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Morse-Theoretic Multiplicity Results for Choquard Equation in Orlicz–Sobolev Spaces Beyond Ambrosetti–Rabinowitz Condition

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We study a nonhomogeneous superlinear Choquard equation driven by the Δ_Φ-operator in the setting of Orlicz–Sobolev spaces. Unlike most existing studies, our analysis does not rely on the Ambrosetti–Rabinowitz condition. By showing that the associated energy functional satisfies the Cerami compactness condition, we employ variational techniques together with Morse theory to compute the critical groups. As a consequence, we prove the existence of at least three distinct nontrivial solutions. These findings extend classical multiplicity results for local elliptic problems to the nonlocal Choquard framework.

Key words: Choquard equation, Orlicz–Sobolev space, Δ_2-condition, Morse theory

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1. Introduction

The Choquard equation, also known as the Schrödinger–Newton equation, plays a central role in mathematical physics. It arises in several physical models, including quantum polarons, Bose–Einstein condensation, and self-gravitating systems. From a mathematical point of view, it constitutes a nonlinear and nonlocal elliptic problem, where convolution-type nonlinearities interact with local differential operators.

A classical starting point in this direction is the p–Laplacian Choquard problem, obtained by considering

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad p > 1,$$

that is, $\phi(t) = p|t|^{p-2}t$ in our framework. In this case, the equation becomes

$$-\Delta_p u + |u|^{p-2}u = \left(\frac{1}{|x|^\mu} * F(u)\right) f(u), \quad u \in W_0^{1,p}(\Omega).$$

This model has been extensively investigated. Existence of ground states was first proved in the whole space by Lieb [14] for $p = 2$, and later extended by Lions [16]. Multiplicity results for p-Laplacian Choquard equations were obtained via variational methods such as the Mountain Pass Theorem, the Nehari manifold, and genus theory (see Ackermann [1], Cingolani–Clapp–Secchi [7], Gao–Yang [12]). Regularity and qualitative properties were studied by Moroz and Van Schaftingen [20, 21], who established existence, symmetry, and decay estimates for solutions in $\mathbb{R}^N$.

In most of these works, the Ambrosetti–Rabinowitz (AR) condition is imposed to ensure the boundedness of Palais–Smale sequences and to overcome compactness difficulties. However, the (AR) condition is restrictive and excludes many nonlinearities of physical relevance. To relax this assumption, several authors (see Figueiredo–Miyagaki–Ruf [8], Biswas–Tiwari [4]) developed new compactness tools and obtained results without relying on (AR).

On the other hand, numerous physical models involve operators with nonstandard growth, going beyond the p-Laplacian. Examples include the (p,q)-Laplacian, as well as operators arising in nonlinear elasticity, plasticity, and minimal surface theory. All these operators can be unified within the general framework of the Φ-Laplacian, whose natural functional setting is provided by Orlicz–Sobolev spaces. These spaces extend the classical Sobolev spaces and allow one to treat very general growth conditions (see Donaldson–Trudinger [10], Fukagai–Ito–Narukawa [11], Mihăilescu–Repovš [19]).

Motivated by these developments, in the present paper we study the following general nonlocal Choquard equation driven by the Φ-Laplacian in a bounded smooth domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\Delta_\Phi u + \phi(u)u = \left(\dfrac{1}{|x|^\mu} * F(u)\right) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < \mu < N$, the function ϕ satisfies suitable structural assumptions ensuring monotonicity and controlled growth between two indices l, m, and f is a Carathéodory nonlinearity with superlinear growth. This setting encompasses, as particular cases, the p-Laplacian, the (p,q)-Laplacian, and more general operators occurring in elasticity and plasticity.

The main objective of this work is to establish the existence of multiple nontrivial weak solutions for this general problem without assuming the Ambrosetti–Rabinowitz condition.

This introduces two major analytical challenges:

- The absence of (AR) makes it harder to prove the compactness of Cerami sequences, a key step in variational methods.
- The lack of (AR) also creates difficulties in the computation of critical groups at infinity, which are essential in Morse-theoretic arguments.

To overcome these obstacles, we rely on the Hardy–Littlewood–Sobolev inequality and a refined analysis of the associated energy functional.

Our contribution can be summarized as follows:

- We prove that the functional satisfies the Cerami compactness condition despite the lack of (AR).
- Using variational methods, we establish the existence of at least two weak solutions with constant sign.
- By applying Morse theory and computing the critical groups, we further prove the existence of a third nontrivial weak solution.

Thus, our results extend the known theory for Choquard equations beyond the p-Laplacian, covering a much broader class of nonlocal problems in the Orlicz–Sobolev framework, while simultaneously relaxing the usual growth restrictions such as the Ambrosetti–Rabinowitz condition.

H(f): *Hypotheses on the nonlinearity f.* We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(x, 0) = 0$ for a.a. $x \in \Omega$, and that the following conditions hold:

(i) **(Subcritical growth).** There exist constants $C > 0$ and $p, q > \frac{m}{2}$ such that

$$|f(x,t)| \leq C\left(|t|^{p-1} + |t|^{q-1}\right) \quad \text{for a.a. } x \in \Omega,\ t \in \mathbb{R}.$$

(ii) **(Superlinear growth at infinity).** There exists $\nu > m$ such that

$$\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{\nu/2}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

(iii) **(Refined superlinearity).** There exist $\tau > 0$ with $m < 2p,\ 2q < \tau < l^*$ and $\beta_0 > 0$ such that

$$\liminf_{|(t,z)|\to\infty} \frac{F(x,t)f(y,z)z - \frac{m}{2}F(x,t)F(y,z)}{|(t,z)|^\tau} \geq \beta_0$$

uniformly for a.a. $(x,y) \in \Omega \times \Omega$.

Remark 1.1.

- *(Superlinearity):* Hypothesis **H(f)**(ii) assumes that the primitive $F(z,\cdot)$ grows faster than $|t|^{\nu/2}$ as $|t| \to +\infty$, thereby ensuring a $\nu/2$-superlinear behavior of the nonlinearity.
- *(Ambrosetti–Rabinowitz (AR) condition):* A classical growth assumption requires that there exist constants $\mu > 0$ and $M > 0$ such that

$$0 < \mu F(x,z) \leq f(x,z)z \quad \text{for a.a. } x \in \Omega,\ |z| \geq M. \tag{1.2}$$

From this we obtain a weaker inequality

$$c_7|z|^\mu \leq F(x,z) \quad \text{for a.a. } x \in \Omega,\ |z| \geq M, \tag{1.3}$$

with some constant $c_7 > 0$. This, in turn, implies the weaker growth condition

$$\lim_{|z|\to\infty} \frac{F(x,z)}{|z|^{\nu/2}} = +\infty \quad \text{uniformly for a.a. } x \in \Omega.$$

Hence, condition (1.3) is weaker than the Ambrosetti–Rabinowitz condition (1.2). Moreover, if the (AR) condition holds, the hypothesis **H(f)**(iii) is also satisfied.

Indeed, choosing $\mu > \tau$, and using the inequality $|(z,t)| > |z||t|$, for $|z| > 1$ and $|t| > 1$, we obtain

$$\begin{aligned}
&\frac{F(x,z)f(y,t)t - \frac{m}{2}F(x,z)F(y,t)}{|(z,t)|^\tau} \\
&\qquad = \frac{F(x,z)f(y,t)t - \mu F(x,z)F(y,t)}{|(z,t)|^\tau} + \frac{(\mu - \frac{m}{2})F(x,z)F(y,t)}{|(z,t)|^\tau} \\
&\qquad \geq c_7 \frac{(\mu - \frac{m}{2})|(z|^\tau |t|^\tau}{|(z,t)|^\tau} \geq (\mu - \frac{m}{2})c_7 c_7^{'},
\end{aligned}$$

which shows that the hypothesis is fulfilled.

H(ϕ): *Hypotheses on the function ϕ.*

(i) **(Monotonicity).** For all $t > 0$, $\phi(t) > 0$ and $(t\phi(t))' > 0$.

(ii) **(Two-sided growth control).** There exist constants $1 < l \leq m < N$ with $m < l^* := \frac{lN}{N-l}$ such that

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} < m \quad \text{for all } t > 0,$$

where $\Phi(t) := \int_0^{|t|} s\phi(s)\, ds$.

In the present paper, the Hardy–Littlewood–Sobolev type inequality [15], which will be stated below, constitutes a key ingredient of our analysis and provides the analytical foundation of our approach.

Proposition 1.2 (Hardy–Littlewood–Sobolev inequality)**.** *Let $t, r > 1$, $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $g \in L^t(\mathbb{R}^N)$, and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,N,r,\mu)$, independent of g,h, such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)h(x)}{|x-y|^\mu}\, dx\, dy \right| \leq C(t,N,r,\mu) \|g\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)}.$$

The paper is organized as follows. In Section 2, we recall some basic notions on Orlicz–Sobolev spaces and introduce the concept of critical groups, together with preliminary hypotheses. In Section 3, we develop the variational framework of the problem and apply the Mountain Pass Theorem to establish the existence of two weak solutions with constant sign (Theorem 1.3). Finally, in Section 4, we use Morse theory to compute the critical groups and to prove our main result, namely the existence of three distinct weak solutions (Theorem 1.4).

Our main result in this paper is the following.

Theorem 1.3. *Assume that conditions* **H(f)** *and* **H(ϕ)** *are satisfied. Then problem* (1.1) *has at least two weak constant-sign solutions.*

Theorem 1.4. *Assume that conditions* **H(f)** *and* **H(ϕ)** *are satisfied. Then problem* (1.1) *has at least three nontrivial solutions.*

2. Mathematical background and hypotheses

2.1. Basics on Orlicz–Sobolev spaces. A continuous function $\Phi : \mathbb{R} \to [0, +\infty[$ is called an N-function if it satisfies the following conditions:

(i) Φ is convex,

(ii) $\Phi(t) = 0 \Leftrightarrow t = 0$,

(iii) $\lim\limits_{t\to 0} \dfrac{\Phi(t)}{t} = 0$ and $\lim\limits_{t\to +\infty} \dfrac{\Phi(t)}{t} = +\infty$,

(iv) Φ is even.

Let Ω be an open subset of $\mathbb{R}^N$ and ϕ be a real-valued function defined in $[0, +\infty)$ with the following properties:

- ϕ is nondecreasing,
- $\phi(0) = 0$, $\phi(t) > 0$ if $t > 0$,
- $\lim\limits_{t\to +\infty} \phi(t) = \infty$, ϕ is right continuous, i.e, $\lim\limits_{s\to t^+} \phi(s) = \phi(t)$.

The function Φ defined by

$$\Phi(t) = \int_0^{|t|} \phi(s)s\, ds$$

is called an N-function.

Δ_2-condition and Orlicz spaces. An N-function Φ satisfies the Δ_2-condition if there exist $t_0 \geq 0$ and $k > 0$ such that

$$\Phi(2t) \leq k\Phi(t), \quad t \geq t_0. \tag{2.1}$$

We suppose that the function

$$t \to \Phi(\sqrt{t}) \quad \text{is convex in } (0, +\infty). \tag{2.2}$$

We define the Orlicz space associated with Φ as

$$L^\Phi(\Omega) = \left\{ u \in L^1_{\text{loc}}(\Omega) : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx < +\infty \quad \text{for some } \lambda > 0 \right\}.$$

This space is a Banach space equipped with the Luxemburg norm

$$\|u\|_\Phi = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

The complementary function $\tilde{\Phi}$ associated with Φ is defined by its Legender transformation

$$\tilde{\Phi}(s) = \max_{t\geq 0}\{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

Young's inequality relates Φ and $\tilde{\Phi}$ as follows:

$$st \leq \Phi(t) + \tilde{\Phi}(s) \quad \text{for } s, t \geq 0.$$

Using this, we derive a Hölder-type inequality:

$$\left| \int_\Omega uv\, dx \right| \leq 2\|u\|_\Phi \|v\|_{\tilde{\Phi}} \quad \text{for all } u \in L^\Phi(\Omega) \text{ and } v \in L^{\tilde{\Phi}}(\Omega).$$

Orlicz–Sobolev spaces. For an N-function Φ, the Orlizc–Sobolev space $W^{1,\Phi}(\Omega)$ is defined as

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega),\ i = 1, \dots, N \right\}.$$

This space is equipped with the norm

$$\|u\| = \|u\|_{\Phi} + \|\nabla u\|_{\Phi}.$$

It is reflexive Banach space under the assumptions of $\mathbf{H}(\phi)$(ii). The completion of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ is denoted by $W_0^{1,\Phi}(\Omega)$ and it can be equivalently renormed by

$$\|u\| = \|\nabla u\|_{\Phi}$$

via a Poincaré-type inequality

$$\|u\|_{\Phi} \le K_0 \|\nabla u\|_{\Phi}, \quad u \in W_0^{1,\Phi}(\Omega), \tag{2.3}$$

where $K_0 > 0$. For more details, see [11].

Unlike the Sobolev spaces, the Orlicz–Sobolev spaces are neither separable nor reflexive in general. The Δ_2-condition (2.1) plays a central role in this regard: it guarantees the separability of both $L^{\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ (see [2]). Furthermore, conditions (2.1) and (2.2) ensure that $L^{\Phi}(\Omega)$ is uniformly convex and hence reflexive (see [18]); consequently, $W_0^{1,\Phi}(\Omega)$ is also a reflexive Banach space.

In addition, assumption $\mathbf{H}(\phi)$(ii) implies that $W_0^{1,\Phi}(\Omega)$ is continuously embedded into the classical Sobolev space $W_0^{l}(\Omega)$. As a result, $W_0^{1,\Phi}(\Omega)$ is continuously and compactly embedded into the Lebesgue space $L^q(\Omega)$ for all $1 \le q < l^*$, where

$$l^* = \begin{cases} \dfrac{Nl}{N-l} & \text{if } l < N, \\ +\infty & \text{if } l \ge N. \end{cases} \tag{2.4}$$

We now state the following lemma, which will play a crucial role in our arguments.

Lemma 2.1 ([5, 9, 18]). *Let*

$$\begin{aligned} \xi_0(t) &= \min\left\{t^l, t^m\right\}, & t \ge 0, \\ \xi_1(t) &= \max\left\{t^l, t^m\right\}, & t \ge 0, \\ \xi_2(t) &= \min\left\{t^{l^*}, t^{m^*}\right\}, & t \ge 0, \\ \xi_3(t) &= \max\left\{t^{l^*}, t^{m^*}\right\}, & t \ge 0. \end{aligned}$$

The following inequalities hold:

$$\xi_0(r)\Phi(t) \le \Phi(rt) \le \xi_1(r)\Phi(t), \quad r, t \ge 0, \tag{2.5}$$

and

$$\xi_0(\|u\|) \le \int_{\Omega} \Phi(|\nabla u|)dx \le \xi_1(\|u\|), \quad u \in W_0^{1,\Phi}(\Omega). \tag{2.6}$$

2.2. Critical groups. In this section, we briefly recall some basic notions and properties of Morse theory. Let X be a Banach space and X^* be its topological dual. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Consider a functional $I \in C^1(X, \mathbb{R})$ and let $c \in \mathbb{R}$. We say that I satisfies the Cerami condition if any sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\, I^{'}(u_n) \to 0 \quad \text{in } X^*, \tag{2.7}$$

as $n \to \infty$, admits a convergent subsequence in X.

This compactness condition is generally weaker than the classical Palais–Smale condition. However, it is still sufficient to establish a deformation theorem from which the minimax characterization of certain critical values of I can be derived. In particular, the following result, usually referred to in the literature as the Mountain Pass Theorem, holds.

Theorem 2.2. *If $I \in C^1(X, \mathbb{R})$ satisfies the Cerami condition, $u_0, u_1 \in X$ are such that $\|u_1 - u_0\| > \hbar > 0$, and*

$$\max\{I(u_0), I(u_1)\} < \inf\{I(u) : \|u - u_0\| = \hbar\} = \eta_\hbar, \quad c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.8}$$

where

$$\Gamma = \big\{\gamma \in C^1([0, 1]; X) : \gamma(0) = u_0, \gamma(1) = u_1\big\},$$

then $c \geq \eta_\hbar$ and c is a critical value of I.

Now, let $I \in C^1(X)$ and let $c \in \mathbb{R}$. We introduce the following notation:

$$\begin{aligned} I^c &= \big\{u \in X : I(u) \leq c\big\}, \\ K_I &= \big\{u \in X : I^{'}(u) = 0\big\}, \\ K_I^c &= \big\{u \in K_I : I^{'}(u) = c\big\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, we denote by $H_k\left(Y_1, Y_2\right)$ the k-th relative singular homology group.

The critical groups of I at an isolated point $u \in K_I$ with $I(u) = c$ are defined as

$$C_k(I, u) := H_k\left(I^c \cap U, (I^c \cap U) \backslash \{u\}\right), \quad k \in \mathbb{N},$$

where U is any neighborhood of u such that $K_I \cap I^c \cap U = \{u\}$. By the excision property of singular homology, this definition is independent of the particular choice of U.

Assume now that $I \in C^1(X)$ satisfies the Cerami condition and that $I(K_I) > -\infty$. For $c < \inf I(K_I)$, the critical groups of I at infinity were introduced by Bartsch–Li [3] as

$$C_k(I, \infty) := H_k\left(X, I^c\right), \quad k \in \mathbb{N}. \tag{2.9}$$

By the second deformation lemma, the right-hand side of (2.9) is independent of the particular choice of $c < \inf I(K_I)$. For further details on Morse theory, we refer the reader to [6, 17]. Assume that K_I is finite. We set

$$M(t, u) = \sum_{k \geq 0} \operatorname{rank} C_k(I, u) t^k, \quad t \in \mathbb{R}, \; u \in K_I,$$

$$P(t,\infty)=\sum_{k\geq 0}\text{rank } C_k(I,\infty)t^k,\quad t\in\mathbb{R}.$$

Proposition 2.3 (Morse inequalities)**.** *Consider a Banach space X and a functional $I\in C^1(X,\mathbb{R})$ satisfy the Cerami condition. Then*

$$\sum_{k=0}^{\infty}M(k,u)t^k=P(t,\infty)+(1+t)Q(t),\tag{2.10}$$

where $Q(t)=\sum_{k=0}^{\infty}a_kt^k$ with $a_k\in\mathbb{N}$.

Critical groups allow us to distinguish critical points, while Morse inequalities provide information on the existence of additional ones.

Proposition 2.4. *Consider a Banach space X and a functional $I\in C^1(X,\mathbb{R})$ satisfy the Cerami condition. Assume further that I has finitely many critical points. Then*

1. *If $C_k(I,\infty)\neq 0$ for some $k\in\mathbb{N}$, then I has a critical point u with $C_k(I,u)\neq 0$.*
2. *If $u=0$ is an isolated critical point of I and $C_k(I,0)\neq C_k(I,\infty)$ for some $k\in\mathbb{N}$, then I admits a nontrivial critical point.*

3. Mountain Pass Approach to two weak constant-sign solutions

3.1. Variational setting. We define the energy functional I associated with problem (1.1) by

$$I(u)=J(u)-\frac{1}{2}\mathbb{K}(u),\quad u\in W_0^{1,\Phi}(\Omega),$$

where

$$J(u)=\int_\Omega\Phi(|\nabla u|)\,dx$$

and

$$\mathbb{K}(u)=\int_\Omega\left(\frac{1}{|x|^\mu}*F(u)\right)F(u)\,dx=\int_\Omega\int_\Omega\frac{F(u(x))F(u(y))}{|x-y|^\mu}\,dx\,dy.$$

The functional I belongs to C^1 with derivative given by

$$\begin{aligned}\langle I^{'}(u),v\rangle=&\int_\Omega\phi(|\nabla u|)\nabla u\nabla v+\int_\Omega\phi(u)uv\,dx\\&-\int_\Omega\int_\Omega\frac{F(x,u(x))f(y,u(y))u(y)}{|x-y|^\mu}\,dx\,dy,\end{aligned}$$

for all $u,v\in W_0^{1,\Phi}(\Omega)$. Hence, the critical points of I correspond to weak solutions of (1.1).

3.2. Cerami condition

Proposition 3.1. *Under hypotheses* **H(f)** *and* **H(ϕ)**, *the functional I satisfies the Cerami condition.*

Proof. Let $\{u_n\} \subset W_0^{1,\Phi}(\Omega)$ be a Cerami sequence for the functional I, that is,

$$I(u_n) \to c, \quad (1 + \|u_n\|)\, \|I'(u_n)\|_* \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Equivalently,

$$I(u_n) = J(u_n) - \frac{1}{2}\mathbb{K}(u_n) \to c \quad \text{and} \quad \left|\langle I'(u_n), v\rangle\right| \le \varepsilon_n \frac{\|v\|}{1 + \|u_n\|}, \tag{3.2}$$

where $v \in W_0^{1,\Phi}(\Omega)$ and $\varepsilon_n \to 0$ as $n \to \infty$.

From this, one immediately obtains

$$I(u_n) = c + o(1), \quad \langle I'(u_n), u_n\rangle = o(1), \tag{3.3}$$

with $o(1) \to 0$ as $n \to \infty$. Which is equivalent to

$$\int_\Omega \Phi(|\nabla u_n|) + \Phi(|u_n|)|u_n|^2\, dx - \frac{1}{2}\int_\Omega \int_\Omega \frac{F(x, u_n(x))F(y, u_n(y))}{|x-y|^\mu}\, dx\, dy = c + o(1) \tag{3.4}$$

and

$$-\int_\Omega \left(\phi(|\nabla u_n|)|\nabla u_n|^2 + \phi(|u_n|)|u_n|^2\right) dx + \int_\Omega \int_\Omega \frac{F(x, u_n(x))f(y, u_n(y))u_n(y)}{|x-y|^\mu}\, dx\, dy = o(1). \tag{3.5}$$

Applying **H(ϕ)**(ii) to (3.5) gives

$$-m\left(\int_\Omega \left(\Phi(|\nabla u_n|)|\nabla u_n|^2 + \Phi(|u_n|)|u_n|^2\right) dx\right) + \int_\Omega \int_\Omega \frac{F(x, u_n(x))f(y, u_n(y))u_n(y)}{|x-y|^\mu}\, dx\, dy \le o(1). \tag{3.6}$$

Multiplying (3.4) by m and adding it to (3.6), we obtain

$$\int_\Omega \int_\Omega \frac{F(x, u_n(x))\left(f(y, u_n(y))u_n(y) - \frac{m}{2}F(y, u_n(y))\right)}{|x-y|^\mu}\, dx\, dy \le c + o(1). \tag{3.7}$$

Using **H(f)**(iii) and (3.7) shows that there exist $\beta_1 \in (0, \beta_0)$ and a positive constant C_1 such that

$$\int_\Omega \int_\Omega \frac{\beta_1 |(u_n(x), u_n(y))|^\tau - C_1}{|x-y|^\mu} dxdy \le c + o(1). \tag{3.8}$$

So, we have

$$\beta_1 \int_\Omega \int_\Omega \frac{|(u_n(x), u_n(y))|^\tau}{|x-y|^\mu}\, dx\, dy \le c + C_1\mathbb{K}(1) + o(1) \le M_1. \tag{3.9}$$

By **H(f)**(i) and (3.9), it follows that

$$
\begin{aligned}
&\left|\int_\Omega\int_\Omega \frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^\mu}\,dx\,dy\right| \\
&\quad\le C_2\int_\Omega\int_\Omega \frac{\big(|u_n(x)|^p+|u_n(x)|^q\big)\big(|u_n(y)|^p+|u_n(y)|^q\big)}{|x-y|^\mu}\,dx\,dy \\
&\quad\le C_2\int_\Omega\int_\Omega \frac{\big(|(u_n(x),u_n(y))|^p+|(u_n(x),u_n(y))|^q\big)^2}{|x-y|^\mu}\,dx\,dy \\
&\quad\le C_3\int_\Omega\int_\Omega \frac{\big(|(u_n(x),u_n(y))|^{2p}+|(u_n(x),u_n(y))|^{2q}\big)}{|x-y|^\mu}\,dx\,dy \\
&\quad= C_3\left[\int_{|(u_n(x),u_n(y))|\le 1} \frac{\big(|(u_n(x),u_n(y))|^{2p}+|(u_n(x),u_n(y))|^{2q}\big)}{|x-y|^\mu}\,dx\,dy\right. \\
&\qquad\left.+\int_{|(u_n(x),u_n(y))|\ge 1} \frac{\big(|(u_n(x),u_n(y))|^{2p}+|(u_n(x),u_n(y))|^{2q}\big)}{|x-y|^\mu}\,dx\,dy\right] \\
&\quad\le C_4+C_5\int_{|(u_n(x),u_n(y))|\ge 1} \frac{|(u_n(x),u_n(y))|^{\tau}}{|x-y|^\mu}\,dx\,dy \\
&\quad\le C_4+C_5\int_\Omega\int_\Omega \frac{|(u_n(x),u_n(y))|^{\tau}}{|x-y|^\mu}\,dx\,dy \le C_4+C_5\frac{M_1}{\beta_1}\le M_2. \qquad (3.10)
\end{aligned}
$$

It follows from (3.4), (3.10) and Lemma 2.1 that

$$
\begin{aligned}
\xi_0(\|u_n\|) &\le \int_\Omega \Phi(|u_n|)+\Phi(|u_n|)|u_n|^2dx \\
&= \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^\mu}dxdy + c + o(1) \le M_3, \qquad (3.11)
\end{aligned}
$$

Consequently, the sequence $\{u_n\}$ is bounded in $W_0^{1,\Phi}(\Omega)$. The Banach space $W_0^{1,\Phi}(\Omega)$ is reflexive. Thus, there exists $u\in W_0^{1,\Phi}(\Omega)$ such that, passing to a subsequence still denoted by $\{u_n\}$, and by the Sobolev embedding theorem, it follows that

$$
\begin{aligned}
u_n &\rightharpoonup u && \text{in } W_0^{1,\Phi}(\Omega),\\
u_n &\to u && \text{in } L^{ps}(\Omega),\ L^{qs}(\Omega) \text{ and, } L_\Phi(\Omega),\\
u_n(x) &\to u(x) && \text{a.e. in } \Omega.
\end{aligned}
$$

Let $v_n = u_n - u$. From condition **H(f)**(i), the boundedness of $\{F(\cdot,u_n(\cdot))\}$, Proposition 1.2, and Hölder's inequality, it follows that

$$
\begin{aligned}
\left|\langle \mathbb{K}'(u_n),v_n\rangle\right| &= \left|\int_\Omega\int_\Omega \frac{F(x,u_n(x))f(y,u_n(y))v_n(y)}{|x-y|^\mu}\,dx\,dy\right| \\
&\le C_1\|F(x,u_n(x))\|_{L^s(\Omega)}\left[\left(\int_\Omega |u_n|^{ps}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |v_n|^{ps}dx\right)^{\frac{1}{p}}\right.
\end{aligned}
$$

$$
+ \left(\int_\Omega |u_n|^{qs} \, dx \right)^{\frac{q-1}{q}} \left(\int_\Omega |v_n|^{qs} dx \right)^{\frac{1}{q}} \Bigg]
$$
$$
\leq C_2 \left(\|u_n\|_{L^{ps}(\Omega)}^{(p-1)s} \|v_n\|_{L^{ps}(\Omega)}^{s} + \|u_n\|_{L^{qs}(\Omega)}^{(q-1)s} \|v_n\|_{L^{qs}(\Omega)}^{s} \right) \to 0
$$

as $n \to +\infty$. Then

$$
\lim_{n\to+\infty} \langle \mathbb{K}^{'}(u_n), v_n \rangle = 0.
$$

In the same way, since $u_n \rightharpoonup u$ in $W_0^{1,\Phi}(\Omega)$, we obtain

$$
\lim_{n\to+\infty} \langle \mathbb{K}^{'}(u), v_n \rangle = 0.
$$

Therefore,

$$
\lim_{n\to+\infty} \langle \mathbb{K}^{'}(u_n) - \mathbb{K}^{'}(u), v_n \rangle = 0.
$$

Moreover, by (3.2), we also have

$$
\lim_{n\to+\infty} \langle I^{'}(u_n) - I^{'}(u), v_n \rangle = 0. \tag{3.12}
$$

Combining (3.2) and (3.12), it follows that

$$
\lim_{n\to+\infty} \|u_n - u\| = 0. \qquad \square \tag{3.13}
$$

We now prove the existence of two nontrivial smooth solutions of problem (1.1), one positive and the other negative, by introducing the positive and negative truncations of $f(x,\cdot)$. More precisely, for every $\zeta \in \mathbb{R}$, we define

$$
\begin{aligned}
\zeta^{\pm} &= \max\{\pm\zeta, 0\}, \\
f_{\pm}(x,\zeta) &= f(x, \pm\zeta^{\pm}), \quad (x,\zeta) \in \Omega \times \mathbb{R}.
\end{aligned} \tag{3.14}
$$

For $u \in W_0^{1,\Phi}(\Omega)$, we define the positive and negative parts by

$$
u^{\pm} = u(\cdot)^{\pm}.
$$

Then, $u^{\pm} \in W_0^{1,\Phi}(\Omega)$, and we have

$$
u = u^+ - u^-, \quad |u| = u^+ + u^-.
$$

We also define

$$
F_{\pm}(x,\zeta) := \int_0^{\zeta} f_{\pm}(x,\tau)\, d\tau. \tag{3.15}
$$

So, we consider the C^1-functionals $I_{\pm} : W_0^{1,\Phi}(\Omega) \to \mathbb{R}$, defined for every $u \in W_0^{1,\Phi}(\Omega)$, by

$$
I_{\pm}(u) := \int_\Omega \Phi(|\nabla u|)\, dx - \frac{1}{2} \int_\Omega \int_\Omega \frac{F_{\pm}(x,u(x))F_{\pm}(y,u(y))}{|x-y|^{\mu}}\, dx\, dy. \tag{3.16}
$$

Lemma 3.2. *We have $I_\pm \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ and the following holds:*

(i) *If $u \in W_0^{1,\Phi}(\Omega)$ is a critical point of $I_\pm$, then $\pm u(x) \geq 0$ a.e. in Ω and u is a critical point of I.*

(ii) *The function $u = 0$ is a local minimizer of both $I_\pm$ and I.*

(iii) *If hypotheses* **H(f)** *and* **H(ϕ)** *hold, then the functionals $I_\pm$ satisfy the Cerami condition.*

Proof. We consider I_+, noting that the argument for I_- is analogous. By hypothesis **H(f)**(i), we have $f(x,0) = 0$ a.e. in Ω, so the mapping $(x,t) \mapsto f(x,t^+)$ is Carathéodory and satisfies a growth condition analogous to **H(f)**(ii). Consequently, $I_+ \in C^1(W_0^{1,\Phi}(\Omega), \mathbb{R})$ with derivative given for all $u, v \in W_0^{1,\Phi}(\Omega)$ by

$$\begin{aligned}\langle I_+'(u), v\rangle = &\int_\Omega \phi(|\nabla u|)\nabla u \nabla v + \phi(|u|)uv\,dx \\ &- \int_\Omega\int_\Omega \frac{F(x,u^+(x))f(y,u^+(y))v(y)}{|x-y|^\mu}\,dx\,dy.\end{aligned}$$

To prove (i), let $u \in W_0^{1,\Phi}(\Omega)$ be a critical point of I_+. We test the derivative of I_+ with $v = -u^- \in W_0^{1,\Phi}(\Omega)$ yielding

$$\begin{aligned}\langle I_+'(u), -u^-\rangle = &-\int_\Omega \phi(|\nabla u|)\nabla u \nabla u^- + \phi(|u|)uu^-\,dx \\ &+ \int_\Omega\int_\Omega \frac{F(x,u^+(x))f(y,u^+(y))u^-(y)}{|x-y|^\mu}\,dx\,dy = 0.\end{aligned}$$

Since $f(y,u^+(y))u^- = 0$ a.e. in Ω, this simplifies to

$$\int_\Omega \phi(|\nabla u^-|)|\nabla u^-|^2 + \phi(|u^-|)|u^-|^2\,dx = 0.$$

The properties of ϕ and Lemma 2.1, then implying $u^- = 0$ a.e. in Ω, show that u is nonnegative a.e. Therefore u is a critical point of I.

To establish (ii), let $u \in W_0^{1,\Phi}(\Omega)$. By hypothesis **H(f)**(i), Proposition 1.2 and the continuous embeddings $W_0^{1,\Phi}(\Omega) \hookrightarrow L^{ks}(\Omega)$ for $k = p, q$, we have

$$\begin{aligned}&\left|\int_\Omega\int_\Omega \frac{F(x,u(x))F(y,u(y))}{|x-y|^\mu}\,dx\,dy\right| \\ &\qquad \leq C_1\|F(u)\|^2_{L^s(\Omega)} \leq C_2\left(\int_\Omega (|u|^{ps} + |u|^{qs})\,dx\right)^{\frac{2}{s}} \\ &\qquad \leq C_3\left(\|u\|^{2p}_{L^{ps}(\Omega)} + \|u\|^{2q}_{L^{qs}(\Omega)}\right) \leq C_4\left(\|u\|^{2p} + \|u\|^{2q}\right), \end{aligned} \tag{3.17}$$

where $C_1, C_2, C_3, C_4 > 0$ are constants independent of u. From Lemma 2.1, it follows that for any $u \in W_0^{1,\Phi}(\Omega)$ with $\|u\| \leq \rho < 1$ we have

$$I_+(u) = \int_\Omega \Phi(|\nabla u|) + \Phi(|u|)\,dx - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(x,u^+(x))F(y,u^+(y))}{|x-y|^\mu}\,dx\,dy$$

$$\geq C_5 \min \left\{ \|u\|^l, \|u\|^m \right\} - C_4/2 \left(\|u\|^{2p} + \|u\|^{2q} \right)$$
$$\geq C_7 \|u\|^m - C_6 \left(\|u\|^{2p} + \|u\|^{2q} \right), \tag{3.18}$$

where $C_i > 0$, $i = 1, 2, \ldots$, are constants independent of u. For a sufficiently small $\rho \in (0, 1)$, noting that $2p, 2q > m$, we have

$$I_+(u) > 0 \quad \text{for all } u \text{ such that } \|u\| \leq \rho. \tag{3.19}$$

Hence, $u = 0$ is a local minimizer of I_+. □

Define $\mathbb{U} = \left\{ u \in W_0^{1,\Phi}(\Omega) : \|u\| = 1 \right\}$.

Proposition 3.3. *Assume that conditions* **H(f)** *and* **H(ϕ)** *hold, and let* $u \in \mathbb{U}$. *Then*

$$\lim_{t \to +\infty} I(tu) = -\infty \text{ and } \lim_{t \to +\infty} I_\pm(tu) = -\infty.$$

Proof. We consider I, noting that the argument for $I_\pm$ is analogous. Let $u \in \mathbb{U}$. By hypothesis **H(f)**(iii), there exist $C_1, C_2 > 0$ such that

$$F(x, z) \geq C_1 |z|^{\frac{\nu}{2}} - C_2 \quad \text{for a.a. } x \in \Omega, \text{ and all } z \in \mathbb{R}.$$

Since $\nu > m$, then $\mathbb{K}(|u|^{\frac{\nu}{2}}) < \infty$ and $\mathbb{L}(u) := \int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\nu}{2}} \, dx \, dy}{|x-y|^\mu} < \infty$. Therefore,

$$\begin{aligned}
I(tu) &= \int_\Omega \Phi(|\nabla tu|) + \Phi(|tu|) \, dx - \frac{1}{2} \int_\Omega \int_\Omega \frac{F(x, tu(x)) F(y, tu(y))}{|x-y|^\mu} \, dx \, dy \\
&\leq \xi_1(\|t \nabla u\|_\Phi) + \xi_1(\|tu\|) - C_3 t^\nu \int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\nu}{2}} |u(y)|^{\frac{\nu}{2}} \, dx \, dy}{|x-y|^\mu} \\
&+ 2C_3 t^{\frac{\nu}{2}} \int_\Omega \int_\Omega \frac{|u(x)|^{\frac{\nu}{2}} \, dx \, dy}{|x-y|^\mu} - C_4 \int_\Omega \int_\Omega \frac{dx \, dy}{|x-y|^\mu} \\
&\leq t^m \left(1 + S_\Phi^m\right) - C_5 t^\nu \mathbb{K}\left(|u|^{\frac{\nu}{2}}\right) + 2C_3 t^{\frac{\nu}{2}} \mathbb{L}(u) - C_4 \mathbb{K}(1) \to -\infty,
\end{aligned}$$

as $t \to +\infty$, and $S_\Phi = \inf_{u \in W_0^{1,\Phi}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_\Phi}{\|u\|_\Phi}$. □

Consider the sets

$$\Lambda_+ = \{u \in K_I : u \geq 0 \; u \neq 0\} \text{ and } \Lambda_- = \{u \in K_I : u \leq 0, \; u \neq 0\},$$

where $K_I = \left\{ u \in W_0^{1,\Phi}(\Omega) : \forall v \in W_0^{1,\Phi}(\Omega) \quad \langle I'(u), v \rangle = 0 \right\}$.

Theorem 3.4. *Assume that hypotheses* **H(f)** *and* **H(ϕ)** *hold. Then problem* (1.1) *possesses at least two distinct nontrivial smooth solutions of constant sign, namely* $u_0 \in \Lambda_+$ *and* $u_1 \in \Lambda_-$.

Proof. From Lemma 3.2(ii), there exists a small number $\gamma \in (0, 1)$ such that

$$\inf_{\{u \in W_0^{1,\Phi}(\Omega) : \|u\| = \gamma\}} I_+(u) = \eta_+ > 0 = I_+(0). \tag{3.20}$$

Moreover, for any $u \in W_0^{1,\Phi}(\Omega)\backslash\{0\}$, Lemma 3.3 yields $\lim_{t\to+\infty} I_+(tu) = -\infty$. Consequently, there exists $t > 0$ large enough such that

$$I_+(tu) \le I_+(0) = 0 < \eta^+, \quad \|tu\| > \eta_+. \tag{3.21}$$

From (3.20) and (3.21), together with the fact that the functional I_+ satisfies the Cerami condition (see Lemma 3.2(iii)), we can apply the Mountain Pass Theorem 2.2. Hence, there exists $u_0 \in W_0^{1,\Phi}(\Omega)$ such that

$$u_0 \text{ is a critical point of } I_+ \text{ and } I_+(u_0) \ge \eta_+ > 0. \tag{3.22}$$

Moreover, by Lemma 3.2(i) and (3.22), it follows that

$$u_0 \text{ is a nontrivial critical point of } I_+, \text{ with } u_0 \ge 0 \text{ and } u_0 \ne 0. \tag{3.23}$$

Therefore, $u_0 \in \Lambda_+$. By an analogous argument as that applied to I_-, we obtain another constant-sign smooth solution $v_0 \in \Lambda_-$. □

4. Three weak solutions for problem (1.1) via Morse theory

4.1. Computation of critical groups.

Proposition 4.1. *Under the assumptions* **H(f)** *and* **H(ϕ)**, *it follows that*

$$C_k(I,\infty) = 0 \quad \textit{and} \quad C_k(I_\pm,\infty) = 0, \quad k \in \mathbb{N}.$$

Proof. We prove the result for I, the case for $I_\pm$ is analogues. Since I satisfies the Cerami condition (see [13]), we can compute $C_q(I,\infty)$.

There exists $\tau > 0$ with $m < 2p, 2q < \tau < l^*$ such that

$$\liminf_{|(t,z)|\to+\infty} \frac{F(x,t)f(y,z)z - \frac{m}{2}F(x,t)F(y,z)}{|(t,z)|^\tau} \ge \beta_0 \text{ uniformly for a.a. } (x,y) \in \Omega\times\Omega.$$

Define

$$A = \mathbb{K}(1) \max_{\overline{\Omega\times\Omega\times[-M,M]}} \left| F(x,u)f(y,u)u - \frac{m}{2}F(x,u)F(y,u)\right| + 1$$

and choose

$$\gamma < \min\left\{ \inf_{\|u\|\le 1} I(u), \frac{1-A}{m}\right\}.$$

Fix $u \in \mathbb{U}$. By Proposition 3.3, there exists $t_0 > 1$ with $I(tu) \le \gamma$ for all $t > t_0$. By **H(f)**(iii), there exist $\beta_1 \in (0,\beta_0)$ and $M > 0$ such that

$$F(x,t)f(y,z)z - \frac{m}{2}F(x,t))F(y,z) \ge \beta_1|(t,z)|^\tau \tag{4.1}$$

for a.a. $(x,y) \in \Omega\times\Omega$ and all $|(t,z)| > M$, where $m < 2p, 2q < \tau < l^*$.

Combining (4.1) with **H(ϕ)**(ii), we deduce

$$t\frac{d}{dt}I(tu) = t\langle I^{'}(tu), u\rangle = \langle I^{'}(tu), tu\rangle = \int_\Omega \phi(|\nabla tu|)|\nabla tu|^2 + \phi(|tu|)|tu|^2\,dx$$

$$
\begin{aligned}
&-\frac{m}{2}\int_\Omega\int_\Omega \frac{F(x,tu(x))F(y,tu(y))}{|x-y|^\mu}\,dx\,dy+\frac{m}{2}\int_\Omega\int_\Omega \frac{F(x,tu(x))F(y,tu(y))}{|x-y|^\mu}\,dx\,dy\\
&-\int_\Omega\int_\Omega \frac{F(x,tu(x))f(y,tu(y))tu(y)}{|x-y|^\mu}dxdy\\
&\le m\left(\int_\Omega \Phi(|\nabla tu|)|\nabla tu|^2+\Phi(|tu|)tu^2dx\right)-\frac{m}{2}\int_\Omega\int_\Omega\frac{F(x,tu(x))F(y,tu(y))}{|x-y|^\mu}dx\,dy\\
&+\int_\Omega\int_\Omega\frac{\frac{m}{2}F(x,tu(x))F(y,tu(y))-F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^\mu}dxdy\\
&\le mI(tu)\\
&+\int_\Omega\int_\Omega\frac{\frac{m}{2}F(x,tu(x))F(y,tu(y))-F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^\mu}dxdy\\
&\le mI(tu)\\
&+\int_{|(tu(x),tu(y))|\le M}\frac{\frac{m}{2}F(x,tu(x))F(y,tu(y))-F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^\mu}dx\,dy\\
&+\int_{|(tu(x),tu(y))|\ge M}\frac{\frac{m}{2}F(x,tu(x))F(y,tu(y))-F(x,tu(x))f(y,tu(y))u(y)}{|x-y|^\mu}dx\,dy\\
&\le m\gamma+A-1-\int_{|(tu(x),tu(y))|\ge M}\frac{\beta_1|(tu(x),tu(y))|^\tau}{|x-y|^\mu}\,dx\,dy<0.\\
&\le m\gamma+A-1-\beta_1M^\tau\int_{|(tu(x),tu(y))|\ge M}\frac{dxdy}{|x-y|^\mu}\,dx\,dy<0. \qquad (4.2)
\end{aligned}
$$

By the implicit function theorem, there exists a unique function $\mu\in C(\mathbb{U})$ satisfying

$$I(\mu(u)u)=\gamma,\quad u\in\mathbb{U}.$$

Define

$$\overline{\mu}(u)=\frac{1}{\|u\|}\mu\left(\frac{u}{\|u\|}\right),\quad u\in W_0^{1,\Phi}(\Omega)\backslash\{0\}.$$

Then $\overline{\mu}\in C(W_0^{1,\Phi}(\Omega)\backslash\{0\})$ and

$$I(\overline{\mu}(u)u)=\gamma,\quad u\in W_0^{1,\Phi}(\Omega)\backslash\{0\}.$$

If $I(u)=\gamma$, we have $\overline{\mu}(u)=1$. Define

$$\widetilde{\mu}(u)=\begin{cases}1 & \text{if } I(u)\le\gamma,\\ \overline{\mu}(u) & \text{if } I(u)>\gamma.\end{cases}\tag{4.3}$$

Clearly, $\widetilde{\mu}\in C(W_0^{1,\Phi}(\Omega)\backslash\{0\})$. The homotopy $h:[0,1]\times W_0^{1,\Phi}(\Omega)\backslash\{0\}\to W_0^{1,\Phi}(\Omega)\backslash\{0\}$ can be defined by $h(t,u)=(1-t)u+t\widetilde{\mu}(u)u$. Then h is continuous and

$$
\begin{aligned}
h(0,u)=u,\quad h(1,u)=\widetilde{\mu}(u),\quad u\in I^\gamma,\\
h(t,u)=u,\quad t\in[0,1],\ u\in I^\gamma.
\end{aligned}
$$

Since I^γ is a strict deformation retract of $W_0^{1,\Phi}(\Omega)\backslash\{0\}$, it is easy to see that $\mathbb{U}$ is a retract of $W_0^{1,\Phi}(\Omega)\backslash\{0\}$, $\mathbb{U}$ and I^γ are homotopy equivalent. Therefore,

$$H_k\left(W_0^{1,\Phi}(\Omega),\mathbb{U}\right)=H_k\left(W_0^{1,\Phi}(\Omega),I^\gamma\right),\quad k\in\mathbb{N},$$

and, consequently,

$$C_k\left(I,\infty\right)=H_k\left(W_0^{1,\Phi}(\Omega),\mathbb{U}\right),\quad k\in\mathbb{N}.$$

As $\mathbb{U}$ is an absolute retract of $W_0^{1,\Phi}(\Omega)$ (see [22]), it is contractible, and thus

$$H_k\left(W_0^{1,\Phi}(\Omega),\mathbb{U}\right)=0,\quad k\in\mathbb{N},$$

yielding

$$C_k\left(I,\infty\right)=0,\quad k\in\mathbb{N}.$$

By analogous arguments, defining

$$\begin{aligned}E_\pm&=\left\{u\in\mathbb{U}:u^\pm\neq0\right\},\\ D_\pm&=\left\{u\in W_0^{1,\Phi}(\Omega):u^\pm\neq0\right\},\end{aligned}$$

we obtain

$$C_k(I_\pm,\infty)=0,\quad k\in\mathbb{N}.$$ □

Corollary 4.2. *Assume that hypotheses* **H(f)** *and* **H(ϕ)** *hold. Then problem* (1.1) *admits at least one nontrivial solution in* $W_0^{1,\Phi}(\Omega)$.

Proof. By Lemma 3.2(ii), we have $C_k(I,0)=\delta_{k,0}\mathbb{Z}$. Combining this with Propositions 2.3, (3.1) and (4.1), it follows that problem (1.1) possesses one nontrivial solution in $W_0^{1,\Phi}(\Omega)$. □

4.2. Proof of Theorem 1.4. It follows from Theorem 3.4 that the problem admits two distinct nontrivial solutions of constant sign denoted by $u_0\in\Lambda_+$ and $v_0\in\Lambda_-$. Let $\theta,\gamma\in\mathbb{R}$ be such that

$$\theta<0=I_+(0)<\gamma<I_+(u_0).\tag{4.4}$$

For convenience, we denote

$$K_I:=\{0,u_0,v_0\}.$$

We first claim that $K_{I_+}=\{0,u_0\}$. If $u\in K_{I_+}$, then by Lemma 3.2(i), we have $u\geq0$, $u\neq0$, and u is a critical point of I. Consequently, $u\in K_I$ and hence $u=u_0$. Let θ and γ be such that (4.4) holds and consider the triple of sets

$$I_+^\theta\subseteq I_+^\gamma\subseteq W_0^{1,\Phi}(\Omega)=W.\tag{4.5}$$

One can obtain the long exact sequence of homological groups corresponding to the above triple of sets

$$\cdots \longrightarrow H_k(W, I_+^\theta) \xrightarrow{i_*} H_k(W, I_+^\gamma) \xrightarrow{\partial_*} H_{k-1}(I_+^\gamma, I_+^\theta) \longrightarrow \cdots . \tag{4.6}$$

Here i_* denotes the homomorphism, induced by the inclusion

$$\left(W, I_+^\theta\right) \to \left(W, I_+^\gamma\right),$$

and ∂_* is the boundary map.

Since $\theta < 0$, $K_{I_+} = \{0, u_0\}$, and in view of (4.4), it follows that we have

$$H_k(W, I_+^\theta) = C_k(I_+, \infty), \quad k \in \mathbb{N}, \tag{4.7}$$

and

$$H_k\left(W, I_+^\gamma\right) = C_k\left(I_+, u_0\right), \quad k \in \mathbb{N}. \tag{4.8}$$

By Proposition 4.1, one has

$$H_k(W, I_+^\theta) = 0, \quad k \in \mathbb{N}. \tag{4.9}$$

Now, choose $\gamma > 0$ in such a way that the only critical value of I in the interval (θ, γ) is 0. Then

$$H_{k-1}\left(I_+^\gamma, I_+^\theta\right) = C_{k-1}(I_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}. \tag{4.10}$$

By the exactness of (4.6) and using (4.10), one obtains

$$H_k(W, I_+^\gamma) \cong H_{k-1}(I_+^\gamma, I_+^\theta) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}.$$

Hence

$$C_k(I_+, u_0) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}.$$

In a similar way, for a critical point v_0 of I_-, we obtain

$$C_k(I_-, v_0) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}. \tag{4.11}$$

Consider now the homotopy

$$[0,1] \times W_0^{1,\Phi}(\Omega) \to \mathbb{R}(t,u) \to h(t,u) = (1-t)I_+(u) + tI(u). \tag{4.12}$$

By the homotopy invariance of critical groups, we deduce

$$C_k(I, u_0) = C_k(I_+, u_0) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}, \tag{4.13}$$

and likewise,

$$C_k(I, v_0) = C_k(I_-, v_0) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}. \tag{4.14}$$

Furthermore, by Lemma 3.2(ii), the function $u = 0$ is a local minimizer of I, and hence

$$C_k(I, 0) = \delta_{k,0}\mathbb{Z}, \quad k \in \mathbb{N}, \tag{4.15}$$

while Proposition 4.1 gives

$$C_k(I,\infty)=0,\quad k\in\mathbb{N}. \tag{4.16}$$

We denote by

$$M_k(I,u)=\operatorname{rank} C_k(I,u),\quad k\in\mathbb{N},\ u\in K_I,$$

the Morse number of I at a critical point u.

From (4.15), (4.13) and (4.14), we have

$$M_0(I,0)=1,\quad M_1(I,u_0)=1,\quad M_1(I,v_0)=1,$$

while all other $M_k(I,u)$ vanish. Therefore, the alternating sum of the Morse numbers is

$$\sum_{\substack{u\in K_I\\ k\in\mathbb{N}}}(-1)^k M_k(I,u)=(-1)^0\cdot 1+(-1)^1\cdot 1+(-1)^1\cdot 1=-1.$$

On the other hand, by Proposition 2.3 (Morse inequalities), the contribution at infinity is zero and, by the Morse relation, the above sum must vanish. This contradiction shows that I admits at least one additional critical point $w\in K_I$.

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Результати кратності в теорії Морса для рівняння Шокара в просторах Орліча–Соболєва поза їхніми межами умови Амбросетті–Рабіновича

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Ми досліджуємо неоднорідне суперлінійне рівняння Шокара з Δ_Φ-оператором у просторах Орліча — Соболєва. На відміну від більшості чинних досліджень, наш аналіз не спирається на умову Амбросетті–Рабіновича. Довівши, що відповідний енергетичний функціонал задовольняє умову компактності Черамі, ми застосовуємо варіаційні методи разом із теорією Морса для обчислення критичних груп. Як наслідок, ми доводимо існування щонайменше трьох різних нетривіальних розв'язків. Ці результати розширюють класичні твердження про кратність для локальних еліптичних задач на невипадковий (нелокальний) контекст рівнянь Шокара.

Ключові слова: рівняння Шокара, простір Орліча–Соболєва, Δ_2-умова, теорія Морса