

Triple Solutions for $p(x)$ -Triharmonic Problems Using Ricceri's Critical Point Theorem

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We study a nonlinear sixth-order elliptic problem governed by the variable-exponent triharmonic operator:

$$\begin{cases} -\Delta_{p(x)}^3 u + |u|^{p(x)-2}u = \lambda f(x, u) + \mu g(x, u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N > 3$) is a smooth bounded domain, $p \in C(\overline{\Omega})$ satisfies $p^- > \max\{3, \frac{N}{3}\}$, and f, g are Carathéodory functions. By applying Ricceri's three critical points theorem and using embeddings of $W^{3,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ into Lebesgue spaces, we prove the existence of at least three bounded weak solutions for an open set of (λ, μ) . This extends multiplicity theory to the triharmonic case with variable exponent and Navier boundary conditions.

Key words: variable-exponent triharmonic operator, sixth-order nonlinear elliptic equation, Ricceri theorem, weak solution, multiplicity, Navier boundary conditions

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1. Introduction

The study of partial differential equations with variable exponent nonlinearities has emerged as a central theme in modern analysis, driven by applications in image processing [23], electrorheological fluids [1], and non-Newtonian flow models [2]. However, while second- and fourth-order problems with $p(x)$ -growth have been extensively studied, sixth-order equations involving the $p(x)$ -triharmonic operator remain largely unexplored, presenting unique analytical challenges that distinguish them fundamentally from their lower-order counterparts.

The $p(x)$ -triharmonic operator $\Delta_{p(x)}^3$ represents a natural but highly non-trivial extension of the well-studied $p(x)$ -Laplacian and $p(x)$ -biharmonic operators. Unlike these simpler cases, the triharmonic setting involves three levels of differentiation combined with variable exponent growth, creating substantial difficulties: the loss of homogeneity prevents standard scaling arguments, the operator's high-order structure complicates boundary value formulations, and the variable exponent introduces additional degeneracy that cannot be handled by classical techniques.

Recent advances in fourth-order problems [19,20] have been primarily focused on constant exponent cases or simpler boundary conditions. The breakthrough contribution of this work lies in addressing the fully variable exponent triharmonic case with Navier boundary conditions $u = \Delta u = \Delta^2 u = 0$, which naturally arise in plate theory and represent the most physically relevant setting for sixth-order problems.

We consider the boundary value problem

$$\begin{cases} -\Delta_{p(x)}^3 u + |u|^{p(x)-2}u = \lambda f(x, u) + \mu g(x, u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 -smooth boundary and $N > 3$. For the variable exponent $p \in C(\overline{\Omega})$, we assume that

$$p^- := \inf_{x \in \Omega} p(x) > \max \left\{ 3, \frac{N}{3} \right\}, \quad p^+ := \sup_{x \in \overline{\Omega}} p(x),$$

and the triharmonic operator is defined by

$$\Delta_{p(x)}^3 u := \operatorname{div} \left(\Delta \left(|\nabla \Delta u|^{p(x)-2} \nabla \Delta u \right) \right).$$

Our main contribution lies in establishing multiplicity results for this challenging problem using Ricceri's critical point theorem [21], working within the variable exponent Sobolev space $X := W^{3,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with an appropriate norm that accounts for both the high-order structure and variable exponent growth.

1.1. Explicit justification of hypotheses (H₁)–(H₃). We now provide a detailed justification of the hypotheses used in our analysis. These conditions ensure the energy functional associated with problem (1.1) satisfies the geometry and compactness requirements of Ricceri's multiplicity result.

(H₁) Positivity Condition:

$$F(x, r) > 0 \quad \text{for all } x \in \Omega \text{ and } r \in]0, \sigma].$$

This condition ensures the functional exhibits positive geometry near the origin, which prevents the trivial solution from being a global minimizer. It plays a key role in satisfying the geometric hypotheses of Ricceri's theorem.

(H₂) Local Growth Near the Origin: Let

$$p_1 : \overline{\Omega} \rightarrow \mathbb{R} \quad \text{with} \quad p_1^- := \min_{x \in \overline{\Omega}} p_1(x) > p^+$$

such that

$$p^+ < p_1^- \leq p_1(x) < p_3^*(x) \quad \text{for all } x \in \overline{\Omega}.$$

We in addition assume

$$\limsup_{r \rightarrow 0} \sup_{x \in \Omega} \frac{F(x, r)}{|r|^{p_1(x)}} < +\infty.$$

This growth restriction guarantees local subcriticality, enabling the use of compact embeddings and ensuring control of the energy functional in a neighborhood of zero.

(H₃) Global Growth Control: There exist constants $w, \sigma > 0$ and a continuous exponent

$$\varepsilon : \overline{\Omega} \rightarrow \mathbb{R} \quad \text{with} \quad 1 < \varepsilon^- \leq \varepsilon^+ < p^-$$

such that

$$|F(x, r)| \leq w \left(1 + |r|^{\varepsilon(x)} \right) \quad \text{for all } x \in \Omega \text{ and } r \in \mathbb{R}.$$

This global condition controls the growth at infinity, ensuring coercivity and the boundedness of Palais–Smale sequences, which are essential for applying variational principles.

1.2. Link with Ricceri’s theorem. Ricceri’s theorem requires three fundamental ingredients:

- (i) local positivity,
- (ii) local subcritical growth, and
- (iii) global coercivity or growth control.

Hypotheses **(H₁)**–**(H₃)** have been formulated to satisfy these criteria explicitly:

(H₁) Guarantees a mountain-pass geometry at the origin.

(H₂) Ensures local subcritical growth needed for compactness.

(H₃) Controls the energy at large scales and ensures boundedness of Palais–Smale sequences.

Together, they create a robust framework for establishing multiplicity results via Ricceri’s critical point theory.

Functional setting and motivation. The analysis of problem (1.1) is conducted in the variable exponent Sobolev space

$$X := W^{3,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

endowed with the norm

$$\|u\|_X := \|\nabla \Delta u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)}.$$

This space accounts for both the higher-order structure of the operator and the nonstandard growth dictated by the function $p(x)$, ensuring a suitable variational formulation for weak solutions.

Note: For compactness and coercivity arguments, we use the Luxemburg-type norm $\|u\|$, which is equivalent to $\|u\|_X$.

Under suitable structural conditions on the nonlinearities f and g , we prove the existence of at least three distinct weak solutions. The paper is organized as follows: Section 2 provides concrete examples, Section 3 states the main result, Section 4 contains essential preliminaries, and Section 5 presents the proof.

2. Examples

Take $\Omega = B(0, 1) \subset \mathbb{R}^N$ to be the unit ball centered at the origin; it is a smooth and bounded domain, and we assume $N > 3$. For the variable exponent choose

$$p(x) = 4 + \frac{\sin(|x|^2)}{10}, \quad p_1(x) = 5, \quad \varepsilon(x) = 3.5,$$

such that $p^- > 3$, in agreement with the standing assumption on $p(\cdot)$.

Prototype nonlinearities. Let

$$f(x, u) = |u|^4 u + a(x) |u|^3 u, \quad g(x, u) = \frac{b(x) |u|^2 u}{1 + |u|},$$

where the auxiliary coefficients $a, b \in C(\overline{\Omega})$ are strictly positive and bounded both above and below. The corresponding primitives are:

$$F(x, u) = \int_0^u f(x, s) ds = \frac{|u|^6}{6} + \frac{a(x) |u|^5}{5}, \quad G(x, u) = \int_0^u g(x, s) ds.$$

Checking the structural hypotheses.

(H₁) *Positivity.* For sufficiently small $r \neq 0$,

$$F(x, r) = \frac{|r|^6}{6} + \frac{a(x) |r|^5}{5} > 0$$

for every $x \in \Omega$; hence the positivity requirement is met.

(H₂) *Local growth near the origin.* A direct computation gives

$$\limsup_{r \rightarrow 0} \sup_{x \in \Omega} \frac{F(x, r)}{|r|^{p_1(x)}} = \frac{\|a\|_\infty}{5} < \infty,$$

so the local subcritical condition holds with $p_1(x) = 5$.

(H₃) *Global control.* Using the bounds on a , we find

$$|F(x, r)| \leq \frac{|r|^6}{6} + \frac{\|a\|_\infty |r|^5}{5} \leq w(1 + |r|^{3.5}) \quad \text{for all } x \in \Omega, r \in \mathbb{R},$$

and a suitable constant $w > 0$. Since $\varepsilon(x) = 3.5 < 4 \leq p^-$, the global growth hypothesis is also satisfied.

3. Main results

We now state and prove the principal multiplicity result for problem (1.1), demonstrating that the $p(x)$ -triharmonic equation admits at least three distinct weak solutions under the specified assumptions.

Definition 3.1. We say that a function $u \in X$ is a *weak solution* of problem (1.1) if it satisfies the following variational identity:

$$\begin{aligned} \int_{\Omega} |\nabla \Delta u|^{p(x)-2} \nabla \Delta u \cdot \nabla \Delta v \, dx + \int_{\Omega} |u|^{p(x)-2} u v \, dx \\ = \lambda \int_{\Omega} f(x, u) v \, dx + \mu \int_{\Omega} g(x, u) v \, dx. \end{aligned} \quad (3.1)$$

Theorem 3.2. Assume that conditions $(\mathbf{H}_1)$ – $(\mathbf{H}_3)$ are fulfilled. Then there exists an open interval $\Gamma \subset (0, \infty)$ and a radius $\eta > 0$ for which the following assertion holds.

Let $\lambda \in \Gamma$ be fixed and let $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy

$$\sup_{(x,r) \in \Omega \times \mathbb{R}} \frac{|g(x,r)|}{1 + |r|^{p_2(x)-1}} < \infty, \quad p_2 \in C(\overline{\Omega}), \quad p_2(x) < p_3^*(x), \quad x \in \overline{\Omega}.$$

Then there exists a non-negative constant θ such that, for every $\mu \in [0, \theta]$, the boundary-value problem (1.1) admits at least three weak solutions lying in the ball $\{u \in X : \|u\|_X < \eta\}$.

This theorem constitutes the focal point of our study: it proves that, given the imposed assumptions, the sixth-order $p(x)$ -triharmonic model admits several weak solutions. Relying on Ricceri's three-critical-points framework, our argument widens the analytical horizon for variable-exponent, high-order nonlinear elliptic equations.

4. Preliminaries

We begin by recalling the basic notation for Lebesgue spaces with variable exponent to facilitate readability. For more details, we refer the reader to [3, 8, 14, 28]. Let $\Omega \subset \mathbb{R}^N$ be bounded. Introduce

$$\mathcal{V}_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for every } x \in \overline{\Omega}\}$$

and abbreviate $p^+ := \max_{x \in \overline{\Omega}} p(x)$, $p^- := \min_{x \in \overline{\Omega}} p(x)$.

Variable-exponent Lebesgue space. For $p \in \mathcal{V}_+(\overline{\Omega})$, define

$$L^{p(\cdot)}(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}.$$

The space $L^{p(\cdot)}(\Omega)$ is endowed with the Luxemburg norm, which is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

With this norm the space $L^{p(\cdot)}(\Omega)$ is complete.

Proposition 4.1 ([8]). The variable-exponent Lebesgue space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is separable, uniformly convex, and hence reflexive. Its dual can be identified with $L^{q(\cdot)}(\Omega)$, where $q(\cdot)$ is determined by $p(x)^{-1} + q(x)^{-1} = 1$ for all $x \in \Omega$. Furthermore, for every $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, one has

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) \|u\|_{p(\cdot)} \|v\|_{q(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}.$$

Variable-exponent Sobolev spaces. For an integer $k \geq 1$, we set

$$W^{k,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) :$$

$$D^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for every multi-index } \alpha \text{ with } |\alpha| \leq k\},$$

where $D^\alpha u = \partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N} u$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. Endowed with the norm $\|u\|_{k,p(\cdot)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(\cdot)}$, the space $W^{k,p(\cdot)}(\Omega)$ is again separable, reflexive, and complete (see [28] for proofs).

If $k \geq 1$ and $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$, define the critical exponent

$$p_k^*(x) = \begin{cases} \frac{N p(x)}{N - k p(x)}, & kp(x) < N, \\ \infty, & kp(x) \geq N. \end{cases}$$

Proposition 4.2 ([3]). *Let $p, q \in C_+(\overline{\Omega})$ and assume $q(x) \leq p_k^*(x)$ for every $x \in \overline{\Omega}$. Then the Sobolev space with variable exponent admits the continuous embedding*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

If the pointwise inequality is strict, that is, $q(x) < p_k^(x)$ for all $x \in \overline{\Omega}$, the embedding becomes compact.*

We intend to demonstrate that problem (1.1) possesses a minimum of three solutions. The result will follow from the auxiliary theorems presented below.

Theorem 4.3 (Riccieri [21, Theorem 1]). *Let X be a reflexive Banach space and let $\mathcal{A} \subset \mathbb{R}$ be an interval. Assume that*

$$I: X \rightarrow \mathbb{R} \quad \text{and} \quad J: X \rightarrow \mathbb{R}$$

are C^1 -functionals such that

- (a) *I is sequentially weakly lower semicontinuous, bounded on bounded subsets of X , and its derivative $I': X \rightarrow X^*$ is a homeomorphism;*
- (b) *J' is compact.*

Suppose, in addition, that

$$(i) \quad \lim_{\|u\| \rightarrow \infty} (I(u) + \lambda J(u)) = +\infty \quad \text{for every } \lambda \in \mathcal{A}; \quad (4.1)$$

(ii) *there exists $t \in \mathbb{R}$ with*

$$\sup_{\lambda \in \mathcal{A}} \inf_{u \in X} (I(u) + \lambda(J(u) + t)) < \inf_{u \in X} \sup_{\lambda \in \mathcal{A}} (I(u) + \lambda(J(u) + t)). \quad (4.2)$$

Then one can find a non-empty open set $\Gamma \subset \mathcal{A}$ and a constant $\eta > 0$ with the following feature:

If $\lambda \in \Gamma$ and $L: X \rightarrow \mathbb{R}$ is any C^1 -functional whose derivative is compact, then there exists $\theta > 0$ such that for every $\mu \in [0, \theta]$ the equation

$$I'(u) + \lambda J'(u) + \mu L'(u) = 0$$

has at least three distinct solutions lying in the ball $\{u \in X : \|u\| < \eta\}$.

Proposition 4.4 (Ricceri [22, Prop.3.1]). *Let $X \neq \emptyset$ and let $I, J : X \rightarrow \mathbb{R}$ be two real-valued functions. Assume there exist $s > 0$ and points $x_0, x_1 \in X$ such that*

$$I(x_0) = 0, \quad J(x_0) = 0, \quad I(x_1) > s, \quad \sup_{I(x) \leq s} [-J(x)] < s \frac{-J(x_1)}{I(x_1)}.$$

If $t \in \mathbb{R}$ is chosen so that

$$\sup_{I(x) \leq s} [-J(x)] < t < s \frac{-J(x_1)}{I(x_1)},$$

then the following strict minimax inequality holds:

$$\sup_{\lambda \geq 0} \inf_{x \in X} (I(x) + \lambda(J(x) + t)) < \inf_{x \in X} \sup_{\lambda \geq 0} (I(x) + \lambda(J(x) + t)).$$

Properties of the $p(x)$ -triharmonic operator. To analyze problem (1.1), we work in the variable-exponent Sobolev framework

$$X := W^{3,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega),$$

where $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Introduce the norm

$$\|u\|_X := \|u\|_{1,p(x)} + \|u\|_{2,p(x)} + \|u\|_{3,p(x)}, \quad u \in X.$$

Whenever $1 < p^- \leq p^+ < \infty$, the pair $(X, \|\cdot\|_X)$ is a separable, reflexive Banach space; moreover, $\|u\|_X$ is equivalent to the quantity $|\nabla \Delta u|_{p(\cdot)}$ (see Rahal [20]). For convenience, we shall work with the Luxemburg-type norm

$$\|u\| := \inf \left\{ \beta > 0 : \int_{\Omega} \left(\left| \frac{\nabla \Delta u}{\beta} \right|^{p(x)} + \left| \frac{u}{\beta} \right|^{p(x)} \right) dx \leq 1 \right\}, \quad u \in X.$$

We begin by introducing the energy functional

$$I : X \longrightarrow \mathbb{R}, \quad I(u) = \int_{\Omega} \frac{|\nabla \Delta u|^{p(x)}}{p(x)} dx + \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx.$$

This mapping is well defined and of class C^1 on X ; its Fréchet derivative acts on $v \in X$ as

$$I'(u)[v] = \int_{\Omega} |\nabla \Delta u|^{p(x)-2} \nabla \Delta u \nabla \Delta v dx + \int_{\Omega} |u|^{p(x)-2} uv dx.$$

Auxiliary functionals. Define

$$J(u) = - \int_{\Omega} F(x, u) dx, \quad L(u) = - \int_{\Omega} G(x, u) dx.$$

Then, for every $u, v \in X$,

$$\langle J'(u), v \rangle = - \int_{\Omega} f(x, u) v dx, \quad \langle L'(u), v \rangle = - \int_{\Omega} g(x, u) v dx.$$

Proposition 4.5 ([5, 10, 28]). *Define the functional*

$$I_{p(x)} : X \longrightarrow \mathbb{R}, \quad I_{p(x)}(u) := \int_{\Omega} |\nabla \Delta u(x)|^{p(x)} + |u(x)|^{p(x)} dx.$$

For $u \in X$ (and for sequences $(u_n) \subset X$ where indicated) the following assertions are valid:

1. If $\|u\| < 1$ (respectively $= 1$ or > 1), then $I_{p(x)}(u) < 1$ (respectively $= 1$ or > 1).
2. Whenever $\|u\| \leq 1$, one has $\|u\|^{p^+} \leq I_{p(x)}(u) \leq \|u\|^{p^-}$.
3. If $\|u\| \geq 1$, the inequalities reverse: $\|u\|^{p^-} \leq I_{p(x)}(u) \leq \|u\|^{p^+}$.
4. For a sequence (u_n) in X , $\|u_n\| \rightarrow 0$ holds iff $I_{p(x)}(u_n) \rightarrow 0$.
5. Likewise, $\|u_n\| \rightarrow \infty$ is equivalent to $I_{p(x)}(u_n) \rightarrow \infty$.

Remark 4.6. The symbol X^* denotes the dual space of X .

Define the operator

$$T := I' : X \longrightarrow X^*.$$

Lemma 4.7. *The mapping T enjoys the following properties:*

- (a) T is bounded, continuous, and strictly monotone;
- (b) T belongs to the (S_+) -class;
- (c) T is a homeomorphism between X and X^* .

Proof. The argument parallels, with minor adjustments, the one employed for Theorem 3.3 in [19].

(a) *Continuity, boundedness, strict monotonicity.* Since $T = I'$ is the Fréchet derivative of the energy functional, it is automatically continuous and bounded on X . Introduce the sets

$$\mathcal{U}_p := \{x \in \Omega : p(x) \geq 2\}, \quad \mathcal{V}_p := \{x \in \Omega : 1 < p(x) < 2\}.$$

For any $x, y \in \mathbb{R}^N$, the elementary inequalities

$$\begin{aligned} |x - y|^p &\leq 2^p (|x|^{p-2}x - |y|^{p-2}y)(x - y), & p \geq 2, \\ |x - y|^2 &\leq \frac{(|x| + |y|)^{2-p}}{p-1} (|x|^{p-2}x - |y|^{p-2}y)(x - y), & 1 < p < 2, \end{aligned}$$

imply, after integration with $x = \nabla \Delta u$ and $y = \nabla \Delta v$, that

$\langle T(u) - T(v), u - v \rangle > 0$ for all $u \neq v$. Thus T is strictly monotone.

(b) *The (S_+) property.* Let $(u_n) \subset X$ be such that $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle T(u_n), u_n - u \rangle \leq 0$. From monotonicity, $\langle T(u_n) - T(u), u_n - u \rangle \geq 0$, whence $\langle T(u_n) - T(u), u_n - u \rangle \rightarrow 0$. Set

$$\begin{aligned} K_n(x) &= (|\nabla \Delta u_n|^{p(x)-2} \nabla \Delta u_n - |\nabla \Delta u|^{p(x)-2} \nabla \Delta u)(\nabla \Delta u_n - \nabla \Delta u), \\ M_n(x) &= (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u)(u_n - u). \end{aligned}$$

Because the embedding $X \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact, $u_n \rightarrow u$ in $L^{p(\cdot)}(\Omega)$ and hence $\int_{\Omega} M_n dx \rightarrow 0$. Consequently, $\int_{\Omega} K_n dx \rightarrow 0$.

Split Ω into $\mathcal{U}_p \cup \mathcal{V}_p$. On $\mathcal{U}_p$ (where $p(x) \geq 2$) it follows that $\|\nabla \Delta u_n - \nabla \Delta u\|_{p(\cdot), \mathcal{U}_p} \rightarrow 0$. On $\mathcal{V}_p$ we apply Young's inequality with a carefully chosen parameter ε (see [19]) to reach the same conclusion. Combining both parts yields

$$\int_{\Omega} (|\nabla \Delta u_n - \nabla \Delta u|^{p(x)} + |u_n - u|^{p(x)}) dx \rightarrow 0,$$

so $u_n \rightarrow u$ in X ; therefore T is of class (S_+) .

(c) *Bijectivity and continuity of the inverse.* Strict monotonicity makes T injective. For $\|u\| > 1$,

$$\frac{\langle T(u), u \rangle}{\|u\|} = \frac{I(u)}{\|u\|} \geq \|u\|^{p^- - 1} \xrightarrow{\|u\| \rightarrow \infty} \infty,$$

so T is coercive. The Minty–Browder theorem therefore guarantees surjectivity. To see that T^{-1} is continuous, take $f_n \rightarrow f$ in X^* and define $u_n = T^{-1}(f_n)$, $u = T^{-1}(f)$. Coercivity yields the boundedness of (u_n) ; pick a subsequence with $u_n \rightharpoonup \hat{u}$. Passing to the limit in the identity $\langle f_n - f, u_n - \hat{u} \rangle = 0$ and using the (S_+) -property gives $u_n \rightarrow \hat{u}$ in X and $T(\hat{u}) = f$. Injectivity of T forces $\hat{u} = u$, proving continuity of T^{-1} . Hence T is a homeomorphism between X and X^* . $\square$

5. Proof of the principal results

Proof of Theorem 3.2. We apply Theorem 4.3 with the functionals:

$$\begin{aligned} I(u) &= \int_{\Omega} \frac{|\nabla \Delta u|^{p(x)} + |u|^{p(x)}}{p(x)} dx, \\ J(u) &= - \int_{\Omega} F(x, u) dx, \\ L(u) &= - \int_{\Omega} G(x, u) dx. \end{aligned}$$

Step 1: Verification of functional properties. By Proposition 4.5 and standard arguments (cf. [19]), I is weakly lower semicontinuous with $I' : X \rightarrow X^*$ being a homeomorphism. The compact embeddings from Proposition 4.2 ensure that J' and L' are compact.

Step 2: Coercivity condition. From hypothesis (H_3) and Proposition 4.5, for $\|u\| \geq 1$, we have

$$I(u) + \lambda J(u) \geq \frac{\|u\|^{p^-}}{p^+} - \lambda w \int_{\Omega} (1 + |u|^{\varepsilon(x)}) dx \geq \frac{\|u\|^{p^-}}{p^+} - C\lambda (1 + \|u\|^{\varepsilon^+}).$$

Since $\varepsilon^+ < p^-$, this tends to $+\infty$ as $\|u\| \rightarrow \infty$ thus establishing condition (i).

Step 3: Geometric condition via Ricceri's auxiliary result. We apply the criterion from [22]: take $u_0 = 0$ (so $I(u_0) = J(u_0) = 0$) and choose $u_1 \in$

$C^2(\overline{\Omega})$ with $u_1 > 0$ in Ω and $u_1 = 0$ on $\partial\Omega$. Then $I(u_1) > 0$ and, by **(H₁)**, $J(u_1) < 0$.

For sufficiently small $s > 0$, hypothesis **(H₂)** ensures

$$\sup_{\{I(u) \leq s\}} [-J(u)] = \sup_{\{I(u) \leq s\}} \int_{\Omega} F(x, u) dx \leq C \sup_{\{I(u) \leq s\}} \|u\|^{p_1^-} \ll s$$

since $p_1^- > p^+$ and small-norm functions satisfy $\|u\|^{p^+} \leq I(u)$ by Proposition 4.5.

This yields $\sup_{\{I(u) \leq s\}} [-J(u)] < s \frac{-J(u_1)}{I(u_1)}$, satisfying the geometric condition (ii).

Step 4: Application of Ricceri's theorem. All hypotheses of Theorem 4.3 are verified. Therefore, there exists an open interval $\Gamma \subset (0, \infty)$ and $\eta > 0$ such that for each $\lambda \in \Gamma$, the growth condition on g ensures L' is compact, and there exists $\theta > 0$ such that for $\mu \in [0, \theta]$, the equation $I'(u) + \lambda J'(u) + \mu L'(u) = 0$ has at least three solutions in $\{u : \|u\| < \eta\}$.

These solutions correspond precisely to weak solutions of problem (1.1) by Definition 3.1. $\square$

Conclusion

In this paper, we demonstrated the existence of three weak solutions to a nonlinear sixth-order boundary value problem driven by the $p(x)$ -triharmonic operator with Navier boundary conditions. Using variational methods and Ricceri's critical point theory, we showed that under suitable structural conditions on the nonlinearities, multiplicity results can be achieved. These findings contribute to the theory of variable exponent elliptic equations and extend known results to higher-order problems.

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Потрійні розв'язки для $p(x)$ -тригармонічних задач за допомогою теореми Річчері про критичні точки

Adnane Belakhdar

Ми досліджуємо нелінійну еліптичну задачу шостого порядку, керовану тригармонічним оператором зі змінним показником:

$$\begin{cases} -\Delta_{p(x)}^3 u + |u|^{p(x)-2}u = \lambda f(x, u) + \mu g(x, u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

де $\Omega \subset \mathbb{R}^N$ ($N > 3$) є гладкою обмеженою областю, $p \in C(\overline{\Omega})$ задовольняє умову $p^- > \max\{3, \frac{N}{3}\}$, а f та g є функціями Каратеодорі. Застосовуючи теорему Річчері про три критичні точки та використовуючи вкладення простору $W^{3,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ у простори Лебега, ми доводимо існування щонайменше трьох обмежених слабких розв'язків для відкритої множини параметрів (λ, μ) . Це розширює теорію кратності на тригармонічний випадок зі змінним показником та крайовими умовами Нав'є.

Ключові слова: тригармонічний оператор зі змінним показником, нелінійне еліптичне рівняння шостого порядку, теорема Річчері, слабкі розв'язки, кратність, крайові умови Нав'є