

A Note on p -Adic Schrödinger Equations with Nonlocal Potentials

Roman Urban

We consider the p -adic Schrödinger equation $i\partial_t u = (\mathbf{D}^\alpha + \mathbf{V}(t))u$ with a time-dependent nonlocal potential $\mathbf{V}$ defined on the field of p -adic numbers $\mathbb{Q}_p$. We construct a solution u in the space $\mathcal{H} = L^2(\mathbb{Q}_p)$ by approximating it with a sequence of functions that solve analogous equations in finite-dimensional subspaces $E_N \subset \mathcal{H}$.

Key words: Vladimirov operator, nonlocal potential, Schrödinger equation, finite-dimensional approximations

Mathematical Subject Classification 2020: 34L15, 34L20, 35R10

1. Introduction

This work concerns problems belonging to the field of p -adic quantum mechanics, which is a branch of p -adic mathematical physics [12, 28] initiated in the 1980s by Volovich [26, 27, 29, 30]. Mathematical physics in a p -adic setting has developed very rapidly to this day. Let us mention here, for example, works on pseudodifferential operators [14, 35], stochastic processes in the field of p -adic numbers, the ring of finite adeles, and the complete ring of adeles [1, 5–7, 22–25, 32, 37], modelling of hierarchical structures of complex systems [5, 6], some of the recent papers on the porous medium equation [4, 11], energy landscapes [16, 36], image processing [33], and on p -adic Laplacian on graphs [9, 10].

Recently, some results have been published on the connections between p -adic quantum mechanics (analysis of Schrödinger operators) on the one hand and quantum networks and quantum computing on the other [34, 38–40]. Certain p -adic Schrödinger equations have been identified with continuous limits of continuous-time quantum walks (CTQWs) [38], including those constructed using adjacency matrices on graphs [39, 40].

All of the facts and examples listed above demonstrate that p -adic quantum mechanics is not only an elegant and mathematically interesting theory, but also—and perhaps above all—has physical meaning.

In this note, we investigate Schrödinger operators with nonlocal potentials. Specifically, we consider the Cauchy problem for the Schrödinger equation with a nonlocal potential over the field of p -adic numbers $\mathbb{Q}_p$. Interest in such equations and operators stems from their utility in simplifying the scattering of a projectile by a many-body target into a one-particle model.

Let $T > 0$ be fixed and let ν be a Borel measure on $\mathbb{Q}_p$. In $L^2(\mathbb{Q}_p)$, we consider the following Cauchy problem for the Schrödinger equation

$$\begin{aligned} i\partial_t u(t) &= (\mathbf{D}^\alpha + \mathbf{V}(t))u(t), \quad 0 \leq s \leq t \leq T, \\ u(s) &= u_s, \end{aligned}$$

where $\mathbf{D}^\alpha$, $\alpha > 1$, is the Vladimirov operator and $\mathbf{V}$ is a nonlocal time-dependent potential of the form

$$\mathbf{V}(t)\phi(x) = \int_{\mathbb{Q}_p} V(t)(x-y)(\phi(x) - \phi(y)) d\nu(y).$$

The purpose of this note is to construct a solution u to the aforementioned Cauchy problem by approximating it with a sequence $\{u_N\}_{N \geq 1}$ of solutions to analogous, truncated problems in finite-dimensional subspaces E_N . Specifically, we provide a sequence of Faedo–Galerkin approximations. Each solution to the truncated equation is expressed as the limit of increasingly long products of unitary evolution operators $U_N(t_n, t_{n-1})$, where $t_{n-1} < t_n$.

This representation of the solution corresponds to the generalized Chernoff formula for evolutions [20, 31], which extends the Trotter product formula for semigroups [18] to a more general setting.

A key component in the proof of our main result is the existence of an orthonormal wavelet basis $\Psi = \{\psi_I\}_I$ of $L^2(\mathbb{Q}_p)$, consisting of eigenfunctions of the Vladimirov operator $\mathbf{D}^\alpha$. This basis was originally constructed by Kozyrev [16]. For a more comprehensive overview of wavelets in p -adic spaces, we refer the reader to [2, 3, 8, 13].

The work of Kozyrev and Khrennikov [17] characterizes the structure of non-local potentials, enabling the definition of appropriate finite-dimensional subspaces E_N .

The construction of the sequence $\{u_N\} \subset E_N$ follows the methodology established by Petersen and Vuillermot [19]. Their work concerns a class of non-autonomous Schrödinger equations which, unlike our current setting, are treated in the Archimedean context.

Although our proof follows the main steps from [19, Theorem on p. 619], our final result is fundamentally different because it deals with non-Archimedean analysis.

Finally, the form of the solutions to the corresponding Cauchy problem facilitates their use in numerical simulations. The reason is that in our basis, the operator $\mathbf{D}^\alpha$ is diagonal and $\mathbf{V}(t)$ is finite-diagonal, which simplifies the computation of their operator exponentials.

1.1. Statement of the results

1.1.1. The Vladimirov operator

Definition 1.1 (The Fourier transform). The Fourier transform operator $\mathcal{F}$ is defined by the following integral,

$$\mathcal{F}\phi(\xi) = \int_{\mathbb{Q}_p} \chi(\xi x)\phi(x) dx,$$

where $\chi(x) = e^{2\pi i\{x\}_p}$ is the additive character on $\mathbb{Q}_p$ and dx denotes the Haar measure on $\mathbb{Q}_p$ normalized so that $\mathbb{Z}_p$ has measure 1.

Definition 1.2 (The Vladimirov operator). For a function $\phi : \mathbb{Q}_p \rightarrow \mathbb{C}$ and $\alpha > 1$, define the multiplication operator $\mathcal{M}_\alpha$ by

$$\mathcal{M}_\alpha \phi(x) = |x|_p^\alpha \phi(x).$$

Let $\mathcal{D}(\mathbb{Q}_p)$ be the Bruhat–Schwartz space of complex-valued functions on $\mathbb{Q}_p$. The Vladimirov operator $\mathbf{D}^\alpha$ is defined initially on $\mathcal{D}(\mathbb{Q}_p)$ by the formula

$$\mathbf{D}^\alpha \phi(x) = (\mathcal{F}^{-1} \mathcal{M}_\alpha \mathcal{F} \phi)(x). \quad (1.1)$$

The following construction is due to Kozyrev [15].

Definition 1.3 ($L^2(\mathbb{Q}_p)$ -basis of eigenfunctions). The orthonormal complete basis of eigenfunctions of the Vladimirov operator $\mathbf{D}^\alpha$ was constructed by Kozyrev [15].

Let

$$\psi_{\gamma n j}(x) = p^{-\gamma/2} \psi_j(p^\gamma x - n),$$

where $x \in \mathbb{Q}_p$, $\gamma \in \mathbb{Z}$, and

$$n \in \mathbb{Q}_p/\mathbb{Z}_p, \quad n = \sum_{i=\beta}^{-1} n_i p^i, \quad n_i \in \{0, 1, \dots, p-1\}, \quad \beta \in \mathbb{Z}_-,$$

and

$$\psi_j(x) = \chi(p^{-1} j x) \Omega(|x|_p), \quad j = 0, 1, \dots, p-1,$$

with

$$\Omega(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ 0, & \text{for } t > 1. \end{cases}$$

We denote the set of all $\{\psi_{\gamma n j}\}$ by Ψ .

Remark 1.4. Note that the support of $\psi_{\gamma n j}$,

$$\text{supp}(\psi_{\gamma n j}) = \{x \in \mathbb{Q}_p \mid |p^\gamma x - n|_p \leq 1\} = \{x \in \mathbb{Q}_p \mid |x - p^{-\gamma} n|_p \leq p^{-\gamma}\}$$

is the ball with center $a = p^{-\gamma} n$ and radius $r = p^{-\gamma}$.

Theorem 1.5 (Kozyrev [15, Theorem 2]). *The functions $\psi_{\gamma n j}$ where $\gamma \in \mathbb{Z}$, $n \in \mathbb{Q}_p/\mathbb{Z}_p$ and $j = 0, 1, \dots, p-1$, are eigenfunctions of the Vladimirov operator on $\mathbb{Q}_p$ with eigenvalues*

$$\lambda_\gamma = p^{-\alpha(\gamma-1)}$$

and form a complete system of vectors in the space $L^2(\mathbb{Q}_p)$.

1.1.2. The Schrödinger equation. Let $T > 0$ be fixed. We consider the following Cauchy problem for the Schrödinger equation

$$\begin{aligned} i\partial_t u(t) &= (\mathbf{D}^\alpha + \mathbf{V}(t))u(t), \quad 0 \leq s \leq t \leq T, \\ u(s) &= u_s \end{aligned} \quad (1.2)$$

defined in a complex Hilbert space $\mathcal{H} = L^2(\mathbb{Q}_p)$, where $\mathbf{V}$ is a nonlocal potential.

Definition 1.6 (Nonlocal potential [17, p. 210]). By the *nonlocal potential* we mean the following operator

$$\mathbf{V}\phi(x) = \int_{\mathbb{Q}_p} V(x-y)(\phi(x) - \phi(y)) d\nu(y), \quad (1.3)$$

where ν is a Borel measure and $V(x-y)$ is a complex-valued integral kernel such that $V(x) = V(-x)$ and V is a locally constant function at $x \neq y$. Moreover, V satisfies the following properties:

- (i) The function V is absolutely integrable at infinity; i.e.,

$$\int_{|x|_p > r} |V(x)| d\nu(x) < +\infty$$

for some $r > 0$.

- (ii) The domains where V is locally constant can be described as follows: on the spheres of radius R centered at zero, T is constant on the balls of diameter $p^{-k}R$, where $k \geq 1$. Thus, in each of these concentric spheres, there are $p^k(1 - p^{-1})$ subballs on which V is a constant.

Remark 1.7. We consider the action of these operators in the space $\mathcal{D}_0(\mathbb{Q}_p)$ of mean zero test functions.

Moreover we will work under the following

Assumption 1.8. The function $V : \mathbb{R}_+ \times \mathbb{Q} \rightarrow \mathbb{R}$ is a real-valued bounded function of the form

$$V(t, q) = a(t)v(q),$$

where a is a bounded and Lipschitz continuous function, i.e. there is a constant C such that for all $t, t' > 0$

$$|a(t) - a(t')| \leq C|t - t'|. \quad (1.4)$$

The collection of all balls with positive diameter in $\mathbb{Q}_p$ is denoted as $\mathcal{T}(\mathbb{Q}_p)$. This collection forms a partially ordered set (poset) under the relation of set inclusion. We can treat this poset as a graph, where each ball is a vertex. In this graph, an edge exists between two vertices I and J if and only if one ball is the immediate predecessor (a maximal subball) of the other, ensuring no other ball exists between them in the inclusion hierarchy.

From now on, we will identify tree $\mathcal{T}(\mathbb{Q}_p)$ with the set of its vertices. So we will say that $\mathcal{T}(\mathbb{Q}_p)$ is a tree, but when we write that $I \in \mathcal{T}(\mathbb{Q}_p)$, we mean that $\mathcal{T}(\mathbb{Q}_p)$ is a set of vertices. This will not lead to misunderstandings.

Theorem 1.9 ([17, Corollary 1]). *The matrix of the operator $\mathbf{V}$ given by (1.3) in a wavelet basis is finite-diagonal; i.e., its elements are nonzero on a finite number of main diagonals. Specifically, for wavelets ψ_I and ψ_J supported in balls I and J , respectively, the matrix element $\langle \psi_J, \mathbf{V}\psi_I \rangle$ is nonzero only for $|IJ| \leq 3k - 3$ (k is from Definition 1.6), where $|IJ|$ is the distance between I and J in the tree $\mathcal{T}(\mathbb{Q}_p)$ (the number of edges along the path joining the tree vertices corresponding to the balls).*

Remark 1.10. If $k = 0$ (the integration kernel V is constant on spheres centered at zero) then it follows from remark after [17, Theorem 1] that the wavelets are the eigenfunctions of the operator $\mathbf{V}$.

Let $\mathcal{L}(\mathcal{H})$ stands for the algebra of all linear bounded operators on $\mathcal{H}$. Under our assumptions, our Schrödinger operator is an additive perturbation of $\mathbf{D}^\alpha$ by a bounded operator $\mathbf{V}(t)$ (see Lemma 2.1) there is a unitary evolution system $U(t, s) \in \mathcal{L}(\mathcal{H})$, $0 \leq s \leq t \leq T$, on $\mathcal{H}$ such that

$$u(t) = U(t, s)u_s$$

solves (1.2) (see, e.g., [18, Ch. 3, Sec. 1, Theorem 1.1]).

1.1.3. The truncated version of (1.2). We denote by $\mathcal{O}$ the root of the tree $\mathcal{T}(\mathbb{Q}_p)$ which corresponds to the set (ball) of radius 1, i.e. $\mathcal{O} = \mathbb{Z}_p$. Let, for $N \in \mathbb{N} \cup \{0\}$,

$$\mathcal{K}_N = \{I \in \mathcal{T}(\mathbb{Q}_p) \mid |\mathcal{O}I| \leq N\}.$$

Remark 1.11. Notice that it follows from Remark 1.4 that there is one-to-one correspondence between vertices I of the tree $\mathcal{T}(\mathbb{Q}_p)$ and supports of $\psi_{\gamma nj} \in \psi$. If $\text{supp}(\psi_{\gamma nj}) = I$ we will often use the notation $\psi_I = \psi_{\gamma nj}$.

Using Remark 1.11 we can define, for each $N \in \mathbb{N}_+$, a finite dimensional subspace of $L^2(\mathbb{Q}_p)$,

$$E_N = \{\psi_{\gamma nj} \in \psi \mid \text{supp}(\psi_{\gamma nj}) = I, |\mathcal{O}I| \leq N\}.$$

Now we consider the truncated version of (1.2),

$$\begin{aligned} i\partial_t u_N(t) &= (\mathbf{D}_N^\alpha + \mathbf{V}_N(t))u_N(t), \quad t \in (0, T], \\ u_N(s) &= u_{s,N}, \end{aligned} \tag{1.5}$$

where $\mathbf{D}_N^\alpha$ and $\mathbf{V}_N(t)$ are linear bounded self-adjoint restrictions of $\mathbf{D}^\alpha$ and $\mathbf{V}(t)$ to $E_N(\mathbb{Q}_p, \mathbb{C})$, respectively, and $u_{N,s}$ stands for the orthogonal projection of u_s onto the subspace $E_N(\mathbb{Q}_p, \mathbb{C})$. Clearly, there is an evolution system $U_N(t, s) \in \mathcal{L}(E_N)$ such that $u_N(t) = U_N(t, s)u_{s,N}$.

We have the following convergence in $L^2(\mathbb{Q}_p)$:

$$U(t, s)\phi = \lim_{N \rightarrow \infty} U_N(t, s)\phi_N \quad \text{for } \phi \in L^2(\mathbb{Q}_p).$$

1.1.4. The main theorem. Let

$$\mathbf{W}_{N,\ell}(s, h) = (\mathbf{V}_N(s + \ell h) + \mathbf{V}_N(s + (\ell + 1)h)) / 2$$

Point (a) below is the Dyson expansion for the evolution (see [21, Theorem X.69]).

Theorem 1.12. *Let $h = (t - s)/n$.*

- (a) *Define, for every $N \in \mathbb{N}_+$, the operators $U_N(t, s)$, $s, t \in [0, T]$ in the space $E_N(\mathbb{Q}_p, \mathbb{C})$ by the formula (the Dyson expansion)*

$$U_N(t, s) = \mathbf{I}_N + \sum_{k=1}^{\infty} (-i)^k \int_s^t d\tau_1 \dots \int_s^{\tau_{k-1}} d\tau_k (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_1)) \dots (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_k)),$$

where $\mathbf{I}_N$ is the identity operator and $\tau_0 = t$. Then, each $U_N(t, s)$ is unitary and satisfies

$$\begin{aligned} U_N(t, t) &= \mathbf{I}_N, \\ U_N(t, s) &= U_N(t, r)U_N(r, s) \end{aligned}$$

for all $r, s, t \in [0, T]$. Moreover, the functions $u_N : [0, T] \rightarrow E_N(\mathbb{Q}_p, \mathbb{C})$ given by

$$u_N(t) = U_N(t, s)u_{s,N}$$

are continuous, once continuously differentiable on $(0, T]$ and satisfy (1.5) identically, thereby providing a sequence of Faedo–Galerkin approximations to (1.2).

Furthermore, for every $N \in \mathbb{N}_+$ and all $s, t \in [0, T]$ with $t \geq s$, the following statements hold:

- (b) *We have*

$$\begin{aligned} U_N(t, s) &= \prod_{\ell=n-1}^0 \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h)) + O_N(h) \\ &= \prod_{\ell=n-1}^0 \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) + O_N(h) \end{aligned} \quad (1.6)$$

as $h \rightarrow 0$.

- (c) *We have*

$$\begin{aligned} U_N(t, s) &= \prod_{\ell=n-1}^0 \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) \exp\left(\frac{h^2}{2}[\mathbf{D}_N^\alpha, \mathbf{W}_{N,\ell}(s, h)]\right) \\ &\quad + O_N(h^2) \\ &= \prod_{\ell=n-1}^0 \exp\left(-i\frac{h}{2}\mathbf{D}_N^\alpha\right) \exp\left(-i\frac{h}{2}\mathbf{W}_{N,\ell}(s, h)\right) \exp\left(-i\frac{h}{2}\mathbf{D}_N^\alpha\right) + O_N(h^2). \end{aligned}$$

2. Proof of Theorem 1.12

2.1. Some lemmas. By Assumption 1.8, $\mathbf{V}(t) = a(t)\mathbf{v}$, where

$$\mathbf{v}\phi(x) = \int_{\mathbb{Q}_p} v(x-y)(\phi(x) - \phi(y)) d\nu(y).$$

Lemma 2.1. *The operator $\mathbf{V}$ is bounded on $L^2(\mathbb{Q}_p)$ with operator norm $\|\mathbf{V}\|_{L^2 \rightarrow L^2} \leq 2\|v\|_{L^1(d\nu)}$.*

Proof. By Minkowski's integral inequality and triangle inequality

$$\begin{aligned} \|\mathbf{v}\phi\|_{L^2} &\leq \int_{\mathbb{Q}_p} \|v(y)(\phi(\cdot) - \phi(\cdot - y))\|_{L^2} d\nu(y) \\ &\leq \int_{\mathbb{Q}_p} |v(y)| \|\phi(\cdot) - \phi(\cdot - y)\|_{L^2} d\nu(y) \leq 2 \int_{\mathbb{Q}_p} |v(y)| d\nu(y) \|\phi\|_{L^2}. \quad \square \end{aligned}$$

Lemma 2.2. *The operator $\mathbf{V}$ is symmetric on $L^2(\mathbb{Q}_p)$.*

Proof. We write

$$\langle \psi_J, \mathbf{V}\psi_I \rangle = \int_{\mathbb{Q}_p} \psi_J(x) \left(\int_{\mathbb{Q}_p} V(x-y)(\psi_I(x) - \psi_I(y)) d\nu(y) \right) d\nu(x).$$

Expanding the terms inside the integral, we obtain:

$$\begin{aligned} \langle \psi_J, \mathbf{V}\psi_I \rangle &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y)\psi_J(x)\psi_I(x) d\nu(y) d\nu(x) \\ &\quad - \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y)\psi_J(x)\psi_I(y) d\nu(y) d\nu(x). \end{aligned} \quad (2.1)$$

Since we assumed that the kernel is symmetric, $V(x-y) = V(y-x)$, we rewrite the above integrals interchanging x and y ,

$$\begin{aligned} \langle \psi_J, \mathbf{V}\psi_I \rangle &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y)\psi_J(y)\psi_I(y) d\nu(x) d\nu(y) \\ &\quad - \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y)\psi_J(y)\psi_I(x) d\nu(x) d\nu(y) \end{aligned} \quad (2.2)$$

We now sum (2.1) and (2.2) and divide by 2 to maintain the equality,

$$\begin{aligned} \langle \psi_J, \mathbf{V}\psi_I \rangle &= \frac{1}{2} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y) \left(\psi_J(x)\psi_I(x) + \psi_J(y)\psi_I(y) \right. \\ &\quad \left. - \psi_J(x)\psi_I(y) - \psi_J(y)\psi_I(x) \right) d\nu(x) d\nu(y). \end{aligned}$$

Finally, we get equality

$$\langle \psi_J, \mathbf{V}\psi_I \rangle = \frac{1}{2} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} V(x-y) (\psi_J(x) - \psi_J(y)) (\psi_I(x) - \psi_I(y)) d\nu(x) d\nu(y) \quad (2.3)$$

from which symmetry follows. $\square$

Lemma 2.3. Let $[\mathbf{V}_N(t)_{I,J}]$ be the matrix of the operator $\mathbf{V}_N(t) : E_N \rightarrow E_N$ in the basis $\{\psi_I\}_{I \in \mathcal{K}_N}$.

- (i) We have $\mathbf{V}_N(t)_{I,J} = a(t)(\mathbf{v}_N)_{I,J}$, $I, J \in \mathcal{K}_N$, where $[(\mathbf{v}_N)_{I,J}]$ is the matrix of $\mathbf{v}$ in the basis $\{\psi_I\}_{I \in \mathcal{K}_N}$.
- (ii) The matrix of the operator $\mathbf{V}$ is symmetric, i.e. $\mathbf{V}_N(t)_{I,J} = \mathbf{V}_N(t)_{J,I}$.
- (iii) For every $N \in \mathbb{N}$, there is $C_N > 0$ such that for all $I, J \in \mathcal{K}_N$, $|\mathbf{V}_N(t)_{I,J}| \leq C_N$ and $\|\mathbf{V}_N(t)\|_{\ell^2 \rightarrow \ell^2} \leq C_N$ uniformly in $t \in [0, T]$.

Proof. (ii) follows from Lemma 2.2. (i) and (iii) are obvious. $\square$

Immediately from Theorem 1.5 we get the following lemma.

Lemma 2.4. Let $[(\mathbf{D}_N^\alpha)_{I,J}]$ be the matrix of the operator $\mathbf{D}_N^\alpha : E_N \rightarrow E_N$ in the basis $\{\psi_I\}_{I \in \mathcal{K}_N}$. Then there exists $c_N > 0$ such that for all $I, J \in \mathcal{K}_N$,

$$(\mathbf{D}_N^\alpha)_{I,J} = \lambda_I \delta_{I,J} \quad \text{and} \quad \|\mathbf{D}_N^\alpha\|_{\ell^2 \rightarrow \ell^2} \leq c_N.$$

uniformly in $t \in [0, T]$.

2.2. Proof of Theorem 1.12. For every $\ell \in \{0, \dots, n-1\}$, define the operators

$$\begin{aligned} U_{N,\ell,n}(t, s) &= U_N(s + (\ell + 1)h, s + \ell h) = \mathbf{I}_N \\ &+ \sum_{k=1}^{\infty} (-i)^k \int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 \dots \int_{s+\ell h}^{\tau_{k-1}} d\tau_k (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_1)) \dots (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_k)) \end{aligned} \quad (2.4)$$

with $\tau_0 = s + (\ell + 1)h$. Clearly,

$$U_N(t, s) = \prod_{\ell=n-1}^0 U_{N,\ell,n}(t, s). \quad (2.5)$$

Lemma 2.5. For all $n, N \in \mathbb{N}_+$ and all $s, t \in [0, T]$ with $t \geq s$ the following statements hold:

(a) We have

$$U_{N,\ell,n}(t, s) = \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h)) + O_N(h^2) \quad (2.6)$$

$$= \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) + O_N(h^2) \quad (2.7)$$

as $h \rightarrow 0$, uniformly in $\ell \in \{0, \dots, n-1\}$.

(b) We have

$$\begin{aligned} U_{N,\ell,n}(t, s) &= \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) \exp\left(-i\frac{h}{2}[\widehat{H}_{0,N}\mathbf{W}_{N,\ell}(s, h)]\right) \\ &\quad + O_N(h^3) \end{aligned} \quad (2.8)$$

$$= \exp\left(-i\frac{h}{2}\mathbf{D}_N^\alpha\right) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) \exp\left(-i\frac{h}{2}\mathbf{D}_N^\alpha\right) + O_N(h^3) \quad (2.9)$$

as $h \rightarrow 0$, uniformly in $\ell \in \{0, \dots, n-1\}$.

Proof. We split (2.4) as

$$U_{N,\ell,n}(t, s) = \mathbf{I}_N - i \int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_1)) + U'_{N,\ell,n}(t, s), \quad (2.10)$$

where

$$\begin{aligned} U'_{N,\ell,n}(t, s) &= \sum_{k=2}^{\infty} (-1)^k \int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 \dots \int_{s+\ell h}^{\tau_{k-1}} d\tau_k (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_1)) \dots (\mathbf{D}_N^\alpha + \mathbf{V}_N(\tau_k)). \end{aligned}$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} \|U'_{N,\ell,n}(t, s)\|_{\ell^2 \rightarrow \ell^2} &= \sum_{k=2}^{\infty} \int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 \dots \int_{s+\ell h}^{\tau_{k-1}} d\tau_k \prod_{j=1}^k (\|\mathbf{D}_N^\alpha\|_{\ell^2 \rightarrow \ell^2} + \|\mathbf{V}_N(\tau_j)\|_{\ell^2 \rightarrow \ell^2}) \\ &= \sum_{k=2}^{\infty} \frac{h^k}{k!} (2c_N)^k = O_N(h^2) \end{aligned} \quad (2.11)$$

as $h \rightarrow 0$. From (2.10), (2.11) we get

$$\begin{aligned} U_{N,\ell,n}(t, s) &- \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h)) \\ &= U_{N,\ell,n}(t, s) - (\mathbf{I}_N - ih(\mathbf{D}_N^\alpha + \mathbf{V}_N(s + \ell h)) + O_N(h^2)) \\ &= i \underbrace{\int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 (\mathbf{V}_N(s + \ell h) - \mathbf{V}_N(\tau_1))}_{=\mathcal{I}(h)} + O_N(h^2) \end{aligned} \quad (2.12)$$

as $h \rightarrow 0$. Now we estimate $\mathcal{I}(h)$,

$$\begin{aligned} \|\mathcal{I}(h)\|_{\ell^2 \rightarrow \ell^2} &\leq \int_{\ell h}^{(\ell+1)h} d\tau \|\mathbf{V}_N(s + \ell h) - \mathbf{V}_N(s + \tau)\|_{\ell^2 \rightarrow \ell^2} \\ &\leq O_N(1) \sum_{K,L \in \mathcal{K}_N} \int_{\ell h}^{(\ell+1)h} d\tau |\mathbf{V}_N(s + \ell h)_{K,L} - \mathbf{V}_N(s + \tau)_{K,L}|. \end{aligned}$$

By Lemma 2.3, $|\mathbf{V}_N(s + \ell h)_{K,L} - \mathbf{V}_N(s + \tau)_{K,L}| = |\mathbf{v}_{K,L}||a(s + \ell h) - a(s + \tau)|$. Thus, by the Lipschitz continuity of a ,

$$\|\mathcal{I}(h)\|_{\ell^2 \rightarrow \ell^2} \leq O_N(1) \sum_{K,L \in \mathcal{K}_N} |\mathbf{v}_{K,L}| \int_{\ell h}^{(\ell+1)h} d\tau |\ell h - \tau| \leq O_N(1)h^2. \quad (2.13)$$

Now, equation (2.6) follows from (2.12) and (2.13).

In order to show (2.7) we replace $\mathbf{V}_N(s + \ell h)$ by $\mathbf{W}_{N,\ell}(s, h)$ in (2.12) and get

$$U_{N,\ell,n}(t, s) - \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h))$$

$$= i \underbrace{\int_{s+\ell h}^{s+(\ell+1)h} d\tau_1 (\mathbf{W}_{N,\ell}(s, h) - \mathbf{V}_N(\tau_1))}_{=\mathcal{J}(h)} + O_N(h^2). \quad (2.14)$$

Similarly to the previous case we write

$$\begin{aligned} \|\mathcal{J}(h)\|_{\ell^2 \rightarrow \ell^2} &\leq O_N(1) \sum_{K, L \in \mathcal{K}_N} \left| \int_{\ell h}^{(\ell+1)h} d\tau (\mathbf{W}_{N,\ell,m,m'}(s, h) - \mathbf{V}_{N,m,m'}(s + \tau)) \right| \\ &= O_N(1) \sum_{K, L \in \mathcal{K}_N} |\mathbf{v}_{K,L}| \left| \frac{h}{2} (a(s + \ell h) + a(s + (\ell + 1)h)) - \int_{\ell h}^{(\ell+1)h} d\tau a(s + \tau) \right|. \end{aligned}$$

By the trapezoidal rule estimate for integration we get that $\mathcal{J}(h) = O_N(h^3)$. Thus, (2.7) holds.

Now we are going to prove (2.8). Exactly in the same way as in [19] we get

$$\begin{aligned} U_{N,\ell,n}(t, s) - \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{W}_{N,\ell}(s, h)) \exp\left(\frac{h^2}{2}[\mathbf{D}_N^\alpha, W_{N,\ell}(s, h)]\right) \\ = i \int_{\ell h}^{(\ell+1)h} d\tau (\mathbf{W}_{N,\ell}(s, h) - \mathbf{V}_N(s + \tau)) \\ + \underbrace{\int_{\ell h}^{(\ell+1)h} d\tau (\mathbf{D}_N^\alpha + \mathbf{V}_N(s + \tau)) \int_{\ell h}^{\tau} d\sigma (\mathbf{W}_{N,\ell}(s, h) - \mathbf{V}_N(s + \sigma))}_{=\mathcal{I}} \\ + \int_{\ell h}^{(\ell+1)h} d\tau (\tau - \ell h) (\mathbf{W}_{N,\ell}(s, h) - \mathbf{V}_N(s + \tau)) (\mathbf{D}_N^\alpha + \mathbf{W}_{N,\ell}(s, h)) \\ + O_N(h^3). \quad (2.15) \end{aligned}$$

The first integral on the right-hand side above is $O_N(h^3)$ since it is $\mathcal{J}(h)$ from (2.14). In the same way as $\mathcal{J}(h)$ one bound $\mathcal{I} = O_N((\tau - \ell h)^3)$. By Lemma 2.3 and Lemma 2.4 $\|\mathbf{D}_N^\alpha + \mathbf{V}_N(s + \tau)\|_{\ell^2 \rightarrow \ell^2} \leq C$. Thus,

$$\|\mathcal{I}\|_{\ell^2 \rightarrow \ell^2} \leq O_N(1) \int_{\ell h}^{(\ell+1)h} d\tau \|\mathcal{J}\|_{\ell^2 \rightarrow \ell^2} \leq O_N(1) \int_{\ell h}^{(\ell+1)h} d\tau (\tau - \ell h)^3 = O_N(h^4)$$

uniformly in ℓ . Finally, for the third term in (2.15) we use trapezoidal rule as previously, and get the bound $O_N(h^3)$, uniformly in ℓ . Thus, (2.8) is proved. The proof of (2.9) is the same. $\square$

Proof of Theorem 1.12. For details see [19]. The point (a) is very well known Dyson formula (see [21, Theorem X.69] for details). So we go to other points. Let us begin with (1.6). By (2.5), we have

$$U_N(t, s) - \prod_{\ell=n-1}^0 \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h))$$

$$= \prod_{\ell=n-1}^0 U_{N,\ell,n}(t, s) - \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h)). \quad (2.16)$$

Since the operators in the product above are unitary with respect to the $\ell^2 \rightarrow \ell^2$ -norm, from (2.16) and (2.6), we get

$$\begin{aligned} & \left\| U_N(t, s) - \prod_{\ell=n-1}^0 \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h)) \right\|_{\ell^2 \rightarrow \ell^2} \\ & \leq \sum_{\ell=0}^{n-1} \|U_{N,\ell,n}(t, s) - \exp(-ih\mathbf{D}_N^\alpha) \exp(-ih\mathbf{V}_N(s + \ell h))\|_{\ell^2 \rightarrow \ell^2} \\ & = nO_N(h^2) = O_N(h) \end{aligned}$$

since the estimates of Lemma 2.5 are all uniform in ℓ . The proofs of the remaining parts of Theorem 1.12 is similar. $\square$

Acknowledgments. The author thanks the reviewer for drawing his attention to the existing connections between the analysis of Schrödinger operators on p -adic spaces and quantum networks. In particular, for pointing out the relevant literature on this subject.

References

- [1] V.A. Aguilar-Arteaga, M. Bocardo-Gaspar, and S. Estala-Arias, *Lévy processes on certain totally disconnected groups*, Bol. Soc. Mat. Mex. (3) **32** (2026), Paper No. 10.
- [2] V.A. Aguilar-Arteaga, S.M. Delfín-Prieto, and S. Estala-Arias, *Finite adelic wavelet bases and a pseudodifferential equation*, p -Adic Numbers Ultrametric Anal. Appl. **16** (2024), 275–288.
- [3] S. Albeverio, A.Yu. Khrennikov, and V.M. Shelkovich, *Harmonic analysis in the p -adic Lizorkin spaces: fractional operators, pseudo-differential equations, p -adic wavelets, Tauberian theorems*, J. Fourier Anal. Appl. **12** (2006), 393–425.
- [4] A.V. Antoniouk, A.Yu. Khrennikov, and A.N. Kochubei, *Multidimensional nonlinear pseudo-differential evolution equation with p -adic spatial variables*, J. Pseudo-Differ. Oper. Appl. **11** (2020), 311–343.
- [5] A. Bendikov, A. Grigor'yan, and S. Molchanov, *Hierarchical Schrödinger type operators: the case of locally bounded potentials*, Operator Theory and Harmonic Analysis (OTHA) 2020, Part II: Probability-Analytical Models, Methods and Applications, **358**, Springer Proc. Math. Stat., Springer, 2021.
- [6] A. Bendikov, A. Grigor'yan, and S. Molchanov, *Hierarchical Schrödinger operators with singular potentials*. Tr. Mat. Inst. Steklova **323** (2023), 17–52 (Russian); Engl. Transl.: Proc. Steklov Inst. Math. **323** (2023), 12–46.
- [7] A. Kh. Bikulov and I.V. Volovich, *p -adic Brownian motion*, Izv. Ross. Akad. Nauk Ser. Mat. **61** (1997), 75–90 (Russian); Engl. Transl.: Izv. Math. **61** (1997), 537–552.

- [8] A.Kh. Bikulov and A. P. Zubarev, *Complete systems of eigenfunctions of the Vladimirov operator in $L^2(B_r)$ and $L^2(\mathbb{Q}_p)$* , Fundam. Prikl. Mat. **21** (2016), 39–56 (Russian); Engl. Transl.: J. Math. Sci. (N.Y.) **237** (2019), 362–374.
- [9] P.E. Bradley and Á.M. Ledezma, *A nonautonomous p -adic diffusion equation on time changing graphs*, Rep. Math. Phys. **95** (2025), 155–180.
- [10] P.E. Bradley and Á.M. Ledezma, *Hearing shapes via p -adic Laplacians*, J. Math. Phys. **64** (2023), Paper No. 113502, 15 pp.
- [11] A.Yu. Khrennikov and A. N. Kochubei, *p -adic analogue of the porous medium equation*, J. Fourier Anal. Appl. **24** (2018), 1401–1424.
- [12] A.Yu. Khrennikov, S.V. Kozyrev, and W.A. Zúñiga-Galindo, *Ultrametric Pseudodifferential Equations and Applications*, Cambridge University Press, Cambridge, 2018.
- [13] S.V. Kozyrev, A.Yu. Khrennikov, and V.M. Shelkovich, *p -adic wavelets and their applications*, Trudy Mat. Inst. Steklova **285** (2014), 166–206 (Russian); Proc. Steklov Inst. Math. **285** (2014), 157–196.
- [14] A.N. Kochubei, *Pseudo-Differential Equations and Stochastics over Non-Archimedean Fields*, Monographs and Textbooks in Pure and Applied Mathematics, **244**, Marcel Dekker, Inc., New York, 2001.
- [15] S.V. Kozyrev, *Wavelet analysis as a p -adic spectral analysis*, Izvestia Akademii Nauk, Seria Math. **66** (2002), 149–158 (Russian); Engl. transl.: Izv. Math. **66** (2002), 367–376.
- [16] S.V. Kozyrev, *Dynamics on rugged landscapes of energy and ultrametric diffusion*, p -Adic Numbers, Ultrametric Anal. Appl. **2** (2010), 122–132.
- [17] S.V. Kozyrev and A. Yu. Khrennikov, *p -adic integral operators in wavelet bases*, Dokl. Akad. Nauk **437** (2011), 457–461 (Russian); Engl. Transl.: Dokl. Math. **83** (2011), 209–212.
- [18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, **44**, Applied Mathematical Sciences, Springer-Verlag, New York, 1983.
- [19] W.P. Petersen and P.A. Vuillermot, *Product approximations for a class of quantum anharmonic oscillators*, Z. Angew. Math. Phys. **65** (2014), 613–643.
- [20] A.S. Plyashechnik, *Feynman formula for Schrödinger-type equations with time- and space-dependent coefficients*, Russ. J. Math. Phys. **19** (2012), 340–359.
- [21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.
- [22] O.G. Smolyanov and N.N. Shamarov, *Hamiltonian Feynman formulas for equations containing the Vladimirov operator with variable coefficients*, Dokl. Akad. Nauk **440** (2011), 597–602 (Russian); Engl. transl.: Dokl. Math. **84** (2011), 689–694.
- [23] S.M. Torba and W.A. Zúñiga-Galindo, *Parabolic type equations and Markov stochastic processes on adeles*, J. Fourier Anal. Appl. **19** (2013), 792–835.
- [24] R. Urban, *The Vladimirov operator with variable coefficients on finite adeles and the Feynman formulas for the Schrödinger equation*, J. Math. Phys. **65** (2024), Paper No. 042103.

- [25] R. Urban, *Feynman formulas for qp - and pq -quantization of some Vladimirov type time-dependent Hamiltonians on finite adeles*, *Anal. Math. Phys.* **14** (2024), Paper No. 105, 38 pp.
- [26] V.S. Vladimirov and I.V. Volovich, *p -adic quantum mechanics*, *Commun. Math. Phys.* **123** (1989), 659–676.
- [27] V.S. Vladimirov and I.V. Volovich, *p -adic Schrödinger-type equation*, *Lett. Math. Phys.* **18** (1989), 43–53.
- [28] V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov, *p -adic analysis and mathematical physics*, Nauka, Moscow, 1994 (Russian); Engl. transl.: Series on Soviet and East European Mathematics, **1**, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.
- [29] I.V. Volovich, *Number theory as the ultimate physical theory*, preprint, Cern-th. 4781/87, Geneva 1987. Available from: <https://cds.cern.ch/record/179558/files/198708102.pdf>.
- [30] I.V. Volovich, *Number theory as the ultimate physical theory*, *p -Adic Numbers Ultrametric Anal. Appl.* **2** (2010), 77–87.
- [31] P.-A. Vuillermot, *A generalization of Chernoff’s product formula for time-dependent operators*, *J. Funct. Anal.* **259** (2010), 2923–2938.
- [32] D. Weisbart, *On infinitesimal generators and Feynman-Kac integrals of adelic diffusion*, *J. Math. Phys.* **62** (2021), Paper No. 103504.
- [33] B.A. Zambrano-Luna and W.A. Zúñiga-Galindo, *p -adic cellular neural networks: applications to image processing*, *Phys. D* **446** (2023), Paper No. 133668.
- [34] B.A. Zambrano-Luna and W.A. Zúñiga-Galindo, *p -adic cellular neural networks*, *J. Nonlinear Math. Phys.* **30** (2023), 34–70.
- [35] W.A. Zúñiga-Galindo, *Pseudodifferential Equations over Non-Archimedean Spaces*, **2174**, Lecture Notes in Mathematics, Springer, Cham, 2016.
- [36] W.A. Zúñiga-Galindo, *Ultrametric diffusion, rugged energy landscapes and transition networks*, *Phys. A* **597** (2022), Paper No. 127221.
- [37] W.A. Zúñiga-Galindo, *p -Adic Analysis — Stochastic Processes and Pseudo-Differential Equations*, De Gruyter, Berlin, 2025.
- [38] W.A. Zúñiga-Galindo, *2-adic quantum mechanics, continuous-time quantum walks, and the space discreteness*, *Fortschr. Phys.* **73** (2025), Paper No. e70019, 15 pp.
- [39] W.A. Zúñiga-Galindo and N.P. Mayes, *p -Adic quantum mechanics, infinite potential wells, and continuous-time quantum walks*, *Rev. Math. Phys.* **38** (2026), No. 06, Paper No. 2550027, 32 pp.
- [40] W.A. Zúñiga-Galindo and L.F. Chacón-Cortés, *Continuous-time quantum Markov chains and discretizations of p -adic Schrödinger equations: comparisons and simulations*, preprint, <https://arxiv.org/abs/2508.06712>, 49 pp.

Received January 8, 2026, revised April 24, 2026.

Roman Urban,

Wrocław University, Pl. Grunwaldzko 2/4, Wrocław, 50-384, Poland,

E-mail: roman.urban@math.uni.wroc.pl

**Нотатка про p -адичні рівняння Шредінгера з
нелокальними потенціалами**

Roman Urban

Ми розглядаємо p -адичне рівняння Шредінгера $i\partial_t u = (\mathbf{D}^\alpha + \mathbf{V}(t))u$ із залежним від часу нелокальним потенціалом $\mathbf{V}$, визначеним на полі p -адичних чисел $\mathbb{Q}_p$. Ми будуємо розв'язок u у просторі $\mathcal{H} = L^2(\mathbb{Q}_p)$, апроксимуючи його послідовністю функцій, які задовольняють аналогічні рівняння у скінченновимірних підпросторах $E_N \subset \mathcal{H}$.

Ключові слова: оператор Владиминова, нелокальний потенціал, рівняння Шредінгера, скінченновимірні апроксимації