

Typically-Real Ellipselike Functions

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In this article, we introduce a new subclass of univalent functions. This class consists of all typically-real functions f such that the image of the unit disk under f is a union of arcs of ellipses with centers on the real axis, tangent to the imaginary axis and constant ratio of axes of the ellipses. We call these functions μ -ellipselike and denote their family by $\mathcal{S}_\mu^c$, where the parameter μ determines the ratio of axes. Moreover, for all z in the upper half of the unit disc the following analytic condition

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}$$

determines whether the function f belongs to the class $\mathcal{S}_\mu^c$. Some examples of functions from $\mathcal{S}_\mu^c$ are given.

Key words: univalence, typically-real functions, ellipse

Mathematical Subject Classification 2020: 30C45, 30C10, 30C75

Introduction

Let $\mathbb{C}$ be the set of complex numbers and let $\mathcal{A}$ denote the space of functions analytic in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ endowed with the usual topology of local uniform convergence. Let $\mathcal{H} \subset \mathcal{A}$ be the class of all functions f normalized by $f(0) = f'(0) - 1 = 0$ and locally univalent, i.e. $f' \neq 0$ in $\mathbb{D}$. Also let $\mathcal{S} \subset \mathcal{H}$ be the class of all functions univalent in $\mathbb{D}$ and let $\mathcal{T}_{\mathbb{R}} \subset \mathcal{H}$ be the class of all typically-real functions f in $\mathbb{D}$, i.e. the set of functions f satisfying the condition $\operatorname{sgn}(\operatorname{Im}(f(z))) = \operatorname{sgn}(\operatorname{Im}(z))$ for every $z \in \mathbb{D}$, where sgn denotes the signum function (see [4, p. 181]).

One of the fundamental parts of the geometric theory of analytic functions is studying of univalence. The development of this theory was associated with the Bieberbach Conjecture. Many subclasses of $\mathcal{S}$ have been introduced and studied such as convex, starlike, and close-to-convex, etc. The problem of univalence remains an active topic of research. New subclasses of $\mathcal{S}$ are introduced and studied (see [1, 3, 6–8]).

The starlike functions map $\mathbb{D}$ onto domains which can be represented as a union of half-lines with an initial point at the origin. Similarly, in [5, Theorem 1.2] we proved for every $f \in \mathcal{T}_{\mathbb{R}}$ that if $f(\mathbb{D})$ can be represented as a union of arcs of circles with centers on the real axis and tangent to the imaginary axis, then $f \in \mathcal{S}$. In this paper, we develop this concept more broadly by generalizing circles to ellipses. Theorem 1.1 shows that this geometric property is equivalent to an appealing analytic condition. We denote the class defined via this analytic

condition by the symbol $\mathcal{S}_\mu^c$, where μ is a parameter dependent on the ratio of axes of the ellipse. We call such functions μ -ellipselike. In [5], we based the proofs on geometric considerations, while, in this paper, we use analytic methods. Later in the article, we give some examples which show that the class $\mathcal{S}_\mu^c$ is nonempty and useful.

To formulate the theorems in the paper, we need some geometric objects. Denote $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. For any $r > 0$, define $\mathbb{D}_r^+ := \{z \in \mathbb{C}^+ : |z| < r\}$ and $\mathbb{T}_r^+ := \{z \in \mathbb{C}^+ : |z| = r\}$. In particular, we denote $\mathbb{D}^+ := \mathbb{D}_1^+$ and $\mathbb{T}^+ := \mathbb{T}_1^+$. Define $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ and $\mathbb{T}(a, r) := \{z \in \mathbb{C} : |z - a| = r\}$ for any $a \in \mathbb{C}$ and $r > 0$. For any $w \in \mathbb{C}^+ \cup \mathbb{R} \setminus \{0\}$ and $\mu \in (-\infty; 1)$, denote

$$\mathbb{T}_\mu^+(w) := \left\{ z \in \mathbb{C}^+ : |z|^2 \operatorname{Re}(w) - |w|^2 \operatorname{Re}(z) = \mu \operatorname{Re}(w) \operatorname{Re}(z) \left(\operatorname{Re}(z) - \operatorname{Re}(w) \right) \right\},$$

which is an upper semiellipse containing the point w with foci at the points $\frac{|w|^2 - \mu \operatorname{Re}^2(w)}{2(1 - \sqrt{\mu}) \operatorname{Re}(w)}$, where $\sqrt{\mu}$ denotes the two complex numbers ν_1 and ν_2 satisfying the equality $\nu_1^2 = \nu_2^2 = \mu$. We denote by $\mathbb{L}_\mu^+(w)$ the open arc of the ellipse joining points 0 and w .

The main idea of this article is to define the class $\mathcal{S}_\mu^c \subset \mathcal{T}_\mathbb{R}$ and then show that it is a class of univalent functions. Each function f belonging to the class $\mathcal{S}_\mu^c$ has the geometric property that for each point $w \in \mathbb{D}^+$, the arc $\mathbb{L}_\mu^+(f(w))$ (which is part of the semiellipse $\mathbb{T}_\mu^+(f(w))$) is contained in the image of the semidisk $f(\mathbb{D}_{|w|}^+)$.

1. Main results

We study typically-real functions for which the image of $\mathbb{D}$ can be represented as a union of arcs of ellipses. The following theorem shows the relationship between ellipselikeness and a given analytic condition.

Theorem 1.1. *Let $f \in \mathcal{T}_\mathbb{R}$ and $\mu \in (-\infty; 1)$. Then the following equivalence holds*

$$\forall_{z \in \mathbb{D}^+} \mathbb{L}_\mu^+(f(z)) \subset f(\mathbb{D}_{|z|}^+) \iff \forall_{z \in \mathbb{D}^+} \operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}. \quad (1.1)$$

Proof. Fix $f \in \mathcal{T}_\mathbb{R}$ and $\mu \in (-\infty; 1)$. Assume that for every $z \in \mathbb{D}^+$,

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}. \quad (1.2)$$

Consider the function

$$(0; 1) \times (0; \pi) \ni (\varrho, \alpha) \mapsto a(\varrho, \alpha) := \frac{(1 - \mu) \operatorname{Re}^2(f(\varrho e^{i\alpha})) + \operatorname{Im}^2(f(\varrho e^{i\alpha}))}{2(1 - \mu) \operatorname{Re}(f(\varrho e^{i\alpha}))}$$

assuming that $\operatorname{Re}(f(\varrho e^{i\alpha})) \neq 0$. Let us notice that the semiellipse $\mathbb{T}_\mu^+(f(re^{i\theta}))$ is contained in the ellipse given by the equation

$$\frac{(x - a(r, \theta))^2}{a^2(r, \theta)} + \frac{y^2}{(1 - \mu)a^2(r, \theta)} = 1.$$

If $\operatorname{Re}(f(re^{i\theta})) > 0$, then the size of the ellipse increases when a increases. Moreover, if $\operatorname{Re}(f(re^{i\theta})) < 0$, then the size of the ellipse decreases when a increases.

Since

$$\frac{d}{d\alpha}a(\varrho, \alpha) = \frac{\mu \operatorname{Re}^2(f(\varrho e^{i\alpha})) \operatorname{Im}(\varrho e^{i\alpha} f'(\varrho e^{i\alpha})) - \operatorname{Im}(z f'(\varrho e^{i\alpha})) \bar{f}^2(\varrho e^{i\alpha})}{2(1 - \mu) \operatorname{Re}^2(f(\varrho e^{i\alpha}))},$$

by (1.2), $\frac{d}{d\alpha}a(\varrho, \alpha) > 0$. Hence, the function a is increasing with respect to α if and only if condition (1.2) holds.

Now let us prove the implication in the direction $(\Leftarrow)$ in (1.1). Fix $r \in (0; 1)$. Assume that there exists $\theta_1 \in (0; \pi)$ such that $\mathbb{L}_\mu^+(f(re^{\theta_1})) \setminus f(\mathbb{D}_r^+) \neq \emptyset$. Then there exists $\theta_2 \in (0; \pi)$ such that $f(re^{i\theta_2}) \in \mathbb{L}_\mu^+(f(re^{i\theta_1})) \setminus f(\mathbb{D}_{|r|}^+)$. Since $f(re^{i\theta_1}), f(re^{i\theta_2}) \in \mathbb{T}_\mu^+(f(re^{i\theta_1}))$, $a(r, \theta_1) = a(r, \theta_2)$. This leads to a contradiction with the monotonicity of a , which ends the proof in the direction $(\Leftarrow)$.

Now let us prove the implication in the direction $(\Rightarrow)$ in (1.1). Assume that for every $z \in \mathbb{D}^+$, $\mathbb{L}_\mu^+(f(z)) \subset f(\mathbb{D}_{|z|}^+)$. Fix $r \in (0; 1)$. There exists $\tilde{\theta} \in (0; \pi)$ such that $\operatorname{Re}(f(re^{i\tilde{\theta}})) = 0$ and for every $\theta \in (0; \tilde{\theta})$, $\operatorname{Re}(f(re^{i\theta})) > 0$. Fix $\theta_1 \in (0; \tilde{\theta})$. Then $\mathbb{L}_\mu^+(f(re^{i\theta_1})) \subset f(\mathbb{D}_r^+)$. Therefore, for every $\theta_2 \in (0; \tilde{\theta}) \setminus \{\theta_1\}$, $f(re^{i\theta_2}) \notin \mathbb{L}_\mu^+(f(re^{i\theta_1}))$. Hence, by the local univalence of f , $f(re^{i\theta_2}) \notin \mathbb{T}_\mu^+(f(re^{i\theta_1}))$ and consequently $a(r, \theta_1) \neq a(r, \theta_2)$. Then a is injective in $(0; \tilde{\theta})$ with respect to second argument. Hence, by continuity of a , a is increasing or decreasing in $(0; \tilde{\theta})$ with respect to second argument. Since $a(r, \theta) \rightarrow +\infty$ as $\theta \rightarrow \tilde{\theta}^-$, a is increasing in $(0; \tilde{\theta})$ with respect to second argument. Analogously, it can be shown that a is increasing in $(\tilde{\theta}; \pi)$ with respect to second argument. This ends the proof in the direction $(\Rightarrow)$. $\square$

The following theorem states that the ellipselikeness of a function implies its univalence.

Theorem 1.2. *Let $f \in \mathcal{T}_{\mathbb{R}}$ and $\mu \in (-\infty; -1)$. If for every $z \in \mathbb{D}^+$, $\mathbb{L}_\mu^+(f(z)) \subset f(\mathbb{D}_{|z|}^+)$, then $f \in \mathcal{S}$.*

Proof. Fix $f \in \mathcal{T}_{\mathbb{R}}$ and $\mu \in (-\infty; -1)$. Assume that for every $z \in \mathbb{D}^+$, $\mathbb{L}_\mu^+(f(z)) \subset f(\mathbb{D}_{|z|}^+)$. Suppose that there exist $r \in (0; 1)$ and $z_1, z_2 \in \mathbb{T}_r^+ \cup [-r, 0) \cup (0, r]$ such that $\mathbb{L}_\mu^+(z_1) \cap \mathbb{L}_\mu^+(z_2) \neq \emptyset$. Since ellipses $\mathbb{T}_\mu^+(f(z_1))$ and $\mathbb{T}_\mu^+(f(z_2))$ are tangent at the origin for any $a \in \mathbb{R} \setminus \{0\}$, we must have $\mathbb{L}_\mu^+(f(z_1)) \subset \mathbb{L}_\mu^+(f(z_2))$ or $\mathbb{L}_\mu^+(f(z_2)) \subset \mathbb{L}_\mu^+(f(z_1))$. Without loss of generality we assume that $\mathbb{L}_\mu^+(f(z_1)) \subset \mathbb{L}_\mu^+(f(z_2))$. Since $\mathbb{L}_\mu^+(f(z_2)) \subset f(\mathbb{D}_{|z_2|}^+)$, it follows that $f(z_1) \in f(\mathbb{D}_{|z_2|}^+)$ or $f(z_1) = f(z_2)$. Then, we have four cases:

- (i) $z_1, z_2 \in [-r; 0) \cup (0; r];$

- (ii) $z_1 \in [-r; 0) \cup (0; r]$ and $z_2 \in \mathbb{T}_r^+$;
- (iii) $z_1 \in \mathbb{T}_r^+$ and $z_2 \in [-r; 0) \cup (0; r]$;
- (iv) $z_1, z_2 \in \mathbb{T}_r^+$.

In case (i), $f(z_1) \notin f(\mathbb{D}_{|z_2|}^+)$ because $\text{Im}(f(z_1)) = 0$ and $f(\mathbb{D}_{|z_2|}^+) \subset \mathbb{C}^+$. Moreover, if $f(z_1) = f(z_2)$, then by the local univalence of f and $f([-r; r]) = [f(-r); f(r)] \subset \mathbb{R}$ we obtain $z_1 = z_2$.

In case (ii), $f(z_1) \notin f(\mathbb{D}_{|z_2|}^+)$ because $\text{Im}(f(z_1)) = 0$ and $f(\mathbb{D}_{|z_2|}^+) \subset \mathbb{C}^+$. Moreover, $f(z_1) \neq f(z_2)$, because $\text{Im}(f(z_1)) = 0$ and $\text{Im}(f(z_2)) > 0$.

In case (iii), $f(z_1) \notin f(\mathbb{D}_{|z_2|}^+)$ follows from $z_1 \notin \mathbb{D}_{|z_2|}^+$ and the maximum principle. Moreover, $f(z_1) \neq f(z_2)$ because $\text{Im}(f(z_1)) > 0$ and $\text{Im}(f(z_2)) = 0$.

In case (iv), $f(z_1) \notin f(\mathbb{D}_{|z_2|}^+)$ follows from $z_1 \notin \mathbb{D}_{|z_2|}^+$ and the maximum principle. Moreover, if $f(z_1) = f(z_2)$ for $z_1 \neq z_2$, then by the local univalence of f , the function maps the arc of $\mathbb{T}_r^+$ between z_1 and z_2 onto a certain closed curve γ . Therefore, there exists $\tilde{z} \in \mathbb{T}_r^+$ belonging to the arc between z_1 and z_2 , such that $\mathbb{T}_\mu^+(f(\tilde{z}))$ intersects γ at two or more points different from $f(z_1)$. Hence, there exist $z_3, z_4 \in \mathbb{T}_r^+$ on the arc between z_1 and z_2 , such that $f(z_3) \in \mathbb{L}_\mu^+(f(z_4))$. Because $\mathbb{L}_\mu^+(f(z_4)) \subset f(\mathbb{D}_{|z_4|}^+) = f(\mathbb{D}_{|z_3|}^+)$, we have $f(z_3) \in f(\mathbb{D}_{|z_3|}^+)$, which is a contradiction. Hence, if $f(z_1) = f(z_2)$, then $z_1 = z_2$.

As a consequence, for all $r \in (0; 1)$ and $z_1, z_2 \in \mathbb{T}_r^+ \cup [-r; 0) \cup (0; r]$, if $\mathbb{L}_\mu^+(f(z_1)) \cap \mathbb{L}_\mu^+(f(z_2)) \neq \emptyset$, then $z_1 = z_2$, which ends the proof of univalence of the function f on every set $\mathbb{T}_r^+ \cup [-r; 0) \cup (0; r]$. Hence, by the maximum principle, we obtain the univalence of f on $\mathbb{D}^+ \cup (-1; 1)$. Since $f \in \mathcal{T}_\mathbb{R}$, we conclude $f \in \mathcal{S}$. $\square$

Remark 1.3. Let us notice that for an arbitrarily fixed $\mu \in (-\infty; 1)$, Theorem 1.1 and Theorem 1.2 describe the subclass of the class $\mathcal{S}$ of all functions $f \in \mathcal{T}_\mathbb{R}$ satisfying an analytic condition

$$\text{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu \frac{\text{Re}^2(f(z)) \text{Im}(zf'(z))}{|f(z)|^4} \quad (1.3)$$

for $z \in \mathbb{D}^+$. Furthermore, these functions are equivalently characterized by a geometric condition

$$\mathbb{L}_\mu^+(f(z)) \subset f(\mathbb{D}_{|z|}^+)$$

for $z \in \mathbb{D}^+$. Hence, the described functions form a new family. We denote this family by $\mathcal{S}_\mu^c$.

Remark 1.4. Let us notice that, if $\mu = 0$, then Theorem 1.1 reduces to Theorem 1.3 in [5]. Moreover, the classes $\mathcal{S}^c$ from [5] and $\mathcal{S}_0^c$ coincide.

The following theorem shows an interesting property of classes $\mathcal{S}_\mu^c$. It states that if the image of $\mathbb{D}$ under function can be represented in two separate ways as a union of arcs of ellipses with two different ratios of axes, then this image can be represented as a union of arcs of ellipses with each intermediate ratio. In a sense, it can be treated as the continuity of the classes $\mathcal{S}_\mu^c$ with respect to the parameter μ .

Theorem 1.5. Let $\mu_1, \mu_2 \in (-\infty; 1)$ and $\mu_1 < \mu_2$. For every $\mu \in (\mu_1; \mu_2)$, if $f \in S_{\mu_1}^c \cap S_{\mu_2}^c$, then $f \in S_{\mu}^c$.

Proof. Fix $\mu_1, \mu_2 \in (-\infty; 1)$ such that $\mu_1 < \mu_2$, $f \in S_{\mu_1}^c \cap S_{\mu_2}^c$ and $z \in \mathbb{D}^+$. Then

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu_1 \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}$$

and

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu_2 \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}.$$

For every $\mu \in (\mu_1; \mu_2)$ there exists $t \in (0; 1)$ such that $\mu = (1 - t)\mu_1 + t\mu_2$. Multiplying the above inequalities by $1 - t$ and t , respectively, we obtain

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < ((1 - t)\mu_1 + t\mu_2) \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}.$$

Hence, $f \in S_{\mu}^c$ which ends the proof. $\square$

We will show examples illustrating that classes $\mathcal{S}_{\mu}^c$ contain some important and interesting functions. In [2], Dmitrishin et al. introduced potentially extremal polynomials p_n . In [5], we proved that $p_2, p_3, p_4 \in \mathcal{S}_0^c$. Moreover, in [5], we also demonstrated that the Koebe function K belongs to $\mathcal{S}_0^c$. In this article we extend this result determining the exact range of the parameter μ such that $K \in \mathcal{S}_{\mu}^c$ (see Example 1.7). Similarly, the following example illustrates the exact range of the parameter μ for a certain family of homographies.

Example 1.6. For any $a \in [-1; 1]$ consider the family of homographies of the form

$$\mathbb{D} \ni z \mapsto f_a(z) := \frac{z}{1 - az}.$$

Since

$$\operatorname{Im}(f_a(z)) = \frac{\operatorname{Im}(z)}{|1 - az|^2},$$

$f_a \in \mathcal{T}_{\mathbb{R}}$ for $a \in [-1; 1]$. Setting $z := re^{i\theta}$ for certain $r \in (0; 1)$ and $\theta \in (0; \pi)$, we obtain

$$\operatorname{Re}(f_a(z)) = \frac{r(\cos(\theta) - ar)}{|1 - are^{i\theta}|^2},$$

$$\operatorname{Im}(zf'_a(z)) = \frac{r \sin(\theta)(1 - a^2r^2)}{|1 - are^{i\theta}|^4}$$

and

$$\operatorname{Im}(zf'_a(z) \overline{f_a^2(z)}) = \frac{-r^3 \sin(\theta)}{|1 - are^{i\theta}|^4}.$$

Therefore condition (1.3) reduces to

$$\mu(\cos(\theta) - ar)^2(1 - a^2r^2) + (1 + a^2r^2 - 2ar \cos(\theta))^2 > 0 \quad (1.4)$$

for a certain $\mu \in (-\infty; 1)$.

Now we prove that for an arbitrarily fixed $a \in (0; 1]$ condition (1.4) is satisfied if and only if $\mu \in [-(1-a)/(1+a); 1]$. Setting $\mu := -(1-a)/(1+a)$ we can rewrite condition (1.4) in the form

$$(3a^3r^2 + 5a^2r^2 + a - 1)\cos^2(\theta) + (-2a^4r^3 - 6a^3r^3 - 6a^2r - 2ar)\cos(\theta) + 2a^4r^4 + 3a^3r^2 + a^2r^2 + a + 1 > 0. \quad (1.5)$$

Consider the function

$$[-1; 1] \times (0; 1) \ni (x, y) \mapsto g(x, y) := (3a^3y^2 + 5a^2y^2 + a - 1)x^2 + (-2a^4y^3 - 6a^3y^3 - 6a^2y - 2ay)x + 2a^4y^4 + 3a^3y^2 + a^2y^2 + a + 1.$$

Then for every $y \in (0; 1)$,

$$g(-1, y) = 2a(1+y)(1+ay)^3 > 0$$

and

$$g(1, y) = 2a(1-y)(1-ay)^3 > 0.$$

Hence, if $3a^3y^2 + 5a^2y^2 + a - 1 \leq 0$, then $g(x, y) > 0$ for $x \in [-1; 1]$. Now assume that $3a^3y^2 + 5a^2y^2 + a - 1 > 0$. Then for fixed $y \in (0; 1)$, g is a quadratic function of the variable x with vertex at the point $(p, g(p, y))$, where

$$p := \frac{a^4y^3 + 3a^3y^3 + 3a^2y + ay}{3a^3y^2 + 5a^2y^2 + a - 1}.$$

Since $g(1, y) > 0$ and

$$p - 1 = \frac{(1-ay)^2(1-a+3ay+a^2y)}{3a^3y^2 + 5a^2y^2 + a - 1} > 0,$$

$g(x, y) > 0$ for $x \in [-1; 1]$. It follows from here and from previous considerations that g is positive in the whole domain, which leads to (1.5). Therefore, $\mu := -(1-a)/(1+a)$ satisfies inequality (1.4) for $a \in (0; 1]$. Since $(\cos(\theta) - ar)^2(1 - a^2r^2) \geq 0$, inequality (1.4) holds for every $\mu \in [-(1-a)/(1+a); 1]$.

It remains to prove that inequality (1.4) is false for any $\mu < -(1-a)/(1+a)$. Fix $\varepsilon > 0$ and $\mu := -(1-a)/(1+a) - \varepsilon$. Setting $\theta := 0$ and $r := 2a/(2a + \varepsilon(1+a))$ we obtain

$$\begin{aligned} & \mu(\cos(\theta) - ar)^2(1 - a^2r^2) + (1 + a^2r^2 - 2ar\cos(\theta))^2 \\ &= \left(-\frac{1-a}{1+a} - \varepsilon\right)(1 - ar)^2(1 - a^2r^2) + (1 + a^2r^2 - 2ar)^2 \\ &= \frac{(1-ar)^3}{1+a} \left(2a(1-r) - \varepsilon(1+a)(1+ar)\right) = -\frac{(1-ar)^3(2a^2 + \varepsilon(1+a))\varepsilon}{2a + \varepsilon(1+a)} < 0 \end{aligned}$$

for $a \in (0; 1]$. Hence, by continuity, there exists $\theta > 0$ such that inequality (1.4) does not hold for $\mu = -(1-a)/(1+a) - \varepsilon$. Since $\varepsilon > 0$ is arbitrarily fixed, inequality (1.4) is false for any $\mu < -(1-a)/(1+a)$. This shows that for $a \in (0; 1]$, $f_a \in \mathcal{S}_\mu^c$ if and only if $\mu \in [-(1-a)/(1+a); 1]$. In an analogous way, it can be proved that for $a \in [-1; 0)$, $f_a \in \mathcal{S}_\mu^c$ if and only if $\mu \in [-(1+a)/(1-a); 1]$. In the case $a = 0$ it is easy to show that $f_0 \in \mathcal{S}_\mu^c$ if and only if $\mu \in [-1; 1]$.

Example 1.7. Consider the Koebe function:

$$\mathbb{D} \ni z \mapsto K(z) := \frac{z}{(1-z)^2}.$$

Since

$$\operatorname{Im}(K(z)) = \frac{\operatorname{Im}(z)(1-|z|^2)}{|1-z|^4},$$

$K \in \mathcal{T}_{\mathbb{R}}$. Setting $z := re^{i\theta}$ for certain $r \in (0; 1)$ and $\theta \in (0; \pi)$, we obtain

$$\begin{aligned} \operatorname{Re}(K(z)) &= \frac{r(\cos(\theta) - 2r + r^2 \cos(\theta))}{|1 - re^{i\theta}|^4}, \\ \operatorname{Im}(zK'(z)) &= \frac{r \sin(\theta)(1 + 2r \cos(\theta) - 6r^2 + 2r^3 \cos(\theta) + r^4)}{|1 - re^{i\theta}|^6}, \\ \operatorname{Im}(zK'(z)\overline{K^2(z)}) &= \frac{-r^3 \sin(\theta)(1 + r^2)}{|1 - re^{i\theta}|^8}. \end{aligned}$$

Therefore, condition (1.3) reduces to

$$\begin{aligned} &-(1+r^2)(1-2r \cos(\theta) + r^2)^3 \\ &< \mu(\cos(\theta) - 2r + r^2 \cos(\theta))^2(1 + 2r \cos(\theta) - 6r^2 + 2r^3 \cos(\theta) + r^4) \end{aligned} \quad (1.6)$$

for a certain $\mu \in (-\infty; 1)$.

Now we prove that condition (1.6) is satisfied if and only if $\mu \in [-1/3; 1)$. Setting $\mu := -1/3$ we can rewrite condition (1.6) in the form

$$\begin{aligned} &2r(1 + 15r^2 + 15r^4 + r^6)(1 - \cos(\theta))^3 \\ &+ (1-r)^2(- (1+r^2)(1 + 8r - 34r^2 + 8r^3 + r^4) \cos^2(\theta) \\ &- 8r(1+r)^2(1+r^2) \cos(\theta) \\ &+ 3 + 4r + 13r^2 - 8r^3 + 13r^4 + 4r^5 + 3r^6) > 0. \end{aligned} \quad (1.7)$$

Consider the function

$$\begin{aligned} [-1; 1] \times (0; 1) \ni (x, y) \mapsto f(x, y) &:= -(1+y^2)(1 + 8y - 34y^2 + 8y^3 + y^4)x^2 \\ &- 8y(1+y)^2(1+y^2)x + 3 + 4y + 13y^2 - 8y^3 + 13y^4 + 4y^5 + 3y^6. \end{aligned}$$

Then for every $y \in (0; 1)$,

$$f(-1, y) = 2(1-y)^2(1 + 2y + 31y^2 - 4y^3 + 31y^4 + 2y^5 + y^6) > 0 \quad (1.8)$$

and

$$f(1, y) = 2(1-y)^8 > 0. \quad (1.9)$$

If $1 + 8y - 34y^2 + 8y^3 + y^4 = 0$, then $f(\cdot, y)$ is a linear function. Hence, by (1.8) and (1.9), $f(x, y) > 0$ for $x \in [-1; 1]$. If $1 + 8y - 34y^2 + 8y^3 + y^4 > 0$, then $f(\cdot, y)$ is a concave quadratic function. Hence, by (1.8) and (1.9), $f(x, y) > 0$ for $x \in$

$[-1; 1]$. If $1 + 8y - 34y^2 + 8y^3 + y^4 < 0$, then $f(\cdot, y)$ is a quadratic function. Let p be the first coordinate of the vertex of the parabola. Since

$$p - 1 = \frac{-(1 - y)^2(y^2 + 14y + 1)}{1 + 8y - 34y^2 + 8y^3 + y^4} > 0,$$

by (1.8) and (1.9), $f(x, y) > 0$ for $x \in [-1; 1]$. Therefore, inequality (1.7) holds.

Setting $\mu := 1$ we can rewrite condition (1.6) in the form

$$2r(1 + r^2) \cos^3(\theta) + (1 + r^2)^2 \cos^2(\theta) - 10r(1 + r^2) \cos(\theta) + 1 + 10r^2 + r^4 > 0. \quad (1.10)$$

Since

$$2r(1 + r^2) \cos^3(\theta) - 10r(1 + r^2) \cos(\theta) = -2r(1 + r^2)(5 - \cos^2(\theta)) \cos(\theta) \geq 0$$

for $\theta \in [\pi/2; \pi)$ and $r \in (0; 1)$, the inequality (1.10) holds in this case. Since

$$\begin{aligned} & 2r(1 + r^2) \cos^3(\theta) + (1 + r^2)^2 \cos^2(\theta) - 10r(1 + r^2) \cos(\theta) + 1 + 10r^2 + r^4 \\ &= 2r(1 + r^2) \cos(\theta)(1 - \cos(\theta))^2 \\ &+ (1 + r^2)(1 + 4r + r^2) \left(\cos(\theta) - \frac{6r}{1 + 4r + r^2} \right)^2 + \frac{(1 - r)^4(1 + 8r + r^2)}{1 + 4r + r^2} > 0 \end{aligned}$$

for $\theta \in (0; \pi/2)$ and $r \in (0; 1)$, the inequality (1.10) holds in this case as well. Hence, the inequality (1.10) holds for $\theta \in (0; \pi)$ and $r \in (0; 1)$.

We proved that the inequality (1.6) holds for $\mu \in \{-1/3, 1\}$. Analogously to proof of Theorem 1.5, one can prove that the inequality (1.6) holds for $\mu \in [-1/3; 1]$. Hence, $K \in \mathcal{S}_\mu^c$ for $\mu \in [-1/3; 1]$.

It remains to prove that for every $\mu < -1/3$, $K \notin \mathcal{S}_\mu^c$. Fix $\varepsilon > 0$ and $\mu := -1/3 - \varepsilon$. Setting $\theta := 0$ and $r := 1/(1 + \varepsilon)$ we obtain

$$\begin{aligned} & -(1 + r^2)(1 - 2r \cos(\theta) + r^2)^3 - \mu(\cos(\theta) - 2r \\ & \quad + r^2 \cos(\theta))^2(1 + 2r \cos(\theta) - 6r^2 + 2r^3 \cos(\theta) + r^4) \\ &= (1 - r)^6 \left(-1 - r^2 + \left(\frac{1}{\varepsilon} + \varepsilon \right) (1 + 4r + r^2) \right) \\ &= \frac{\varepsilon(1 - r)^6(3\varepsilon^2 + 16\varepsilon + 18)}{3(1 + \varepsilon)^2} > 0. \end{aligned}$$

Hence, by continuity, there exists $\theta > 0$ such that inequality (1.6) does not hold for $\mu = -1/3 - \varepsilon$. Since $\varepsilon > 0$ is arbitrarily fixed, inequality (1.6) is false for any $\mu < -1/3$. This shows that $K \in \mathcal{S}_\mu^c$ if and only if $\mu \in [-1/3; 1]$.

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Received November 12, 2025, revised February 23, 2026.

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Типово дійсні еліпсоподібні функції

Szymon Ignaciuk and Maciej Parol

У цій статті ми вводимо новий підклас однолистих функцій. Цей клас складається з усіх типово дійсних функцій f , для яких образ одиничного диска при відображенні f є об'єднанням дуг еліпсів із центрами на дійсній осі, дотичних до уявної осі, та з постійним співвідношенням осей еліпсів. Ми називаємо такі функції μ -еліпсоподібними та позначаємо їхнє сімейство через $\mathcal{S}_\mu^c$, де параметр μ визначає співвідношення осей. Крім того, для всіх z у верхній півплощині одиничного диска наступна аналітична умова

$$\operatorname{Im} \left(\frac{zf'(z)}{f^2(z)} \right) < \mu \frac{\operatorname{Re}^2(f(z)) \operatorname{Im}(zf'(z))}{|f(z)|^4}$$

визначає, чи належить функція f до класу $\mathcal{S}_\mu^c$. Наведено деякі приклади функцій із $\mathcal{S}_\mu^c$.

Ключові слова: однолистність, типово дійсні функції, еліпс