$$=\frac{\xi}{2}\tau'(t)\int_\Omega\int_0^1\left[z_1^2+z_2^2\right]d\rho\,dx+\xi\tau(t)\int_\Omega\int_0^1\left[z_1z_{1t}+z_2z_{2t}\right]d\rho\,dx.$$

From the transport equations in (2.5), we have

$$z_{1t}=-\frac{1-\tau'(t)\rho}{\tau(t)}z_{1\rho}\text{ and }z_{2t}=-\frac{1-\tau'(t)\rho}{\tau(t)}z_{2\rho}.$$

Since $2z_1z_{1\rho}=\frac{\partial}{\partial\rho}(z_1^2)$, we get

$$\xi\tau(t)\int_0^1 z_1z_{1t}\,d\rho=-\xi\int_0^1(1-\tau'(t)\rho)z_1z_{1\rho}\,d\rho=-\frac{\xi}{2}\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z_1^2)\,d\rho.$$

Integrating by parts in ρ, we obtain

$$\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}(z_1^2)\,d\rho=\left[(1-\tau'(t)\rho)z_1^2\right]_0^1+\int_0^1\tau'(t)z_1^2\,d\rho$$

$$= (1-\tau'(t))z_1^2(x,1,t) - z_1^2(x,0,t) + \tau'(t)\int_0^1 z_1^2\,d\rho.$$

Recalling that $z_1(x,0,t) = w(x,t)$, we get

$$\xi\tau(t)\int_0^1 z_1 z_{1t}\,d\rho = -\frac{\xi}{2}\left[(1-\tau'(t))z_1^2(x,1,t) - w^2(x) + \tau'(t)\int_0^1 z_1^2\,d\rho\right].$$

A similar expression holds for the z_2 term. Combining all parts, the derivative of the delay energy is

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\xi}{2}\tau(t)\int_\Omega\int_0^1 (z_1^2+z_2^2)\,d\rho\,dx\right)\\ &\quad= \frac{\xi}{2}\int_\Omega\left[w^2+z^2\right]dx - \frac{\xi}{2}(1-\tau'(t))\int_\Omega\left[z_1^2(x,1)+z_2^2(x,1)\right]dx. \qquad (3.8)\end{aligned}$$

Notice the terms involving $\tau'(t)$ and the integrals over ρ cancel out perfectly.

Now, we add the dissipation from the main energy (3.7) and the delay energy derivative (3.8) to get the total $E'(t)$:

$$\begin{aligned}E'(t) = &-\left(\alpha_1-\frac{\xi}{2}\right)\|w\|_2^2 - \left(\beta_1-\frac{\xi}{2}\right)\|z\|_2^2 - \frac{\xi}{2}(1-\tau'(t))\int_\Omega\left[z_1^2(1)+z_2^2(1)\right]dx\\ &- \alpha_2\int_\Omega w z_1(x,1,t)\,dx - \beta_2\int_\Omega z z_2(x,1,t)\,dx.\end{aligned}$$

We now estimate the cross terms using Young's inequality. For the first cross term, we have

$$\left|\alpha_2\int_\Omega w z_1(x,1,t)dx\right| \le \frac{|\alpha_2|}{2\sqrt{1-d}}\|w\|_2^2 + \frac{|\alpha_2|\sqrt{1-d}}{2}\|z_1(\cdot,1)\|_2^2.$$

Similarly, we obtain

$$\left|\beta_2\int_\Omega z z_2(x,1,t)dx\right| \le \frac{|\beta_2|}{2\sqrt{1-d}}\|z\|_2^2 + \frac{|\beta_2|\sqrt{1-d}}{2}\|z_2(\cdot,1)\|_2^2.$$

Also, since $\tau'(t) \le d < 1$, we have $-(1-\tau'(t)) \le -(1-d) < 0$. Therefore,

$$-\frac{\xi}{2}(1-\tau'(t)) \ge \frac{\xi}{2}(1-d).$$

Substituting these inequalities into the expression for $E'(t)$, we get

$$\begin{aligned}E'(t) \le &-\left(\alpha_1-\frac{\xi}{2}-\frac{|\alpha_2|}{2\sqrt{1-d}}\right)\|w\|_2^2 - \left(\beta_1-\frac{\xi}{2}-\frac{|\beta_2|}{2\sqrt{1-d}}\right)\|z\|_2^2\\ &- \left(\frac{\xi}{2}(1-d) - \frac{|\alpha_2|\sqrt{1-d}}{2}\right)\|z_1(\cdot,1)\|_2^2\\ &- \left(\frac{\xi}{2}(1-d)\frac{|\beta_2|\sqrt{1-d}}{2}\right)\|z_2(\cdot,1)\|_2^2. \qquad (3.9)\end{aligned}$$

The conditions in (3.3) are chosen precisely so that each coefficient is non-negative. Thus, there exists a constant $C > 0$ (e.g., the minimum of the four coefficients) such that

$$E'(t) \leq -C \left(\|w\|_2^2 + \|z\|_2^2 + \|z_1(\cdot, 1)\|_2^2 + \|z_2(\cdot, 1)\|_2^2 \right) \leq 0,$$

which completes the proof. □

Theorem 3.2 (Finite-Time Blow-Up). *Assume the conditions of the energy dissipation lemma hold and that the initial energy is negative, i.e., $E(0) < 0$. Then the solution of system (2.5)–(2.6) blows up in finite time $T^* > 0$, i.e.,*

$$\lim_{t \to T^{*-}} \left(\|w\|_2^2 + \|\nabla u\|_2^2 + \|z\|_2^2 + \|\nabla v\|_2^2 \right) = \infty.$$

Proof. The proof proceeds by constructing a Lyapunov functional $\mathcal{K}(t)$ whose derivative satisfies a differential inequality leading to finite-time explosion. Since $E'(t) \leq 0$ and $E(0) < 0$, it follows that $E(t) \leq E(0) < 0$ for all $t \geq 0$ in the interval of existence. Define the positive function

$$H(t) := -E(t) > 0.$$

From the energy dissipation inequality, we also have:

$$H'(t) = -E'(t) \geq C \left(\|w\|_2^2 + \|z\|_2^2 + \|z_1(\cdot, 1, t)\|_2^2 + \|z_2(\cdot, 1, t)\|_2^2 \right) \geq 0, \quad (3.10)$$

so $H(t)$ is non-decreasing.

Define the following functional:

$$\mathcal{K}(t) := H^{1-\alpha}(t) + \epsilon \int_\Omega (uw + vz)\, dx + \frac{\epsilon}{2} \left(\alpha_1 \|u\|_2^2 + \beta_1 \|v\|_2^2 \right), \quad (3.11)$$

where $0 < \alpha < 1$ and $\epsilon > 0$ are constants to be chosen later. The term $\frac{\epsilon}{2} \left(\alpha_1 \|u\|_2^2 + \beta_1 \|v\|_2^2 \right)$ is added to compensate for the damping terms and simplify the analysis.

First, we show that $\mathcal{K}(t)$ is equivalent to $H^{1-\alpha}(t)$. By Young's inequality,

$$\left| \epsilon \int_\Omega (uw + vz) dx \right| \leq \frac{\epsilon}{2} \int_\Omega (u^2 + w^2)\, dx + \frac{\epsilon}{2} \int_\Omega (v^2 + z^2)\, dx.$$

From the definition of $H(t)$ and the Poincaré inequality, we have

$$\|u\|_2^2 \leq C_P \|\nabla u\|_2^2 \leq C_1 H(t), \quad \|v\|_2^2 \leq C_P \|\nabla v\|_2^2 \leq C_1 H(t),$$

and

$$\|u_t\|_2^2 \leq C_2 H(t), \quad \|v_t\|_2^2 \leq C_2 H(t).$$

Therefore,

$$\left| \epsilon \int_\Omega (uw + vz) dx \right| \leq \epsilon C_3 H(t).$$

Similarly,

$$\frac{\epsilon}{2}\left(\alpha_1\|u\|_2^2+\beta_1\|v\|_2^2\right)\le \epsilon C_4 H(t).$$

Thus, for sufficiently small ϵ, there exist constants $C_5, C_6>0$ such that

$$C_5 H^{1-\alpha}(t)\le \mathcal{K}(t)\le C_6 H^{1-\alpha}(t). \tag{3.12}$$

Differentiate $\mathcal{K}(t)$ term by term:

$$\frac{d}{dt}\left[H^{1-\alpha}(t)\right]=(1-\alpha)H^{-\alpha}(t)H'(t).$$

$$\frac{d}{dt}\left[\epsilon\int_\Omega (uw+vz)\,dx\right]=\epsilon\int_\Omega (w^2+uu_{tt})\,dx+\epsilon\int_\Omega (z^2+vv_{tt})\,dx.$$

$$\frac{d}{dt}\left[\frac{\epsilon}{2}(\alpha_1\|u\|_2^2+\beta_1\|v\|_2^2)\right]=\epsilon\alpha_1\int_\Omega uw\,dx+\epsilon\beta_1\int_\Omega vz\,dx.$$

Summing these, we obtain

$$\begin{aligned}\mathcal{K}'(t)=(1-\alpha)H^{-\alpha}(t)H'(t)+\epsilon\int_\Omega (w^2+uu_{tt})\,dx+\epsilon\int_\Omega (z^2+vv_{tt})\,dx\\ +\epsilon\alpha_1\int_\Omega uw\,dx+\epsilon\beta_1\int_\Omega vz\,dx.\end{aligned}$$

Now, using (2.5) and after expansion and simplification (noting the cancellation of the damping terms $\alpha_1 uw$ and $\beta_1 vz$), we arrive at

$$\begin{aligned}\mathcal{K}'(t)&=(1-\alpha)H^{-\alpha}(t)H'(t)+\epsilon\left(\|w\|_2^2+\|z\|_2^2\right)-\epsilon\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\ &\quad-\epsilon\alpha_2\int_\Omega uz_1(x,1,t)\,dx-\epsilon\beta_2\int_\Omega vz_2(x,1,t)\,dx+2\epsilon\int_\Omega |u|^p|v|^p\ln(|uv|^k)\,dx.\end{aligned}$$

Using the relation between the source term and the energy $H(t)$:

$$\begin{aligned}\int_\Omega |u|^p|v|^p\ln(|uv|^k)\,dx&=pH(t)+\frac{p}{2}\left(\|u_t\|_2^2+\|\nabla u\|_2^2+\|v_t\|_2^2+\|\nabla v\|_2^2\right)\\ &\quad+\frac{k}{p}\int_\Omega |u|^p|v|^p\,dx+\frac{p\xi}{2}\tau(t)\int_\Omega\int_0^1[z_1^2+z_2^2]\,d\rho\,dx,\end{aligned}$$

we obtain the expression

$$\begin{aligned}\mathcal{K}'(t)&=(1-\alpha)H^{-\alpha}(t)H'(t)+2\epsilon pH(t)+\epsilon p\xi\tau(t)\int_\Omega\int_0^1[z_1^2+z_2^2]\,d\rho\,dx\\ &\quad+\epsilon(1+p)\left(\|w\|_2^2+\|z\|_2^2\right)+\epsilon(p-1)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\ &\quad-\epsilon\alpha_2\int_\Omega uz_1(x,1,t)\,dx-\epsilon\beta_2\int_\Omega vz_2(x,1,t)\,dx+\frac{2\epsilon k}{p}\int_\Omega |u|^p|v|^p\,dx.\end{aligned} \tag{3.13}$$

We estimate the delay terms using Young's inequality. For the first delay term, we get

$$\left|\epsilon\alpha_2\int_\Omega uz_1(x,1,t)dx\right|\le \epsilon\frac{\delta_1}{2}\|u\|_2^2+\epsilon\frac{|\alpha_2|^2}{2\delta_1}\|z_1(\cdot,1,t)\|_2^2,$$

where $\delta_1 > 0$ is a constant to be chosen.

We now relate these norms to controlled quantities. First, by the Poincaré inequality, we have $\|u\|_2^2 \le C_P \|\nabla u\|_2^2$, from the energy dissipation inequality, we get

$$H'(t) \ge C_8 \left(\|u_t\|_2^2 + \|v_t\|_2^2 + \|z_1(\cdot, 1, t)\|_2^2 + \|z_2(\cdot, 1, t)\|_2^2 \right),$$

so

$$\|z_1(\cdot, 1, t)\|_2^2 \le \frac{1}{C_8} H'(t).$$

Substituting these bounds, we obtain

$$\left| \epsilon \alpha_2 \int_\Omega u z_1(x, 1, t)\, dx \right| \le \epsilon \frac{\delta_1 C_P}{2} \|\nabla u\|_2^2 + \epsilon \frac{|\alpha_2|^2}{2\delta_1 C_8} H'(t).$$

Then we get

$$- \epsilon \alpha_2 \int_\Omega u z_1(x, 1, t)\, dx \ge -\epsilon \frac{\delta_1 C_P}{2} \|\nabla u\|_2^2 - \epsilon \frac{|\alpha_2|^2}{2\delta_1 C_8} H'(t). \tag{3.14}$$

An identical estimate holds for the term $-\epsilon \beta_2 \int_\Omega v z_2(x, 1, t) dx$, we have

$$- \epsilon \beta_2 \int_\Omega v z_2(x, 1, t)\, dx \ge -\epsilon \frac{\delta_2 C_P}{2} \|\nabla v\|_2^2 - \epsilon \frac{|\beta_2|^2}{2\delta_2 C_8} H'(t). \tag{3.15}$$

For simplicity, we take $\delta_1 = \delta_2 = 2\delta$.

Let $M = \frac{|\alpha_2|^2 + |\beta_2|^2}{4\delta C_8}$. Substituting estimates (3.14) and (3.15) into (3.13), and noting that the terms:

$$\frac{2\epsilon k}{p} \int_\Omega |u|^p |v|^p \, dx \quad \text{and} \quad \epsilon p \xi \tau(t) \int_\Omega \int_0^1 [z_1^2 + z_2^2]\, d\rho\, dx,$$

are non-negative and can be dropped for a lower bound, we obtain

$$\begin{aligned} K'(t) \ge{} & \left[(1-\alpha) H^{-\alpha}(t) - \epsilon M \right] H'(t) + \epsilon (1+p) \left(\|w\|_2^2 + \|z\|_2^2 \right) \\ & + \epsilon (p - 1 - \delta C_P) \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) + 2\epsilon p H(t). \end{aligned}$$

Now, we have to choose coefficients carefully. Choosing ϵ small enough such that

$$\left[(1-\alpha) H^{-\alpha}(t) - \epsilon M \right] \ge 0.$$

Next, taking $\delta = \frac{p-1}{2C_P} > 0$, we get

$$\epsilon (p - 1 - \delta C_P) = \epsilon \left(p - 1 - \frac{p-1}{2} \right) = \epsilon \frac{p-1}{2} > 0.$$

Define the minimum coefficient as follows

$$\lambda_1 = \epsilon \frac{p-1}{2}, \quad \lambda_2 = \epsilon (1+p), \quad \lambda_3 = 2\epsilon p,$$

and

$$\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\} = \epsilon\frac{p-1}{2}.$$

Therefore, we obtain

$$\mathcal{K}'(t) \geq \lambda\left(H(t) + \|w\|_2^2 + \|\nabla u\|_2^2 + \|z\|_2^2 + \|\nabla v\|_2^2\right). \tag{3.16}$$

From the equivalence (3.12), we have

$$H(t) \geq C_9 \mathcal{K}^{\frac{1}{1-\alpha}}(t).$$

Also, from the definition of $H(t)$, we have

$$\|w\|_2^2 + \|\nabla u\|_2^2 + \|z\|_2^2 + \|\nabla v\|_2^2 \geq C_{10} H(t) \geq C_{11} \mathcal{K}^{\frac{1}{1-\alpha}}(t).$$

Therefore,

$$H(t) + \|w\|_2^2 + \|\nabla u\|_2^2 + \|z\|_2^2 + \|\nabla v\|_2^2 \geq C_{12} \mathcal{K}^{\frac{1}{1-\alpha}}(t).$$

Substituting into (3.16) yields

$$\mathcal{K}'(t) \geq \Lambda \mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{3.17}$$

where $\Lambda = \lambda C_{12} > 0$.

The inequality (3.17) implies that $\mathcal{K}(t)$ is increasing and satisfies

$$\frac{d}{dt}\left[\mathcal{K}^{-\frac{\alpha}{1-\alpha}}(t)\right] = -\frac{\alpha}{1-\alpha}\mathcal{K}^{-\frac{1}{1-\alpha}}(t)\mathcal{K}'(t) \leq -\frac{\alpha}{1-\alpha}\Lambda.$$

Integrating from 0 to t, we obtain

$$\mathcal{K}^{-\frac{\alpha}{1-\alpha}}(t) - \mathcal{K}^{-\frac{\alpha}{1-\alpha}}(0) \leq -\frac{\alpha\Lambda}{1-\alpha}t.$$

Thus,

$$\mathcal{K}^{-\frac{\alpha}{1-\alpha}}(t) \leq \mathcal{K}^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\alpha\Lambda}{1-\alpha}t.$$

The right-hand side becomes zero at time

$$T^* = \frac{1-\alpha}{\alpha\Lambda}\mathcal{K}^{-\frac{\alpha}{1-\alpha}}(0) < \infty.$$

Therefore, $\mathcal{K}(t) \to \infty$ as $t \to T^{*-}$. By the equivalence (3.12), $H(t) \to \infty$, which implies that the solution blows up in finite time. This completes the proof. □

4. Conclusion

This paper provides a comprehensive analysis of a coupled nonlinear wave system incorporating both time-varying delay and logarithmic nonlinearity. We successfully established the local well-posedness of solutions through semigroup theory and delay reformulation techniques. Our main result demonstrates finite-time blow-up for solutions with negative initial energy, generalizing previous scalar-case results to the coupled system framework. The analysis reveals how the interplay between delay effects and nonlinear coupling generates new mathematical challenges not present in single-equation models. This work establishes foundational results for coupled systems with delay-nonlinearity interactions, opening several directions for future research on more complex multi-component systems.

References

[1] M.M. Al-Gharabli, A.M. Al-Mahdi, and M. Kafini, *Global existence and new decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities*. AIMS Mathematics, **6** (2021), 10105–10129.

[2] L. Bashir, J. Hao, S. Boulaaras, M.F. Aslam, and T. Zhang, *Exponential decay and blow-up results of a logarithmic nonlinear wave equation having infinite memory and strong time-varying delay*, Bound. Value Probl. **2025** (2025), Paper No. 83.

[3] I. Bialynicki-Birula and J. Mycielski, *Wave equations with logarithmic nonlinearities*, Bull. Acad. Pol. Sci. **23** (1975), 461–466.

[4] T. Cazenave and A. Haraux, *Équations d'évolution avec non linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math. **5** (1980), 21–51.

[5] A. Choucha and D. Ouchenane, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with time-varying delay*, Stud. Univ. Babes-Bolyai Math. **68** (2023), 171–187.

[6] A. Choucha, S. Boulaaras, A. Allahem, A. Alharbi, and R. Jan, *Results from a nonlinear wave equation with acoustic and fractional boundary conditions coupling by logarithmic source and delay terms: global existence and asymptotic behavior of solutions*, Mathematics, **12** (2024), Paper No. 2616.

[7] L. Fischer and R. Racke, *Ill-posedness of coupled systems with delay*, Evolution Equations, Cambridge Univ. Press, 2017, 151–160.

[8] M. Houasni and A. Kelleche, *Nonlinear delayed porous-elastic system with microtemperatures- Existence, uniqueness and general decay*, Nonlinear Studies, **32**, (2025), 851–871.

[9] M. Kafini and S. A. Messaoudi, *Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay*, Appl. Anal. **98** (2018), 530–547.

[10] I. Kanwal, J. Hao, L. Bashir, and M.F. Aslam, *Blow-up phenomena of boundary delay effects in nonlinear logarithmic wave dynamics*, Evol. Equ. Control Theory **15** (2026), 207–230.

[11] T. Kato, *Linear and quasilinear equations of evolution of hyperbolic type*, Hyperbolicity. C.I.M.E. Summer Schools, **72**, Springer, Berlin, Heidelberg, 2011, 125–191.

[12] M. Kirane, B. Said-Houari, and M. N. Anwar. *Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks*, Commun. Pure Appl. Anal. **10** (2011), 667–686.

[13] V. Komornik, *Exact Controllability and Stabilization: The Multiplier Method*, Masson-John Wiley, 1994.

[14] S. Nicaise, C. Pignotti, and J. Valein, *Exponential stability of the wave equation with boundary time-varying delay*, Discrete Contin. Dyn. Syst. Ser. **4** (2011), 693–722.

[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[16] F. Zeng, P. Shi, and M. Jiang, *Global existence and finite time blow-up for a class of fractional p-Laplacian Kirchhoff type equations with logarithmic nonlinearity*. AIMS Mathematics, **6** (2021), 2559–2578.

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Про вибух спряжених хвильових рівнянь із часозалежним запізненням та логарифмічною нелінійністю

Mohamed Houasni and Abdelkarim Kelleche

У цій роботі досліджується система нелінійних хвильових рівнянь із часозалежним запізненням та логарифмічною нелінійністю. Спочатку, використовуючи теорію напівгруп, ми доводимо локальне існування та єдиність розв'язків. Наш основний результат встановлює, що розв'язки з від'ємною початковою енергією вибухають за скінченний час, що узагальнює скалярний випадок [5, 9]. Доведення спирається на ретельно побудований функціонал Ляпунова та нелінійні диференціальні нерівності, адаптуючи методи з [4, 14] для роботи зі спряженою структурою. Ця робота поширює теорію вибуху на спряжені системи із запізненням.

Ключові слова: спряжена хвильова система, часозалежне запізнення, логарифмічна нелінійність, локальне існування, вибух за скінченний час, функціонал Ляпунова, теорія напівгруп