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Constant Mean Curvature Surfaces with Harmonic Gauss Map in Three-Dimensional Lie Groups

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We describe constant mean curvature surfaces with harmonic left-invariant Gauss map in three-dimensional unimodular Lie groups endowed with left-invariant metrics that are also right-invariant with respect to one-dimensional Lie subgroups, as well as in the hyperbolic space.

Key words: left-invariant metric, Gauss map, harmonic map, CMC surface

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1. Introduction

A well-known result [12] of E.A. Ruh and J. Vilms states that the Gauss map of a submanifold in the Euclidean space is harmonic if and only if its mean curvature vector field is parallel. The notion of the Gauss map can be naturally generalized to submanifolds in any Lie group G. For each point $p \in M$, the differential of the left translation $L_{p^{-1}}$ maps the tangent space T_pM to an n-dimensional subspace of the Lie algebra $\mathfrak{g}$ of G. Hence one obtains a map $p \mapsto dL_{p^{-1}}(T_pM)$ from M to the Grassmannian of n-dimensional subspaces in $\mathfrak{g}$. In particular, for hypersurfaces this map takes values in the projective space. Considering a left-invariant metric on the Lie group and the standard symmetric space structure on the Grassmannian, we can establish the harmonicity of this left-invariant Gauss map. In [4], it was proved that a Gauss map of a hypersurface M in a Lie group with a bi-invariant metric is harmonic if and only if M has constant mean curvature (CMC). However, for general left-invariant metrics, this equivalence does not hold. For example, it was shown in [11] that the class of CMC hypersurfaces in the $(2m+1)$-dimensional Heisenberg group with the harmonic Gauss map is rather narrow: such complete hypersurfaces are precisely cylinders over CMC hypersurfaces in the $2m$-dimensional Euclidean space. As for higher codimensions, generalizations of the Ruh–Vilms theorem do not take place even for bi-invariant metrics and totally geodesic submanifolds, as was shown in [10]. In this work, we will use a general harmonicity criterion for the Gauss map from that paper, but will restrict ourselves to the case of surfaces in some classes of three-dimensional Lie groups.

Motivated by the case of bi-invariant metrics, we consider a weaker condition on left-invariant metrics of unimodular groups: right-invariance with respect to a one-dimensional Lie subgroup. Note that replacing this subgroup with a two-dimensional one yields a bi-invariant metric. Using a classical description of three-dimensional unimodular Lie group structures due to J. Milnor [8], we prove Theorem 3.1, which resembles the results of [11] and generalizes them in the case of the three-dimensional Heisenberg group. We give a complete list of CMC surfaces with harmonic Gauss map in such groups and discuss several examples in detail, including surfaces in the universal coverings of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{E}(2)$ (the group of orientation-preserving Euclidean isometries). For the non-unimodular case, we prove a similar result (Theorem 4.1) for connected surfaces in three-dimensional hyperbolic space with its Lie group structure: such surfaces have CMC and harmonic Gauss map if and only if they are totally umbilic.

For general left-invariant metrics on three-dimensional Lie groups (unimodular or non-unimodular), the harmonicity conditions for the Gauss map become more complicated, and no general relationship with mean curvature can be expected.

2. Preliminaries

A harmonic map $\Phi\colon M\to N$ between two Riemannian manifolds (M,g) and (N,h) is usually defined as a stationary point of the *energy functional* $\Phi\mapsto E(\Phi)=\int\limits_M e(\Phi)\,dV_g$, where $e(\Phi)=\frac{1}{2}\operatorname{Tr}_g(\Phi^*h)$ is the *energy density function*, dV_g is the volume form of g, and M is compact (see [3]). The corresponding Euler–Lagrange equation has the form $\tau_\Phi=0$, where the *tension field* τ_Φ of Φ can be defined as the trace of the *second fundamental form* B_Φ of Φ with respect to g:

$$\tau_\Phi=\operatorname{Tr}_g B_\Phi=\sum_{1\leqslant i\leqslant n}(\nabla_h)_{d\Phi(E_i)}d\Phi(E_i)-d\Phi\left((\nabla_g)_{E_i}E_i\right).$$

Here $\{E_1,\dots,E_n\}$ is a (local) orthonormal frame on M, ∇_g and ∇_h are the Riemannian connections of g and h, respectively. We will call a map Φ *harmonic* if $\tau_\Phi=0$, and *harmonic at a point* $p\in M$ if $\tau_\Phi(p)=0$. This will allow us to extend the notion of harmonicity to maps from non-compact manifolds. In particular, constant maps are always harmonic.

Let M be an immersed smooth hypersurface in an $(n+1)$-dimensional Lie group G with the Lie algebra $\mathfrak{g}$ and a left-invariant Riemannian metric $\langle\cdot,\cdot\rangle$. Define the (left-invariant) *Gauss map* Φ of M as follows:

$$\Phi\colon M\to\mathbb{R}P^n,\quad p\mapsto dL_{p^{-1}}(N_pM). \tag{2.1}$$

Here, a point p is identified with its image under the immersion, N_pM is the normal line of M at p, and $dL_{p^{-1}}\colon T_pG\to T_eG=\mathfrak{g}$ is the differential of the left translation in G. So, Φ maps M to the space of one-dimensional vector subspaces of $(n+1)$-dimensional $\mathfrak{g}$, i.e., the projective space $\mathbb{R}P^n$. This construction is a natural generalization of the Gauss map of a hypersurface in the Euclidean space $\mathbb{E}^{n+1}$. For an orientable M, we can also consider the *oriented Gauss map* $p\mapsto dL_{p^{-1}}(\eta(p))$, where η is a unit normal field of M. As left translations are isometries, the target space of this map is the sphere S^n of unit vectors in $\mathfrak{g}$ (with respect to $\langle\cdot,\cdot\rangle$). In particular, the oriented Gauss map of M is harmonic if and only if the Gauss map (2.1) is harmonic. We consider the induced metric on M and the standard metrics on $\mathbb{R}P^n$ and S^n, respectively. In [10], we obtained necessary and sufficient conditions for the harmonicity of these maps. Let us introduce additional notation to state these conditions.

Let p be a point in M, $\{Y_1,\dots,Y_n\}$ be an orthonormal basis of the tangent space $T_pM \subset T_pG$, and Y_{n+1} be a unit normal vector at p. From now on, we use Y_i to denote both the vectors in the tangent space T_pG and their left-invariant extensions corresponding to elements of $\mathfrak{g}$. Denote by ∇, $\mathrm{R}(\cdot,\cdot)\cdot$, and $\mathrm{Ric}(\cdot,\cdot)$ the Riemannian connection of the metric $\langle\cdot,\cdot\rangle$, its curvature tensor, and its Ricci tensor, respectively. Let b_{ij}, where $1\leqslant i,j\leqslant n$, be the coefficients of the second fundamental form B of M at p with respect to the basis $\{Y_i\}$, and let H denote the mean curvature function of M. The following is a special case of Theorem 1 in [10]:

Proposition 2.1. *The Gauss map of a hypersurface M in a Lie group G with a left-invariant metric $\langle\cdot,\cdot\rangle$ is harmonic at a point $p\in M$ if and only if*

$$\begin{aligned} &Y_k(nH) + nH\left\langle \nabla_{Y_{n+1}}Y_{n+1}, Y_k\right\rangle - 2\sum_{1\leqslant i,j\leqslant n} b_{ij}\left\langle \nabla_{Y_i}Y_j, Y_k\right\rangle \\ &- \sum_{1\leqslant i\leqslant n}\left\langle \nabla_{\nabla_{Y_i}Y_i}Y_{n+1} + \nabla_{Y_i}\nabla_{Y_i}Y_{n+1}, Y_k\right\rangle - \mathrm{Ric}(Y_k,Y_{n+1}) = 0 \end{aligned} \tag{2.2}$$

for all $1\leqslant k\leqslant n$.

Recall that a Lie group G is called *unimodular* if its left-invariant Haar measure is also right-invariant. For connected Lie groups, this is equivalent to the condition that the Lie algebra of G is *unimodular*, i.e., the traces of all its adjoint representation operators vanish: $\operatorname{Tr}\operatorname{ad}_X = 0$ for each $X\in\mathfrak{g}$. A well-known application of this property was given in [8]: for any three-dimensional unimodular G with a left-invariant metric $\langle\cdot,\cdot\rangle$, there exists an orthonormal basis $\{E_1,E_2,E_3\}$ of $\mathfrak{g}$ with the Lie brackets defined by

$$[E_2,E_3] = \lambda_1 E_1, \quad [E_3,E_1] = \lambda_2 E_2, \quad [E_1,E_2] = \lambda_3 E_3 \tag{2.3}$$

for some real λ_1, λ_2, λ_3. Moreover, different signatures (i.e., sets of signs up to permutations and simultaneous changes of non-zero signs) of these three coefficients correspond bijectively to the isomorphism classes of three-dimensional unimodular Lie algebras, and hence to those of three-dimensional unimodular simply connected Lie groups. Therefore, there are six such classes. They are listed in [8]; see also the discussion after Theorem 3.1 in the next section. The following consequences of (2.3) are also taken from [8]. The Koszul formula implies that the Riemannian connection of $\langle\cdot,\cdot\rangle$ is given by

$$\begin{aligned} &\nabla_{E_1}E_2 = \mu_1 E_3, \quad \nabla_{E_2}E_1 = -\mu_2 E_3, \quad \nabla_{E_3}E_1 = \mu_3 E_2, \quad \nabla_{E_1}E_3 = -\mu_1 E_2, \\ &\nabla_{E_2}E_3 = \mu_2 E_1, \quad \nabla_{E_3}E_2 = -\mu_3 E_1, \quad \nabla_{E_1}E_1 = \nabla_{E_2}E_2 = \nabla_{E_3}E_3 = 0, \end{aligned} \tag{2.4}$$

where $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) - \lambda_i$ for $1 \leqslant i \leqslant 3$. The Ricci tensor has the form

$$\begin{aligned} \operatorname{Ric}(E_1, E_1) &= 2\mu_2\mu_3, \qquad & \operatorname{Ric}(E_2, E_2) &= 2\mu_3\mu_1, \\ \operatorname{Ric}(E_3, E_3) &= 2\mu_1\mu_2, \qquad & \operatorname{Ric}(E_i, E_j) &= 0,\ i \neq j. \end{aligned} \tag{2.5}$$

Note also that $\langle\cdot,\cdot\rangle$ is bi-invariant on a connected G if and only if it is an invariant form: $\langle [X, Y], Z\rangle = \langle X, [Y, Z]\rangle$ for all $X, Y, Z \in \mathfrak{g}$. This implies unimodularity (because adjoint representation operators $\operatorname{ad}_X = [X, \cdot]$ are then skew-adjoint with respect to $\langle\cdot,\cdot\rangle$) and is clearly equivalent to the equality $\lambda_1 = \lambda_2 = \lambda_3$ in (2.3). Now, let only two of these coefficients be equal:

Lemma 2.2. *A left-invariant metric on a connected three-dimensional unimodular Lie group G is right-invariant with respect to a connected one-dimensional Lie subgroup $K \subset G$ if and only if there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of its Lie algebra $\mathfrak{g}$ with the Lie brackets of the form (2.3) such that $\lambda_1 = \lambda_2$, where the Lie algebra $\mathfrak{k}$ of K is spanned by E_3.*

Proof. If $\lambda_1 = \lambda_2$ in (2.3) for a left-invariant metric $\langle\cdot,\cdot\rangle$ and some orthonormal basis $\{E_i\}$, then ad_{E_3} is skew-adjoint. Consequently $\langle\cdot,\cdot\rangle$ is right-invariant with respect to the connected one-dimensional subgroup K corresponding to the Lie subalgebra $\mathfrak{k}$ spanned by E_3.

On the other hand, let $\langle\cdot,\cdot\rangle$ be right-invariant with respect to a one-dimensional K and let $\{E_1, E_2, E_3\}$ be any orthonormal basis of $\mathfrak{g}$ such that $E_3 \in \mathfrak{k}$. Then ad_{E_3} is skew-adjoint, so its matrix with respect to $\{E_i\}$ is skew-symmetric. Together with $\operatorname{ad}_{E_3} E_3 = 0$, this gives the Lie brackets $[E_3, E_1] = \lambda\, E_2$ and $[E_3, E_2] = -\lambda\, E_1$ for some λ. Since G is unimodular, $\langle [E_1, E_2], E_2\rangle + \langle [E_1, E_3], E_3\rangle = \operatorname{Tr} \operatorname{ad}_{E_1} = 0$, thus $\langle [E_1, E_2], E_2\rangle = 0$ and, similarly, $\langle [E_1, E_2], E_1\rangle = 0$, so $[E_1, E_2] = \mu\, E_3$ for some μ. □

Note that for the necessary condition here to hold the connectedness of G or K is not required. This proof also implies that if $\langle\cdot,\cdot\rangle$ is right-invariant with respect to a two-dimensional subgroup K, then it is bi-invariant (for a connected group G). Indeed, if we choose $\{E_i\}$ such that $\{E_2, E_3\}$ forms a basis of $\mathfrak{k}$, then ad_{E_2} is also skew-adjoint, and consequently $\lambda = \mu$.

Therefore, for a left-invariant metric that is right-invariant with respect to a one-dimensional subgroup, but not bi-invariant, we have $\lambda_1 = \lambda_2 = \lambda \neq \mu = \lambda_3$ for some λ and μ in (2.3) and thus $\mu_1 = \mu_2 = \frac{\mu}{2}$, $\mu_3 = \lambda - \frac{\mu}{2}$ in (2.4) and (2.5). We call a left-invariant vector field *horizontal* if it is orthogonal to E_3. We can rewrite (2.3) as

$$[X, Y] = \mu\langle J(X), Y\rangle\, E_3, \quad [X, E_3] = -\lambda\, J(X) \tag{2.6}$$

for any horizontal fields X and Y, where $J(X)$ is the right-angle rotation of X in the orthogonal complement of E_3 with respect to the orientation defined by $\{E_i\}$. In particular, $J^2(X) = -X$. It then follows from (2.4) that, for horizontal X and Y,

$$\nabla_X Y = \frac{1}{2}[X, Y] = \frac{\mu}{2}\langle J(X), Y\rangle\, E_3, \qquad \nabla_X E_3 = -\frac{\mu}{2} J(X),$$

$$\nabla_{E_3} X = \left(\lambda - \frac{\mu}{2}\right) J(X), \qquad \nabla_{E_3} E_3 = 0, \tag{2.7}$$

in particular, $\nabla_X X = 0$.

It was also shown in [8] that if a three-dimensional Lee group G with a left-invariant metric $\langle\cdot,\cdot\rangle$ is non-unimodular, then there exists an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ whose Lie brackets are defined by

$$[E_1, E_2] = \alpha\, E_2 + \beta\, E_3, \quad [E_1, E_3] = \gamma\, E_2 + \delta\, E_3, \quad [E_2, E_3] = 0 \tag{2.8}$$

for some real α, β, γ, δ such that $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. Here $\{E_2, E_3\}$ spans the *unimodular kernel*, i.e., the kernel of $X \mapsto \operatorname{Tr} \operatorname{ad}_X$, which is a two-dimensional abelian ideal. Note that any such group is solvable. Similarly to Lemma 2.2, the metric $\langle\cdot,\cdot\rangle$ here is right-invariant with respect to a connected one-dimensional Lie subgroup $K \subset G$ if and only if there exists an orthonormal basis $\{E_i\}$ of $\mathfrak{g}$ such that $\gamma = \delta = 0$ in (2.8), where E_3 spans $\mathfrak{k}$. Indeed, $\operatorname{ad}_{E_3} = 0$ in this case is obviously skew-adjoint. On the other hand, if ad_{E_3} is skew-adjoint, then E_3 belongs to the unimodular kernel. Completing the orthonormal basis of this kernel with E_2, we have $[E_1, E_3] = 0$ from $[E_2, E_3] = 0$ and the skew-adjointness of ad_{E_3}. In particular, if $\beta = \gamma = \delta = 0$, then $\mathfrak{g}$ is the direct sum of ideals spanned by $\{E_1, E_2\}$ and E_3, thus simply connected G must be the direct product of the corresponding simply connected Lie groups with left invariant metrics, one of which is two-dimensional with the Gaussian curvature $-\alpha^2$. So, $\alpha = 1$, $\beta = \gamma = \delta = 0$ leads to the product $\mathbb{H}^2 \times \mathbb{R}$ of the hyperbolic plane and the Euclidean line.

We obtain another interesting example by setting $\alpha = \delta = 1$, $\beta = \gamma = 0$ in (2.8). Changing the notation, consider for such a group an orthonormal basis of $\mathfrak{g}$ with the Lie brackets

$$[E_1, E_2] = 0, \quad [E_3, E_1] = E_1, \quad [E_3, E_2] = E_2. \tag{2.9}$$

Again, we call a left-invariant vector field orthogonal to E_3 *horizontal*. In this notation, the brackets (2.9) become

$$[X, Y] = 0, \quad [E_3, X] = X$$

for a horizontal field X. From the Koszul formula, we then obtain

$$\nabla_X Y = \langle X, Y\rangle\, E_3, \quad \nabla_X E_3 = -X, \quad \nabla_{E_3} X = \nabla_{E_3} E_3 = 0, \tag{2.10}$$

where X and Y are arbitrary horizontal fields. It follows that G has constant sectional curvature -1. Therefore, if it is simply connected, then it should be isometric to the hyperbolic space $\mathbb{H}^3$. Indeed, the usual metric $\frac{1}{z^2}\left(dx^2 + dy^2 + dz^2\right)$ of the upper half-space model $\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ is left-invariant with respect to a Lie group product $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + z_1 x_2, y_1 + z_1 y_2, z_1 z_2)$, and $dL_{(x,y,z)} = z \operatorname{Id}$, so from an orthonormal basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ at the identity $(0, 0, 1)$ we get a left-invariant orthonormal frame

$$E_1 = z\, \frac{\partial}{\partial x}, \quad E_2 = z\, \frac{\partial}{\partial y}, \quad E_3 = z\, \frac{\partial}{\partial z} \tag{2.11}$$

with the Lie brackets (2.9).

3. Mean curvature and the Gauss map of surfaces in unimodular groups

Here, we establish the connection between constant mean curvature (CMC) and the harmonicity of the Gauss map for a surface in a three-dimensional unimodular Lie group endowed with a type of metric discussed in the previous section:

Theorem 3.1. *Let a left-invariant metric on a three-dimensional unimodular Lie group G be right-invariant with respect to some one-dimensional Lie subgroup $K \subset G$, but not bi-invariant, and let M be a connected surface in G. Then, any two of the following statements imply the third:*

1. *M has constant mean curvature;*
2. *the Gauss map of M is harmonic;*
3. *M is contained either in a union of leaves of the one-dimensional left-invariant foliation generated by K (is vertical) or in a leaf of the two-dimensional left-invariant foliation generated by a Lie subgroup transversal to K.*

As we will see, the case of a non-vertical M occurs if and only if $\lambda \neq 0$ and $\lambda\mu \leqslant 0$ in the notation introduced before (2.6). Then M is minimal and orthogonal to a left-invariant vector field of the form $\sqrt{\frac{-\mu}{\lambda-\mu}}\, X + \sqrt{\frac{\lambda}{\lambda-\mu}}\, E_3$, where X is an arbitrary unit horizontal left-invariant field. We will call a surface *left-invariant* if it is contained in a leaf of a two-dimensional left-invariant foliation (here this foliation is also minimal, see, e.g., [1]). Note that, by (2.1), the Gauss map of such a surface is constant and therefore harmonic. Hence, this theorem describes, in particular, all connected left-invariant CMC surfaces.

Proof. We use the notation from the previous section. According to Lemma 2.2, the structure of $\mathfrak{g}$ is defined by (2.6). To apply Proposition 2.1, for any point $p \in M$ we choose a left-invariant orthonormal frame $\{Y_1, Y_2, Y_3\}$ such that Y_3 is normal to M at p. Set $c = \cos\varphi$ and $d = \sin\varphi$, where $\varphi \in \left[0, \frac{\pi}{2}\right]$ is the angle between Y_3 and the plane spanned by $\{E_1, E_2\}$. Then, by multiplying Y_3 by -1 if necessary, we have

$$Y_1 = J(X), \quad Y_2 = -d\, X + c\, E_3, \quad Y_3 = c\, X + d\, E_3, \tag{3.1}$$

where X is a unit horizontal left-invariant field. Since c and d are constant, it follows from (2.7) that

$$\begin{aligned}
&\nabla_{Y_1} Y_1 = 0, \qquad \nabla_{Y_1} Y_2 = \frac{\mu}{2}\, Y_3, && \nabla_{Y_1} Y_3 = -\frac{\mu}{2}\, Y_2,\\
&\nabla_{Y_2} Y_1 = \left(\frac{\mu}{2} - \lambda\right) c\, X - \frac{\mu}{2} d\, E_3, && \nabla_{Y_2} Y_2 = (\mu - \lambda) cd\, Y_1,\\
&\nabla_{Y_2} Y_3 = \left(\frac{\mu}{2}(d^2 - c^2) + \lambda c^2\right) Y_1, && \nabla_{Y_3} Y_1 = \left(\frac{\mu}{2} - \lambda\right) d\, X + \frac{\mu}{2} c\, E_3,\\
&\nabla_{Y_3} Y_2 = \left(\frac{\mu}{2}(d^2 - c^2) - \lambda d^2\right) Y_1, && \nabla_{Y_3} Y_3 = (\lambda - \mu) cd\, Y_1,
\end{aligned} \tag{3.2}$$

thus the expressions in (2.2) become

$$\nabla_{\nabla_{Y_1}Y_1}Y_3 = 0, \qquad \nabla_{\nabla_{Y_2}Y_2}Y_3 = \frac{\mu}{2}(\lambda-\mu)cd\,Y_2,$$
$$\nabla_{Y_1}\nabla_{Y_1}Y_3 = -\frac{\mu^2}{4}Y_3, \quad \nabla_{Y_2}\nabla_{Y_2}Y_3 = \left(\frac{\mu}{2}(d^2-c^2)+\lambda c^2\right)\left(\left(\frac{\mu}{2}-\lambda\right)cX - \frac{\mu}{2}dE_3\right).$$

Finally, from (2.5) we have

$$\mathrm{Ric}(Y_1,Y_3)=0, \quad \mathrm{Ric}(Y_2,Y_3)=(\mu-\lambda)\mu cd,$$

and, after some simplifications, equations (2.2) take the form

$$\begin{aligned} Y_1(H) + (\lambda-\mu)cd\,(H+b_{22}) &= 0,\\ Y_2(H) - (\lambda-\mu)cd\left(\frac{1}{2}(\lambda-\mu)c^2 + b_{12}\right) &= 0. \end{aligned} \tag{3.3}$$

For a vertical surface, we have $d=0$ at any p, so the harmonicity conditions (3.3) become $Y_1(H)=Y_2(H)=0$ and therefore are satisfied everywhere if and only if a connected surface M has CMC. If M is left-invariant, then $\{Y_1,Y_2\}$ forms a global left-invariant orthonormal frame on its tangent bundle, and Y_3 is a global unit normal field for some constant c and d in (3.1). Since

$$[Y_1,Y_2] = \lambda c\,X + \mu d\,E_3 \tag{3.4}$$

from (2.6), the integrability condition $\langle [Y_1,Y_2],Y_3\rangle = 0$ is equivalent to $\lambda c^2 + \mu d^2 = 0$. For M transversal to K, we must have $d\neq 0$, and thus $\lambda \neq 0$. Together with $c^2+d^2=1$, this yields $c=\sqrt{\frac{-\mu}{\lambda-\mu}}$, $d=\sqrt{\frac{\lambda}{\lambda-\mu}}$, where $\lambda\mu \leqslant 0$. By (3.2), the coefficients of the second fundamental form of M are

$$b_{11} = \langle \nabla_{Y_1}Y_1, Y_3\rangle = 0, \quad b_{12} = \langle \nabla_{Y_1}Y_2, Y_3\rangle = \frac{\mu}{2}, \quad b_{22} = \langle \nabla_{Y_2}Y_2, Y_3\rangle = 0,$$

thus $H=\frac{1}{2}(b_{11}+b_{22})=0$, and M is minimal. As noted above, its Gauss map is harmonic because it is constant. This is also consistent with (3.3).

Now let M be a connected surface with CMC H and the harmonic Gauss map. From now on, we consider $c=\cos\varphi$, $d=\sin\varphi$ as functions on M. If $cd=0$ everywhere, then, taking into account $c^2+d^2=1$ and continuity, either $d=0$, hence M is vertical, or $c=0$, in which case M is contained in a leaf of the two-dimensional integral foliation of the left-invariant distribution generated by $\{E_1,E_2\}$ (which, of course, occurs only if the distribution is integrable, as shown above to be equivalent to $\mu=0$). Therefore, we can assume that $cd\neq 0$ at a point $q\in M$ and hence on some neighborhood U of q. Taking a smaller U if necessary, let $\{F_1,F_2\}$ be an orthonormal tangent frame and F_3 be a unit normal field of M on U such that $F_i(p)=Y_i$ for each $p\in U$, $1\leqslant i\leqslant 3$, where the vectors $\{Y_i\}$ are chosen at p according to (3.1). Then, by (3.1), $F_3 = -\frac{c}{d}\,F_2 + \frac{1}{d}\,E_3$. From the harmonicity of the Gauss map and (3.3), it follows that on U

$$b_{11}=3H, \quad b_{12}=\frac{1}{2}(\mu-\lambda)c^2, \quad b_{22}=-H, \tag{3.5}$$

where b_{ij} are now the coefficients of the second fundamental form B with respect to the moving frame $\{F_i\}$. On the other hand, they can be obtained from the Gauss decompositions

$$\begin{aligned} \nabla_{F_1} F_1 &= \kappa_1 F_2 + b_{11} F_3, & \nabla_{F_1} F_2 &= -\kappa_1 F_1 + b_{12} F_3, \\ \nabla_{F_2} F_1 &= -\kappa_2 F_2 + b_{12} F_3, & \nabla_{F_2} F_2 &= \kappa_2 F_1 + b_{22} F_3, \end{aligned} \tag{3.6}$$

where κ_1 and κ_2 denote the geodesic curvatures of the integral curves of F_1 and F_2, respectively. Namely, for any pair $1 \leqslant i, j \leqslant 2$, we have

$$\begin{aligned} b_{ij} &= \langle \nabla_{F_i} F_j, F_3 \rangle = - \langle F_j, \nabla_{F_i} F_3 \rangle = - \left\langle F_j, \nabla_{F_i} \left(-\frac{c}{d} F_2 + \frac{1}{d} E_3 \right) \right\rangle \\ &= F_i \left(\frac{c}{d} \right) \langle F_j, F_2 \rangle + \frac{c}{d} \langle F_j, \nabla_{F_i} F_2 \rangle - F_i \left(\frac{1}{d} \right) \langle F_j, E_3 \rangle - \frac{1}{d} \langle F_j, \nabla_{F_i} E_3 \rangle . \end{aligned}$$

Substituting (3.5) and (3.6) into the above, taking into account the construction of $\{F_i\}$, and using (2.7) to calculate $\nabla_{F_i} E_3$, we derive

$$\begin{aligned} \kappa_1 &= -\frac{3d}{c} H, & \kappa_2 &= \frac{1}{2} (\mu - \lambda) cd + \frac{\mu d}{2c}, \\ F_1(c) &= \frac{1}{2} (\mu - \lambda) c^2 d - \frac{\mu d}{2}, & F_2(c) &= -dH. \end{aligned} \tag{3.7}$$

Consider one of the Codazzi equations for M:

$$\langle \mathrm{R}(F_1, F_2) F_1, F_3 \rangle = \nabla_{F_1} B(F_2, F_1) - \nabla_{F_2} B(F_1, F_1), \tag{3.8}$$

where

$$\begin{aligned} \nabla_{F_1} B(F_2, F_1) &= F_1(B(F_2, F_1)) - B\left(\nabla^T_{F_1} F_2, F_1\right) - B\left(F_2, \nabla^T_{F_1} F_1\right) \\ &= (\mu - \lambda) c F_1(c) + 4H\kappa_1 \end{aligned}$$

by (3.5) and (3.6). Here $\nabla^T_{F_i} F_j$ denotes the tangent component of $\nabla_{F_i} F_j$ (the induced connection). Similarly,

$$\nabla_{F_2} B(F_1, F_1) = F_2(B(F_1, F_1)) - 2B\left(\nabla^T_{F_2} F_1, F_1\right) = (\mu - \lambda) c^2 \kappa_2 .$$

From this and (3.7), the right-hand side of (3.8) becomes $-\frac{12d}{c} H^2 + \mu(\lambda - \mu) cd$. For the left-hand side, at each point of U we have, by (3.2) and (3.4),

$$\begin{aligned} \langle \mathrm{R}(F_1, F_2) F_1, F_3 \rangle &= \langle \mathrm{R}(Y_1, Y_2) Y_1, Y_3 \rangle \\ &= \langle \nabla_{Y_1} \nabla_{Y_2} Y_1 - \nabla_{Y_2} \nabla_{Y_1} Y_1 - \nabla_{[Y_1, Y_2]} Y_1, Y_3 \rangle \\ &= \left\langle -\frac{\mu^2}{4} d\, X + \frac{\mu}{2} \left(\lambda - \frac{\mu}{2} \right) c\, E_3 + \mu \left(\lambda - \frac{\mu}{2} \right) d\, X \right. \\ &\qquad \left. - \frac{\lambda\mu}{2} c\, E_3, c\, X + d\, E_3 \right\rangle = \mu(\lambda - \mu) cd. \end{aligned}$$

Therefore, (3.8) yields $H = 0$, and hence $b_{11} = b_{22} = \kappa_1 = 0$ by (3.5) and (3.7). In particular, $\nabla_{F_1} F_1 = 0$ from (3.6), i.e., the integral curves of F_1 are geodesics in G. Take any $p \in U$ and choose Y_1 as above. Since $F_1(p) = Y_1$ and the integral curves of a left-invariant field Y_1 are geodesics by (3.2), the uniqueness of geodesics implies $F_1 = Y_1$ everywhere on U; in other words, F_1 is left-invariant. Then, from (2.7) and (3.5), we have

$$\frac{1}{2}(\mu - \lambda)c^2 = b_{12} = \langle \nabla_{F_2} F_1, F_3 \rangle = \langle \nabla_{F_2} Y_1, F_3 \rangle = \left(\frac{\mu}{2} - \lambda\right) c^2 - \frac{\mu}{2} d^2,$$

that is, $\lambda c^2 + \mu d^2 = 0$. It follows from this and $c^2 + d^2 = 1$ that $c = \sqrt{\frac{-\mu}{\lambda - \mu}} > 0$ and $d = \sqrt{\frac{\lambda}{\lambda - \mu}} > 0$ are constant on U (if $\lambda\mu < 0$; otherwise, a contradiction arises at this point). Hence, F_2 and F_3 are also left-invariant.

So, the continuous function c takes constant value $0 < \sqrt{\frac{-\mu}{\lambda-\mu}} < 1$ on U and on a neighborhood of any other point, where $cd \neq 0$, and, apart from that, can take only values 0 and 1. Since the surface M is connected, it follows that $c = \sqrt{\frac{-\mu}{\lambda-\mu}}$ and thus $d = \sqrt{\frac{\lambda}{\lambda-\mu}}$ everywhere on M; therefore, the normal distribution of M is left-invariant. Hence, this surface is contained in a leaf of the two-dimensional left-invariant foliation generated by a subgroup transversal to K. $\square$

Note that the derivation of (3.3) in the proof remains valid for $\lambda = \mu$, i.e., for a bi-invariant metric. In that case, the conditions reduce to $Y_1(H) = Y_2(H) = 0$, so a surface has CMC if and only if its Gauss map is harmonic. Therefore, we get the result of [4] for a three-dimensional group. Of all isomorphism classes of simply connected three-dimensional unimodular groups listed in [8], such metrics exist on $\mathbb{R}^3$ and $\mathrm{SU}(2) \cong \mathbb{S}^3$ (with the signatures $(0, 0, 0)$ and $(+, +, +)$ of $\{\lambda_i\}$, respectively, see the remark after (2.3)), and they are the standard metrics of constant curvatures $\frac{\lambda^2}{4}$. Note that SU(2) also allows metrics of the type considered in Theorem 3.1, but, since $\lambda\mu > 0$, CMC surfaces with harmonic Gauss map here can be only vertical.

If $\lambda = 0$, $\mu \neq 0$, then the signature is $(0, 0, +)$, and, if G is simply connected, it is isomorphic to the three-dimensional Heisenberg group Nil. According to Lemma 2.2, all left-invariant metrics on such a group are right-invariant with respect to its center K, but not bi-invariant. In fact, it follows from (2.3) that such a metric is unique up to scaling. In this case, Theorem 3.1, which essentially coincides with Theorem 8 from [9], again yields only vertical surfaces. This result was generalized in [11], where an explicit description of CMC hypersurfaces with harmonic Gauss map in the $(2m + 1)$-dimensional Heisenberg group was given. In particular, Nil, being three-dimensional, nilpotent, and simply connected, can be described as $\mathbb{R}^3$ with global coordinates (x, y, z), and an orthonormal left-invariant frame from (2.3) can be given as

$$E_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad E_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad E_3 = \frac{1}{\mu}\frac{\partial}{\partial z}.$$

In this description, connected vertical CMC surfaces appear to be subsets of vertical Euclidean planes and round cylinders. Their Gauss maps are then harmonic by Theorem 3.1. See also [13], where a similar result was obtained for another notion of the Gauss map.

The case $\lambda \neq 0$, $\mu = 0$ corresponds to the signature $(+,+,0)$ and to the simply connected group $\widetilde{\mathrm{E}(2)}$, i.e., the universal covering of the group $\mathrm{E}(2)$ of orientation-preserving Euclidean plane isometries. This group is solvable, so its universal cover $\widetilde{\mathrm{E}(2)}$ can also be identified with $\mathbb{R}^3$. In general, a left-invariant metric on $\widetilde{\mathrm{E}(2)}$ is right-invariant with respect to a one-dimensional Lie subgroup K (i.e., $\lambda_1 = \lambda_2 = \lambda \neq 0 = \lambda_3$ in (2.3) by Lemma 2.2, where E_3 spans the Lie algebra of K) if and only if it is Euclidean, and again such a metric is unique up to scaling. Leaves of the left-invariant foliation generated by K are the integral curves of E_3 and hence are geodesics by (2.7); in particular, they are straight lines in the Euclidean case. The direction of these lines, which we call vertical, corresponds to the rotation angle of a plane isometry. Since $\mu = 0$, in Theorem 3.1, the left-invariant minimal surfaces transversal to vertical lines are orthogonal to them and therefore are contained in horizontal Euclidean planes. Theorem 3.1 thus implies the following:

Corollary 3.2. *A connected surface in $\widetilde{\mathrm{E}(2)}$ with the Euclidean metric has CMC and the harmonic Gauss map if and only if it is a subset of a vertical Euclidean plane, a vertical Euclidean round cylinder, or a horizontal Euclidean plane.*

This corollary illustrates an important difference: the left-invariant Gauss map (2.1) (with respect to the Lie group structure of $\widetilde{\mathrm{E}(2)}$) behaves differently from the usual Euclidean Gauss map (with respect to the abelian Lie group structure of $\mathbb{R}^3$) for surfaces in $\mathbb{E}^3$. Indeed, recall that the latter is harmonic if and only if M has CMC [12].

Metrics with non-zero λ and μ of opposite signs, and hence with the signature $(+,+,-)$, exist on groups isomorphic (in the simply connected case) to the universal covering $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\mathrm{SL}(2,\mathbb{R})$. In particular, $\lambda = 1$ and $\mu = -1$ yield the standard metric on this group (a Thurston geometry). In this case, it can also be viewed as the universal covering of the unit tangent bundle $T_1\mathbb{H}^2$ of the hyperbolic plane, endowed with the Sasaki metric (see [14]). This group can be identified with the half-space $\{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$ using the upper half-plane model of $\mathbb{H}^2$, with metric $\frac{1}{y^2}\left(dx^2 + dy^2 + (dx + y\,dz)^2\right)$. The covering map $\widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{SL}(2,\mathbb{R})$ in these coordinates is defined by (cf. [6])

$$(x,y,z) \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\frac{z}{2} & \sin\frac{z}{2} \\ -\sin\frac{z}{2} & \cos\frac{z}{2} \end{pmatrix},$$

and the group structure is induced by matrix multiplication in $\mathrm{SL}(2,\mathbb{R})$. From this, it can be derived that the orthonormal left-invariant frame (2.3) has the form

$$E_1 = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z},$$

$$E_2 = -y\sin z\frac{\partial}{\partial x} + y\cos z\frac{\partial}{\partial y} + \sin z\frac{\partial}{\partial z}, \qquad E_3 = \frac{\partial}{\partial z}, \tag{3.9}$$

so the integral trajectories of E_3 are again vertical straight lines. It follows from Proposition 4.1 in [6] (see also Example 3.1 in [15]) that the mean curvature of a vertical surface parameterized by $(s,t) \mapsto (x(s), y(s), t)$ is equal, up to multiplication by a constant, to the geodesic curvature of the curve $s \mapsto (x(s), y(s))$ with respect to the standard hyperbolic metric $\frac{1}{y^2}\left(dx^2 + dy^2\right)$ of the half-plane model. Then, from Theorem 3.1 and its proof, we obtain the following:

Corollary 3.3. *A connected surface in* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *with the standard metric has CMC and the harmonic Gauss map if and only if it is a subset of a vertical Euclidean plane, a vertical Euclidean round cylinder, or a left-invariant surface orthogonal to a vector field of the form*

$$Y_3 = \frac{\cos\psi}{\sqrt{2}}\,E_1 + \frac{\sin\psi}{\sqrt{2}}\,E_2 + \frac{1}{\sqrt{2}}\,E_3 \tag{3.10}$$

for some constant angle ψ.

Recall that left-invariant surfaces in this setting must be minimal. In particular, for $\psi = \pm\frac{\pi}{2}$, such complete connected surfaces were described in [5]. It follows from Theorem 3 of that paper that they are Euclidean half-planes $z = \pm\frac{\pi}{2} + 2\pi k$, where $k \in \mathbb{Z}$, and "helicoidal" surfaces with parameterizations

$$(s,t) \mapsto (x_0 \mp s - s\sin t, s\cos t, t), \tag{3.11}$$

where $s \in (0, +\infty)$ and $t \in \left(-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k\right)$, $k \in \mathbb{Z}$. For a general angle ψ, a left-invariant surface orthogonal to Y_3 in (3.10) can be obtained by applying to one of these surfaces an isometry that fixes the group identity $(0,1,0)$ and whose differential at that point acts on $\mathfrak{g}$ as a rotation around the axis spanned by E_3. Note that the use of the term "vertical" in [5] differs from our convention: there it refers to surfaces tangent to the field E_1, since its orthogonal distribution is horizontal for a sub-Riemannian structure considered in that paper.

According to Lemma 2.2, a general left-invariant metric on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, which is right-invariant with respect to a one-dimensional Lie subgroup, after a possible scaling corresponds to $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \mu < 0$ in (2.3). We can use the same coordinate description of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ as before, now multiplying E_1 and E_2 by $\sqrt{-\mu}$ in the orthonormal left-invariant frame (3.9), so this general metric takes the form $\frac{1}{-\mu\, y^2}\left(dx^2 + dy^2 - \mu(dx + y\,dz)^2\right)$. Corollary 3.3 remains valid for this general metric, but the normal field of a left-invariant surface now takes the form

$$Y_3 = \frac{\sqrt{-\mu}\cos\psi}{\sqrt{1-\mu}}\,E_1 + \frac{\sqrt{-\mu}\sin\psi}{\sqrt{1-\mu}}\,E_2 + \frac{1}{\sqrt{1-\mu}}\,E_3,$$

instead of a particular case (3.10). The description of left-invariant surfaces from the previous paragraph, including half-planes $z = \pm\frac{\pi}{2} + 2\pi k$, "helicoids" (3.11), and their isometric images, carries over without change.

Remark 3.4. The case where the coefficients $\{\lambda_i\}$ in (2.3) are pairwise distinct is considerably more difficult. One can still derive conditions analogous to (3.3) from Proposition 2.1, but they become much more complicated. Such metrics exist on all isomorphism classes of groups listed above except $\mathbb{R}^3$ and Nil. Moreover, for the simply connected solvable group Sol with signature $(+,-,0)$, all left-invariant metrics are of this type. For example, $\lambda_1 = -\lambda_2 = 1$ and $\lambda_3 = 0$ yield the standard metric on Sol, which is a Thurston geometry. Together with $\mathbb{E}^3$, $\mathbb{S}^3$, Nil, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and the non-unimodular Lie groups $\mathbb{H}^2\times\mathbb{R}$ and $\mathbb{H}^3$ described in the previous section, these exhaust all three-dimensional Thurston geometries defined by left-invariant metrics (see [14]).

4. Mean curvature and the Gauss map of surfaces in the hyperbolic space

Consider a surface in the three-dimensional hyperbolic space $\mathbb{H}^3$ and its left-invariant Gauss map (2.1) with respect to the Lie group structure described at the end of Section 2. It turns out that a result similar to Theorem 3.1 can be proved in this setting:

Theorem 4.1. *A connected surface in $\mathbb{H}^3$ has constant mean curvature and the harmonic Gauss map if and only if it is totally umbilic.*

Proof. We proceed as in the proof of Theorem 3.1 and use the same notation. For a point $p \in M$, choose $\{Y_i\}$ as in (3.1):

$$Y_1 = J(X), \quad Y_2 = -d\,X + c\,E_3, \quad Y_3 = c\,X + d\,E_3.$$

Here $c = \cos\varphi$, $d = \sin\varphi$, where φ is the angle between Y_3 and the plane spanned by $\{E_1, E_2\}$, X and $J(X)$ are a pair of orthogonal unit horizontal left-invariant fields. Together with (2.10), this implies

$$\begin{aligned}
&\nabla_{Y_1}Y_1 = E_3, && \nabla_{Y_1}Y_2 = -c\,Y_1, && \nabla_{Y_1}Y_3 = -d\,Y_1,\\
&\nabla_{Y_2}Y_1 = 0, && \nabla_{Y_2}Y_2 = d\,Y_3, && \nabla_{Y_2}Y_3 = -d\,Y_2,\\
&\nabla_{Y_3}Y_1 = 0, && \nabla_{Y_3}Y_2 = -c\,Y_3, && \nabla_{Y_3}Y_3 = c\,Y_2,
\end{aligned} \tag{4.1}$$

and hence, in (2.2), we now have

$$\nabla_{\nabla_{Y_1}Y_1}Y_3 = 0,\ \nabla_{\nabla_{Y_2}Y_2}Y_3 = cd\,Y_2,\ \nabla_{Y_1}\nabla_{Y_1}Y_3 = -d\,E_3,\ \nabla_{Y_2}\nabla_{Y_2}Y_3 = -d^2\,Y_3.$$

The Ricci tensor of $\mathbb{H}^3$ is proportional to the metric, so $\mathrm{Ric}(Y_1, Y_3) = \mathrm{Ric}(Y_2, Y_3) = 0$, and the conditions (2.2) reduce to

$$Y_1(H) + c\,b_{12} = 0, \quad Y_2(H) + c\,(H - b_{11}) = 0. \tag{4.2}$$

Any connected totally umbilic surface in a three-dimensional constant curvature manifold has CMC H, so the coefficients of its second fundamental form with respect to any orthonormal frame are $b_{11} = b_{22} = H$, $b_{12} = 0$. Hence, the Gauss map is harmonic by (4.2). On the other hand, if these conditions are satisfied everywhere and H is constant, then either $c = \cos\varphi = 0$ on M (in which case M is left-invariant and orthogonal to E_3), or $c \neq 0$ at some point q, and hence on a neighborhood U of q, with $b_{11} = H = b_{22}$, $b_{12} = 0$ there; thus, M is umbilic on U. In the former case, it follows from (2.11) that, in the half-space model, M is contained in a plane $z = z_0$, i.e. in a horosphere parallel to the sphere at infinity. From (4.1), we then have $b_{11} = b_{22} = d = 1$ and $b_{12} = 0$ (see also the description of totally umbilic surfaces below). In the latter case, since the umbilicity condition $K_{ext} = H^2$, where K_{ext} is an extrinsic Gaussian curvature of M, is true on U, and connected M is real analytic as a CMC surface in $\mathbb{H}^3$, it follows that $K_{ext} = H^2$ everywhere on M. Therefore, M is totally umbilic in either case. □

It is well known (see, e.g., [2]) that, since the half-space model of $\mathbb{H}^3$ is locally conformally equivalent to the Euclidean space $\mathbb{E}^3$, connected totally umbilic surfaces in $\mathbb{H}^3$ have the same appearance as in $\mathbb{E}^3$. They are subsets of Euclidean planes and spheres, corresponding to hyperbolic planes (totally geodesic surfaces), their equidistant surfaces, spheres, and horospheres. In particular, such surfaces have CMC in $\mathbb{E}^3$. By this conformal equivalence, together with (2.1) and the discussion preceding (2.11), the Gauss map considered in Theorem 4.1 is precisely the Euclidean Gauss map of a surface M. Thus, by [12], it is harmonic if and only if M has CMC in $\mathbb{E}^3$. Accordingly, Theorem 4.1 can be reformulated as follows:

Corollary 4.2. *A connected surface in the upper half-space model of $\mathbb{H}^3$ has CMC in both $\mathbb{H}^3$ and $\mathbb{E}^3$ if and only if it is totally umbilic (both in $\mathbb{H}^3$ and $\mathbb{E}^3$).*

There are other natural notions of the Gauss map of a surface in $\mathbb{H}^3$. A survey of these definitions and their connections with the mean curvature of a surface can be found, e.g., in [7].

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Поверхні постійної середньої кривини з гармонічним гаусовим відображенням у тривимірних групах Лі

Eugene Petrov and Iryna Savchuk

Ми даємо опис поверхонь постійної середньої кривини з гармонічним лівоінваріантним гаусовим відображенням у тривимірних унімодулярних групах Лі з лівоінваріантними метриками, що також є правоінваріантними відносно одновимірних підгруп Лі, а також у гіперболічному просторі.

Ключові слова: лівоінваріантна метрика, гаусове відображення, гармонічне відображення, поверхня постійної середньої кривини