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# On a Nonclassical Boundary Problem for the Laplace Equation in a Cracked Plane

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We consider the Laplace equation in a cracked plane with nonclassical boundary conditions. This problem arises as a model of the flow in the fractured media. The main result is the theorem of existence and uniqueness of a solution in weighted Sobolev spaces.

*Key words:* a priori estimates, crack, weighted Sobolev spaces

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## 1. Introduction

Consider the plane containing a crack $\Sigma = \{(x_1, x_2) : x_1 < 0, x_2 = 0\}$. Let us denote by $\gamma_+$ and $\gamma_-$ the trace operators on $\Sigma$ from the upper and lower sides of $\Sigma$ respectively. This paper is concerned with the problem of finding a pair $(\mathrm{p}(x), \mathrm{q}(x_1))$ such that

$$\Delta \mathrm{p}(x) = 0 \qquad \text{in } \Omega = \mathbb{R}^2 \setminus \Sigma, \tag{1.1}$$

$$\gamma_+\mathrm{p} + \gamma_-\mathrm{p} - 2\mathrm{q} = 0 \qquad \text{on } \Sigma, \tag{1.2}$$

$$\gamma_+(\kappa_1 \mathrm{p}_{x_2} - \mathrm{p}) + \gamma_-(\kappa_1 \mathrm{p}_{x_2} + \mathrm{p}) = 0 \qquad \text{on } \Sigma, \tag{1.3}$$

$$-\mathrm{q}_{x_1x_1} - \kappa_2(\gamma_+\mathrm{p}_{x_2} - \gamma_-\mathrm{p}_{x_2}) = \mathrm{f} \qquad \text{on } \Sigma, \tag{1.4}$$

$$\mathrm{q}_{x_1}(0) = 0, \tag{1.5}$$

$$\mathrm{p}, \mathrm{q} \to 0 \qquad \text{as } |x| \to \infty, \tag{1.6}$$

where $\kappa_1 > 0$ and $\kappa_2 > 0$.

We may consider (1.1)–(1.6) as a model problem for the flow in the fractured media (cf. system (4.1) in [22] with parameter $\xi = 1/2$). In this context, q is the pressure of a fluid in the fracture $\Sigma$ and p is a pressure in the surrounding media $\Omega$ (cf. [1]).

Different aspects of the theory of linear elliptic boundary value problems in non-smooth domains were discussed in [5,12,17,20,26] and, in a more special case of cracks, in [2,6,8,14–16,23]. The papers [19,24,25] are devoted to the application of the potential theory to the third boundary problem for the Laplace equation on a bounded planar domain with inside crack.

In [28], the authors undertook a detailed study of the Poisson equation in infinite planar angle under the Dirichlet boundary condition on the one side of the angle and the oblique derivative boundary condition with a complex parameter on the other side of the angle. Using the Mellin transform, they reduced that problem to a first-order difference equation in the complex plane. Their approach provides a unified way of treating a whole variety of elliptic problems with nonhomogeneous symbol in corner domains (see [3,4,9,13,29] and references therein). Our argument is an adaptation of some ideas in [3, 28]. We give the material from [28] without proof, thus making our exposition self-contained. We emphasize that the main difficulty is in verifying boundary condition (1.5).

The main result of this paper is the following theorem (see Section 2 for the definitions of the weighted Sobolev spaces $H^k_\mu(\Omega)$, $H^{k+1/2}_\mu(\Sigma)$).

**Theorem 1.1.** *Let $k \geq 0$, $\mu \in \mathbb{R}$, and*

$$0 < \mu - k - 1 < \frac{1}{2}. \tag{1.7}$$

*Let $\mathrm{f} \in H^{k+1/2}_\mu(\Sigma)$ and, additionally,*

$$\int_{-\infty}^{0} \mathrm{f}(x_1)\, dx_1 = 0. \tag{1.8}$$

*Then there exists a unique solution $(\mathrm{p}(x), \mathrm{q}(x_1))$ to the problem (1.1)–(1.6) such that*

$$\|\mathrm{p}\|_{H^{k+2}_\mu(\Omega)} + \|\mathrm{q}\|_{H^{k+3/2}_\mu(\Sigma)} + \|\mathrm{q}_x\|_{H^{k+3/2}_\mu(\Sigma)} \leq C\|\mathrm{f}\|_{H^{k+1/2}_\mu(\Sigma)}, \tag{1.9}$$

*where $C$ does not depend on $f$.*

The paper is organized as follows. In Section 2, we define functional spaces and give some auxiliary results. The proof of the main result is split into two parts. Section 3 is devoted to the integral representation of the solution and estimate (1.9). In Section 4, we verify that the obtained solution satisfies the boundary condition (1.5). Appendices contain the proof of some technical results. Boundary conditions (1.2)–(1.4) are verified in Appendix A. Appendix B contains the proof of a key estimate, which we use in Section 4.

Throughout the paper, the symbol $C$ denotes a generic positive constant depending only on the structural quantities of the problem.

## 2. Functional spaces. Auxiliary results

Let us define the functional spaces used below (cf. [11,17,18,26]). For $k \in \mathbb{N}$, we define the Hilbert space $H^k(-\pi,\pi)$ with the norm

$$\|u\|_{H^k(-\pi,\pi)} = \left( \sum_{l=0}^{k} \int_{-\pi}^{\pi} \left| \frac{d^l u(\theta)}{d\,\theta^l} \right|^2 d\theta \right)^{\frac{1}{2}}.$$

Define the space $H^k_\mu(\Omega)$ as the closure of infinitely differentiable functions defined in $\Omega$, vanishing near the origin and for large $|x|$, with respect to the norm ($k \in \mathbb{N} \cup 0, \mu \in \mathbb{R}$),

$$\|\mathrm{p}\|_{H^k_\mu(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_\Omega |x|^{2(\mu - k + |\alpha|)} |D^\alpha_x \mathrm{p}(x)|^2\, dx \right)^{\frac{1}{2}}. \tag{2.1}$$

Passing to polar coordinates ($r > 0$, $\theta \in (-\pi, \pi]$),

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad \mathrm{p}(x) = p(r,\theta), \tag{2.2}$$

we define the norm

$$\|p\|_{k,\mu} = \left( \int_0^\infty r^{2(\mu-k)+1} \sum_{l=0}^{k} \left\| \left( r\frac{\partial}{\partial r} \right)^l p \right\|^2_{H^{k-l}(-\pi,\pi)} dr \right)^{\frac{1}{2}}, \tag{2.3}$$

which is equivalent to the norm (2.1) (see formula (6.1.4) in [18]).

Define also $H^{k+1/2}_\mu(\mathbb{R}_+)$ as the closure of $C^\infty$ functions with compact support in the norm

$$\|h\|_{H^{k+1/2}_\mu(\mathbb{R}_+)} = \left( \sum_{l=0}^{k} \int_0^\infty r^{2(\mu-k+l)-1} \left| \frac{d^l h(r)}{dr^l} \right|^2 dr + \|h\|_{L^{k+1/2}_\mu(\mathbb{R}_+)} \right)^{\frac{1}{2}}, \tag{2.4}$$

where

$$\|h\|_{L^{k+1/2}_\mu(\mathbb{R}_+)} = \left( \int_0^\infty r^{2\mu}\, dr \int_0^r \left| \frac{d^k h(r+\rho)}{dr^k} - \frac{d^k h(r)}{dr^k} \right|^2 \frac{d\rho}{\rho^2} \right)^{\frac{1}{2}}. \tag{2.5}$$

Let us say that a function $\mathrm{h}(x_1)$, defined on $\Sigma$, belongs to $H^{k+1/2}_\mu(\Sigma)$ if $h(r) = \mathrm{h}(-r)$ belongs to $H^{k+1/2}_\mu(\mathbb{R}_+)$.

Define the Mellin transformation and its inverse

$$\begin{aligned} \widetilde{h}(s) &= \int_0^\infty r^{s-1} h(r)\, dr, \quad s = s_1 + is_2, \\ h(r) &= \frac{1}{2\pi i} \int_{s_1-i\infty}^{s_1+i\infty} r^{-s} \widetilde{h}(s)\, ds \end{aligned} \tag{2.6}$$

(see [28, 30] for theoretical background).

We will use the Parceval formula

$$\int_0^\infty |h(r)|^2 r^{2a-1}\, dr = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left|\widetilde{h}(s)\right|^2 \frac{ds}{i} \tag{2.7}$$

and the Mellin convolution formula

$$\int_0^\infty h_1\left(\frac{r}{t}\right) h_2(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s} \widetilde{h_1}(s) \widetilde{h_2}(s)\, ds \tag{2.8}$$

(see (3.1.15) and (3.1.13) in [27]).

Integrating by parts, one can prove that for any $h \in C_0^\infty(\mathbb{R}_+)$,

$$\left( \widetilde{\frac{d^k h}{dr^k}} \right)(s) = (-1)^k (s-1)\cdots(s-k)\widetilde{h}(s-k), \tag{2.9}$$

$$\left(\widetilde{\left(r\frac{\partial}{\partial r}\right)^k h}\right)(s) = s^k \widetilde{h}(s). \tag{2.10}$$

For $h \in H_\mu^{k+1/2}(\mathbb{R}_+)$, we have (see Proposition 3.2 in [28]):

$$c_1(\mu)\int_{\mu-k-i\infty}^{\mu-k+i\infty} \left|\widetilde{h}(s)\right|^2 (1+|s|)^{2k+1}\frac{ds}{i} \le \|h\|^2_{H_\mu^{k+1/2}(\mathbb{R}_+)}$$
$$\le c_2(\mu)\int_{\mu-k-i\infty}^{\mu-k+i\infty} \left|\widetilde{h}(s)\right|^2 (1+|s|)^{2k+1}\frac{ds}{i}, \tag{2.11}$$

where $0 < c_1(\mu) \le c_2(\mu)$.

## 3. Integral representation of the solution and its estimate

Let us set

$$q(r) = \mathrm{q}(-x), \quad f(r) = \mathrm{f}(-x). \tag{3.1}$$

Next, we rewrite (1.1)–(1.6) by adopting polar coordinates (2.2) and (3.1),

$$p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} = 0, \qquad r>0,\ \theta \in (-\pi,\pi), \tag{3.2}$$
$$p(r,\pi) + p(r,-\pi) - 2q(r) = 0, \qquad r>0, \tag{3.3}$$
$$\frac{\kappa_1}{r}p_\theta(r,\pi) + p(r,\pi) + \frac{\kappa_1}{r}p_\theta(r,-\pi) - p(r,-\pi) = 0, \qquad r>0, \tag{3.4}$$
$$-q_{rr}(r) + \frac{\kappa_2}{r}(p_\theta(r,\pi) - p_\theta(r,-\pi)) = f(r), \quad r>0, \tag{3.5}$$
$$q_r(0) = 0, \tag{3.6}$$
$$p, q \to 0, \qquad r \to \infty. \tag{3.7}$$

The Mellin transformation with respect to $r$ and formulas (2.9), (2.10) leads (3.2)–(3.5) to the system

$$\widetilde{p}(s,\theta) + s^2\widetilde{p}(s,\theta) = 0, \tag{3.8}$$
$$\widetilde{p}(s,\pi) + \widetilde{p}(s,-\pi) - 2\widetilde{q}(s) = 0, \tag{3.9}$$
$$\kappa_1[\widetilde{p}_\theta(s-1,\pi) + \widetilde{p}_\theta(s-1,-\pi)] + \widetilde{p}(s,\pi) - \widetilde{p}(s,-\pi) = \widetilde{q}(s), \tag{3.10}$$
$$-(s-1)(s-2)\widetilde{q}(s-2) + \kappa_2[\widetilde{p}_\theta(s-1,\pi) - \widetilde{p}_\theta(s-1,-\pi)] = \widetilde{f}(s). \tag{3.11}$$

It seems appropriate to seek for $\widetilde{p}$, $\widetilde{q}$ in the form

$$\widetilde{p}(s,\theta) = \frac{\cos s\theta}{\sin s\pi}d(s), \quad \widetilde{q}(s) = \cot s\pi d(s), \tag{3.12}$$

with unknown $d(s)$. We have

$$\widetilde{p}(s,\pm\pi) = \cot s\pi d(s),$$
$$\widetilde{p}_\theta(s,\theta) = -\, s\frac{\sin s\theta}{\sin s\pi}d(s), \tag{3.13}$$

$$\widetilde{p}_\theta(s, \pm\pi) = \mp sd(s).$$

This allows us to satisfy (3.8)–(3.10) simultaneously and convert (3.11) to the following difference equation:

$$-(s-1)(s-2)\cot(s-2)\pi d(s-2) - 2\kappa_2(s-1)d(s-1) = \widetilde{f}(s). \tag{3.14}$$

Set

$$\kappa_0 = 2\kappa_2, s = \lambda + 2, \omega(\lambda) = \lambda \cot \lambda\pi. \tag{3.15}$$

Then equation (3.14) takes the form

$$\kappa_0 d(\lambda+1) + \omega(\lambda)d(\lambda) = -\frac{\widetilde{f}(\lambda+2)}{\lambda+1}. \tag{3.16}$$

This equation was extensively studied in [28] in a more general situation. Let us sum up some key results.

One of the solutions to the homogeneous equation

$$\kappa_0 d(\lambda+1) + \omega(\lambda)d(\lambda) = 0 \tag{3.17}$$

is

$$d_0(\lambda) = \exp(i\lambda\pi)(\kappa_0\pi)^{1/2-\lambda}K(\lambda), \tag{3.18}$$

where

$$K(\lambda) = \prod_{n=1}^{\infty} \frac{\Gamma(n-\frac{1}{2}+\lambda)\Gamma(n+1-\lambda)}{\Gamma(n+\frac{1}{2}-\lambda)\Gamma(n+\lambda)} \left(\frac{n}{n-1/2}\right)^{2\lambda-1}, \tag{3.19}$$

and $\Gamma(\cdot)$ is the Euler gamma function. The function $\Gamma(\lambda)$ has the simple poles at $\lambda = -m$, $m \in \mathbb{N} \cup 0$, and, for example, $\Gamma(\lambda)\Gamma(\lambda+1)$ have the simple pole at $\lambda = 0$ and the pole of the second order at $\lambda = -1$. Therefore, $K(\lambda)$ has the poles of order $m+1$ at $\lambda = -\frac{1}{2} - m$, $\lambda = 2 + m$ and the zeroes of order $m+1$ at $\lambda = \frac{3}{2} + m$, $\lambda = -1 - m$, $m \in \mathbb{N} \cup 0$. Thus $K(\lambda)$ is analytic in the strip (see Figure 3.1),

$$-\frac{1}{2} < \operatorname{Re}\lambda < 2. \tag{3.20}$$


Fig. 3.1: Poles (black circles) and zeroes (white circles) of $K$.

The function $\frac{1}{K(\lambda)}$ is analytic in the strip (see Figure 3.2),

$$-1 < \operatorname{Re}\lambda < \frac{3}{2}. \tag{3.21}$$

The function $K_0(\lambda) = \frac{1}{(\lambda+1)K(\lambda+1)}$ is analytic in the strip (see Figure 3.3),

$$-1 < \operatorname{Re}\lambda < \frac{1}{2}. \tag{3.22}$$


Fig. 3.2: Poles (black circles) and zeroes (white circles) of $1/K$.

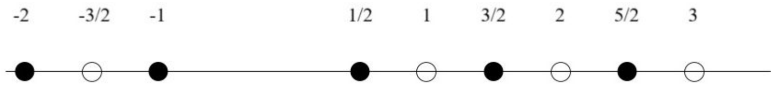


Fig. 3.3: Poles (black circles) and zeroes (white circles) of $K_0$.

Corollary to Proposition 5.1 in [28, p. 222] states the convergence of the infinite product $K(\lambda)$ except the poles of $\Gamma(n - \frac{1}{2} + \lambda)$ and $\Gamma(n + 1 - \lambda)$ for $n \in \mathbb{N}$. Proposition 5.2 in [28] gives the following asymptotic formula:

$$K(\lambda) = |\omega(\lambda)|^{\lambda_1 - 1/2} \exp(-\lambda_2 \arg \omega(\lambda) + r_1(\lambda)), \quad \lambda = \lambda_1 + i\lambda_2, \tag{3.23}$$

where $r_1(\lambda) = O(1)$ in the strip

$$-\frac{1}{2} < \operatorname{Re} \lambda < 2.$$

By definition (3.15), we have (cf. (6.7) in [28]):

$$\begin{aligned} \omega(\lambda) &= \lambda_1 + i\lambda_2) \left\{ \frac{\sin 2\lambda_1\pi}{\cosh 2\lambda_2\pi - \cos 2\lambda_1\pi} - i \frac{\sinh 2\lambda_2\pi}{\cosh 2\lambda_2\pi - \cos 2\lambda_1\pi} \right\}, \\ \operatorname{Re} \omega(\lambda) &= \frac{\lambda_1 \sin 2\lambda_1\pi + \lambda_2 \sinh 2\lambda_2\pi}{\cosh 2\lambda_2\pi - \cos 2\lambda_1\pi}, \\ \operatorname{Im} \omega(\lambda) &= \frac{\lambda_2 \sin 2\lambda_1\pi - \lambda_1 \sinh 2\lambda_2\pi}{\cosh 2\lambda_2\pi - \cos 2\lambda_1\pi}. \end{aligned} \tag{3.24}$$

Hence, by consecutive steps, we obtain

$$\operatorname{Re} \omega(\lambda) > 0, \quad |\arg \omega(\lambda)| < \frac{\pi}{2} \tag{3.25}$$

$$\arg \omega(\lambda) = \arctan \frac{\lambda_2 \sin 2\pi\lambda_1 - \lambda_1 \sinh 2\pi\lambda_2}{\lambda_1 \sin 2\pi\lambda_1 + \lambda_2 \sinh 2\pi\lambda_2}, \tag{3.26}$$

$$|\arg \omega(\lambda)| \le \frac{C}{|\lambda_2|} \tag{3.27}$$

for sufficiently large $|\lambda_2|$ (see [28, §6, p. 228, §7, p. 231–232]).

**Remark 3.1.** We emphasize that, as opposed to [28], there is a simple pole at $\lambda = -1$ in the right-hand side of (3.16).

As in [28], the solution of (3.16) is sought in the form

$$d_1(\lambda) = d_0(\lambda) y(\lambda). \tag{3.28}$$

Substitution of (3.28) in (3.16) gives

$$\kappa_0 d_0(\lambda+1)y(\lambda+1)+\omega(\lambda)d_0(\lambda)y(\lambda)=-\frac{\widetilde{f}(\lambda+2)}{\lambda+1},$$
$$\kappa_0 d_0(\lambda+1)y(\lambda+1)-\kappa_0 d_0(\lambda+1)y(\lambda)=-\frac{\widetilde{f}(\lambda+2)}{\lambda+1}$$

and, to this end,

$$y(\lambda+1)-y(\lambda)=h(\lambda),\quad h(\lambda)=-\frac{\widetilde{f}(\lambda+2)}{\kappa_0(\lambda+1)d_0(\lambda+1)}. \tag{3.29}$$

Let $\mathcal{L}_{l,\varepsilon}$, $l=-1,0$, be the contour in the complex plane consisting of three parts

$$\mathcal{L}_{l,\varepsilon}=\{\operatorname{Re}\lambda=l,\ \varepsilon<\operatorname{Im}\lambda<\infty\}\cup\{\operatorname{Re}\lambda>l,\ |\lambda-l|=\varepsilon\}$$
$$\cup\{\operatorname{Re}\lambda=l,-\infty<\operatorname{Im}\lambda<-\varepsilon\}.$$

**Remark 3.2.** The function $h(\lambda)$ is analytic in the strip (3.22) instead of $-2<\operatorname{Re}\lambda<\frac{1}{2}$ in [28].

Taking into account this observation, we can apply Corollary to Proposition 6.1 in [28] to conclude that the function

$$y(\lambda)=\frac{1}{2i}\int_{\mathcal{L}_{-1,\varepsilon}}h(\lambda+\zeta)[\cot\zeta\pi+i]d\zeta \tag{3.30}$$

is a solution of equation (3.29) for all $\lambda$ such that

$$\operatorname{Re}\lambda\in(0,1/2), \tag{3.31}$$

instead of $\operatorname{Re}\lambda\in(-1/2,1/2)$ in [28] (see Remark 3.2). Here we use the same argument as in Statement 6.1 in [28]. We consider the difference $y(\lambda+1)-y(\lambda)$, change the variable in the integral for $y(\lambda+1)$ and verify that $h(\lambda+\zeta)$ is an analytic function with respect to $\zeta$ in a domain bounded by the contours $\mathcal{L}_{-1,\varepsilon}$ and $\mathcal{L}_{0,\varepsilon}$. Finally, we use the Cauchy residue theorem.

By (3.28), (3.30), we find a solution of (3.16),

$$\begin{aligned}d_1(\lambda)&=d_0(\lambda)y(\lambda)\\&=-\frac{1}{2i}\int_{\mathcal{L}_{-1,\varepsilon}}\frac{d_0(\lambda)}{\kappa_0(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}\widetilde{f}(\lambda+\zeta+2)[\cot\zeta\pi+i]d\zeta.\end{aligned} \tag{3.32}$$

One can see that $y(\lambda)$ and therefore $d_1(\lambda)$ are analytic in the strip

$$0<\operatorname{Re}\lambda<\frac{3}{2} \tag{3.33}$$

(cf. (3.20)).

Estimate (7.3) in [28] gives us (see also (2.11)):

$$\int_{\operatorname{Re}\lambda=\mu-k-1}|d_1(\lambda)|^2(1+|\lambda|)^{2(k+1)+1}\frac{d\lambda}{i}$$

$$\leq C \int_{\operatorname{Re}\lambda=\mu-k-1} \left|\frac{f(\lambda+1)}{\lambda}\right|^2 (1+|\lambda|)^{2(k+1)+1}\frac{d\lambda}{i}$$
$$\leq C \int_{\operatorname{Re}\lambda=\mu-k-1} |f(\lambda+1)|^2 (1+|\lambda|)^{2k+1}\frac{d\lambda}{i}$$
$$\leq C \int_{\operatorname{Re}\lambda=\mu-k} |f(\lambda)|^2 (1+|\lambda|)^{2k+1}\frac{d\lambda}{i} \leq C\|f\|_{H_\mu^{k+1/2}(\mathbb{R}_+)}. \tag{3.34}$$

It should be mentioned that the right-hand term in (3.24) is $\frac{\widetilde{f}(\lambda+2)}{(\lambda+1)}$ instead of $\widetilde{g}(\lambda+1)$ in the corresponding equation (4.7) in [28].

A general solution of (3.29) has the form

$$d_2(\lambda) = d_1(\lambda) + D_0(\lambda),$$

where $D_0(\lambda)$ is a general solution of homogeneous equation (3.17). We check at once that

$$\frac{D_0(\lambda+1)}{d_0(\lambda+1)} = \frac{\omega(\lambda)D_0(\lambda)}{\omega(\lambda)d_0(\lambda)} = \frac{D_0(\lambda)}{d_0(\lambda)},$$

i.e., $E(\lambda) = \frac{D_0(\lambda)}{d_0(\lambda)}$ is a periodic function with period one. One can show that an entire periodic function with period 1 is either constant or grows no slower as $\exp(2\pi|\operatorname{Im}\lambda|)$ as $\lambda_2 \to \pm\infty$ (see Theorem 4.10 in [21]). Thus the integral

$$\int_{\operatorname{Re}\lambda=\mu-k-1} |D_0(\lambda)|^2(1+|\lambda|)^{2(k+1)+1}\frac{d\lambda}{i}$$

is bounded only if $E(\lambda) \equiv 0$ (cf. (3.34)). In the same way as in [28, p. 228–229], this proves that $d_1$ is a unique solution of (3.16) (see also Lemma 3.1.7 in [13]).

Next, we return to the functions (3.12). We set

$$\widetilde{p}(s,\theta) = \frac{\cos s\theta}{\sin s\pi} d_1(s), \quad \widetilde{q}(s) = \cot s\pi d_1(s). \tag{3.35}$$

Clearly, these functions are analytic in the strip $0 < \operatorname{Re} s < \frac{1}{2}$.

Let us turn to estimate (1.9). First, we will estimate norms of the function $q$. Let us notice that

$$\operatorname{Re}\lambda = \mu - k - 1. \tag{3.36}$$

By (2.11) and (3.34), we have

$$\|q\|_{H_\mu^{k+\frac{3}{2}}(\mathbb{R}_+)} \leq C \int_{\mu-k-1-i\infty}^{\mu-k-1+i\infty} |\widetilde{q}(\lambda)|^2(1+|\lambda|)^{2(k+1)+1}\frac{d\lambda}{i} \leq C\|f\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)}. \tag{3.37}$$

From equation (3.5) one can expect at least that $q_{xx} \in H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)$. By (A.2) (see Appendix A), we have

$$\frac{1}{r}(p_\theta(r,-\pi) - p_\theta(r,\pi))$$

$$= \frac{1}{\pi i} \int_{\operatorname{Re}\lambda=\mu-k-1} r^{-\lambda-1}\lambda d_1(\lambda)\,d\lambda = \frac{1}{\pi i}\int_{\operatorname{Re}\lambda=\mu-k} r^{-\lambda}(\lambda-1)d_1(\lambda-1)\,d\lambda.$$

Therefore, the Mellin transform of

$$\frac{1}{r}(p_\theta(r,-\pi)-p_\theta(r,\pi))$$

is $2(\lambda-1)\lambda d_1(\lambda-1)$. By (3.34) and (2.11), we obtain

$$\int_{\mu-k-i\infty}^{\mu-k+i\infty} |(\lambda-1)d_1(\lambda-1)|^2(1+|\lambda|)^{2k+1}\frac{d\lambda}{i} \\ \leq C\int_{\mu-k-1-i\infty}^{\mu-k-1+i\infty} |d_1(\lambda)|^2(1+|\lambda|)^{2(k+1)+1}\frac{d\lambda}{i} \leq C\|f\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)},$$

and hence

$$\left\|\frac{1}{r}(p_\theta(r,-\pi)-p_\theta(r,\pi))\right\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)} \leq C\|f\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)}. \tag{3.38}$$

By (3.5), (3.38), we deduce that $q_{rr}\in H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)$ and

$$\|q_{rr}\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)} \leq C\|f\|_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)}. \tag{3.39}$$

We are to prove that $q_r\in H_\mu^{k+\frac{3}{2}}(\mathbb{R}_+)$, as it is asserted in (1.9). In fact, definition (2.4) and estimate (2.11) give

$$\begin{aligned}\|q_r\|^2_{H_\mu^{k+3/2}(\mathbb{R}_+)} &= \sum_{l=0}^{k+1}\int_0^\infty r^{2(\mu-k-1+l)-1}\left|\frac{d^{l+1}q(r)}{dr^{l+1}}\right|^2 dr + \left\|\frac{d^{k+2}q}{dr^{k+2}}\right\|_{L_\mu^{1/2}(\mathbb{R}_+)} \\ &= \int_0^\infty r^{2(\mu-k-1)-1}|q_r(r)|^2\,dr \\ &\quad + \sum_{j=0}^{k}\int_0^\infty r^{2(\mu-k-1+j+1)-1}\left|\frac{d^{j+2}q(r)}{dr^{j+2}}\right|^2 dr + \left\|\frac{d^{k+2}q}{dr^{k+2}}\right\|_{L_\mu^{1/2}(\mathbb{R}_+)} \\ &= \int_0^\infty r^{2(\mu-k-1)-1}|q_r(r)|^2dr + \|q_{rr}\|^2_{H_\mu^{k+1/2}(\mathbb{R}_+)}.\end{aligned} \tag{3.40}$$

The Hardy inequality (see (0.32) in [20]) gives

$$\int_0^\infty r^{2(\mu-k)-1-2}|q_r(r)|^2dr \leq C\int_0^\infty r^{2(\mu-k)-1}|q_{rr}(r)|^2dr \leq C\|q_{rr}\|^2_{H_\mu^{k+1/2}(\mathbb{R}_+)}, \tag{3.41}$$

for $2(\mu-k)-1>1$, i.e., $\mu-k-1>0$ (see (1.7)). Then (3.40) and (3.41) lead to

$$\|q_r\|^2_{H_\mu^{k+3/2}(\mathbb{R}_+)} \leq C\|f\|^2_{H_\mu^{k+1/2}(\mathbb{R}_+)}. \tag{3.42}$$

Now, consider the norm (2.3). Notice that $2(\mu-k-2)+1 = 2(\mu-k-1)-1$. The Parceval formula (2.7) and (2.10) give

$$\begin{aligned}\|p\|^2_{k+2,\mu} &= \sum_{l=0}^{k+2}\int_0^\infty r^{2(\mu-k-2)+1}\int_{-\pi}^{\pi}\left|\left(r\frac{\partial}{\partial r}\right)^{(l)}\partial_\theta^{(k+2-l)}p(r,\theta)\right|^2 d\theta\, dr\\ &= \sum_{l=0}^{k+2}\int_{\operatorname{Re}s=\mu-k-1}\frac{ds}{i}\int_{-\pi}^{\pi}\left|s^l\,\partial_\theta^{(k+2-l)}\widetilde{p}(s,\theta)\right|^2 d\theta. \end{aligned} \tag{3.43}$$

We use (3.35) and (3.34) to obtain

$$\begin{aligned}\|p\|^2_{k+2,\mu} &\le C\int_{\operatorname{Re}s=\mu-k-1}|s|^{2l}|d_1(s)|^2\frac{ds}{i}\int_{-\pi}^{\pi}|s|^{2(k+2-l)}\frac{|\cos s\theta|^2+|\sin\theta s|^2}{|\sin s\pi|^2}\,d\theta\\ &\le C\int_{\operatorname{Re}s=\mu-k-1}|s|^{2(k+1)}|d_1(s)|^2\frac{ds}{i}\int_{-\pi}^{\pi}|s|^2\frac{|\cos s\theta|^2+|\sin s\theta|^2}{|\sin s\pi|^2}\,d\theta.\end{aligned}$$

By estimate (see p.230 in [28])

$$\int_{-\pi}^{\pi}|s|^2\frac{|\cos s\theta|^2+|\sin s\theta|^2}{|\sin s\pi|^2}\,d\theta \le C(1+|s|)$$

and (3.34), (2.11), we get

$$\|p\|^2_{k+2,\mu} \le C\int_{\operatorname{Re}s=\mu-k-1}(1+|s|)^{2(k+1)+1}|d_1(s)|^2\frac{ds}{i} \le C\|f\|^2_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)}. \tag{3.44}$$

Summing up (3.37), (3.42), and (3.44), we deduce estimate (1.9).

## 4. Boundary conditions

Boundary conditions (1.2)–(1.4) are considered in Appendix A. This section is devoted to (1.5).

In brief outline, here are the main steps of the proof. First, we split $q$ into two terms $q_1$ and $q_2$ (see (4.2), (4.6) below). Second, we prove that $q_{1,r}(0)=0$ under assumption (1.8). Third, the term $q_{2,r}$ (see (4.15) below) is estimated by the Cauchy inequality (see (4.18) below). Here we use that $f\in H_\mu^{k+1/2}(\mathbb{R}_+)$, and hence

$$\int_0^\infty |f(\rho)|^2\rho^{2(\mu-k)-1}\,d\rho \le \|f\|^2_{H_\mu^{k+1/2}(\mathbb{R}_+)}.$$

To this end, we use the appropriate changes of the contour of integration with respect to $s$ in order to estimate integrals with the kernel $\Phi(s,\zeta)$. The kernel $\Phi(s,\zeta)$ itself is estimated in Appendix B.

We proceed by converting (3.30) to a more appropriate form. The Cauchy residue theorem gives

$$y(\lambda) = -\frac{1}{2i}\int_{\mathcal{L}_{0,\varepsilon}}\frac{\widetilde{f}(\lambda+\zeta+2)}{\kappa_0(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}[\cot\zeta\pi+i]\,d\zeta$$

$$\begin{aligned}&+\pi \operatorname*{Res}_{\zeta=0} \frac{\widetilde{f}(\lambda+\zeta+2)}{\kappa_0(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}\cot\zeta\pi,\\
&= -\frac{1}{2i}\int_{\mathcal{L}_{0,\varepsilon}} \frac{\widetilde{f}(\lambda+\zeta+2)}{\kappa_0(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}[\cot\zeta\pi+i]d\zeta\\
&+\frac{\widetilde{f}(\lambda+2)}{\kappa_0(\lambda+1)d_0(\lambda+1)},\end{aligned}\tag{4.1}$$

where $\operatorname{Re}\lambda \in (0,\frac{1}{2})$. By (3.12), (3.17), (3.30), and (3.32), we have

$$\begin{aligned}\widetilde{q}(\lambda) &= \cot\pi\lambda d_1(\lambda) = \cot\pi\lambda d_0(\lambda)y(\lambda) = -\frac{\kappa_0 d_0(\lambda+1)}{\lambda}y(\lambda) = -\frac{\widetilde{f}(\lambda+2)}{\lambda(\lambda+1)}\\
&+\frac{1}{2i}\int_{\mathcal{L}_{0,\varepsilon}} \frac{d_0(\lambda+1)\widetilde{f}(\lambda+\zeta+2)}{\lambda(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}[\cot\zeta\pi+i]d\zeta = \widetilde{q}_1(\lambda)+\widetilde{q}_2(\lambda).\end{aligned}\tag{4.2}$$

Formula (3.18) yields

$$\begin{aligned}&\frac{1}{2i}\frac{d_0(\lambda+1)}{(\lambda+\zeta+1)d_0(\lambda+\zeta+1)}[\cot\zeta\pi+i]\\
&\qquad = (\kappa_0\pi)^{\zeta}\frac{K(\lambda+1)}{(\lambda+\zeta+1)K(\lambda+\zeta+1)}\frac{1}{e^{i\pi\zeta}-e^{-i\pi\zeta}}.\end{aligned}\tag{4.3}$$

Set

$$\Phi(\lambda,\zeta) = (\kappa_0\pi)^{\zeta}\frac{K(\lambda)}{(\lambda+\zeta)K(\lambda+\zeta)}\frac{1}{e^{i\pi\zeta}-e^{-i\pi\zeta}}.\tag{4.4}$$

Then we get

$$\widetilde{q}_2(\lambda) = \int_{\mathcal{L}_{0,\varepsilon}} \frac{1}{\lambda}\Phi(\lambda+1,\zeta)\widetilde{f}(\lambda+2+\zeta)\,d\zeta.\tag{4.5}$$

Set $l_0 = \mu-k-1$, and then $l_0 \in (0,1/2)$. By (2.6), we have

$$\begin{aligned}q(r) &= \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda}\widetilde{q}(\lambda)\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda}\widetilde{q}_1(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda}\widetilde{q}_2(\lambda)\,d\lambda\\
&= q_1(r)+q_2(r).\end{aligned}\tag{4.6}$$

Consider $q_{1,r}(r)$ and $q_{2,r}(r)$ separately. For the first term we have

$$q_{1,r}(r) = \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda-1}\frac{\widetilde{f}(\lambda+2)}{\lambda+1}\,d\lambda = \frac{1}{2\pi i}\int_{\operatorname{Re}s=l_0+1} r^{-s}\frac{\widetilde{f}(s+1)}{s}\,ds.\tag{4.7}$$

It is clear that if $u(r) = rf(r)$, then $\widetilde{u}(s) = \widetilde{f}(s+1)$. By formula (7.1.2) in [10], if $\widetilde{v}(s) = s^{-1}$, $\operatorname{Re}s>0$, then

$$v(r) = \begin{cases}1, & 0<x<1,\\ 0, & 1<x.\end{cases}$$

The Mellin convolution formula (2.8) gives

$$q_{1,r}(r) = \int_0^1 \frac{r}{t} f\left(\frac{r}{t}\right) \frac{dt}{t} = \int_r^\infty f(\rho)\, d\rho. \tag{4.8}$$

By virtue of (1.8), we conclude that

$$q_{1,r}(0) = 0. \tag{4.9}$$

Next, we consider $q_{2,r}(r)$. Substituting (4.5) into (4.6), we have

$$q_{2,r}(r) = -\frac{1}{2\pi i} \int_{\operatorname{Re}\lambda = l_0} r^{-\lambda-1}\, d\lambda \int_{\mathcal{L}_{0,\varepsilon}} \Phi(\lambda+1,\zeta)\widetilde{f}(\lambda+2+\zeta)\, d\zeta. \tag{4.10}$$

The point $\lambda = 0$ is not a singular point of the last expression (compare with (4.5)). Therefore we are able to deduce

$$\begin{aligned} q_{2,r}(r) &= -\frac{1}{2\pi i} \int_{\operatorname{Re} s = l_0+1} r^{-s}\, ds \int_{\mathcal{L}_{0,\varepsilon}} \Phi(s,\zeta)\widetilde{f}(s+1+\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_{\operatorname{Re} s = l_0} r^{-s}\, ds \int_{\mathcal{L}_{0,\varepsilon}} \Phi(s,\zeta)\widetilde{f}(s+1+\zeta)\, d\zeta. \end{aligned} \tag{4.11}$$

Remark that $\Phi(s,\zeta)$ in (4.11) is analytic with respect to $s$ if $\operatorname{Re} s \in (l_0, l_0+1)$, $\zeta \in \mathcal{L}_{0,\varepsilon}$ and $\varepsilon$ is sufficiently small ($\varepsilon < \frac{1}{2} - l_0$). Let us notice that demanding of smallness of $\varepsilon$ was implicitly presented in [28, p. 228]. Below we will change the contours of integration with respect to $s$, $\zeta$, keeping $\Phi(s,\zeta)$ away from singular points. However, we need to be sure that

$$0 < \operatorname{Re}\zeta < 1, \quad 0 < \operatorname{Re}(s+\zeta) < \frac{3}{2}, \quad -\frac{1}{2} < \operatorname{Re} s < \frac{3}{2}, \tag{4.12}$$

(cf. (3.20)–(3.22)).

Set $s = \sigma + i\tau$, $\zeta = \vartheta + i\eta$. Fix any $\alpha \in (1,2)$ and put

$$\vartheta = l_0 + \frac{\alpha - 1}{2}. \tag{4.13}$$

This choice will be explained below. Then

$$0 < \vartheta < 1, \quad 0 < \vartheta + l_0 = 2l_0 + \frac{\alpha-1}{2} < \frac{3}{2}, \tag{4.14}$$

i.e., (4.12) holds for $l_0 = \operatorname{Re} s$, $\theta = \operatorname{Re}\zeta$. Hence we get

$$q_{2,r}(r) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = l_0} r^{-s}\, ds \int_{\operatorname{Re}\zeta = \vartheta} \Phi(s,\zeta)\widetilde{f}(s+1+\zeta)\, d\zeta. \tag{4.15}$$

By the Mellin convolution formula (2.8), we deduce

$$q_{2,r}(r) = \frac{1}{2\pi i} \int_0^\infty \frac{dt}{t} \int_{\operatorname{Re}\zeta = \vartheta} d\zeta \int_{\operatorname{Re} s = l_0} f\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^{1+\zeta} t^{-s}\Phi(s,\zeta)\, ds$$

$$= \frac{1}{2\pi i}\int_0^\infty f\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^{1+\theta}\frac{dt}{t}\int_{\operatorname{Re}\zeta=\vartheta}d\zeta\int_{\operatorname{Re}s=l_0}\left(\frac{r}{t}\right)^{i\eta}t^{-s}\Phi(s,\zeta)\,ds$$
$$= \int_0^\infty f\left(\frac{r}{t}\right)\left(\frac{r}{t}\right)^{1+\vartheta}\mathcal{G}(t,\vartheta)\frac{dt}{t}, \tag{4.16}$$

where

$$\mathcal{G}(t,\vartheta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{r}{t}\right)^{i\eta}d\eta\int_{\operatorname{Re}s=l_0}t^{-s}\Phi(s,\vartheta+i\eta)\,ds \tag{4.17}$$

The Cauchy inequality yields

$$|q_2'(r)| \le C\left[\int_0^\infty \left|f\left(\frac{r}{t}\right)\right|^2\left(\frac{r}{t}\right)^{2(1+\vartheta)}t^{\alpha-2}\,dt\right]^{\frac12}\left[\int_0^\infty|\mathcal{G}(t,\vartheta)|^2t^{-\alpha}\,dt\right]^{\frac12}$$
$$\le CI_1^{\frac12}I_2^{\frac12}. \tag{4.18}$$

Following the choice of $\vartheta$ in (4.13), we obtain

$$I_1 = \int_0^\infty \left|f\left(\frac{r}{t}\right)\right|^2\left(\frac{r}{t}\right)^{2(1+l_0)+\alpha-1}t^{\alpha-2}\,dt$$
$$= \int_0^\infty |f(\rho)|^2\rho^{2l_0+1+\alpha}\left(\frac{r}{\rho}\right)^{\alpha-2}\frac{r\,d\rho}{\rho^2} = r^{\alpha-1}\int_0^\infty |f(\rho)|^2\rho^{2(\mu-k-1)+1}\,d\rho$$
$$= r^{\alpha-1}\int_0^\infty |f(\rho)|^2\rho^{2(\mu-k)-1}\,d\rho \le Cr^{\alpha-1}\|f\|^2_{H_\mu^{k+\frac12}(\mathbb{R}_+)}. \tag{4.19}$$

The second integral $I_2$ can be split into a sum of two integrals

$$I_2 = \int_0^\infty|\mathcal{G}(t,\vartheta)|^2t^{-\alpha}\,dt = \int_0^1|\mathcal{G}(t,\vartheta)|^2t^{-\alpha}\,dt + \int_1^\infty|\mathcal{G}(t,\vartheta)|^2t^{-\alpha}\,dt.$$

Let $\sigma_1$, $\sigma_2$ be chosen to satisfy (compare with (4.12)),

$$\sigma_1 > 0,\ 0 < -\sigma_1+\vartheta < \frac32,\ -\frac12 < -\sigma_1 < \frac32,\ 0 < \sigma_2+\vartheta < \frac32,\ 0 < \sigma_2 < \frac32. \tag{4.20}$$

Then we derive

$$|I_2| \le C\Bigg(\int_0^1 t^{2\sigma_1-\alpha}\left|\int_{-\infty}^{+\infty}d\eta\int_{\operatorname{Re}s=-\sigma_1}|\Phi(s,\vartheta+i\eta)|\,ds\right|^2dt$$
$$+\int_1^\infty t^{-2\sigma_2-\alpha}\left|\int_{-\infty}^{+\infty}d\eta\int_{\operatorname{Re}s=\sigma_2}|\Phi(s,\vartheta+i\eta)|\,ds\right|^2dt\Bigg). \tag{4.21}$$

We claim that

$$|\Phi(s,\zeta)| \le C(M,\sigma,\vartheta)\exp\left(-\frac{\pi}{4}|\eta|\right)\left[\frac{1}{(1+|\tau|)^{1+\vartheta}}+\exp\left(-\frac{\pi}{4}|\tau|\right)\right] \tag{4.22}$$

under assumptions (cf. (4.12)) that

$$\vartheta\in(0,1),\quad 0<\operatorname{Re}s+\vartheta<\frac32,\quad -\frac12<\operatorname{Re}s<\frac32. \tag{4.23}$$

The proof of (4.22) is given in Appendix B. Estimate (4.22) yields

$$\int_{-\infty}^{+\infty} d\eta \int_{\operatorname{Re} s=\sigma_0} |\Phi(s,\vartheta+i\eta)|\, ds \le C(\sigma_0,\vartheta) \tag{4.24}$$

for any $\sigma_0 = \operatorname{Re} s$ and $\vartheta$ satisfying (4.23).

In addition to (4.20), we need

$$2\sigma_1 - \alpha > -1, \quad -2\sigma_2 - \alpha < -1. \tag{4.25}$$

It can be easily checked that (4.20), (4.24), and (4.25) guarantee the convergence of integrals in (4.21).

Now we proceed to the choice of $\sigma_1$, $\sigma_2$. Let us remind that $\theta$ is already chosen. It is easily seen that any

$$\sigma_2 \in \left(0, \frac{3}{2} - \vartheta\right) \tag{4.26}$$

satisfies both (4.20) and (4.25). The parameter $\sigma_1$ satisfies (4.20) and (4.25) if

$$\frac{\alpha-1}{2} < \sigma_1, \quad \sigma_1 < \vartheta, \quad \sigma_1 < \frac{1}{2}. \tag{4.27}$$

We check at once that

$$\frac{\alpha-1}{2} < \frac{1}{2}, \quad \frac{\alpha-1}{2} < \vartheta = l_0 + \frac{\alpha-1}{2},$$

i.e., the interval $\left(\frac{\alpha-1}{2}, \min\{\frac{1}{2},\vartheta\}\right)$ is not empty. Hence we can take any

$$\sigma_1 \in \left(\frac{\alpha-1}{2}, \min\left\{\frac{1}{2},\vartheta\right\}\right). \tag{4.28}$$

This allows us finally to conclude

$$I_2 \le C(\sigma_1,\sigma_2,\theta). \tag{4.29}$$

Substitution (4.19) and (4.29) in (4.18) gives

$$|q_{2,r}(r)| \le C r^{\frac{\alpha-1}{2}} \|f\|^2_{H_\mu^{k+\frac{1}{2}}(\mathbb{R}_+)}. \tag{4.30}$$

Combining (4.9) and (4.30), we can assert that $q_r(0) = 0$, which justifies (1.5). The proof of Theorem 1.1 is finished.

**Remark 4.1.** Let $\widehat{f}$ be a polynomial of degree $m$,

$$\widehat{f}(x_1) = \sum_{k=0}^{m} f_k x_1^k.$$

One can easily check that

$$\widehat{q}(x_1) = q_0 + q_1 x_1 - \sum_{k=0}^{m} \frac{f_k}{(k+1)(k+2)} x_1^{k+2},$$
$$\widehat{p}(x) = q_0 + q_1 x_1 - \sum_{k=0}^{m} \frac{f_k}{(k+1)(k+2)} \operatorname{Re}\left[(x_1 + i x_2)^{k+2}\right]$$

satisfy (1.1)–(1.4). In this context, it might be the subject of a future paper to consider problem (1.1)–(1.6) without artificial condition $\int_{-\infty}^{0} \widehat{f}(x_1)dx_1 = 0$, in weighted Sobolev spaces with nonhomogeneous norms (cf. [7], [29], [26]).

## A. Appendix

We have immediately from (3.35),

$$p(r,\theta) = \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda}\frac{\cos\lambda\theta}{\sin\lambda\pi} d_1(\lambda)\, d\lambda,$$
$$q(r) = \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda} \cot\lambda\pi d_1(\lambda)\, d\lambda, \tag{A.1}$$

where $l_0 = \mu - k - 1 \in (0, 1/2)$. Then it follows that

$$p(r,\pi) + p(r,-\pi) - 2q(r)$$
$$= \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda}[\cot\lambda\pi + \cot\lambda\pi - 2\cot\lambda\pi] d_1(\lambda) d\lambda = 0. \tag{A.2}$$

This gives (1.2).

Direct computation shows that

$$p_\theta(r,\theta) = -\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda}\frac{\sin\lambda\theta}{\sin\lambda\pi}\lambda d_1(\lambda)\, d\lambda,$$
$$p_\theta(r,\pm\pi) = \mp\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda}\lambda d_1(\lambda)\, d\lambda \tag{A.3}$$

and

$$p(r,\pi) - p(r,-\pi) = 0.$$

This gives (3.4) and hence (1.3). Let us notice that, as it is easily seen, $p_\theta(r,\pi) - p_\theta(r,-\pi) \neq 0$. Thus, equation (3.5) is not reduced to $-q_{rr}(r) = f(r)$.

Next, we proceed to condition (3.5). According to the definition, we have

$$B[p,q](r) = -q_{rr}(r) + \frac{\kappa_2}{r}\left(p_\theta(r,\pi) - p_\theta(r,-\pi)\right)$$
$$= -\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda-2}\cot\lambda\pi\lambda(\lambda+1) d_1(\lambda)\, d\lambda$$
$$- \frac{2\kappa_2}{2\pi i}\int_{\operatorname{Re}\lambda = l_0} r^{-\lambda-1}\lambda d_1(\lambda)\, d\lambda.$$

Since $d_1(\lambda)$ solves (3.16), we can continue

$$\begin{aligned}
B[p,q](r) &= \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda-2}\widetilde{f}(\lambda+2)\,d\lambda \\
&\quad+\frac{\kappa_0}{2\pi i}\int_{\operatorname{Re}\lambda=l_0}\Big[r^{-\lambda-2}(\lambda+1)d_1(\lambda+1)d\lambda-r^{-\lambda-1}\lambda d_1(\lambda)\Big]\,d\lambda \\
&=\frac{1}{2\pi i}\int_{\operatorname{Re}\lambda=l_0+2} r^{-\lambda}\widetilde{f}(\lambda)\,d\lambda \\
&\quad+\frac{\kappa_0}{2\pi i}\Big[\int_{\operatorname{Re}\lambda=l_0+1} r^{-\lambda-1}\lambda d_1(\lambda)\,d\lambda-\int_{\operatorname{Re}\lambda=l_0} r^{-\lambda-1}\lambda d_1(\lambda)\,d\lambda\Big]=f(r),
\end{aligned}$$

where the expression in the square brackets is equal to zero since $d_1(\lambda)$ is analytic in the strip

$$\operatorname{Re}\lambda\in(l_0,l_0+1)$$

(cf. (3.33)). This gives (3.5) and hence (1.4).

## B. Appendix

Preliminarily, in the plane $(\eta,\tau)$ we consider the sets:

$$\begin{aligned}
Q_1 &= \Big\{\eta\le 0,\ -2\eta<\tau\Big\}\cup\Big\{\eta>0,\ \tau>-\frac{2}{3}\eta\Big\},\\
Q_2 &= \Big\{\eta\le 0,\ \tau<-\frac{2}{3}\eta\Big\}\cup\Big\{\eta>0,\ \tau<-2\eta\Big\},\\
Q_3 &= \Big\{\tau>0,\ -\frac{2}{3}\eta<\tau<-2\eta\Big\},\\
Q_4 &= \Big\{\tau<0,\ -2\eta<\tau<-\frac{2}{3}\eta\Big\}.
\end{aligned}\tag{B.1}$$

It follows immediately that

$$\begin{aligned}
|\tau| &\le 2|\eta+\tau|, \quad (\eta,\tau)\in Q_1\cup Q_2,\\
|\tau| &\ge 2|\eta+\tau|, \quad (\eta,\tau)\in Q_3\cup Q_4.
\end{aligned}\tag{B.2}$$

To deal with estimates of $\Phi(s,\zeta)$, we partition the plane $(\eta,\tau)$ into the following sets (see Fig. B.1):

$$\begin{aligned}
B_0 &= \{(\eta,\tau):\ |\eta|\le 3M,\ |\tau|\le 2M\},\\
B_1 &= \{(\eta,\tau):\ \eta>3M,\ |\tau|\le 2M\},\\
B_{1.1} &= \Big\{(\eta,\tau):\ -\frac{3}{2}\tau<\eta,\ \tau<-2M\Big\},\\
B_{1.2} &= \{(\eta,\tau):\ \eta>0,\ \tau>2M\},\\
B_{1.3} &= \Big\{(\eta,\tau):\ -\frac{\tau}{2}<\eta<0,\ \tau>2M\Big\},\\
B_{3.1} &= \Big\{(\eta,\tau):\ M-\tau<-\frac{\tau}{2},\ \tau>2M\Big\},\\
B_{3.2} &= \{(\eta,\tau):\ -M-\tau<M-\tau,\ \tau>2M\},
\end{aligned}$$

$$B_{3.3} = \Big\{(\eta,\tau):\ -\frac{3}{2}\tau < \eta < -M-\tau,\ \tau > 2M\Big\},$$
$$B_{2.1} = \Big\{(\eta,\tau):\ \eta < -\frac{3}{2}\tau,\ \tau > 2M\Big\},$$
$$B_{2} = \{(\eta,\tau):\ \eta < -3M,\ |\tau| < 2M\},$$
$$B_{2.2} = \{(\eta,\tau):\ \eta < 0,\ \tau < -2M\},$$
$$B_{2.3} = \Big\{(\eta,\tau):\ 0 < \eta < -\frac{\tau}{2},\ \tau < -2M\Big\},$$
$$B_{4.1} = \Big\{(\eta,\tau):\ -\frac{\tau}{2} < \eta < -\tau - M, \tau < -2M\Big\},$$
$$B_{4.2} = \{(\eta,\tau):\ -\tau - M < \eta < M-\tau,\ \tau < -2M\},$$
$$B_{4.3} = \Big\{(\eta,\tau):\ M-\tau < \eta < -\frac{3}{2}\tau,\ \tau < -2M\Big\}.$$

Here we choose $M > 0$ sufficiently large to neglect $r_1(\lambda)$ in (3.23) and use (3.27). This construction was motivated by [4, 9].

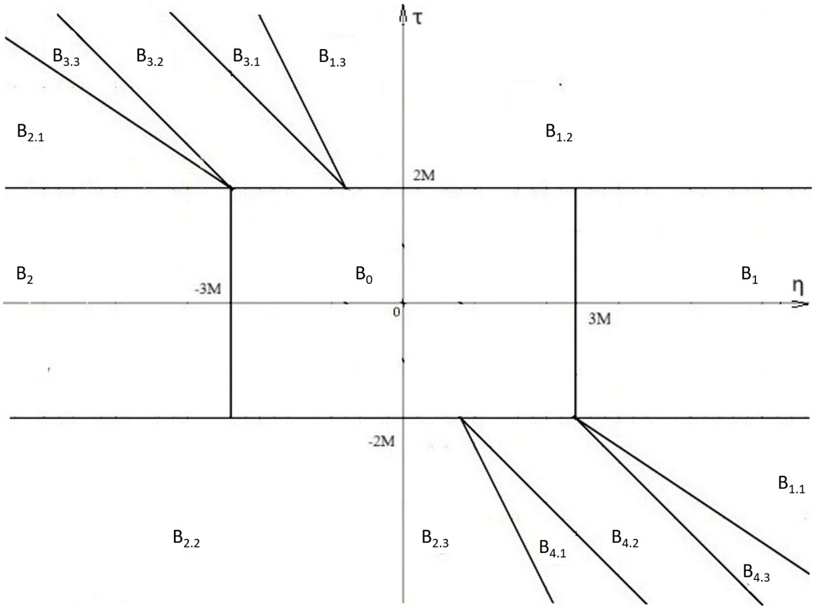


Fig. B.1

By assumptions (4.23), we clearly have

$$|\Phi(s,\zeta)| \le C(M,\vartheta) \quad \text{in } B_0. \tag{B.3}$$

Now estimate (4.22) will follow from the four lemmas below.

**Lemma B.1.** *Under conditions* (4.23), *we have*

$$|\Phi(s,\zeta)| \le C(M,\vartheta)\exp(-\pi|\eta|) \quad \textit{in } B_1, B_2. \tag{B.4}$$

*Proof.* It is clear that

$$|K(s)| \le C(M), \qquad \left|\frac{1}{e^{i\zeta\pi} - e^{-i\zeta\pi}}\right| \le C(M,\vartheta)\exp(-\pi|\eta|) \tag{B.5}$$

with fixed $\vartheta \in (0,1)$ in $B_1$, $B_2$. Now (3.23) and (B.5) give

$$|\Phi(s,\zeta)| \le C(M,\vartheta)|s+\zeta|^{-1/2-\sigma-\vartheta} \exp((\tau+\eta)\arg\omega(s+\zeta) - \pi|\eta|) \quad \text{in } B_1, B_2. \tag{B.6}$$

Notice that $|\tau+\eta| \to \infty$, when $|\eta| \to \infty$, and, by (4.23), (3.27), we get

$$\begin{aligned} \exp((\tau+\eta)\arg\omega(s+\zeta)) &\le C(M), \\ |s+\zeta|^{-1/2-\sigma-\vartheta} &\le C(M). \end{aligned} \tag{B.7}$$

From the above there follows (B.4). □

Since $|\tau|$ is bounded in $B_1$, $B_2$, (B.4) implies (4.22) in these sets.

**Lemma B.2.** *Under conditions* (4.23)*, we have*

$$|\Phi(s,\zeta)| \le C(M,\sigma,\vartheta)\tau^{-1-\vartheta} \exp(-(\pi-\varepsilon)|\eta|) \ \textit{in } B_{1.1}, B_{1.2}, B_{1.3}, B_{2.1}, B_{2.2}, B_{2.3}, \tag{B.8}$$

*where $\varepsilon$ is sufficiently small and depends on $M$.*

*Proof.* Our proof starts with observation that, by definition,

$$|\tau+\eta| \ge M \tag{B.9}$$

and, by (B.2),

$$\frac{1}{|\tau+\eta|} \le \frac{2}{|\tau|} \quad \text{in } B_{1.1}, B_{1.2}, B_{1.3}, B_{2.1}, B_{2.2}, B_{2.3}. \tag{B.10}$$

We consider separately the set $B_{1.2}$, where $\operatorname{sgn}\tau = \operatorname{sgn}\eta$, and the set $B_{1.3}$, where $\operatorname{sgn}\tau = -\operatorname{sgn}\eta$.

From (3.23), we deduce that

$$\begin{aligned} |\Phi(s,\zeta)| \le C(M,\vartheta)\exp(-\pi|\eta|)\left|\frac{\omega(s)}{\omega(s+\zeta)}\right|^{\sigma-1/2} &\frac{1}{|\omega(s+\eta|^{1+\vartheta}} \\ \times \exp(-\tau\arg\omega(s) + (\tau+\eta)&\arg\omega(s+\zeta)). \end{aligned} \tag{B.11}$$

We have

$$\begin{aligned} &\exp(-\tau\arg\omega(s) + (\tau+\eta)\arg\omega(s+\zeta)) \\ &\qquad = \exp(\eta\arg\omega(s+\zeta))\exp(\tau(\arg\omega(s+\zeta) - \arg\omega(s))). \end{aligned} \tag{B.12}$$

We claim that $|\eta\arg\omega(s+\zeta)|$ and $|\tau(\arg\omega(s+\zeta) - \arg\omega(s))|$ are uniformly bounded in $B_{1.2}$, $B_{1.3}$.

For the first expression, estimate (3.27) implies

$$|\eta\arg\omega(s+\zeta)| \le C(M)\frac{|\eta|}{|\eta+\tau|} \quad \text{in } B_{1.2}, B_{1.3}. \tag{B.13}$$

In $B_{1.2}$, we get

$$\frac{|\eta|}{|\eta+\tau|} \le 1 \tag{B.14}$$

since $\operatorname{sgn}\tau = \operatorname{sgn}\eta$. In $B_{1.3}$, we have $|\eta| \leq \frac{|\tau|}{2}$ and use (B.10) to obtain

$$\frac{|\eta|}{|\eta+\tau|} \leq \frac{|\tau|}{2|\eta+\tau|} \leq \frac{|\tau|}{|\tau|} = 1. \tag{B.15}$$

In order to estimate $|\tau(\arg\omega(s+\zeta) - \arg\omega(s))|$, we use the formula

$$\arctan a - \arctan b = \arctan \frac{a-b}{1+ab} \tag{B.16}$$

and (3.27). Then

$$\begin{aligned} &\arg\omega(s) = \arctan \frac{\Psi_1(s)}{\Psi_2(s)}, && \arg\omega(s+\zeta) = \arctan \frac{\Psi_1(s+\zeta)}{\Psi_2(s+\zeta)}, \\ &\Psi_1(s) = \tau \sin 2\pi\sigma - \sigma \sinh 2\pi\tau, && \Psi_2(s) = \sigma \sin 2\pi\sigma + \tau \sinh 2\pi\tau. \end{aligned} \tag{B.17}$$

Along this way, in $B_{1.2}$, $B_{1.3}$, we deduce

$$\begin{aligned} &\arg\omega(s+\zeta) - \arg\omega(s) = \arctan \Psi_0(s, s+\zeta), \\ &\Psi_0(s, s+\zeta) = \frac{\Psi_1(s)\Psi_2(s+\zeta) - \Psi_1(s+\zeta)\Psi_2(s)}{\Psi_2(s)\Psi_1(s+\zeta) + \Psi_1(s)\Psi_2(s+\zeta)}. \end{aligned} \tag{B.18}$$

A detailed analysis of the function $\Psi_0(s, s+\zeta)$ shows that the following estimate holds:

$$\begin{aligned} |\Psi_0(s, s+\zeta)| &\leq C_1(M,\sigma,\vartheta) \frac{|(\sigma+\vartheta)\tau - \sigma(\tau+\eta)|}{|\tau||\tau+\eta|} + \frac{C_2(M,\sigma,\vartheta)}{|\tau|} \\ &\leq C_1(M,\sigma,\vartheta) \left[ \frac{1}{|\tau+\eta|} + \frac{|\eta|}{|\tau||\tau+\eta|} \right] + \frac{C_2(M,\sigma,\vartheta)}{|\tau|}. \end{aligned} \tag{B.19}$$

In $\Sigma_2'$ we have $\tau > 2M$, $\eta > 0$ and therefore

$$|\Psi_0(s, s+\zeta)| \leq C_1(M,\sigma,\vartheta) \left[ \frac{1}{|\tau|} + \frac{|\eta|}{|\tau||\eta|} \right] + \frac{C_2(M,\sigma,\vartheta)}{|\tau|} \leq \frac{C(M,\sigma,\tau)}{|\tau|}. \tag{B.20}$$

By (B.2), we have $\frac{1}{|\eta+\tau|} \leq \frac{2}{|\tau|}$ in $\Sigma_2''$ and, besides, $-|\eta| > -\frac{|\tau|}{2}$. Thus we get

$$|\Psi_0(s, s+\zeta)| \leq C_1(M,\sigma,\vartheta) \frac{2}{|\tau|} \left[ 1 + \frac{|\eta|}{|\tau|} \right] + \frac{C_2(M,\sigma,\vartheta)}{|\tau|} \leq \frac{C(M,\sigma,\tau)}{|\tau|}. \tag{B.21}$$

Combining of (B.18)–(B.20) yields

$$|\tau(\arg\omega(s+\zeta) - \arg\omega(s))| \leq C(M,\sigma,\vartheta) \quad \text{in } B_{1.2}, B_{1.3}. \tag{B.22}$$

From (B.12), (B.14), (B.15), and (B.22), we obtain

$$|\exp(-\tau \arg\omega(s) + (\tau+\eta)\arg\omega(s+\zeta))| \leq C(M,\sigma,\vartheta). \tag{B.23}$$

Taking into account (B.11) and (B.23), we conclude that (B.8) is valid in $B_{1.2}$, $B_{1.3}$. In a similar way, one can consider the sets $B_{1.1}$, $B_{2.1}$, $B_{2.2}$, $B_{2.3}$. □

**Lemma B.3.** *We have*

$$|\Phi(s,\eta)| \leq C(M,\sigma,\vartheta)\exp(-(\frac{\pi}{2}-\varepsilon)|\eta|)\exp(-\frac{\pi}{4}|\tau|) \text{ in } B_{3.1}, B_{3.3}, B_{4.1}, B_{4.3}, \tag{B.24}$$

*under assumptions* (4.23).

*Proof.* In $B_{3.1}, B_{3.3}, B_{4.1}, B_{4.3}$, the inequality $|\tau| \leq 2|\eta+\tau|$ no longer works as in the previous lemmas, but we can still suggest that $|\tau|$, $\eta$, $|\tau+\eta|$ are large and use asymptotics (3.23),

$$|\Phi(s,\zeta)| \leq C(M,\sigma,\vartheta)\exp(-\pi|\eta|)\frac{|\eta|^{\sigma-1/2}}{|\omega(s+\zeta)|^{1+\vartheta}} \times \exp(-\tau\arg\omega(s)+(\tau+\eta)\arg\omega(s+\zeta)). \tag{B.25}$$

With the help of calculations as above, we infer

$$\begin{aligned} |(\tau+\eta)\arg\omega(s+\zeta)-\arg\omega(s)| &= |(\tau+\eta)(\arg\omega(s+\zeta)-\tau\arg\omega(s))+\eta\arg\omega(s)| \\ &\leq C(M,\sigma,\vartheta)|\tau+\eta|\left[\frac{|\tau|}{|\tau||\tau+\eta|}+\frac{|\eta|}{|\tau||\tau+\eta|}\right] \\ &\leq C(M,\sigma,\vartheta)\left(1+\frac{|\eta|}{|\tau|}\right). \end{aligned} \tag{B.26}$$

By definitions of considered sets, we have

$$\frac{|\eta|}{|\tau|} \leq \frac{3}{2}, \exp(-|\eta|) \leq C(M)\exp\left(-\frac{|\tau|}{2}\right). \tag{B.27}$$

Applying (B.25)–(B.27) gives

$$\begin{aligned} |\Phi(s,\zeta)| &\leq C(M,\sigma,\vartheta)|\eta|^{\sigma-1/2}\exp\left(-\frac{\pi}{2}|\eta|\right)\exp\left(-\frac{\pi}{4}|\tau|\right) \\ &\leq C(M,\sigma,\vartheta)|\exp\left(-(\frac{\pi}{2}-\varepsilon)|\eta|\right)\exp\left(-\frac{\pi}{4}|\tau|\right). \end{aligned} \tag{B.28}$$

This completes the proof of (B.24). □

**Lemma B.4.** *Let $\sigma$, $\vartheta$ satisfy* (4.23). *Then*

$$|\Phi(s,\zeta)| \leq C(M,\sigma,\vartheta)\exp(-(\frac{\pi}{2}-\varepsilon)|\tau|)\exp(-\frac{\pi}{2}|\eta|) \quad \text{in } B_{3.2}, B_{4.2}. \tag{B.29}$$

*Proof.* We observe that $|\tau+\eta| \leq M$ in $B_{3.2}$, $B_{4.2}$, or, in more details,

$$\begin{aligned} -M-|\tau| < -|\eta| < M-|\tau| \quad &\text{in } B_{3.2}, \\ |\tau|-M < |\eta| < M+|\tau| \quad &\text{in } B_{4.2}, \end{aligned} \tag{B.30}$$

and, consequently,

$$\exp(-|\eta|) \leq C(M)\exp(-|\tau|). \tag{B.31}$$

By (3.23), (3.25), it can be easily verified that

$$\begin{aligned} |\Phi(s,\zeta)| &\le C(M,\sigma,\vartheta)\exp(-\pi|\eta|)|\omega(s)|^{\sigma-1/2}\exp(-\tau\arg\omega(s)) \\ &\le C(M,\sigma,\vartheta)\exp\Big(-\frac{\pi}{2}|\eta|\Big)\exp\Big(-(\frac{\pi}{2}-\varepsilon)|\tau|\Big) \end{aligned} \tag{B.32}$$

in $B_{3.2}$, $B_{4.2}$. □

Summing up estimates (B.3), (B.4), (B.8), (B.24), and (B.29), we finish the proof of (4.22).

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## **Некласична гранична задача для рівняння Лапласа на площині з розрізом**

Mykola Krasnoshchok

Розглянемо рівняння Лапласа на площині з розрізом з некласичними граничними умовами. Така задача виникає як модель потоку у тріщинуватому середовищі. Основним результатом є теорема існування та єдиності розв'язку у вагових просторах Соболева.

*Ключові слова:* апріорні оцінки, тріщина, вагові простори Соболева