

The Initial-Boundary Value Problem for the General Matrix NLS Equation on a Finite Interval via Fokas Method

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In this paper, we analyze the initial-boundary value problem for the integrable general matrix nonlinear Schrödinger (NLS) equation with a 4×4 Lax pair on a finite interval, utilizing the original Fokas unified transform method. The generalizations include the defocusing matrix NLS system, the focusing matrix NLS system and two mixed-sign matrix NLS equations, which are less explored in the literature. We treat the corresponding 4×4 matrix Lax pair equations as 2×2 block matrix problems, performing spectral analysis on both components of the Lax pair simultaneously. By analyzing the algebraic connection between the spectral functions associated with appropriately normalized solutions of the Lax equations, we derive a “global relation”. This relation is crucial for characterizing the unknown boundary values in terms of the given data. This approach allows the systematic investigation of non-linearized boundary conditions and characterizing the mapping between the Dirichlet and Neumann boundary values. It is also proved that as the interval length L tends to infinity, the solution formula for the initial-boundary value problem on a finite interval reduces to that on the half-line.

Key words: general matrix nonlinear Schrödinger equation, initial-boundary value problem, Riemann–Hilbert problem, global relation, Fokas method

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1. Introduction

In 1967, Gardner, Greene, Kruskal, and Miura first proposed the inverse scattering transform (IST) [22]. They successfully applied it to solve the initial value problem for the KdV equation by relating it to the one-dimensional linear Schrödinger equation. Subsequently, the IST was employed to solve numerous integrable nonlinear evolution equations possessing Lax pairs. In the early 1990s, Deift and Zhou [10] developed the IST by introducing a powerful nonlinear steepest-descent method for the analytical study of long-time asymptotics of the Cauchy problem for (1+1)-dimensional integrable nonlinear evolution equations, such as the mKdV and nonlinear Schrödinger (NLS) equations. In the late 1990s, Fokas [13, 14] further extended this framework by proposing a unified transform method for studying initial-boundary value problems of linear and nonlinear integrable equations with Lax pairs. This approach analyzes initial-boundary value

(IBV) problems by performing spectral analysis on both components of the Lax pair simultaneously and by utilizing the global relations that connect the resulting spectral functions. This differs from the classical IST, which involves spectral analysis of only one part of the Lax pair. The Fokas method effectively integrates the key ideas of IST with the framework of the Riemann–Hilbert problem [14,34].

As a novel and effective method, the Fokas method has been applied to study the initial boundary value problems for linear partial differential equations with 2×2 Lax pairs and some integrable nonlinear evolution equations on the half-line and in finite intervals, such as the NLS equation [13, 18–20, 25], the sine-Gordon equation [16,33], the KdV equation [17,37], the modified KdV equation [3–5, 7, 8, 15, 21, 27, 30, 31]. Recently, Lenells [28] further developed the Fokas method to analyze the IBV problems of integrable nonlinear evolution equations with 3×3 Lax pairs on the half-line. Subsequently, the improved method was applied to the IBV problems for other integrable nonlinear evolution equations with 3×3 Lax pairs on the half-line or in finite intervals, such as the Degasperis–Procesi equation [6,9,29], the Sasa–Satsuma equation [39], the Manakov equation [2,23,41], and the Ostrovsky–Vakhnenko equation [42]. Yan [43] explored the IBV problem for an integrable spin-1 Gross–Pitaevskii system with a 4×4 Lax pair on the finite interval by extending the Fokas unified transform approach.

In recent years, based on the research work of Deift and Zhou [10], Guo et al. [24], and Geng et al. [23] have respectively applied the nonlinear steepest descent method to a 3×3 spectral problem model and studied the long-time asymptotic behavior of the solution. Fan et al. [40] investigated a derivative NLS model with a gradient initial value condition and obtained the long-time asymptotic properties. Wang et al. [38] examined the NLS model under non-zero boundary value conditions and also obtained the corresponding long-term asymptotic properties. In addition, Tian et al. [36] analyzed the Gerdjikov–Ivanov-type derivative nonlinear Schrödinger equation and analyzed the long-time asymptotics of the solution to its initial-boundary value problem.

In the past two decades, multi-component Bose–Einstein condensates (BECs) have been a very active field of research. These systems can be analyzed within the framework of mean-field theory, with the static and dynamic properties of their nonlinear excitations well described by the coupled Gross–Pitaevskii equations [12, 26, 32, 35], which are a variant of the so-called defocusing vector NLS equation [1, 11]. For $F = 1$, spinor BECs can be described by the matrix NLS equation,

$$iQ_t + Q_{xx} - 2vQQ^\dagger Q = 0_{2 \times 2}. \quad (1.1)$$

We generalize the above matrix NLS equation, that is,

$$iQ_t + Q_{xx} - 2Q\Sigma Q^\dagger \Omega Q = 0_{2 \times 2}, \quad (1.2)$$

where $Q = Q(x, t)$ is a 2×2 complex symmetric matrix-valued function and $Q^\dagger$ denotes the Hermitian conjugate of Q . For any 2×2 Hermitian matrix Σ and Ω , equation (1.2) is integrable. The latter can be chosen in the standard form without loss of generality, that is, the diagonal and off-diagonal terms are

equal to 0 or ± 1 . Since we are interested in a fully coupled system rather than a triangular system, we assume that Σ and Ω are 2×2 diagonal matrices with elements ± 1 . Let $\Sigma = \text{diag}(\sigma_{11}, \sigma_{22})$, $\Omega = \text{diag}(\omega_{11}, \omega_{22})$, and let $\sigma_{11}^2 = \sigma_{22}^2 = \omega_{11}^2 = \omega_{22}^2 = 1$. The compatibility requirement for the off-diagonal terms in the system (1.2) is $\sigma_{11}\omega_{22} = \sigma_{22}\omega_{11}$.

Case 1:

$$\Sigma = \Omega = I_2$$

(or equivalently, $\Sigma = \Omega = -I_2$), which yields:

$$\begin{cases} i\partial_t q_1 + \partial_x^2 q_1 - 2q_1 [|q_1|^2 + 2|q_0|^2] - 2q_0^2 q_{-1}^* = 0, \\ i\partial_t q_{-1} + \partial_x^2 q_{-1} - 2q_{-1} [|q_{-1}|^2 + 2|q_0|^2] - 2q_0^2 q_1^* = 0, \\ i\partial_t q_0 + \partial_x^2 q_0 - 2q_0 [|q_1|^2 + |q_0|^2 + |q_{-1}|^2] - 2q_1 q_0^* q_{-1} = 0, \end{cases}$$

which is a matrix NLS system of defocusing type.

Case 2:

$$\Sigma = -\Omega = I_2$$

(or equivalently, $\Omega = -\Sigma = I_2$), which yields:

$$\begin{cases} i\partial_t q_1 + \partial_x^2 q_1 + 2q_1 [|q_1|^2 + 2|q_0|^2] + 2q_0^2 q_{-1}^* = 0, \\ i\partial_t q_{-1} + \partial_x^2 q_{-1} + 2q_{-1} [|q_{-1}|^2 + 2|q_0|^2] + 2q_0^2 q_1^* = 0, \\ i\partial_t q_0 + \partial_x^2 q_0 + 2q_0 [|q_1|^2 + |q_0|^2 + |q_{-1}|^2] + 2q_1 q_0^* q_{-1} = 0, \end{cases}$$

which is a matrix NLS system of focusing type.

Case 3:

$$\Sigma = \Omega = \sigma_3$$

(or equivalently, $\Sigma = \Omega = -\sigma_3$), which yields:

$$\begin{cases} i\partial_t q_1 + \partial_x^2 q_1 - 2q_1 [|q_1|^2 - 2|q_0|^2] - 2q_0^2 q_{-1}^* = 0, \\ i\partial_t q_{-1} + \partial_x^2 q_{-1} - 2q_{-1} [|q_{-1}|^2 - 2|q_0|^2] - 2q_0^2 q_1^* = 0, \\ i\partial_t q_0 + \partial_x^2 q_0 - 2q_0 [|q_1|^2 - |q_0|^2 + |q_{-1}|^2] + 2q_1 q_0^* q_{-1} = 0. \end{cases}$$

Case 4:

$$\Sigma = -\Omega = \sigma_3$$

(or equivalently, $\Sigma = -\Omega = \sigma_3$), which yields:

$$\begin{cases} i\partial_t q_1 + \partial_x^2 q_1 + 2q_1 [|q_1|^2 - 2|q_0|^2] + 2q_0^2 q_{-1}^* = 0, \\ i\partial_t q_{-1} + \partial_x^2 q_{-1} + 2q_{-1} [|q_{-1}|^2 - 2|q_0|^2] + 2q_0^2 q_1^* = 0, \\ i\partial_t q_0 + \partial_x^2 q_0 + 2q_0 [|q_1|^2 - |q_0|^2 + |q_{-1}|^2] - 2q_1 q_0^* q_{-1} = 0. \end{cases}$$

Let $\tilde{\Sigma} = \text{diag}(1, \beta)$, $\tilde{\Omega} = \text{diag}(\alpha, \alpha\beta)$ ($\alpha^2 = \beta^2 = 1$), then the equations can be written as

$$\begin{cases} i\partial_t q_1 + \partial_x^2 q_1 - 2\alpha q_1 [|q_1|^2 + 2\beta|q_0|^2] - 2\alpha q_0^2 q_{-1}^* = 0, \\ i\partial_t q_{-1} + \partial_x^2 q_{-1} - 2\alpha q_{-1} [|q_{-1}|^2 + 2\beta|q_0|^2] - 2\alpha q_0^2 q_1^* = 0, \\ i\partial_t q_0 + \partial_x^2 q_0 - 2\alpha q_0 [|q_1|^2 + \beta|q_0|^2 + |q_{-1}|^2] - 2\alpha\beta q_1 q_0^* q_{-1} = 0 \end{cases} \quad (1.3)$$

with the IBV condition

$$\left\{ \begin{array}{l} \text{Initial data:} \\ \qquad q_j(x, 0) = q_{0j}(x), \quad 0 < x < L, \\ \text{Dirichlet boundary conditions:} \\ \qquad q_j(0, t) = f_{0j}(t), q_j(L, t) = g_{0j}(t), \quad 0 < t < T, \\ \text{Neumann boundary conditions:} \\ \qquad q_{jx}(0, t) = f_{1j}(t), q_{jx}(L, t) = g_{1j}(t), \quad 0 < t < T, \end{array} \right.$$

for $j = 1, 0, -1$ in the finite region,

$$\bar{\Omega} = \{(x, t) \mid x \in [0, L], t \in [0, T]\}. \quad (1.4)$$

If $L > 0$ and $T > 0$ are fixed finite length and time, then the initial data $q_{0j}(x)$, $j = 1, 0, -1$, and the boundary data $f_{0j}(t), f_{1j}(t)$, $j = 1, 0, -1$, are sufficiently smooth and compatible at the point $(x, t) = (0, 0)$. Among them, the initial data $q_{0j}(x)$, $j = 1, 0, -1$, and the boundary data $f_{sj}(t), g_{sj}(t)$, $s = 0, 1$, $j = 1, 0, -1$, are respectively sufficiently smooth and compatible at the points $(x, t) = (0, 0), (L, 0)$.

The IBV condition is written in matrix form as follows:

$$\left\{ \begin{array}{l} \text{Initial data:} \\ \qquad Q(x, 0) = Q_0(x), \quad 0 < x < L, \\ \text{Dirichlet boundary conditions:} \\ \qquad F(0, t) = F_0(t), \quad G(L, t) = G_0(t), \quad 0 < t < T, \\ \text{Neumann boundary conditions:} \\ \qquad F_x(0, t) = F_1(t), \quad G_x(L, t) = G_1(t), \quad 0 < t < T. \end{array} \right. \quad (1.5)$$

This paper focuses on the IBV problem for the four cases mentioned above by employing the Fokas method. This method was previously applied to scalar integrable NLS equations with a 2×2 Lax pair defined on a finite interval in [19]. Although the authors of [43] investigated the same problem for the integrable spin-1 Gross–Pitaevskii equations, several distinctions exist between their work and the present paper. First, the equations studied in [43] correspond to Cases 1 and 2 in this paper, namely the focusing and defocusing matrix NLS equations. In contrast, we additionally address Cases 3 and 4, i.e., the mixed-sign matrix NLS equations. Second, we decompose the 4×4 matrix into 2×2 block form and then proceed within the framework of the original Fokas unified transform method, which constitutes a distinct strategy from that adopted in [43]. Finally, via analysis of the global relations, we examine nonlinear boundary conditions, construct the mapping between Dirichlet and Neumann boundary data, derive a full characterization of such data, and further represent the spectral functions as solutions to a system of nonlinear integral equations jointly constrained by the prescribed initial and boundary data.

The remainder of this paper is structured as follows. In Section 2, we introduce the 4×4 Lax pair associated with equation (1.3) and conduct its spectral

analysis, including eigenfunctions, jump matrices and global relations, and exhibit the associated 4×4 matrix RH problem. Section 3 demonstrates how the global relations induce the mapping between Dirichlet and Neumann boundary values. In particular, the formulas derived for finite-interval boundary value problems reduce to those defined on the half-line as the interval length approaches infinity.

2. A Riemann–Hilbert problem formulation

2.1. The general matrix NLS equation and its Lax pair. As is well-known, the matrix NLS equation corresponds to the compatibility condition associated with the Lax pair of the potential matrix $Q(x, t)$. Consequently, the integrable system (1.3) admits the following 4×4 Lax pair:

$$\begin{cases} \psi_x + ik\sigma_4\psi = U(x, t)\psi, \\ \psi_t + 2ik^2\sigma_4\psi = V(x, t, k)\psi, \end{cases} \quad (2.1)$$

where $\psi \equiv \psi(x, t, k)$ is a 4×4 matrix-valued function, $k \in \mathbb{C}$ is a spectral parameter, and

$$\begin{aligned} U(x, t) &= \begin{pmatrix} 0_{2 \times 2} & Q \\ R & 0_{2 \times 2} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} I_2 & 0_{2 \times 2} \\ 0_{2 \times 2} & -I_2 \end{pmatrix}, \\ V(x, t, k) &= 2kU(x, t) + i\sigma_4(U_x - U^2) \\ &= \begin{pmatrix} iQR & 2kQ + iQ_x \\ 2kR - iR_x & iRQ \end{pmatrix}, \end{aligned} \quad (2.2)$$

where

$$Q(x, t) = \begin{pmatrix} q_1(x, t) & q_0(x, t) \\ q_0(x, t) & q_{-1}(x, t) \end{pmatrix}, \quad R = \tilde{\Sigma}Q^\dagger\tilde{\Omega} = \begin{pmatrix} \alpha\bar{q}_1(x, t) & \alpha\beta\bar{q}_0(x, t) \\ \alpha\beta\bar{q}_0(x, t) & \alpha\bar{q}_{-1}(x, t) \end{pmatrix}. \quad (2.3)$$

We introduce the new eigenfunction $\mu(x, t, k)$ defined by

$$\mu(x, t, k) = \psi(x, t, k)e^{i(kx+2k^2t)\sigma_4}. \quad (2.4)$$

Then the Lax pair (2.1) can be expressed as:

$$\begin{cases} \mu_x + ik[\sigma_4, \mu] = U(x, t)\mu, \\ \mu_t + 2ik^2[\sigma_4, \mu] = V(x, t, k)\mu. \end{cases} \quad (2.5)$$

Denote $\hat{\sigma}_4\mu = [\sigma_4, \mu] = \sigma_4\mu - \mu\sigma_4$. It is known that $e^{\hat{\sigma}_4}\mu = e^{\sigma_4}\mu e^{-\sigma_4}$. Then the Lax pair (2.5) can be written in the total derivative form:

$$d \left[e^{i(kx+2k^2t)\hat{\sigma}_4} \mu(x, t, k) \right] = W(x, t, k), \quad (2.6)$$

where $W(x, t, k)$ is explicitly defined as follows:

$$W(x, t, k) = e^{i(kx+2k^2t)\hat{\sigma}_4} [U(x, t)\mu(x, t, k)dx + V(x, t, k)\mu(x, t, k)dt]. \quad (2.7)$$

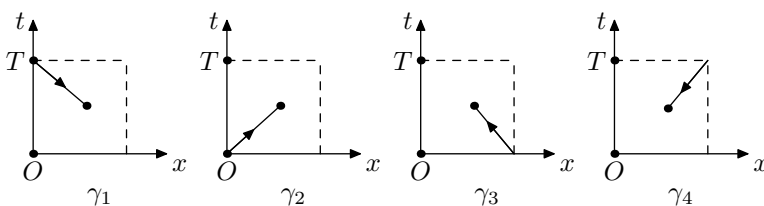


Fig. 2.1: The contours γ_j ($j = 1, 2, 3, 4$) exhibited in the finite region $\bar{\Omega}$.

2.2. Eigenfunctions and their relations. For any point $(x, t) \in \bar{\Omega} = \{(x, t) \mid x \in [0, L], t \in [0, T]\}$, we introduce a family of contours $\{\gamma_j\}_1^4$. Each γ_j links one vertex of the rectangular domain $\bar{\Omega}$, namely $(0, T), (0, 0), (L, 0), (L, T)$, to the interior point (x, t) (see Fig. 2.1). Then for all $(\xi, \tau) \in \gamma_j$, $j = 1, 2, 3, 4$, the following identities along these contours are satisfied:

$$\begin{aligned}\gamma_1 : x - \xi &\geq 0, \quad t - \tau \leq 0, \\ \gamma_2 : x - \xi &\geq 0, \quad t - \tau \geq 0, \\ \gamma_3 : x - \xi &\leq 0, \quad t - \tau \geq 0, \\ \gamma_4 : x - \xi &\leq 0, \quad t - \tau \leq 0.\end{aligned}\tag{2.8}$$

We assume the system (1.3) possesses smooth complex-valued solutions $q_j(x, t)$ for $j = 1, 0, -1$ throughout $\bar{\Omega}$. Based on the Lax pair (2.5), we introduce four sectionally analytic functions $\mu_j(x, t, k)$ defined on the contour family $\{\gamma_j\}_1^4$:

$$\begin{aligned}\mu_j(x, t, k) &= \mathbb{I} + \int_{\gamma_j} e^{-i(kx+2k^2t)\hat{\sigma}_4} W_j(\xi, \tau, k) \\ &= \mathbb{I} + \int_{(x_j, t_j)}^{(x, t)} e^{-i(kx+2k^2t)\hat{\sigma}_4} W_j(\xi, \tau, k), \quad j = 1, 2, 3, 4.\end{aligned}\tag{2.9}$$

In terms of the Volterra integral equation (2.9), where $\mathbb{I} = \text{diag}(1, 1, 1, 1)$, and the integration is performed along a piecewise smooth curve connecting (x_j, t_j) to (x, t) . The function $W_j(x, t, k)$ is specified in equation (2.7) by substituting $\mu_j(x, t, k)$ for $\mu(x, t, k)$. As $W_j(x, t, k)$ admits a closed-form expression, $\mu_j(x, t, k)$ is path-independent. When the integration path is chosen parallel to the x - and t -axes. Equation (2.9) reduces to the simplified form:

$$\begin{aligned}\mu_j(x, t, k) &= \mathbb{I} + \int_{x_j}^x e^{-ik(x-\xi)\hat{\sigma}_4} (U\mu_j)(\xi, t, k) d\xi \\ &\quad + e^{-ik(x-x_j)\hat{\sigma}_4} \int_{t_j}^t e^{-2ik^2(t-\tau)\hat{\sigma}_4} (V\mu_j)(x_j, \tau, k) d\tau, \quad j = 1, 2, 3, 4.\end{aligned}\tag{2.10}$$

Equation (2.10) indicates that each of the two columns of $\mu_j(x, t, k)$ involves the exponential expressions listed below:

$$[\mu_j(x, t, k)]_s : e^{2i[k(x-\xi)+2k^2(t-\tau)]}, \quad s = 1, \quad j = 1, 2, 3, 4, \tag{2.11a}$$

$$[\mu_j(x, t, k)]_s : e^{-2i[k(x-\xi)+2k^2(t-\tau)]}, \quad s = 2, \quad j = 1, 2, 3, 4. \tag{2.11b}$$

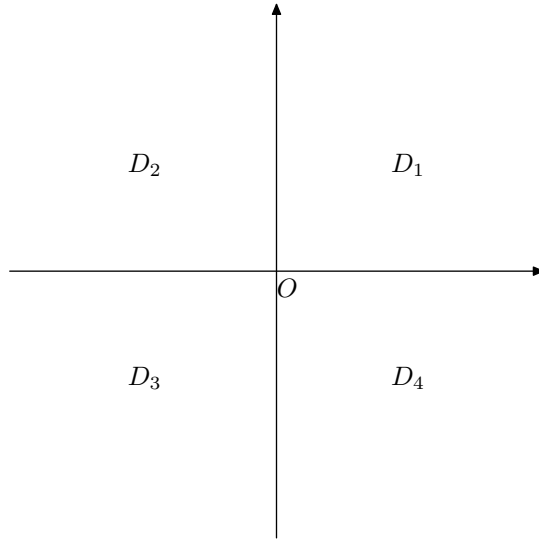


Fig. 2.2: The regions $\{D_j\}_1^4$ separating the complex k -plane.

To characterize the boundedness domains of the eigenfunctions $\{\mu_j(x, t, k)\}_1^4$, we introduce the curve $K = \{k \in \mathbb{C} \mid (\Re f(k))(\Re g(k)) = 0, f(k) = ik, g(k) = ik^2\}$, which separates the complex k -plane into four distinct regions (see Fig. 2.2):

$$\begin{aligned} D_1 &= \{k \in \mathbb{C} \mid \Re f(k) < 0 \text{ and } \Re g(k) < 0\}, \\ D_2 &= \{k \in \mathbb{C} \mid \Re f(k) < 0 \text{ and } \Re g(k) > 0\}, \\ D_3 &= \{k \in \mathbb{C} \mid \Re f(k) > 0 \text{ and } \Re g(k) < 0\}, \\ D_4 &= \{k \in \mathbb{C} \mid \Re f(k) > 0 \text{ and } \Re g(k) > 0\}. \end{aligned} \quad (2.12)$$

This indicates that D_1 and D_3 (D_2 and D_4) are symmetric about the origin of the complex k -plane.

Therefore, from equations (2.8), (2.11), and (2.12), the regions where the different columns of eigenfunctions $\mu_j(x, t, k)$ are bounded and analytic on the complex k -plane are as follows:

$$\begin{aligned} \mu_1(x, t, k) &: (f_- \cap g_+, f_+ \cap g_-) =: (D_2, D_3), \\ \mu_2(x, t, k) &: (f_- \cap g_-, f_+ \cap g_+) =: (D_1, D_4), \\ \mu_3(x, t, k) &: (f_+ \cap g_-, f_- \cap g_+) =: (D_3, D_2), \\ \mu_4(x, t, k) &: (f_+ \cap g_+, f_- \cap g_-) =: (D_4, D_1), \end{aligned} \quad (2.13)$$

where $f_+ := \Re f(k) > 0$, $f_- := \Re f(k) < 0$, $g_+ := \Re g(k) > 0$, $g_- := \Re g(k) < 0$.

The eigenfunctions $\mu_j(x, t, k)$ are related to one another via three matrices $s(k)$, $S(k)$, and $S_L(k)$ as follows:

$$\mu_3(x, t, k) = \mu_2(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_4} s(k), \quad (2.14a)$$

$$\mu_1(x, t, k) = \mu_2(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_4} S(k), \quad (2.14b)$$

$$\mu_4(x, t, k) = \mu_3(x, t, k) e^{-i[k(x-L)+2k^2t]\hat{\sigma}_4} S_L(k), \quad (2.14c)$$

and

$$\mu_1(0, T, k) = \mathbb{I}, \quad \mu_2(0, 0, k) = \mathbb{I}, \quad \mu_3(L, 0, k) = \mathbb{I}, \quad \mu_4(L, T, k) = \mathbb{I}. \quad (2.15)$$

Evaluating equation (2.14a) at $(x, t) = (0, 0)$ yields $s(k) = \mu_3(0, 0, k)$. From equation (2.14b) evaluated at $(x, t) = (0, 0)$, we have $S(k) = \mu_1(0, 0, k)$. Evaluating the same equation at $(x, t) = (0, T)$ gives $S(k) = e^{2ik^2T\hat{\sigma}_4}\mu_2^{-1}(0, T, k)$. For equation (2.14c), evaluation at $(x, t) = (L, 0)$ leads to $S_L(k) = \mu_4(L, 0, k)$, while evaluation at $(x, t) = (L, T)$ yields $S_L(k) = e^{2ik^2T\hat{\sigma}_4}\mu_3^{-1}(L, T, k)$. Evaluating equations (2.14a) and (2.14c) together, we arrive at the relations below:

$$\begin{aligned} \mu_4(x, t, k) &= \mu_2(x, t, k)e^{-i(kx+2k^2t)\hat{\sigma}_4}(s(k)e^{ikL\hat{\sigma}_4}S_L(k)) \\ &= \mu_2(x, t, k)e^{-i(kx+2k^2t)\hat{\sigma}_4}\mathbb{S}(k), \end{aligned} \quad (2.16)$$

where

$$\mathbb{S}(k) = s(k)e^{ikL\hat{\sigma}_4}S_L(k). \quad (2.17)$$

From the above, we conclude that

$$S(k) = \mu_1(0, 0, k) = e^{2ik^2T\hat{\sigma}_4}\mu_2^{-1}(0, T, k), \quad (2.18a)$$

$$s(k) = \mu_3(0, 0, k) = e^{ikL\hat{\sigma}_4}\mu_2^{-1}(L, 0, k), \quad (2.18b)$$

$$S_L(k) = \mu_4(L, 0, k) = e^{2ik^2T\hat{\sigma}_4}\mu_3^{-1}(L, T, k), \quad (2.18c)$$

$$\mathbb{S}(k) = \mu_4(0, 0, k) = e^{i(kL+2k^2T)\hat{\sigma}_4}\mu_2^{-1}(L, T, k). \quad (2.18d)$$

These matrices satisfy certain symmetry properties, so they can be expressed as

$$\begin{aligned} s(k) &= \begin{pmatrix} \overline{s_{22}(k)} & s_{12}(k) \\ \alpha s_{12}(k) & s_{22}(k) \end{pmatrix}, & S(k) &= \begin{pmatrix} \overline{S_{22}(k)} & S_{12}(k) \\ \alpha S_{12}(k) & S_{22}(k) \end{pmatrix}, \\ S_L(k) &= \begin{pmatrix} \overline{S_{L22}(k)} & S_{L12}(k) \\ \alpha S_{L12}(k) & S_{L22}(k) \end{pmatrix}, & \mathbb{S}(k) &= \begin{pmatrix} \overline{\mathbb{S}_{22}(k)} & \mathbb{S}_{12}(k) \\ \alpha \mathbb{S}_{12}(k) & \mathbb{S}_{22}(k) \end{pmatrix}. \end{aligned} \quad (2.19)$$

From the properties of $\mu_j(x, t, k)$, it is known that the functions $S(k)$, $s(k)$, $S_L(k)$, and $\mathbb{S}(k)$ have the following boundedness:

$$\begin{aligned} S(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_1 \cup D_3), \\ s(k) &\text{ is bounded for } k \in (D_3 \cup D_4, D_1 \cup D_2), \\ S_L(k) &\text{ is bounded for } k \in (D_2 \cup D_4, D_1 \cup D_3), \\ \mathbb{S}(k) &\text{ is bounded for } k \in (D_4, D_1). \end{aligned} \quad (2.20)$$

2.3. The global relation. Based on the definitions of the spectral functions $s(k)$, $S(k)$, $S_L(k)$, and $\mathbb{S}(k)$ introduced above, one can readily see their mutual connections. Combining equations (2.14) with equation (2.16), we obtain the following relations:

$$\mu_4(x, t, k) = \mu_2(x, t, k)e^{-i(kx+2k^2t)\hat{\sigma}_4}\mathbb{S}(k)$$

$$\begin{aligned}
&= \mu_2(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_4} \left[s(k) e^{ikL\hat{\sigma}_4} S_L(k) \right] \\
&= \mu_1(x, t, k) e^{-i(kx+2k^2t)\hat{\sigma}_4} \left[S^{-1}(k) s(k) e^{ikL\hat{\sigma}_4} S_L(k) \right]. \quad (2.21)
\end{aligned}$$

Evaluating equation (2.16) at $(x, t) = (0, T)$ and substituting $\mu_1(0, T, k) = \mathbb{I}$, we obtain the global relation

$$c(T, k) = e^{-2ik^2T\hat{\sigma}_4} \left[S^{-1}(k) s(k) e^{ikL\hat{\sigma}_4} S_L(k) \right]. \quad (2.22)$$

where $c(T, k) = \mu_4(0, T, k)$.

This relation plays a vital role in expressing the unknown boundary values via the prescribed boundary and initial data. Using this method, we can systematically analyze nonlinear boundary conditions and describe the mapping relation between Dirichlet and Neumann boundary data in Section 4.

2.4. Symmetry of eigenfunctions. Let

$$\tilde{U}(x, t, k) = -ik\sigma_4 + U(x, t, k), \quad \tilde{V}(x, t, k) = -2ik^2\sigma_4 + V(x, t, k), \quad (2.23)$$

since

$$P_{\pm} \tilde{U}^{\dagger}(x, t, \bar{k}) P_{\pm} = -\tilde{U}(x, t, k), \quad P_{\pm} \tilde{V}^{\dagger}(x, t, \bar{k}) P_{\pm} = -\tilde{V}(x, t, k), \quad (2.24)$$

where

$$P_{\pm} = \begin{pmatrix} \pm \tilde{\Sigma} & 0 \\ 0 & \mp \tilde{\Omega} \end{pmatrix}, \quad P_{\pm}^2 = \mathbb{I}. \quad (2.25)$$

Proposition 2.1. *The eigenfunction $\psi(x, t, k)$ of the Lax pair (2.1) and $\mu_j(x, t, k)$ of the Lax pair (2.5) both possess the same symmetry relation*

$$\begin{aligned}
\psi^{-1}(x, t, k) &= P_{\pm} \psi^{\dagger}(x, t, \bar{k}) P_{\pm}, \\
\mu_j^{-1}(x, t, k) &= P_{\pm} \mu_j^{\dagger}(x, t, \bar{k}) P_{\pm}, \quad j = 1, 2, 3, 4. \quad (2.26)
\end{aligned}$$

Moreover, in the domains where μ_j is bounded, we have

$$\mu_j(x, t, k) = \mathbb{I} + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow \infty, \quad j = 1, 2, 3, 4, \quad (2.27)$$

and

$$\det[\mu_j(x, t, k)] = 1, \quad j = 1, 2, 3, 4, \quad (2.28)$$

since the traces of the matrices $U(x, t)$ and $V(x, t, k)$ are zero.

2.5. The jump conditions. Let $M(x, t, k)$ be defined by

$$\begin{aligned}
M_+ &= ([\mu_2]_1, [\mu_4]_1 \mathbb{S}_{22}^{-1}(k)), & \arg k &\in \left[0, \frac{\pi}{2}\right], \\
M_- &= ([\mu_1]_2 \overline{d^{-1}(\bar{k})}, [\mu_3]_2 s_{22}^{-1}(k)), & \arg k &\in \left[\frac{\pi}{2}, \pi\right], \\
M_+ &= ([\mu_3]_3 \overline{s_{22}^{-1}(\bar{k})}, [\mu_1]_3 d^{-1}(k)), & \arg k &\in \left[\pi, \frac{3\pi}{2}\right],
\end{aligned}$$

$$M_- = \begin{pmatrix} [\mu_4]_4 \overline{\mathbb{S}_{22}^{-1}(\bar{k})} & [\mu_2]_4 \end{pmatrix}, \quad \arg k \in \left[\frac{3\pi}{2}, 2\pi \right]. \quad (2.29)$$

where the matrices $d(k)$ and $\mathbb{S}(k)$ are defined in (2.35) and (2.36). These definitions imply

$$\det M(x, t, k) = 1. \quad (2.30)$$

and

$$M(x, t, k) = I + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow \infty. \quad (2.31)$$

Proposition 2.2. *Let $M(x, t, k)$ be defined by equations (2.29), where $\mu_j(x, t, k)$ are defined by equations (2.10). Then M satisfies the jump condition*

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}, \quad (2.32)$$

where the 4×4 block matrix J is defined by

$$J = \begin{cases} J_2, & \arg k = 0, \\ J_1, & \arg k = \frac{\pi}{2}, \\ J_4 \equiv J_3 J_2^{-1} J_1, & \arg k = \pi, \\ J_3, & \arg k = \frac{3\pi}{2}, \end{cases} \quad (2.33)$$

and

$$\begin{aligned} J_1 &= \begin{pmatrix} I_2 - \alpha \gamma(k) \overline{S_{12}(\bar{k})} d^{-1}(\bar{k}) & -\gamma(k) e^{-2i\theta} \\ \alpha \overline{S_{12}(\bar{k})} \cdot \overline{d^{-1}(\bar{k})} e^{2i\theta} & I_2 \end{pmatrix}, \\ J_2 &= \begin{pmatrix} I_2 - \alpha \mathbb{S}_{12}(k) \mathbb{S}_{22}^{-1}(k) \overline{\mathbb{S}_{12}(\bar{k})} \mathbb{S}_{22}^{-1}(\bar{k}) & -\mathbb{S}_{12}(k) \mathbb{S}_{22}^{-1}(k) e^{-2i\theta} \\ \overline{\mathbb{S}_{12}(\bar{k})} \mathbb{S}_{22}^{-1}(\bar{k}) e^{2i\theta} & I_2 \end{pmatrix}, \\ J_3 &= \begin{pmatrix} I_2 - \alpha \overline{S_{12}(k)} d^{-1}(k) \gamma(\bar{k}) & -S_{12}(k) d^{-1}(k) e^{-2i\theta} \\ \alpha \gamma(\bar{k}) e^{2i\theta} & I_2 \end{pmatrix} \end{aligned} \quad (2.34)$$

for $\theta(x, t, k) = kx + 2k^2 t$,

$$\begin{aligned} \mathbb{S}_{22}(k) &= s_{22}(k) S_{L22}(k) + \alpha \overline{s_{12}(\bar{k})} S_{L12}(k) e^{2ikL}, \\ \mathbb{S}_{12}(k) &= s_{12}(k) S_{L22}(k) + \overline{s_{22}(\bar{k})} S_{L12}(k) e^{2ikL}, \end{aligned} \quad (2.35)$$

$$\begin{aligned} d(k) &= S_{22}(k) - \alpha \overline{s_{12}(\bar{k})} s_{22}(\bar{k}) S_{12}(k), \\ \gamma(k) &= \mathbb{S}_{12}(k) \mathbb{S}_{22}^{-1}(k) - s_{12} s_{22}^{-1}. \end{aligned} \quad (2.36)$$

Proof. Writing equations (2.14a), (2.14b), and (2.16) in vector form, we find

$$[\mu_3]_3 = [\mu_2]_1 \overline{s_{22}^{-1}(\bar{k})} + [\mu_2]_4 \alpha \overline{s_{12}(\bar{k})} e^{2i\theta}, \quad (2.37a)$$

$$[\mu_3]_2 = [\mu_2]_1 s_{12}(\bar{k}) e^{-2i\theta} + [\mu_2]_4 s_{22}(k), \quad (2.37b)$$

$$[\mu_1]_2 = [\mu_2]_1 \overline{S_{22}^{-1}(\bar{k})} + [\mu_2]_4 \overline{\alpha S_{12}(\bar{k})} e^{2i\theta}, \quad (2.37c)$$

$$[\mu_1]_3 = [\mu_2]_1 \overline{S_{12}(\bar{k})} e^{-2i\theta} + [\mu_2]_4 S_{22}(k), \quad (2.37d)$$

$$[\mu_4]_4 = [\mu_2]_1 \overline{S_{22}^{-1}(\bar{k})} + [\mu_2]_4 \overline{\alpha S_{12}(\bar{k})} e^{2i\theta}, \quad (2.37e)$$

$$[\mu_4]_1 = [\mu_2]_1 \overline{S_{12}(\bar{k})} e^{-2i\theta} + [\mu_2]_4 S_{22}(k). \quad (2.37f)$$

By rearranging equation (2.37f) and substituting it into equation (2.37e), the jump condition J_2 at $\arg k = 0$ can be obtained.

To derive the jump condition J_1 at $\arg k = \frac{\pi}{2}$, we first eliminate $[\mu_2]_4$ by using equations (2.37f) and (2.37c):

$$[\mu_1]_2 = [\mu_2]_1 \left(\overline{d(\bar{k})} - \alpha \gamma(k) \overline{S_{12}(\bar{k})} \right) + [\mu_4]_1 \alpha \overline{S_{22}^{-1}(k)} \overline{S_{12}(\bar{k})} e^{2i\theta}. \quad (2.38)$$

We then eliminate $[\mu_2]_4$ by using equation (2.37f) and (2.37b):

$$[\mu_3]_2 = -[\mu_2]_1 \gamma(k) s_{22}(k) e^{-2i\theta} + [\mu_4]_1 \overline{S_{22}^{-1}(k)} s_{22}(k). \quad (2.39)$$

By further utilizing equations (2.38) and (2.39), the jump condition J_1 at $\arg k = \frac{\pi}{2}$ can be derived.

Finally, from equation (2.37a), we have

$$[\mu_2]_1 = [\mu_3]_3 \overline{s_{22}^{-1}(\bar{k})} - [\mu_2]_4 \overline{\alpha s_{12}(\bar{k})} \overline{s_{22}^{-1}(\bar{k})} e^{2i\theta}. \quad (2.40)$$

Substituting equation (2.40) into equation (2.37d) yields

$$[\mu_2]_4 = [\mu_1]_3 d^{-1}(k) - [\mu_3]_3 \overline{s_{22}^{-1}(\bar{k})} S_{12}(k) d^{-1}(k) e^{-2i\theta}. \quad (2.41)$$

Finally, substitute equations (2.40) and (2.41) into equation (2.37e), we get

$$\begin{aligned} [\mu_4]_4 &= [\mu_3]_3 \overline{s_{22}^{-1}(\bar{k})} \left(I_2 - \alpha S_{12}(k) d^{-1}(k) \gamma(\bar{k}) \right) \overline{S_{22}(\bar{k})} \\ &\quad + [\mu_1]_3 d^{-1}(k) \overline{\gamma(\bar{k}) S_{22}(\bar{k})} e^{2i\theta}. \end{aligned} \quad (2.42)$$

From equations (2.41) and (2.42), the jump condition J_3 at $\arg k = \frac{3\pi}{2}$ can be derived. $\square$

2.6. The residue conditions. Let us prove the following proposition.

Proposition 2.3. *Let $S_{22}(k)$ and $d(k)$ be the spectral functions defined by equations (2.35) and (2.36) in Proposition 2.1, and $s_{22}(k)$ be the spectral function defined by the equation (2.19). Assume that*

- $|S_{22}(k)|$ has simple zeros $\{\nu_j\}$, $\arg \nu_j \in (0, \frac{\pi}{2})$, and has no zeros at $\arg k = 0$ and $\arg k = \frac{\pi}{2}$;
- $|d(k)|$ has simple zeros $\{\lambda_j\}$, $\arg \lambda_j \in (\pi, \frac{3\pi}{2})$, and has no zeros at $\arg k = \pi$ and $\arg k = \frac{3\pi}{2}$;
- none of the zeros of $|S_{22}(k)|$ for $\arg \nu_j \in (0, \frac{\pi}{2})$ coincides with any of the zeros of $|s_{22}(k)|$;

- none of the zeros of $|d(k)|$ for $\arg \lambda_j \in (\pi, \frac{3\pi}{2})$ coincides with any of the zeros of $|s_{22}(k)|$.

Let $[M]_1$ and $[M]_2$ denote the first and the second column of the matrix M . Then

$$\text{Res}_{k=\nu_j}[M(x, t, k)]_1 = c_j^{(1)} e^{-2iv_j x - 4iv_j^2 t} [M(x, t, \nu_j)]_2, \quad (2.43a)$$

$$\text{Res}_{k=\bar{\nu}_j}[M(x, t, k)]_2 = \overline{\alpha c_j^{(1)}} e^{2i\bar{\nu}_j x + 4i\bar{\nu}_j^2 t} [M(x, t, \bar{\nu}_j)]_1, \quad (2.43b)$$

$$\text{Res}_{k=\lambda_j}[M(x, t, k)]_1 = c_j^{(2)} e^{2i\lambda_j x + 4i\lambda_j^2 t} [M(x, t, \lambda_j)]_2, \quad (2.43c)$$

$$\text{Res}_{k=\bar{\lambda}_j}[M(x, t, k)]_2 = \overline{\alpha c_j^{(2)}} e^{-2i\bar{\lambda}_j x - 4i\bar{\lambda}_j^2 t} [M(x, t, \bar{\lambda}_j)]_1, \quad (2.43d)$$

where

$$\begin{aligned} c_j^{(1)} &= \frac{[s_{22}(\nu_j)\mathbb{S}_{12}(\nu_j) - s_{12}(\nu_j)\mathbb{S}_{22}(\nu_j)] s_{22}^*(\nu_j)}{|\dot{\mathbb{S}}_{22}(\nu_j)| |s_{22}(\nu_j)|}, \\ c_j^{(2)} &= \frac{\alpha s_{22}^*(\lambda_j) \overline{\mathbb{S}_{12}(\bar{\lambda}_j)}}{|\dot{d}^{-1}(\bar{\lambda}_j)| |s_{22}(\lambda_j)|}, \end{aligned} \quad (2.44)$$

and X^* is the adjoint matrix of matrix X , $|X| = \det(X)$ is the determinant of matrix X , and it holds that $XX^* = X^*X = |X|I_2$.

Proof. Decompose the jump matrix $J_1(x, t, k)$ defined by equation (2.34) as

$$J_1(x, t, k) = \begin{pmatrix} I_2 & -\gamma(k)e^{-2i\theta} \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ \alpha S_{12}(\bar{k})d^{-1}(\bar{k})e^{2i\theta} & I_2 \end{pmatrix}. \quad (2.45)$$

Using the decomposition (2.45), the jump condition $M_- = M_+ J_1$ becomes the following:

$$\begin{aligned} \left([\mu_1]_2 \overline{d^{-1}(\bar{k})}, [\mu_3]_2 s_{22}^{-1}(k) \right) & \begin{pmatrix} I_2 & 0 \\ -\alpha S_{12}(\bar{k})d^{-1}(\bar{k})e^{2i\theta} & I_2 \end{pmatrix} \\ &= ([\mu_2]_1, [\mu_4]_1 \mathbb{S}_{22}^{-1}(k)) \begin{pmatrix} I_2 & -\gamma(k)e^{-2i\theta} \\ 0 & I_2 \end{pmatrix}. \end{aligned} \quad (2.46)$$

Evaluating the second column of this equation at $k = \nu_j$, we find that

$$0 = [\mu_4]_1(\nu_j) \mathbb{S}_{22}^{-1}(\nu_j) s_{22}(\nu_j) - [\mu_2]_1(\nu_j) (s_{22}(\nu_j) \mathbb{S}_{12}(\nu_j) \mathbb{S}_{22}^{-1}(\nu_j) - s_{22}(\nu_j)) e^{2i\theta(\nu_j)}, \quad (2.47)$$

where for the convenience of notation we have suppressed the x and t dependence of $[\mu_2]_1$, $[\mu_4]_1$, and θ . Hence,

$$\begin{aligned} \text{Res}_{k=\nu_j}[M(x, t, k)]_1 &= \frac{[\mu_4]_1(\nu_j)}{|\dot{\mathbb{S}}_{22}(\nu_j)|} \\ &= \frac{[\mu_2]_1(\nu_j) [s_{22}(\nu_j) \mathbb{S}_{12}(\nu_j) - s_{12}(\nu_j) \mathbb{S}_{22}(\nu_j)] s_{22}^*(\nu_j)}{|\dot{\mathbb{S}}_{22}(\nu_j)| |s_{22}(\nu_j)| e^{2i\theta(\nu_j)}}. \end{aligned} \quad (2.48)$$

By using $[\mu_2]_1(\nu_j) = [M(x, t, \nu_j)]_2$ we find equation (2.43a).

Similarly, evaluating the first column of equation (2.46) at $k = \lambda_j$, we find

$$0 = [\mu_1]_2(\lambda_j) \overline{d^{-1}(\bar{\lambda}_j)} + [\mu_3]_2(\lambda_j) \alpha s_{22}^{-1}(\lambda_j) \overline{S_{12}(\bar{\lambda}_j)} e^{2i\theta(\lambda_j)} \overline{d^{-1}(\bar{\lambda}_j)}. \quad (2.49)$$

Hence,

$$\text{Res}_{k=\lambda_j} [M(x, t, k)]_1 = \frac{[\mu_1]_2(\lambda_j)}{|\dot{d}^{-1}(\bar{\lambda}_j)|} = \frac{[\mu_3]_2(\lambda_j) \alpha s_{22}^*(\lambda_j) \overline{S_{12}(\bar{\lambda}_j)} e^{2i\theta(\lambda_j)}}{|\dot{d}^{-1}(\bar{\lambda}_j)| |s_{22}(\lambda_j)|}, \quad (2.50)$$

which yields (2.43c).

Equations (2.43b) and (2.43d) follow from equations (2.43a) and (2.43c) using symmetry considerations. $\square$

3. Existence under assumption that the global relation is valid

3.1. The spectral functions. The analysis of Section 2 motivates the following definitions and results for the spectral matrix functions. The relevant rigorous analysis can be found in [18].

Definition 3.1 (The spectral matrix functions $s_{22}(k), s_{21}(k)$). Given the smooth matrix function $Q_0(x)$, define the vector $\phi(x, k) = (\phi_1, \phi_2)^\dagger$ as the unique solution of

$$\begin{cases} \phi_{1x} + 2ik\phi_1 = Q_0(x)\phi_2, \\ \phi_{2x} = \tilde{\Sigma}Q_0^\dagger(x)\tilde{\Omega}\phi_1, & 0 < x < L, \quad k \in \mathbb{C}, \\ \phi(L, k) = (0, 1)^\dagger. \end{cases} \quad (3.1)$$

Given $\phi(x, k)$, define the matrix functions $s_{22}(k)$ and $s_{21}(k)$ by

$$s_{22}(k) = \phi_2(0, k), \quad s_{21}(k) = \phi_1(0, k), \quad k \in \mathbb{C}. \quad (3.2)$$

Properties of $s_{22}(k)$ and $s_{21}(k)$:

- $s_{22}(k), s_{21}(k)$ are entire matrix functions;
- $|s_{22}(k)| |\overline{s_{22}(\bar{k})}| - \lambda |s_{21}(k)| |\overline{s_{21}(\bar{k})}| = 1, \quad k \in \mathbb{C};$
- $s_{22}(k) = I_2 + O\left(\frac{1+e^{2ikL}}{k}\right), \quad s_{21}(k) = O\left(\frac{1+e^{2ikL}}{k}\right)$ as $k \rightarrow \infty$.

In particular,

$$s_{22}(k), s_{21}(k), \overline{s_{22}(\bar{k})} e^{2ikL}, \overline{s_{21}(\bar{k})} e^{2ikL} \text{ are bounded for } \arg k \in [0, \pi]. \quad (3.3)$$

We shall also assume that $|s_{22}(k)|$ has at most simple zeros $\{k_j\}$ for $\text{Im } k_j > 0$ and has no zeros for $\text{Im } k = 0$.

Remark 3.2. Definition 3.1 gives rise to the map

$$\mathbb{S} : \{Q_0(x)\} \rightarrow \{s_{22}(k), s_{21}(k)\}. \quad (3.4a)$$

The inverse of this map,

$$\mathbb{Q} : \{s_{22}(k), s_{21}(k)\} \rightarrow \{Q_0(x)\} \quad (3.4b)$$

can be defined as follows:

$$Q_0(x) = 2i \lim_{k \rightarrow \infty} (kM^{(x)}(x, k))_{12}, \quad (3.5)$$

where $M^{(x)}(x, k)$ is the unique solution of the following RH problem:

•

$$M^{(x)}(x, k) = \begin{cases} M_-^{(x)}(x, k), & \text{Im } k \leq 0, \\ M_+^{(x)}(x, k), & \text{Im } k \geq 0, \end{cases}$$

is a sectionally meromorphic function with unit determinant;

•

$$M_-^{(x)}(x, k) = M_+^{(x)}(x, k)J^{(x)}(x, k), \quad k \in \mathbb{R},$$

where

$$J_2 = \begin{pmatrix} I_2 - \alpha s_{12}(k) \frac{s_{22}^{-1}(k) \overline{s_{12}(\bar{k}) s_{22}^{-1}(\bar{k})}}{s_{12}(\bar{k}) s_{22}^{-1}(\bar{k}) e^{2ikx}} & -s_{12}(k) s_{22}^{-1}(k) e^{-2ikx} \\ \frac{s_{12}(\bar{k}) s_{22}^{-1}(\bar{k}) e^{2ikx}}{I_2} & I_2 \end{pmatrix};$$

•

$$M^{(x)}(x, k) = I_4 + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow \infty.$$

It can be shown (see, for example, [18]) that

$$\mathbb{S}^{-1} = \mathbb{Q}. \quad (3.6)$$

Definition 3.3 (The spectral functions matrix $S_{22}(k), S_{21}(k)$). Let

$$\begin{aligned} V^{(0)}(t, k) = & 2k \begin{pmatrix} 0 & F_0(t) \\ \tilde{\Sigma} F_0^\dagger(t) \tilde{\Omega} & 0 \end{pmatrix} - i\sigma_4 \begin{pmatrix} 0 & F_1(t) \\ \tilde{\Sigma} F_1^\dagger(t) \tilde{\Omega} & 0 \end{pmatrix} \\ & - i\sigma_4 \begin{pmatrix} 0 & F_0(t) \\ \tilde{\Sigma} F_0^\dagger(t) \tilde{\Omega} & 0 \end{pmatrix}^2. \end{aligned} \quad (3.7)$$

Given the smooth functions $G_0(t), G_1(t)$, define the vector $\Phi(t, k) = (\Phi_1, \Phi_2)^\dagger$ as the unique solution of

$$\begin{cases} \Phi_{1t} + 4ik^2 \Phi_1 = V_{11}^{(0)} \Phi_1 + V_{12}^{(0)} \Phi_2, \\ \Phi_{2t} = V_{21}^{(0)} \Phi_1 + V_{22}^{(0)} \Phi_2, & 0 < t < T, \quad k \in \mathbb{C}, \\ \Phi(0, k) = (0, 1)^\dagger. \end{cases} \quad (3.8)$$

Given $\Phi(t, k)$, define the functions $S_{22}(k)$ and $S_{21}(k)$ by

$$S_{22}(k) = \overline{\Phi_2(T, \bar{k})}, \quad S_{21}(k) = -\Phi_1(T, k) e^{4ik^2 T}. \quad (3.9)$$

Properties of $S_{22}(k)$ and $S_{21}(k)$:

- $S_{22}(k), S_{21}(k)$ are entire matrix functions;
- $|S_{22}(k)| \left| \overline{S_{22}(\bar{k})} \right| - \lambda |S_{21}(k)| \left| \overline{S_{21}(\bar{k})} \right| = 1, k \in \mathbb{C}$;
- $S_{22}(k) = I_2 + O\left(\frac{1+e^{4ik^2T}}{k}\right), \quad s_{21}(k) = O\left(\frac{1+e^{4ik^2T}}{k}\right)$ as $k \rightarrow \infty$.

In particular,

$$S_{22}(k), S_{21}(k) \text{ are bounded for } \arg k \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right].$$

We shall also assume that $|S_{22}(k)|$ has at most simple zeros, $\{K_j\}$, for $\arg K_j \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ and has no zeros for $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Remark 3.4. Definition 3.3 gives rise to the map

$$\mathbb{S}^{(0)} : \{G_0(t), G_1(t)\} \rightarrow \{S_{22}(k), S_{21}(k)\}. \quad (3.10a)$$

The inverse of this map,

$$\mathbb{Q}^{(0)} : \{S_{22}(k), S_{21}(k)\} \rightarrow \{G_0(t), G_1(t)\} \quad (3.10b)$$

can be defined as follows:

$$\begin{cases} G_0(x) = 2i \lim_{k \rightarrow \infty} (kM^{(0)}(x, k))_{12}, \\ G_1(x) = 4 \lim_{k \rightarrow \infty} \{(k^2 M^{(0)}(x, k))_{12} + 2iG_0(t)(kM^{(0)}(x, k))_{22}\}, \end{cases} \quad (3.11)$$

where $M^{(0)}(x, k)$ is the unique solution of the following RH problem:

•

$$M^{(0)}(x, k) = \begin{cases} M_-^{(0)}(x, k), & \arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}], \\ M_+^{(0)}(x, k), & \arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi], \end{cases}$$

is a sectionally meromorphic function with unit determinant;

•

$$M_-^{(0)}(x, k) = M_+^{(0)}(x, k)J^{(0)}(x, k), \quad k \in \mathbb{R} \cup i\mathbb{R},$$

where

$$J_2 = \begin{pmatrix} I_2 - \alpha S_{12}(k) S_{22}^{-1}(k) \overline{S_{12}(\bar{k}) S_{22}^{-1}(\bar{k})} & -S_{12}(k) S_{22}^{-1}(k) e^{-4ik^2x} \\ \frac{S_{12}(\bar{k}) S_{22}^{-1}(\bar{k}) e^{4ik^2x}}{I_2} & I_2 \end{pmatrix};$$

•

$$M^{(x)}(x, k) = I_4 + O\left(\frac{1 + e^{2ikL}}{k}\right) \quad \text{as } k \rightarrow \infty.$$

It can be shown (see, for example, [18]), that

$$\left(\mathbb{S}^{(0)}\right)^{-1} = \mathbb{Q}^{(0)}. \quad (3.12)$$

Definition 3.5 (The spectral matrix functions $\mathbb{S}_{22}(k), \mathbb{S}_{21}(k)$). Let $Q^{(L)}(t, k)$ be defined by an equation similar to (3.7) with $F_0(t), F_1(t)$ replaced by $G_0(t), G_1(t)$. Given the smooth functions $G_0(t), G_1(t)$, define the vector $\Psi(t, k)$ by equations similar to (3.8) with $Q^{(0)}(t, k)$ replaced by $Q^{(L)}(t, k)$. Given $\Psi(t, k)$, define $\mathbb{S}_{22}(k)$ and $\mathbb{S}_{21}(k)$ by

$$\mathbb{S}_{22}(k) = \overline{\Psi_2(T, \bar{k})}, \quad \mathbb{S}_{21}(k) = -\Psi_1(T, k)e^{4ik^2T}. \quad (3.13)$$

Properties of $\mathbb{S}_{22}(k)$ and $\mathbb{S}_{21}(k)$: Identical to those of $S_{22}(k), S_{21}(k)$.

Remark 3.6. Definition 3.5 gives rise to the map

$$\mathbb{S}^{(L)} : \{G_0(t), G_1(t)\} \rightarrow \{\mathbb{S}_{22}(k), \mathbb{S}_{21}(k)\}. \quad (3.14a)$$

The inverse of this map,

$$\mathbb{Q}^{(L)} : \{\mathbb{S}_{22}(k), \mathbb{S}_{21}(k)\} \rightarrow \{G_0(t), G_1(t)\} \quad (3.14b)$$

is defined exactly as in Remark 3.4, using the notation

$$M^{(L)}(t, k), J^{(L)}(t, k) \text{ instead of } M^{(0)}(t, k), J^{(0)}(t, k). \quad (3.15)$$

In analogy with equation (3.12) we find

$$\left(\mathbb{S}^{(L)}\right)^{-1} = \mathbb{Q}^{(L)}. \quad (3.16)$$

3.2. The Riemann–Hilbert problem.

Proposition 3.7. Let $Q_0(x)$ be a smooth function. Suppose that the set of functions $F_0(t), F_1(t), G_0(t)$, and $G_1(t)$ is admissible with respect to $Q_0(x)$. Define the spectral functions $s_{22}(k), s_{12}(k), S_{22}(k), S_{12}(k), \mathbb{S}_{22}(k)$, and $\mathbb{S}_{12}(k)$ in terms of $Q_0(x), F_0(t), F_1(t), G_0(t)$, and $G_1(t)$ according to Definitions 3.1, 3.3, and 3.5. Assume that

- $|s(k)|$ has at most simple zeros at points $\{k_j\}$ for $\text{Im } k_j > 0$ and has no zeros for $\text{Im } k = 0$;
- $|S(k)|$ has at most simple zeros at points $\{K_j\}$ for $\arg K_j \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ and has no zeros for $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$;
- $|\mathbb{S}_{22}(k)|$ has at most simple zeros at points $\{\mathbb{K}_j\}$ for $\arg \mathbb{K}_j \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2})$ and has no zeros for $\arg k = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$;
- $|d(k)|$ has at most simple zeros at points $\{\lambda_j\}$ for $\arg \lambda_j \in (\frac{\pi}{2}, \pi)$ and has no zeros for $\arg k = \frac{\pi}{2}$ and $\arg k = \pi$;
- none of the zeros of $|s(k)|$ for $\arg k \in (\frac{\pi}{2}, \pi)$ coincides with a zero of $|d(k)|$;
- none of the zeros of $|d(k)|$ for $\arg k \in (\frac{\pi}{2}, \pi)$ coincides with a zero of $|\overline{S(\bar{k})}|$ or a zero of $|\overline{\mathbb{S}(\bar{k})}|$.

Define $M(x, t, k)$ as the solution of the following 4×4 matrix RH problem:

- M is sectionally meromorphic in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, and has unit determinant;
-

$$M_-(x, t, k) = M_+(x, t, k)J(x, t, k), \quad k \in \mathbb{R} \cup i\mathbb{R}. \quad (3.17)$$

where M is M_- for $\arg k \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$, M is M_+ for $\arg k \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, and J is defined in terms of s_{22} , s_{12} , S_{22} , S_{12} , $\mathbb{S}_{22}$, $\mathbb{S}_{12}$, by equations (2.33) and (2.34);

-

$$M(x, t, k) = I_4 + O\left(\frac{1}{k}\right) \quad \text{as } k \rightarrow \infty; \quad (3.18)$$

- residue conditions (2.43).

Then $M(x, t, k)$ exists and is unique. Define $Q(x, t)$ in terms of $M(x, t, k)$ by

$$Q(x, t) = 2i \lim_{k \rightarrow \infty} (kM(x, t, k))_{12} \quad (3.19)$$

Then $Q(x, t)$ solves the NLS equation (1.3) with

$$\begin{aligned} Q(x, 0) &= Q_0(x), & Q(0, t) &= F_0(t), & Q_x(0, t) &= F_1(t), \\ Q(L, t) &= G_0(t), & Q_x(L, t) &= G_1(t). \end{aligned} \quad (3.20)$$

4. Nonlinearizable boundary conditions

The main conclusion of this article exhibits the unknown boundary condition on basis of the prescribed boundary condition and the initial condition in terms of the solution of a system of nonlinear integral equations.

By evaluating equation (2.16) at the point $(x, t) = (0, t)$ and using t instead of T in equation (2.18c), we find

$$\begin{aligned} c(t, k) &= \mu_2(0, t, k)e^{-2ik^2t\hat{\sigma}_4} \left[s(k)e^{ikL\hat{\sigma}_4}e^{2ik^2t\hat{\sigma}_4}\mu_3^{-1}(L, t, k) \right] \\ &= \mu_2(0, t, k) \left[e^{-2ik^2t\hat{\sigma}_4}s(k) \right] \left[e^{ikL\hat{\sigma}_4}\mu_3^{-1}(L, t, k) \right]. \end{aligned} \quad (4.1)$$

Let $\mu_2(0, t, k) = (\Phi_{ij}(t, k))_{2 \times 2}$ and $\mu_3(L, t, k) = (\Psi_{ij}(t, k))_{2 \times 2}$. By using symmetry, equation (4.1) can be expanded as

$$\begin{aligned} [c(t, k)]_{r1} &= \left(\Phi_{r1}(t, k)s_{11}(k) + \Phi_{r2}(t, k)s_{21}(k)e^{4ik^2t} \right) \left(\tilde{\Sigma}\Psi_{11}^\dagger(t, \bar{k})\tilde{\Sigma} \right) \\ &\quad - \left(\Phi_{r1}(t, k)s_{12}(k)e^{-4ik^2t} + \Phi_{r2}(t, k)s_{22}(k) \right) \left(\tilde{\Omega}\Psi_{12}^\dagger(t, \bar{k})\tilde{\Sigma} \right) e^{-2ikL}, \\ &\quad r = 1, 2. \end{aligned} \quad (4.2a)$$

$$\begin{aligned} [c(t, k)]_{r2} &= \left(\Phi_{r1}(t, k)s_{12}(k)e^{-4ik^2t} + \Phi_{r2}(t, k)s_{22}(k) \right) \left(\tilde{\Omega}\Psi_{22}^\dagger(t, \bar{k})\tilde{\Omega} \right) \\ &\quad - \left(\Phi_{r1}(t, k)s_{11}(k) + \Phi_{r2}(t, k)s_{21}(k)e^{4ik^2t} \right) \left(\tilde{\Sigma}\Psi_{21}^\dagger(t, \bar{k})\tilde{\Omega} \right) e^{2ikL}, \\ &\quad r = 1, 2. \end{aligned} \quad (4.2b)$$

Thus, the column vectors $[c(t, k)]_{r1}$ are analytic and bounded in D_4 away from the possible zeros of $\det(\tilde{\mathbb{S}}_{22}(\bar{k}))$ and of order $O\left(\frac{1+e^{-2ikL}}{k}\right)$ as $k \rightarrow \infty$, and the column vectors $[c(t, k)]_{r2}$ are analytic and bounded in D_1 away from the possible zeros of $\det(\mathbb{S}_{22}(k))$ and are of order $O\left(\frac{1+e^{-2ikL}}{k}\right)$ as $k \rightarrow \infty$.

4.1. The asymptotic behaviors of eigenfunctions. It follows from the Lax pair (2.5) that the eigenfunctions $\{\mu_j\}_1^4$ possess the following asymptotics as $k \rightarrow \infty$:

$$\begin{aligned} \mu_j(x, t, k) &= \mathbb{I} + \sum_{l=1}^2 \frac{1}{k^l} \begin{pmatrix} \mu_{j,11}^{(l)} & \mu_{j,12}^{(l)} \\ \mu_{j,21}^{(l)} & \mu_{j,22}^{(l)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ &= \mathbb{I} + \frac{1}{k} \begin{pmatrix} \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} & -\frac{i}{2}Q \\ \frac{i}{2}R & \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} \end{pmatrix} \\ &\quad + \frac{1}{k^2} \begin{pmatrix} \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(2)} & \mu_{j,12}^{(2)} \\ \mu_{j,21}^{(2)} & \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right). \end{aligned} \quad (4.3)$$

where we have introduced the following functions:

$$\begin{aligned} \Delta_{11}^{(1)} &= \frac{i}{2}QRdx + \frac{1}{2}(QR_x - Q_xR)dt, \\ \Delta_{22}^{(1)} &= -\frac{i}{2}RQdx + \frac{1}{2}(RQ_x - R_xQ)dt, \\ \mu_{j,12}^{(2)} &= \frac{1}{4}Q_x - \frac{i}{2}Q\mu_{j,22}^{(1)} = \frac{1}{4}Q_x - \frac{i}{2}Q \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)}, \\ \mu_{j,21}^{(2)} &= \frac{1}{4}R_x + \frac{i}{2}R\mu_{j,11}^{(1)} = \frac{1}{4}R_x + \frac{i}{2}R \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)}, \\ \Delta_{11}^{(2)} &= \left[\frac{1}{4}QR_x + \frac{i}{2}QR\mu_{j,11}^{(1)} \right] dx \\ &\quad + \left[\frac{i}{4}(Q_xR_x - QRQR) + \frac{1}{2}(QR_x - Q_xR)\mu_{j,11}^{(1)} \right] dt \\ &= \left[\frac{1}{4}QR_x + \frac{i}{2}QR \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} \right] dx \\ &\quad + \left[\frac{i}{4}(Q_xR_x - QRQR) + \frac{1}{2}(QR_x - Q_xR) \int_{(x_j, t_j)}^{(x, t)} \Delta_{11}^{(1)} \right] dt, \\ \Delta_{22}^{(2)} &= \left[\frac{1}{4}QR_x + \frac{i}{2}QR\mu_{j,22}^{(1)} \right] dx \\ &\quad + \left[\frac{i}{4}(Q_xR_x - QRQR) + \frac{1}{2}(QR_x - Q_xR)\mu_{j,22}^{(1)} \right] dt \\ &= \left[\frac{1}{4}RQ_x - \frac{i}{2}RQ \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} \right] dx \\ &\quad + \left[-\frac{i}{4}(R_xQ_x - RQ RQ) + \frac{1}{2}(RQ_x - R_xQ) \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)} \right] dt, \end{aligned} \quad (4.4)$$

where the matrix-valued functions $\{\mu_{jl}^{(i)} = \mu_{jl}^{(i)}(x, t)\}_{l=1}^2$, $i = 1, 2$, are independent of k .

We introduce the matrix-valued function $\mu_2(0, t, k) = (\Phi_{ij}(t, k))_{2 \times 2}$ as follows:

$$\mu_2(0, t, k) = \begin{pmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{pmatrix} = \mathbb{I} + \sum_{l=1}^2 \frac{1}{k^l} \begin{pmatrix} \Phi_{11}^{(l)} & \Phi_{12}^{(l)} \\ \Phi_{21}^{(l)} & \Phi_{22}^{(l)} \end{pmatrix} + O\left(\frac{1}{k^3}\right). \quad (4.5)$$

By the asymptotic expansion in equation (4.3) and the boundary data at $x = 0$, we find

$$\Phi_{12}^{(1)} = -\frac{i}{2}F_0(t), \quad \Phi_{12}^{(2)} = \frac{1}{4}F_1(t) - \frac{i}{2}F_0(t)\Phi_{22}^{(1)}. \quad (4.6)$$

Thus, we have the Dirichlet–Neumann boundary data at $x = 0$ by using the spectral function:

$$F_0(t) = 2i\Phi_{12}^{(1)}, \quad F_1(t) = 4\Phi_{12}^{(2)} + 2iF_0(t)\Phi_{22}^{(1)}. \quad (4.7)$$

Similarly, we assume that the asymptotic formula of $\mu_3(L, t, k) = (\Psi_{ij}(t, k))_{2 \times 2}$ is of the form

$$\mu_3(L, t, k) = \begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t) \\ \Psi_{21}(t) & \Psi_{22}(t) \end{pmatrix} = \mathbb{I} + \sum_{l=1}^2 \frac{1}{k^l} \begin{pmatrix} \Psi_{11}^{(l)} & \Psi_{12}^{(l)} \\ \Psi_{21}^{(l)} & \Psi_{22}^{(l)} \end{pmatrix} + O\left(\frac{1}{k^3}\right). \quad (4.8)$$

We find

$$G_0(t) = 2i\Psi_{12}^{(1)}, \quad G_1(t) = 4\Psi_{12}^{(2)} + 2iG_0(t)\Psi_{22}^{(1)}. \quad (4.9)$$

For the vanishing initial values, it follows from equations (4.2) that we have the following asymptotics of $c_{12}(t, k)$ and $c_{21}(t, k)$.

Proposition 4.1. *Let the initial and Dirichlet boundary data be compatible at points $x = 0$ and $x = L$ ($F(0, 0) = F_0(0)$ at $x = 0$ and $G(L, 0) = G_0(0)$ at $x = L$), respectively. Then, the global relation (4.1) in the vanishing initial data case implies that the large k behaviors of $c_{12}(t, k)$ and $c_{21}(t, k)$ can be given as follows:*

$$\begin{aligned} c_{12}(t, k) &= \frac{\Phi_{12}^{(1)}}{k} + \frac{\Phi_{12}^{(2)} + \Phi_{12}^{(1)}\tilde{\Omega}\Psi_{22}^{\dagger(1)}\tilde{\Omega}}{k^2} + O\left(\frac{1}{k^3}\right) \\ &\quad - \left[\frac{\tilde{\Sigma}\Psi_{21}^{\dagger(1)}\tilde{\Omega}}{k} + \frac{\tilde{\Sigma}\Psi_{21}^{\dagger(2)}\tilde{\Omega} + \Phi_{11}^{(1)}\tilde{\Sigma}\Psi_{21}^{\dagger(1)}\tilde{\Omega}}{k^2} + O\left(\frac{1}{k^3}\right) \right] e^{2ikL} \text{ as } k \rightarrow \infty, \end{aligned} \quad (4.10a)$$

$$\begin{aligned} c_{21}(t, k) &= \frac{\Phi_{21}^{(1)}}{k} + \frac{\Phi_{21}^{(2)} + \Phi_{21}^{(1)}\tilde{\Sigma}\Psi_{11}^{\dagger(1)}\tilde{\Sigma}}{k^2} + O\left(\frac{1}{k^3}\right) \\ &\quad - \left[\frac{\tilde{\Omega}\Psi_{12}^{\dagger(1)}\tilde{\Sigma}}{k} + \frac{\tilde{\Omega}\Psi_{12}^{\dagger(2)}\tilde{\Sigma} + \Phi_{22}^{(1)}\tilde{\Omega}\Psi_{12}^{\dagger(1)}\tilde{\Sigma}}{k^2} + O\left(\frac{1}{k^3}\right) \right] e^{-2ikL} \text{ as } k \rightarrow \infty. \end{aligned} \quad (4.10b)$$

Proof. It follows from the global relation (4.1) under the vanishing initial data that:

$$[c(t, k)]_{12} = \left(\Phi_{11}(t, k)s_{12}(k)e^{-4ik^2t} + \Phi_{12}(t, k)s_{22}(k) \right) \left(\tilde{\Omega}\Psi_{22}^{\dagger}(t, \bar{k})\tilde{\Omega} \right)$$

$$- \left(\Phi_{11}(t, k) s_{11}(k) + \Phi_{12}(t, k) s_{21}(k) e^{4ik^2 t} \right) \left(\tilde{\Sigma} \Psi_{21}^\dagger(t, \bar{k}) \tilde{\Omega} \right) e^{2ikL}, \quad (4.11a)$$

$$\begin{aligned} [c(t, k)]_{21} = & \left(\Phi_{21}(t, k) s_{11}(k) + \Phi_{22}(t, k) s_{21}(k) e^{4ik^2 t} \right) \left(\tilde{\Sigma} \Psi_{11}^\dagger(t, \bar{k}) \tilde{\Sigma} \right) \\ & - \left(\Phi_{21}(t, k) s_{12}(k) e^{-4ik^2 t} + \Phi_{22}(t, k) s_{22}(k) \right) \left(\tilde{\Omega} \Psi_{12}^\dagger(t, \bar{k}) \tilde{\Sigma} \right) e^{-2ikL}. \end{aligned} \quad (4.11b)$$

From $s(k) = \mu_3(0, 0, k)$ and (4.3), we conclude that

$$\begin{aligned} \begin{pmatrix} s_{11} & s_{12} \\ s_{11} & s_{22} \end{pmatrix} = & \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \\ & + \frac{1}{2ik} \begin{pmatrix} 2i \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)}(0, 0) & Q(0, 0) \\ -R(0, 0) & 2i \int_{(x_j, t_j)}^{(x, t)} \Delta_{22}^{(1)}(0, 0) \end{pmatrix} + O\left(\frac{1}{k^2}\right). \end{aligned}$$

Then, combining with (4.5) and (4.8), the proof can be completed.

Recalling the t -part of the Lax pair (2.5)

$$\mu_t + 2ik^2[\sigma_4, \mu] = V(x, t, k)\mu. \quad (4.12)$$

It follows from equation (4.12) that the first column of this equation with $\mu = \mu_2(0, t, k) = \Phi(t, k)$ is

$$\begin{cases} \Phi_{11,t}(t, k) = -iF_0 \tilde{\Sigma} F_0^\dagger \tilde{\Omega} \Phi_{11} + [2kF_0 + iF_1] \Phi_{21}, \\ \Phi_{21,t}(t, k) = \left(2k \tilde{\Sigma} F_0^\dagger \tilde{\Omega} - i \tilde{\Sigma} F_1^\dagger \tilde{\Omega} \right) \Phi_{11} + i \tilde{\Sigma} F_0^\dagger \tilde{\Omega} F_0 \Phi_{21} + 4ik^2 \Phi_{21}. \end{cases} \quad (4.13)$$

The second column of equation (4.12) with $\mu = \mu_2(0, t, k) = \Phi(t, k)$ yields

$$\begin{cases} \Phi_{12,t}(t, k) = -iF_0 \tilde{\Sigma} F_0^\dagger \tilde{\Omega} \Phi_{12} + [2kF_0 + iF_1] \Phi_{22} - 4ik^2 \Phi_{21}, \\ \Phi_{22,t}(t, k) = \left(2k \tilde{\Sigma} F_0^\dagger \tilde{\Omega} - i \tilde{\Sigma} F_1^\dagger \tilde{\Omega} \right) \Phi_{12} + i \tilde{\Sigma} F_0^\dagger \tilde{\Omega} F_0 \Phi_{22}. \end{cases} \quad (4.14)$$

We suppose

$$\begin{pmatrix} \Phi_{11} \\ \Phi_{21} \end{pmatrix} = \left(\varepsilon_0(t) + \frac{\varepsilon_1(t)}{k} + \frac{\varepsilon_2(t)}{k^2} + \dots \right) + \left(\eta_0(t) + \frac{\eta_1(t)}{k} + \frac{\eta_2(t)}{k^2} + \dots \right) e^{-4ik^2 t}, \quad (4.15)$$

where vector functions $\varepsilon_l(t)$ and $\eta_l(t) (l \geq 0)$ are independent of k . To determine these coefficients, we substitute the above equation into equations (4.13) and (4.14) and use the initial conditions

$$\varepsilon_0(0) + \eta_0(0) = (I_2, 0_{2 \times 2})^T, \quad \varepsilon_1(0) + \eta_1(0) = (0_{2 \times 2}, 0_{2 \times 2})^T.$$

We find

$$\begin{pmatrix} \Phi_{11} \\ \Phi_{21} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{11}^{(1)} \\ \Phi_{21}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{11}^{(2)} \\ \Phi_{21}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right)$$

$$+ \left[\frac{1}{k} \begin{pmatrix} 0 \\ -\frac{i}{2} \tilde{\Sigma} F_0^\dagger(0) \tilde{\Omega} \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{4ik^2t}, \quad (4.16a)$$

$$\begin{pmatrix} \Phi_{12} \\ \Phi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Phi_{12}^{(1)} \\ \Phi_{22}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Phi_{11}^{(2)} \\ \Phi_{21}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ + \left[\frac{1}{k} \begin{pmatrix} \frac{i}{2} F_0(0) \\ 0 \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{-4ik^2t}. \quad (4.16b)$$

Similarly, at $x = L$, the function $\mu(x, t, k) = \mu_3(L, t, k)$ satisfies the t -part of the Lax pair (2.5). Therefore, when $\mu = \mu_3(L, t, k) = \Psi(t, k)$, the two columns of equation (4.12) are

$$\begin{cases} \Psi_{11,t}(t, k) = -iG_0 \tilde{\Sigma} G_0^\dagger \tilde{\Omega} \Psi_{11} + [2kG_0 + iG_1] \Psi_{21}, \\ \Psi_{21,t}(t, k) = \left(2k \tilde{\Sigma} G_0^\dagger \tilde{\Omega} - i \tilde{\Sigma} G_1^\dagger \tilde{\Omega} \right) \Psi_{11} + i \tilde{\Sigma} G_0^\dagger \tilde{\Omega} G_0 \Psi_{21} + 4ik^2 \Psi_{21}, \end{cases} \quad (4.17a)$$

$$\begin{cases} \Psi_{12,t}(t, k) = -iG_0 \tilde{\Sigma} G_0^\dagger \tilde{\Omega} \Psi_{12} + [2kG_0 + iG_1] \Psi_{22} - 4ik^2 \Psi_{21}, \\ \Psi_{22,t}(t, k) = \left(2k \tilde{\Sigma} G_0^\dagger \tilde{\Omega} - i \tilde{\Sigma} G_1^\dagger \tilde{\Omega} \right) \Psi_{12} + i \tilde{\Sigma} G_0^\dagger \tilde{\Omega} G_0 \Psi_{22}. \end{cases} \quad (4.17b)$$

Similarly, we can find the asymptotic formulas of $\Psi(t, k)$

$$\begin{pmatrix} \Psi_{11} \\ \Psi_{21} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Psi_{11}^{(1)} \\ \Psi_{21}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Psi_{11}^{(2)} \\ \Psi_{21}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ + \left[\frac{1}{k} \begin{pmatrix} 0 \\ -\frac{i}{2} \tilde{\Sigma} G_0^\dagger(0) \tilde{\Omega} \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{4ik^2t}, \quad (4.18a)$$

$$\begin{pmatrix} \Psi_{12} \\ \Psi_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} + \frac{1}{k} \begin{pmatrix} \Psi_{12}^{(1)} \\ \Psi_{22}^{(1)} \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} \Psi_{11}^{(2)} \\ \Psi_{21}^{(2)} \end{pmatrix} + O\left(\frac{1}{k^3}\right) \\ + \left[\frac{1}{k} \begin{pmatrix} \frac{i}{2} G_0(0) \\ 0 \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right] e^{-4ik^2t}. \quad (4.18b)$$

Substituting these formulas into equations (4.11a) and (4.11b), and assuming compatibility at $x = 0$ and $x = L$, we obtain the asymptotic results (4.10a) and (4.10b) for $c(t, k)$ as $k \rightarrow \infty$. $\square$

4.2. The map between Dirichlet and Neumann problems. In what follows, we prove that the functions $S(k)$ and $S_L(k)$ can be expressed through the solutions of integral equations, using the given Dirichlet and Neumann boundary conditions as well as initial conditions. For simplicity, we define the following symbols:

$$W_\pm(t, k) = W(t, k) \pm W(t, -k); \quad \Delta_\pm = \Delta_\pm(k) = e^{2ikL} \pm e^{-2ikL}. \quad (4.19)$$

The symbol ∂D_j , $j = 1, \dots, 4$, denotes the boundary of the j -th quadrant D_j , and the direction of ∂D_j is such that D_j lies on the left. ∂D_3^0 denotes the boundary contour that does not contain the zeros of $\Delta_-(k)$, and $\partial D_3^0 = -\partial D_1^0$.

Proposition 4.2. Let $Q_0(x) = Q(x, t = 0) \in C^\infty[0, L]$ be the initial condition of equation (1.3) in the finite interval $x \in [0, L]$ with $T < \infty$.

For the Dirichlet problem, the boundary conditions $F_0(t)$ and $G_0(t)$ on the interval $t \in [0, T]$ need to be sufficiently smooth and compatible with the initial condition $Q_0(x)$ at the points $(x_2, t_2) = (0, 0)$ and $(x_3, t_3) = (L, 0)$, i.e., $F_0(0) = Q_0(0)$, $G_0(0) = Q_0(L)$.

For the Neumann problem, the boundary conditions $F_1(t)$ and $G_1(t)$ on the interval $t \in [0, T]$ need to be sufficiently smooth and compatible with the initial condition $Q_0(x)$ at the points $(x_2, t_2) = (0, 0)$ and $(x_3, t_3) = (L, 0)$, i.e., $F_1(0) = \partial_x Q_0(0)$, $G_1(0) = \partial_x Q_0(L)$.

Let $\det(\mathbb{S}_{11}(k))$ have no zeros in the domain D_1 . Then the spectral functions $S(k)$ and $S_L(k)$ are defined by

$$S(k) = \begin{pmatrix} \tilde{\Sigma}\Phi_{11}^\dagger(T, \bar{k})\tilde{\Sigma} & \tilde{\Sigma}\Phi_{21}^\dagger(T, \bar{k})\tilde{\Omega}e^{4ik^2T} \\ -\tilde{\Omega}\Phi_{12}^\dagger(T, \bar{k})\tilde{\Sigma}e^{-4ik^2T} & \tilde{\Omega}\Phi_{22}^\dagger(T, \bar{k})\tilde{\Omega} \end{pmatrix}, \quad (4.20a)$$

$$S_L(k) = \begin{pmatrix} \tilde{\Sigma}\Psi_{11}^\dagger(T, \bar{k})\tilde{\Sigma} & \tilde{\Sigma}\Psi_{21}^\dagger(T, \bar{k})\tilde{\Omega}e^{4ik^2T} \\ -\tilde{\Omega}\Psi_{12}^\dagger(T, \bar{k})\tilde{\Sigma}e^{-4ik^2T} & \tilde{\Omega}\Psi_{22}^\dagger(T, \bar{k})\tilde{\Omega} \end{pmatrix}, \quad (4.20b)$$

and the complex-valued functions $\{\Phi_{ij}(t, k)\}_{i,j=1}^2$ can be expressed using the following system of integral equations

$$\begin{cases} \Phi_{11}(t, k) = I_2 + \int_0^t \left\{ -iF_0\tilde{\Sigma}F_0^\dagger\tilde{\Omega}\Phi_{11} + [2kF_0 + iF_1]\Phi_{21} \right\}(\tau, k)d\tau, \\ \Phi_{12}(t, k) = \int_0^t e^{-4ik^2(t-\tau)} \left\{ -iF_0\tilde{\Sigma}F_0^\dagger\tilde{\Omega}\Phi_{12} + [2kF_0 + iF_1]\Phi_{22} \right\}(\tau, k)d\tau, \\ \Phi_{21}(t, k) = \int_0^t e^{4ik^2(t-\tau)} \left\{ [2k\tilde{\Sigma}F_0^\dagger\tilde{\Omega} - i\tilde{\Sigma}F_1^\dagger\tilde{\Omega}]\Phi_{11} + i\tilde{\Sigma}F_0^\dagger\tilde{\Omega}F_0\Phi_{21} \right\}(\tau, k)d\tau, \\ \Phi_{22}(t, k) = I_2 + \int_0^t \left\{ [2k\tilde{\Sigma}F_0^\dagger\tilde{\Omega} - i\tilde{\Sigma}F_1^\dagger\tilde{\Omega}]\Phi_{12} + i\tilde{\Sigma}F_0^\dagger\tilde{\Omega}F_0\Phi_{22} \right\}(\tau, k)d\tau. \end{cases} \quad (4.21)$$

The functions $\{\Psi_{ij}(t, k)\}_{i,j=1}^2$ satisfy the same integral equations as in (4.21) with $\{F_s, F_s^\dagger, \Phi_{ij}\}$ replaced by $\{G_s, G_s^\dagger, \Psi_{ij}\}$, $i, j = 1, 2$, $s = 0, 1$.

- (i) For the given Dirichlet boundary data, the unknown Neumann boundary data $F_1(t)$ and $G_1(t)$ can be obtained as

$$\begin{aligned} F_1(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{2\Delta_+}{i\Delta_-} (k\Phi_{12-} + iF_0) - \frac{4}{i\Delta_-} \left(k\tilde{\Sigma}\Psi_{21-}^\dagger\tilde{\Omega} + iG_0^\dagger \right) \right] dk \\ & + \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Phi_{11}s_{12} \left(\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega} \right) e^{-4ik^2t} \right. \right. \\ & \left. \left. + \Phi_{12} \left(s_{22}\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega} - I_2 \right) \right] e^{-2ikL} \right\}_- dk \\ & - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\left(\Phi_{11}s_{11} + \Phi_{12}s_{21}e^{4ik^2t} - I_2 \right) \tilde{\Sigma}\Psi_{21}^\dagger\tilde{\Omega} \right] \right\}_- dk \end{aligned}$$

$$+ \frac{1}{\pi} \int_{\partial D_3^0} 2F_0 \Phi_{22-} dk \quad (4.22)$$

and

$$\begin{aligned} G_1(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{2\Delta_+}{i\Delta_-} (k\Psi_{12-} + iG_0) + \frac{4}{i\Delta_-} \left(k\tilde{\Sigma}\Phi_{21-}^\dagger \tilde{\Omega} + iF_0^\dagger \right) \right] dk \\ & + \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\left(\Psi_{11}\tilde{\Sigma}s_{11}^\dagger \tilde{\Sigma} - I_2 \right) \tilde{\Sigma}\Phi_{21}^\dagger \tilde{\Omega} \right. \right. \\ & \left. \left. + \Psi_{11}\tilde{\Sigma}s_{21}^\dagger \Phi_{22}^\dagger \tilde{\Omega} e^{-4ik^2t} \right] \right\}_- dk \\ & - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega}s_{12}^\dagger \Phi_{21}^\dagger \tilde{\Omega} e^{4ik^2t} \tilde{\Omega}s_{22}^\dagger \Phi_{22}^\dagger \tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_- dk \\ & + \frac{1}{\pi} \int_{\partial D_3^0} 2G_0 \Psi_{22-} dk. \end{aligned} \quad (4.23)$$

- (ii) For the given Neumann boundary data, the unknown Dirichlet boundary data $F_0(t)$ and $G_0(t)$ can be found as

$$\begin{aligned} F_0(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left(\frac{\Delta_+}{\Delta_-} \Phi_{12+} - \frac{2}{\Delta_-} \tilde{\Sigma}\Psi_{21+}^\dagger \tilde{\Omega} \right) dk \\ & + \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Phi_{11}s_{12} e^{-4ik^2t} + \Phi_{12} \left(s_{22}\tilde{\Omega}\Psi_{22}^\dagger \tilde{\Omega} - I_2 \right) \right] e^{-2ikL} \right\}_+ dk \\ & - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Phi_{11}s_{11} + \Phi_{12}s_{21} e^{4ik^2t} - I_2 \right) \tilde{\Sigma}\Psi_{21}^\dagger \tilde{\Omega} \right] \right\}_+ dk \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} G_0(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left(-\frac{\Delta_+}{\Delta_-} \Psi_{12+} + \frac{2}{\Delta_-} \tilde{\Sigma}\Phi_{21+}^\dagger \tilde{\Omega} \right) dk \\ & + \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Psi_{11}\tilde{\Sigma}s_{11}^\dagger \tilde{\Sigma} - I_2 \right) \tilde{\Sigma}\Phi_{21}^\dagger \tilde{\Omega} \right. \right. \\ & \left. \left. + \Psi_{11}\tilde{\Sigma}s_{21}^\dagger \Phi_{22}^\dagger \tilde{\Omega} e^{-4ik^2t} \right] \right\}_+ dk \\ & - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega}s_{12}^\dagger \Phi_{21}^\dagger \tilde{\Omega} e^{4ik^2t} + \tilde{\Omega}s_{22}^\dagger \Phi_{22}^\dagger \tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_+ dk, \end{aligned} \quad (4.25)$$

where $\Phi_{rj} = \Phi_{rj}(t, k)$, $\Phi_{rj}^\dagger = \Phi_{rj}^\dagger(t, \bar{k})$ and other functions have similar expressions.

Proof. We can verify equations (4.20a) and (4.20b) by replacing T with t in equations (2.18a) and (2.18c), i.e., $S(k) = e^{2ik^2t\hat{\sigma}_4}\mu_2^{-1}(0, t, k)$ and $S_L(k) = e^{2ik^2t\hat{\sigma}_4}\mu_3^{-1}(L, t, k)$. Then, by using the symmetric relation (2.26), we can obtain equations (4.20a) and (4.20b). In addition, the equation (4.21) for $\Phi_{ij}(t, k)$

$(i, j = 1, 2)$ can be represented by the Volterra integral equation of $\mu_2(0, t, k)$. Similarly, the expression for $\Psi_{ij}(t, k)$, $i, j = 1, 2$, can be obtained from the Volterra integral equation of $\mu_3(L, t, k)$.

In the following we will prove equations (4.22)–(4.25).

(i) Applying Cauchy's theorem to equation (4.5), we find

$$\begin{aligned} -\frac{i\pi}{2}\Phi_{22}^{(1)}(t) &= \int_{\partial D_2} [\Phi_{22} - I]dk = \int_{\partial D_4} [\Phi_{22} - I]dk, \\ -\frac{i\pi}{2}\Phi_{12}^{(2)}(t) &= \int_{\partial D_2} [k\Phi_{12} + \frac{iF_0}{2}]dk = \int_{\partial D_4} [k\Phi_{12} + \frac{i}{2}F_0]dk. \end{aligned} \quad (4.26)$$

We further find

$$\begin{aligned} i\pi\Phi_{22}^{(1)}(t) &= -\left(\int_{\partial D_2} + \int_{\partial D_4}\right) [\Phi_{22} - I]dk = \left(\int_{\partial D_1} + \int_{\partial D_3}\right) [\Phi_{22} - I]dk \\ &= \int_{\partial D_3} [\Phi_{22} - I]dk - \int_{\partial D_1} [\Phi_{22} - I]dk = \int_{\partial D_3} \Phi_{22-} dk, \end{aligned} \quad (4.27)$$

$$\begin{aligned} i\pi\Phi_{12}^{(2)}(t) &= \left(\int_{\partial D_1} + \int_{\partial D_3}\right) \left[k\Phi_{12} + \frac{iF_0}{2}\right] dk \\ &= \int_{\partial D_3} \left[k\Phi_{12} + \frac{iF_0}{2}\right]_- dk \\ &= \int_{\partial D_3^0} \left\{ k\Phi_{12} + \frac{iF_0}{2} + \frac{2e^{-2ikL}}{\Delta_-} \left[k\Phi_{12} + \frac{iF_0}{2}\right] \right\}_- dk + C_1(t) \\ &= \int_{\partial D_3^0} \frac{\Delta_+(k)}{\Delta_-(k)} (k\Phi_{12-} + iF_0) dk + C_1(t), \end{aligned} \quad (4.28)$$

where we have introduced the function $C_1(t)$ in the form

$$C_1(t) = - \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta_-} \left[k\Phi_{12} + \frac{iF_0}{2}\right] \right\}_- dk.$$

We use the global relation (4.11a) to further reduce $C_1(t)$ as

$$\begin{aligned} C_1(t) &= - \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta_-} \left[k\Phi_{12} + \frac{iF_0}{2}\right] \right\}_- dk \\ &= \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta_-} \left[\Phi_{12}^{(1)} - kc_{12}(t, k) - \tilde{\Sigma}\Psi_{21}^{\dagger(1)}\tilde{\Omega}e^{2ikL} \right] \right\}_- dk \\ &\quad - \int_{\partial D_3^0} \left\{ \frac{2e^{-2ikL}}{\Delta_-} \left[\left(k\tilde{\Sigma}\Psi_{21}^{\dagger}\tilde{\Omega} - \tilde{\Sigma}\Psi_{21}^{\dagger(1)}\tilde{\Omega} \right) e^{2ikL} \right] \right\}_- dk \\ &\quad + \int_{\partial D_3^0} \left\{ \frac{2ke^{-2ikL}}{\Delta_-} \left[\Phi_{11}s_{12}(\tilde{\Omega}\Psi_{22}^{\dagger}\tilde{\Omega})e^{-4ik^2t} + \Phi_{12}(s_{22}\tilde{\Omega}\Psi_{22}^{\dagger}\tilde{\Omega} - I_2) \right] \right\}_- dk \\ &\quad - \int_{\partial D_3^0} \left\{ \frac{2ke^{-2ikL}}{\Delta_-} \left[(\Phi_{11}s_{11} + \Phi_{12}s_{21}e^{4ik^2t} - I_2)\tilde{\Sigma}\Psi_{21}^{\dagger}\tilde{\Omega}e^{2ikL} \right] \right\}_- dk. \end{aligned} \quad (4.29)$$

By applying Cauchy's theorem and asymptotic form (4.10a) to equation (4.29), we find that the term on the right side of equation (4.29) has the following form:

$$\begin{aligned}
C_1(t) = & -i\pi\Phi_{12}^{(2)} - \int_{\partial D_3^0} \left[+\frac{2}{\Delta_-} \left(k\tilde{\Sigma}\Psi_{21}^\dagger\tilde{\Omega} - iG_0^\dagger \right) \right] dk \\
& + \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[\Phi_{11}s_{12}(\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega})e^{-4ik^2t} + \Phi_{12}(s_{22}\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega} - I_2) \right] e^{-2ikL} \right\}_- dk \\
& - \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[(\Phi_{11}s_{11} + \Phi_{12}s_{21}e^{4ik^2t} - I_2)\tilde{\Sigma}\Psi_{21}^\dagger\tilde{\Omega} \right] \right\}_- dk. \tag{4.30}
\end{aligned}$$

It follows from equations (4.28) and (4.30) that

$$\begin{aligned}
\Phi_{12}^{(2)}(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{\Delta_+}{2i\Delta_-} (k\Phi_{12-} + iF_0) - \frac{1}{i\Delta_-} (k\tilde{\Sigma}\Psi_{21}^\dagger\tilde{\Omega} + iG_0^\dagger) \right] dk \\
& + \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{k}{i\Delta_-} \left[\Phi_{11}s_{12}(\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega})e^{-4ik^2t} + \Phi_{12}(s_{22}\tilde{\Omega}\Psi_{22}^\dagger\tilde{\Omega} - I_2) \right] e^{-2ikL} \right\}_- dk \\
& - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{k}{i\Delta_-} \left[(\Phi_{11}s_{11} + \Phi_{12}s_{21}e^{4ik^2t} - I_2)\tilde{\Sigma}\Psi_{21}^\dagger\tilde{\Omega} \right] \right\}_- dk. \tag{4.31}
\end{aligned}$$

From equation (4.27), we obtain

$$\Phi_{22}^{(1)}(t) = \frac{1}{i\pi} \int_{\partial D_3} \Phi_{22-} dk. \tag{4.32}$$

Substituting equations (4.31) and (4.32) into the second formula of equation (4.7), we obtain (4.22).

To derive equation (4.23), we first need to find the functions $\Psi_{22}^{(1)}(t, k)$ and $\Psi_{12}^{(2)}(t, k)$, and then use the second formula in (4.9). Applying Cauchy's theorem to equation (4.8), we have

$$\Psi_{22}^{(1)}(t) = \frac{1}{i\pi} \int_{\partial D_3} \Psi_{22-} dk \tag{4.33}$$

and

$$\begin{aligned}
i\pi\Psi_{12}^{(2)}(t) = & \int_{\partial D_3} \left[k\Psi_{12} - \Psi_{12}^{(1)} \right]_- dk \\
= & \int_{\partial D_3^0} \left[k\Psi_{12} - \Psi_{12}^{(1)} - \frac{2e^{2ikL}}{\Delta_-} \left(k\Psi_{12} - \Psi_{12}^{(1)} \right) \right]_- dk + C_2(t) \tag{4.34} \\
= & \int_{\partial D_3^0} -\frac{\Delta_+}{\Delta_-} [k\Psi_{12-} + iG_0] dk + C_2(t),
\end{aligned}$$

where we have introduced the function $C_2(t)$ in the form

$$C_2(t) = \int_{\partial D_3^0} \left\{ \frac{2e^{2ikL}}{\Delta_-} \left(k\Psi_{12} - \Psi_{12}^{(1)} \right) \right\}_- dk.$$

We apply Cauchy's theorem, asymptotic (4.10b), and the global relation (4.11b) to $C_2(t)$ to have

$$\begin{aligned}
C_2(t) &= \int_{\partial D_3^0} \left\{ -\frac{2}{\Delta_-} \left[k\tilde{\Sigma}c_{21}^\dagger(t, k)\tilde{\Omega} - \tilde{\Sigma}\Phi_{21}^{\dagger(1)}\tilde{\Omega} + \Psi_{12}^{(1)}e^{2ikL} \right] \right\}_- dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[+ \left(k\tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} - \tilde{\Sigma}\Phi_{21}^{\dagger(1)}\tilde{\Omega} \right) \right] \right\}_- dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[\left(\Psi_{11}\tilde{\Sigma}s_{11}^\dagger\tilde{\Sigma} - I_2 \right) \tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} + \Psi_{11}\tilde{\Sigma}s_{21}^\dagger\Phi_{22}^\dagger\tilde{\Omega}e^{-4ik^2t} \right] \right\}_- dk \\
&- \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega}s_{12}^\dagger\Phi_{21}^\dagger\tilde{\Omega}e^{4ik^2t} + \tilde{\Omega}s_{22}^\dagger\Phi_{22}^\dagger\tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_- dk \\
&= -i\pi\Psi_{12}^{(2)} + \int_{\partial D_3^0} \left[+\frac{2}{\Delta_-} \left(k\tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} + iF_0^\dagger \right) \right] dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[\left(\Psi_{11}\tilde{\Sigma}s_{11}^\dagger\tilde{\Sigma} - I_2 \right) \tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} + \Psi_{11}\tilde{\Sigma}s_{21}^\dagger\Phi_{22}^\dagger\tilde{\Omega}e^{-4ik^2t} \right] \right\}_- dk \\
&- \int_{\partial D_3^0} \left\{ \frac{2k}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega}s_{12}^\dagger\Phi_{21}^\dagger\tilde{\Omega}e^{4ik^2t} + \tilde{\Omega}s_{22}^\dagger\Phi_{22}^\dagger\tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_- dk.
\end{aligned} \tag{4.35}$$

It follows from equations (4.34) and (4.35) that we find

$$\begin{aligned}
\Psi_{12}^{(2)}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{\Delta_+}{2i\Delta_-(k)} (k\Psi_{12-} + iG_0) + \frac{1}{i\Delta_-} (k\tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} + iF_0^\dagger) \right] dk \\
&+ \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{k}{i\Delta_-} \left[\left(\Psi_{11}\tilde{\Sigma}s_{11}^\dagger\tilde{\Sigma} - I_2 \right) \tilde{\Sigma}\Phi_{21}^\dagger\tilde{\Omega} \right. \right. \\
&+ \left. \left. \Psi_{11}\tilde{\Sigma}s_{21}^\dagger\Phi_{22}^\dagger\tilde{\Omega}e^{-4ik^2t} \right] \right\}_- dk \\
&- \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{k}{i\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega}s_{12}^\dagger\Phi_{21}^\dagger\tilde{\Omega}e^{4ik^2t} + \tilde{\Omega}s_{22}^\dagger\Phi_{22}^\dagger\tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_- dk.
\end{aligned} \tag{4.36}$$

Substituting equations (4.33) and (4.36) into the second formula of equation (4.9), we obtain (4.23).

(ii) We now derive the Dirichlet boundary data (4.24) at $x = 0$ from the known Neumann boundary value problem. From the first equation in (4.7), we know that F_0 can be expressed by $\Phi_{12}^{(1)}$. Applying Cauchy's theorem to equation (4.5), we can obtain:

$$\begin{aligned}
i\pi\Phi_{12}^{(1)}(t) &= \left(\int_{\partial D_1} + \int_{\partial D_3} \right) \Phi_{12} dk = \int_{\partial D_3} \Phi_{12-} dk \\
&= \int_{\partial D_3^0} \left[\Phi_{12-} + \frac{2}{\Delta_-} (e^{-2ikL}\Phi_{12})_+ \right] dk + C_3(t) \\
&= \int_{\partial D_3^0} \frac{\Delta_+}{\Delta_-} \Phi_{12+} dk + C_3(t),
\end{aligned} \tag{4.37}$$

where we have introduced the function $C_3(t)$ in the form

$$\begin{aligned}
C_3(t) &= - \int_{\partial D_3^0} \left[\frac{2}{\Delta_-} (e^{-2ikL} \Phi_{12})_+ \right] dk \\
&= \int_{\partial D_3^0} -\frac{2}{\Delta_-} \left[c_{12}(t, k) e^{-2ikL} + \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} \right]_+ dk \\
&\quad + \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Phi_{11} s_{12} e^{-4ik^2 t} + \Phi_{12} \left(s_{22} \tilde{\Omega} \Psi_{22}^\dagger \tilde{\Omega} - I_2 \right) \right] e^{-2ikL} \right\}_+ dk \\
&\quad - \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Phi_{11} s_{11} + \Phi_{12} s_{21} e^{4ik^2 t} - I_2 \right) \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} \right] \right\}_+ dk \\
&= -i\pi \Phi_{12}^{(1)} - \int_{\partial D_3^0} \frac{2}{\Delta_-} \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} dk \\
&\quad + \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Phi_{11} s_{12} e^{-4ik^2 t} + \Phi_{12} \left(s_{22} \tilde{\Omega} \Psi_{22}^\dagger \tilde{\Omega} - I_2 \right) \right] e^{-2ikL} \right\}_+ dk \\
&\quad - \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Phi_{11} s_{11} + \Phi_{12} s_{21} e^{4ik^2 t} - I_2 \right) \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} \right] \right\}_+ dk. \quad (4.38)
\end{aligned}$$

It follows from equations (4.37) and (4.38) that we find

$$\begin{aligned}
\Phi_{12}^{(1)} &= \frac{1}{2i\pi} \int_{\partial D_3^0} \left(\frac{\Delta_+}{\Delta_-} \Phi_{12+} - \frac{2}{\Delta_-} \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} \right) dk \\
&\quad + \frac{1}{2i\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Phi_{11} s_{12} e^{-4ik^2 t} + \Phi_{12} \left(s_{22} \tilde{\Omega} \Psi_{22}^\dagger \tilde{\Omega} - I_2 \right) \right] e^{-2ikL} \right\}_+ dk \\
&\quad - \frac{1}{2i\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Phi_{11} s_{11} + \Phi_{12} s_{21} e^{4ik^2 t} - I_2 \right) \tilde{\Sigma} \Psi_{21}^\dagger \tilde{\Omega} \right] \right\}_+ dk. \quad (4.39)
\end{aligned}$$

Substituting equation (4.39) into the first formula of equation (4.7), then we obtain (4.24).

We now derive the Dirichlet boundary data (4.25) at $x = L$ from the known Neumann boundary value problem. By applying Cauchy's theorem to equation (4.8), we obtain

$$\begin{aligned}
i\pi \Psi_{12}^{(1)}(t) &= \left(\int_{\partial D_1} + \int_{\partial D_3} \right) \Psi_{12} dk = \int_{\partial D_3} \Psi_{12-} dk \\
&= \int_{\partial D_3^0} \left[\Psi_{12-} - \frac{2}{\Delta_-} (e^{2ikL} \Psi_{12})_+ \right] dk + C_4(t) \\
&= \int_{\partial D_3^0} -\frac{\Delta_+}{\Delta_-} \Psi_{12+} dk + C_4(t), \quad (4.40)
\end{aligned}$$

where we have introduced the function $C_4(t)$ in the form

$$C_4(t) = \int_{\partial D_3^0} \left[\frac{2}{\Delta_-} (e^{2ikL} \Psi_{12})_+ \right] dk$$

$$\begin{aligned}
&= \int_{\partial D_3^0} \frac{2}{\Delta_-} \left[-\tilde{\Sigma} c_{21}^\dagger(t, \bar{k}) \tilde{\Omega} + \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} \right]_+ dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Psi_{11} \tilde{\Sigma} s_{11}^\dagger \tilde{\Sigma} - I_2 \right) \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} + \Psi_{11} \tilde{\Sigma} s_{21}^\dagger \Phi_{22}^\dagger \tilde{\Omega} e^{-4ik^2 t} \right] \right\}_+ dk \\
&- \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega} s_{12}^\dagger \Phi_{21}^\dagger \tilde{\Omega} e^{4ik^2 t} + \tilde{\Omega} s_{22}^\dagger \Phi_{22}^\dagger \tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_+ dk \\
&= -i\pi \Psi_{12}^{(1)} + \int_{\partial D_3^0} \frac{2}{\Delta_-} \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} dk \\
&+ \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Psi_{11} \tilde{\Sigma} s_{11}^\dagger \tilde{\Sigma} - I_2 \right) \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} + \Psi_{11} \tilde{\Sigma} s_{21}^\dagger \Phi_{22}^\dagger \tilde{\Omega} e^{-4ik^2 t} \right] \right\}_+ dk \\
&- \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega} s_{12}^\dagger \Phi_{21}^\dagger \tilde{\Omega} e^{4ik^2 t} + \tilde{\Omega} s_{22}^\dagger \Phi_{22}^\dagger \tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_+ dk.
\end{aligned} \tag{4.41}$$

It follows from equations (4.40) and (4.41) that we find

$$\begin{aligned}
\Psi_{12}^{(1)} &= \frac{1}{2i\pi} \int_{\partial D_3^0} \left(-\frac{\Delta_+}{\Delta_-} \Psi_{12+} + \frac{2}{\Delta_-} \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} \right) dk \\
&+ \frac{1}{2i\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\left(\Psi_{11} \tilde{\Sigma} s_{11}^\dagger \tilde{\Sigma} - I_2 \right) \tilde{\Sigma} \Phi_{21}^\dagger \tilde{\Omega} + \Psi_{11} \tilde{\Sigma} s_{21}^\dagger \Phi_{22}^\dagger \tilde{\Omega} e^{-4ik^2 t} \right] \right\}_+ dk \\
&- \frac{1}{2i\pi} \int_{\partial D_3^0} \left\{ \frac{2}{\Delta_-} \left[\Psi_{12} \left(\tilde{\Omega} s_{12}^\dagger \Phi_{21}^\dagger \tilde{\Omega} e^{4ik^2 t} + \tilde{\Omega} s_{22}^\dagger \Phi_{22}^\dagger \tilde{\Omega} - I_2 \right) e^{2ikL} \right] \right\}_+ dk.
\end{aligned} \tag{4.42}$$

Substituting equation (4.42) into the first formula of equation (4.9), then we obtain (4.25). $\square$

4.3. The effective characterizations. For given Dirichlet boundary data $\{F_0(t), G_0(t)\}$, substituting equations (4.22)–(4.25) into equation (4.21), and the similar expressions for $\{\Psi_{ij}(t, k)\}_{i,j=1}^2$, we obtain a system of coupled nonlinear integral equations for $\{\Phi_{ij}(t, k), \Psi_{ij}(t, k)\}_{i,j=1}^2$. This system effectively characterizes the spectral functions $\{\Phi(t, k), \Psi(t, k)\}$ for the given Dirichlet problem. In the following, we use the method of perturbation expressions to represent them in detail.

Substituting these perturbation expressions:

$$\begin{aligned}
\Phi_{ij}(t, k) &= \Phi_{ij}^{[0]}(t, k) + \epsilon \Phi_{ij}^{[1]}(t, k) + \epsilon^2 \Phi_{ij}^{[2]}(t, k) + \cdots, \quad i, j = 1, 2, \\
\Psi_{ij}(t, k) &= \Psi_{ij}^{[0]}(t, k) + \epsilon \Psi_{ij}^{[1]}(t, k) + \epsilon^2 \Psi_{ij}^{[2]}(t, k) + \cdots, \quad i, j = 1, 2, \\
s_{ij}(k) &= s_{ij}^{[0]}(k) + \epsilon s_{ij}^{[1]}(k) + \epsilon^2 s_{ij}^{[2]}(k) + \cdots, \quad i, j = 1, 2, \\
F_0(t) &= \epsilon F_0^{[1]}(t) + \epsilon^2 F_0^{[2]}(t) + \epsilon^3 F_0^{[3]}(t) + \cdots, \\
G_0(t) &= \epsilon G_0^{[1]}(t) + \epsilon^2 G_0^{[2]}(t) + \epsilon^3 G_0^{[3]}(t) + \cdots, \\
F_1(t) &= \epsilon F_1^{[1]}(t) + \epsilon^2 F_1^{[2]}(t) + \epsilon^3 F_1^{[3]}(t) + \cdots,
\end{aligned}$$

$$G_1(t) = \epsilon G_1(t)^{[1]}(t) + \epsilon^2 G_1(t)^{[2]}(t) + \epsilon^3 G_1(t)^{[3]}(t) + \cdots, \quad (4.43)$$

into equations (4.13) and (4.14), where $\epsilon > 0$ is a small parameter, we obtain the $O(1)$ and $O(\epsilon)$ order terms of $\Phi_{ij}(t, k)$ as follows:
 $O(1)$:

$$O(1) : \quad \begin{cases} \Phi_{jj}^{[0]} = 1, & j = 1, 2, \\ \Phi_{ij}^{[0]} = 0, & i, j = 1, 2, i \neq j, \end{cases} \quad (4.44)$$

$$O(\epsilon) : \quad \begin{cases} \Phi_{11}^{[1]}(t, k) = \Phi_{22}^{[1]}(t, k) = 0, \\ \Phi_{12}^{[1]}(t, k) = \int_0^t e^{-4ik^2(t-\tau)} (2kF_0^{[1]} + iF_1^{[1]})(\tau, k) d\tau, \\ \Phi_{21}^{[1]}(t, k) = \int_0^t e^{-4ik^2(t-\tau)} (2k\tilde{\Sigma}F_0^{\dagger[1]}\tilde{\Omega} + i\tilde{\Sigma}F_1^{\dagger[1]}\tilde{\Omega})(\tau, k) d\tau, \end{cases} \quad (4.45)$$

and other $O(\epsilon^s)$ ($s > 1$) order terms of $\Phi_{ij}(t, k)$, which are omitted here. Similarly, through the boundary values at $x = L$, we can also obtain similar expressions for $\{\Psi_{ij}^{[s]}(t, k)\}_{i,j=1}^2, s = 0, 1$. If we assume that $\det(\mathbb{S}_{22})$ has no zeros, then we expand equations (4.22) and (4.23) to obtain the following:

$$\begin{aligned} F_1^{[1]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{2\Delta_+}{i\Delta_-} (k\Phi_{12-}^{[1]} + iF_0^{[1]}) - \frac{4}{i\Delta_-} \left(k\tilde{\Sigma}\Psi_{21-}^{\dagger[1]}\tilde{\Omega} + iG_0^{\dagger[1]} \right) \right] dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Phi_{21}^{[1]} e^{-2ikL} + \left(\Phi_{21}^{[1]} s_{21}^{[1]} e^{4ik^2t} - I_2 \right) \tilde{\Sigma}\Psi_{21}^{\dagger[1]}\tilde{\Omega} \right] \right\}_- dk, \\ G_1^{[1]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{2\Delta_+}{i\Delta_-} (k\Psi_{12-}^{[1]} + iG_0^{[1]}) + \frac{4}{i\Delta_-} (k\tilde{\Sigma}\Phi_{21-}^{\dagger[1]}\tilde{\Omega} + iF_0^{\dagger[1]}) \right] dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\tilde{\Sigma}\Phi_{21}^{\dagger[1]}\tilde{\Omega} - \Psi_{12}^{[1]} \left(\tilde{\Omega}s_{12}^{\dagger[1]}\Phi_{21}^{\dagger[1]}\tilde{\Omega} e^{4ik^2t} - I_2 \right) e^{2ikL} \right] \right\}_- dk. \end{aligned} \quad (4.46)$$

According to equation (4.45), we further derive

$$\begin{aligned} \Phi_{12-}^{[1]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} F_0^{[1]}(\tau, k) d\tau, \\ \Phi_{21-}^{[1]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Sigma}F_0^{\dagger[1]}\tilde{\Omega}(\tau, k) d\tau. \end{aligned} \quad (4.47)$$

Similarly can be obtained

$$\begin{aligned} \Psi_{12-}^{[1]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} G_0^{[1]}(\tau, k) d\tau, \\ \Psi_{21-}^{[1]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Sigma}G_0^{\dagger[1]}\tilde{\Omega}(\tau, k) d\tau. \end{aligned} \quad (4.48)$$

Therefore, the unknown Neumann boundary value problem can be solved by the perturbation method as follows. For the given $\det(\mathbb{S}_{22})$ with no zeros and

the Dirichlet boundary data $F_0^{[1]}$ and $G_0^{[1]}$ at $x = 0$, we can find the functions $\{\Phi_{12,-}^{[1]}, \Phi_{21,-}^{[1]}, \Psi_{12}^{[1]}, \Psi_{21}^{[1]}\}$ according to equations (4.47) and (4.48). Then, we can obtain $F_1^{[1]}(t)$ and $G_1^{[1]}(t)$ from equation (4.46). Finally, we obtain $\Phi_{ij}^{[1]}$ from equation (4.45). Similarly, we can also find $\Psi_{ij}^{[1]}$.

If we assume that $\det(\mathbb{S}_{22})$ has no zeros, then we expand equations (4.24) and (4.25) to obtain

$$\begin{aligned} F_0^{[1]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{\Delta_+}{\Delta_-} \Phi_{12+}^{[1]} - \frac{2}{\Delta_-} \tilde{\Sigma} \Psi_{21+}^{\dagger[1]} \tilde{\Omega} \right] dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Phi_{21}^{[1]} e^{-2ikL} + \left(\Phi_{21}^{[1]} s_{21}^{[1]} e^{4ik^2t} - I_2 \right) \tilde{\Sigma} \Psi_{21}^{\dagger[1]} \tilde{\Omega} \right] \right\}_- dk, \\ G_0^{[1]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{\Delta_+}{i\Delta_-} \Psi_{12+}^{[1]} + \frac{2}{\Delta_-} \tilde{\Sigma} \Phi_{21+}^{\dagger[1]} \tilde{\Omega} \right] dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\tilde{\Sigma} \Phi_{21}^{\dagger[1]} \tilde{\Omega} - \Psi_{12}^{[1]} \left(\tilde{\Omega} s_{12}^{\dagger[1]} \Phi_{21}^{\dagger[1]} \tilde{\Omega} e^{4ik^2t} - I_2 \right) e^{2ikL} \right] \right\}_- dk. \end{aligned} \quad (4.49)$$

According to equation (4.45), we further derive

$$\begin{aligned} \Phi_{12+}^{[1]}(t, k) &= 2i \int_0^t e^{-4ik^2(t-\tau)} F_1^{[1]}(\tau, k) d\tau, \\ \Phi_{21+}^{[1]}(t, k) &= -2i \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Sigma} F_1^{\dagger[1]} \tilde{\Omega}(\tau, k) d\tau. \end{aligned} \quad (4.50)$$

Similarly,

$$\begin{aligned} \Psi_{12+}^{[1]}(t, k) &= 2i \int_0^t e^{-4ik^2(t-\tau)} G_1^{[1]}(\tau, k) d\tau, \\ \Psi_{21+}^{[1]}(t, k) &= -2i \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Sigma} G_1^{\dagger[1]} \tilde{\Omega}(\tau, k) d\tau. \end{aligned} \quad (4.51)$$

Thus, the unknown Dirichlet boundary value problem can now be solved by the perturbation method as follows: For the given $\det(\mathbb{S}_{22})$ with no zeros and the Neumann boundary data $F_1^{[1]}$ and $G_1^{[1]}$ at $x = L$, we can determine the functions $\{\Psi_{12,+}^{[1]}, \Psi_{21,+}^{[1]}, \Phi_{12}^{[1]}, \Phi_{21}^{[1]}\}$ from equations (4.50) and (4.51). In addition, we can further find $F_0^{[1]}(t)$ and $G_0^{[1]}(t)$ from equation (4.49). Finally, we can obtain $\Psi_{ij}^{[1]}$ from equation (4.45).

In fact, the recurrence formulas obtained above can be extended infinitely. We assume that they hold for all $0 \leq j \leq n-1$. Then, for $n > 0$, substituting equation (4.43) into equations (4.22) and (4.23) will yield terms of order ϵ^n :

$$\begin{aligned} F_1^{[n]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{2\Delta_+}{i\Delta_-} (k\Phi_{12-}^{[n]} + iF_0^{[n]}) - \frac{4}{i\Delta_-} \left(k\tilde{\Sigma} \Psi_{21-}^{\dagger[n]} \tilde{\Omega} + iG_0^{\dagger[n]} \right) \right] dk \\ &\quad - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Phi_{21}^{[n]} e^{-2ikL} + \left(\Phi_{21}^{[n]} s_{21}^{[n]} e^{4ik^2t} - I_2 \right) \tilde{\Sigma} \Psi_{21}^{\dagger[n]} \tilde{\Omega} \right] \right\}_- dk \end{aligned}$$

$$\begin{aligned}
& + O(\kappa^n), \\
G_1^{[n]}(t) = & \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{2\Delta_+}{i\Delta_-} (k\Psi_{12-}^{[n]} + iG_0^{[n]}) + \frac{4}{i\Delta_-} (k\tilde{\Sigma}\Phi_{21-}^{\dagger[n]}\tilde{\Omega} + iF_0^{\dagger[n]}) \right] dk \\
& - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\tilde{\Sigma}\Phi_{21}^{\dagger[n]}\tilde{\Omega} - \Psi_{12}^{[n]} \left(\tilde{\Omega}s_{12}^{\dagger[n]}\Phi_{21}^{\dagger[n]}\tilde{\Omega}e^{4ik^2t} - I_2 \right) e^{2ikL} \right] \right\}_- dk \\
& + O(\kappa^n), \tag{4.52}
\end{aligned}$$

where $O(\kappa^n)$ stands for the result involving known terms of lower order. The terms of $O(\epsilon^n)$ for Φ_{ij} in equation(4.21) and the similar equations for Ψ_{ij} yield

$$\begin{aligned}
\Phi_{12}^{[n]}(t, k) &= \int_0^t e^{-4ik^2(t-\tau)} (2kF_0^{[n]} + iF_1^{[n]})(\tau, k) d\tau + O(\kappa^n), \\
\Phi_{21}^{[n]}(t, k) &= \int_0^t e^{-4ik^2(t-\tau)} (2k\tilde{\Sigma}F_0^{\dagger[n]}\tilde{\Omega} + i\tilde{\Sigma}F_1^{\dagger[n]}\tilde{\Omega})(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{12}^{[n]}(t, k) &= \int_0^t e^{-4ik^2(t-\tau)} (2kG_0^{[n]} + iG_1^{[n]})(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{21}^{[n]}(t, k) &= \int_0^t e^{-4ik^2(t-\tau)} (2k\tilde{\Sigma}G_0^{\dagger[n]}\tilde{\Omega} + i\tilde{\Sigma}G_1^{\dagger[n]}\tilde{\Omega})(\tau, k) d\tau + O(\kappa^n), \tag{4.53}
\end{aligned}$$

which leads to

$$\begin{aligned}
\Phi_{12-}^{[n]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} F_0^{[n]}(\tau, k) d\tau + O(\kappa^n), \\
\Phi_{21-}^{\dagger[n]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Omega}F_0^{[n]}\tilde{\Sigma}(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{12-}^{[n]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} G_0^{[n]}(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{21-}^{\dagger[n]}(t, k) &= 4k \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Omega}G_0^{[n]}\tilde{\Sigma}(\tau, k) d\tau + O(\kappa^n). \tag{4.54}
\end{aligned}$$

From the system of equations (4.54), for the known Dirichlet boundary data $F_0^{[n]}$ and $G_0^{[n]}$, we can obtain $\Phi_{12}^{[n]}$, $\Phi_{21}^{[n]}$, $\Psi_{12-}^{[n]}$, $\Psi_{21-}^{[n]}$ at each step. In this way, we can find the Neumann boundary data $F_1^{[n]}$ at $x = 0$ and the Neumann boundary data $G_0^{[n]}$ at $x = L$ through equation (4.52).

Similarly, substituting equation (4.43) into equations (4.24) and (4.25) will yield terms of order ϵ^n :

$$\begin{aligned}
F_0^{[n]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[\frac{\Delta_+}{\Delta_-} \Phi_{12+}^{[n]} - \frac{2}{\Delta_-} \tilde{\Sigma}\Psi_{21+}^{\dagger[n]}\tilde{\Omega} \right] dk \\
& - \frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\Phi_{21}^{[n]}e^{-2ikL} + \left(\Phi_{21}^{[n]}s_{21}^{[n]}e^{4ik^2t} - I_2 \right) \tilde{\Sigma}\Psi_{21}^{\dagger[n]}\tilde{\Omega} \right] \right\}_- dk \\
& + O(\kappa^n), \\
G_0^{[n]}(t) &= \frac{1}{\pi} \int_{\partial D_3^0} \left[-\frac{\Delta_+}{i\Delta_-} \Psi_{12+}^{[n]} + \frac{2}{\Delta_-} \tilde{\Sigma}\Phi_{21+}^{\dagger[n]}\tilde{\Omega} \right] dk
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\pi} \int_{\partial D_3^0} \left\{ \frac{4k}{i\Delta_-} \left[\tilde{\Sigma} \Phi_{21}^{\dagger[n]} \tilde{\Omega} - \Psi_{12}^{[n]} \left(\tilde{\Omega} s_{12}^{\dagger[n]} \Phi_{21}^{\dagger[n]} \tilde{\Omega} e^{4ik^2 t} - I_2 \right) e^{2ikL} \right] \right\}_- dk \\
& + O(\kappa^n). \tag{4.55}
\end{aligned}$$

We find

$$\begin{aligned}
\Phi_{12+}^{[n]}(t, k) &= 2i \int_0^t e^{-4ik^2(t-\tau)} F_1^{[n]}(\tau, k) d\tau + O(\kappa^n), \\
\Phi_{21+}^{\dagger[n]}(t, k) &= -2i \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Omega} F_1^{[n]} \tilde{\Sigma}(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{12+}^{[n]}(t, k) &= 2i \int_0^t e^{-4ik^2(t-\tau)} G_1^{[n]}(\tau, k) d\tau + O(\kappa^n), \\
\Psi_{21+}^{\dagger[n]}(t, k) &= -2i \int_0^t e^{-4ik^2(t-\tau)} \tilde{\Omega} G_1^{[n]} \tilde{\Sigma}(\tau, k) d\tau + O(\kappa^n). \tag{4.56}
\end{aligned}$$

From the system of equation (4.56), according to the known Neumann boundary data $F_1^{[n]}$ at $x = 0$ and $G_0^{[n]}$ at $x = L$, we can determine $\Phi_{12,+}^{[n]}$, $\Phi_{21,+}^{[n]}$, $\Psi_{12,+}^{[n]}$, $\Psi_{21,+}^{[n]}$ at each step. In this way, we know that we can obtain the Dirichlet boundary data $F_0^{[n]}$ at $x = 0$ and $G_0^{[n]}$ at $x = L$ through equation (4.55).

4.4. The large L limit. The formulae for $F_0(t)$, G_0 , $F_1(t)$, and G_1 of Proposition 4.2 in the limit $L \rightarrow \infty$ can be reduced to the corresponding ones on the half-line. In the case of $L \rightarrow \infty$, we have

$$G_0 \rightarrow 0, \quad G_1 \rightarrow 0, \quad \Psi_{ij} \rightarrow I_{ij}, \quad s_{ij} \rightarrow I_{ij}, \quad \frac{\Delta_+(k)}{\Delta_-(k)} \rightarrow 1 \quad \text{as } k \rightarrow \infty. \tag{4.57}$$

According to equation (4.57), the $L \rightarrow \infty$ limits of equations (4.22)–(4.25) for the given Dirichlet boundary data, the unknown Neumann boundary data have the form

$$F_1(t) = -\frac{1}{\pi} \int_{\partial D_3^0} [2i(k\Phi_{12-}(t, k) + iF_0) - 2F_0\Phi_{22-}(t, k)] dk, \tag{4.58}$$

$$G_1(t) = \frac{1}{\pi} \int_{\partial D_3^0} [2i(k\Psi_{12-}(t, k) + iG_0) + 2G_0\Psi_{22-}(t, k)] dk \tag{4.59}$$

for the given Neumann boundary data, the unknown Dirichlet boundary data have the form

$$F_1(t) = -\frac{1}{\pi} \int_{\partial D_3^0} [2i(k\Phi_{12-}(t, k) + iF_0) - 2F_0\Phi_{22-}(t, k)] dk, \tag{4.60}$$

$$G_1(t) = \frac{1}{\pi} \int_{\partial D_3^0} [2i(k\Psi_{12-}(t, k) + iG_0) + 2G_0\Psi_{22-}(t, k)] dk. \tag{4.61}$$

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Початково-крайова задача для загального матричного нелінійного рівняння Шредінгера на скінченному інтервалі за допомогою методу Фокаса

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У цій роботі ми аналізуємо початково-крайову задачу для інтегровного загального матричного нелінійного рівняння Шредінгера (НРШ) з пари Лакса розміром 4×4 на скінченному інтервалі, використовуючи оригінальний єдиний метод перетворення Фокаса. Узагальнення включають дефокусуючу матричну систему НРШ, фокусуючу матричну систему НРШ та дві матричні системи НРШ зі змішаними знаками, які менше досліджені в літературі. Ми розглядаємо відповідні рівняння пари Лакса у вигляді 4×4 матриць як проблеми з блочними матрицями розміром 2×2 , виконуючи спектральний аналіз обох компонент пари Лакса одночасно. Аналізуючи алгебраїчний зв'язок між спектральними функціями, пов'язаними з відповідним чином нормованими розв'язками рівнянь Лакса, ми виводимо “глобальне співвідношення”. Це співвідношення є вирішальним для характеристики невідомих крайових значень через задані дані. Такий підхід дозволяє систематично досліджувати нелінійнізовані крайові умови та характеризувати відображення між крайовими значеннями Діріхле та Неймана. Також доведено, що коли довжина інтервалу L прямує до нескінченності, формула розв'язку початково-крайової задачі на скінченному інтервалі зводиться до формули на напівпрямій.

Ключові слова: загальне матричне нелінійне рівняння Шредінгера, початково-крайова задача, задача Рімана–Гільберта, глобальне співвідношення, метод Фокаса