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Actions on the Four-Dimensional Pseudo-Euclidean Space $\mathbb{R}^{2,2}$ with a Three-Dimensional Orbit

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In this paper, we classify the connected Lie groups up to conjugacy within $\mathrm{Iso}(\mathbb{R}^{2,2})$, which act isometrically on the four-dimensional pseudo-Euclidean space $\mathbb{R}^{2,2}$ in such a way that there is a three-dimensional induced orbit in $\mathbb{R}^{2,2}$. Then we give the list of the acting groups in both cases, proper and nonproper actions. When the action is proper, we determine the explicit representation of the acting group in $SO(2,2) \ltimes \mathbb{R}^{2,2}$ and then we specify the orbits and the orbit spaces.

Key words: cohomogeneity one, isometric action, pseudo-Euclidean space

Mathematical Subject Classification 2010: 57S25, 53C30

1. Introduction and preliminaries

The concept of a cohomogeneity one action on a manifold was introduced by P.S. Mostert in his paper [17] in 1956. According to this paper, an action of a Lie grpoup G on a manifold M is said to be of *cohomogeneity one* if there exists an induced orbit in M of codimension one. He assumed that the acting Lie group G is a compact subgroup of $\mathrm{Diff}(M)$, the group of diffeomorphisms of M, and then studied the orbit space M/G. He showed that the space is homeomorphic to one of (i) a circle, (ii) an open interval, (iii) a half-open interval or (iv) a closed interval. The compactness of the acting group was crucial in the study since it implied that the induced orbits were closed embedded submanifolds and the orbit space was Hausdorff. The generalization of the study was done by L. Berard Bergery [10] for closed subgroups of a Riemannian manifold. More precisely, he assumed that G is a closed Lie subgroup of $\mathrm{Iso}(M)$, where M is a Riemannian manifold, acting on M isometrically and with cohomogeneity one. Then he got the same results about the orbit space. Thereafter, the topic "cohomogeneity one Riemannian manifolds" has been studied by many mathematicians with different approaches (see, for instance, [3, 10, 12–14, 16, 18–23]). However, if the metric on M is indefinite, i.e., M is a pseudo-Riemannian manifold, then the closedness of the acting group in the isometry group $\mathrm{Iso}(M)$ does not imply that the orbits are closed and the quotient space is Haussdorff.

Let G be a Lie group which acts on a connected smooth manifold M. For each point x in M, $G(x)$ denotes the orbit of x and G_x is the stabilizer in G of x. The action is said to be *proper* if the mapping $\varphi : G\times M \to M\times M$, where $\varphi((g,x)) = (g.x,x)$, is proper. Equivalently, for any sequences x_n in M and g_n in G, $g_n x_n \to y$ and $x_n \to x$ imply that g_n has a convergent subsequence. The action of G on M is *nonproper* if it is not proper. Equivalently, there are sequences g_n in G and x_n in M such that x_n and $g_n x_n$ converge in M and $g_n \to \infty$, i.e., g_n leaves compact subsets. For instance, if G is compact, the action is obviously proper. The orbit space M/G of a proper action of G on M is Hausdorff, the orbits are closed embedded submanifolds, and the stabilizers are compact (see [1]).

By a result of D.V. Alekseevsky in [2], for a Lie group G, there exists a complete G invariant Riemannian metric g on M if and only if the action of G on M is proper. Furthermore, in this case the Lie group G would be a closed subgroup of the Riemannian isometry group $\mathrm{Iso}(M,g)$. This theorem makes a link between proper actions and Riemannian G-manifolds. Also, it clarifies the importance of cohomogeneity one proper actions on pseudo-Riemannian manifolds.

This paper which deals with cohomogeneity one actions on four-dimensional pseudo-Euclidean space $\mathbb{R}^{2,2}$, the real vector space $\mathbb{R}^4$ with the quadratic form $\mathfrak{q} = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$, is the continuation of our study of cohomogeneity one pseudo-Riemannian manifolds (see, for instance, [4–9, 15]). In the main Theorem 2.5 of the paper, we determine all the Lie subgroups of $\mathrm{Iso}(\mathbb{R}^{2,2})$ acting with cohomogeneity one on $\mathbb{R}^{2,2}$ up to conjugacy. As a remarkable consequence of Theorem 2.5, we prove that any connected Lie subgroup of $\mathrm{Iso}(\mathbb{R}^{2,2})$ acting linearly and with cohomogeneity one preserves a maximal totally isotropic subspace (Corollary 2.7). Then we specify the subgroups acting properly and those acting nonproperly in Theorem 3.1. When the action is proper, we give the explicit representation of the acting group in $\mathrm{Iso}(\mathbb{R}^{2,2})$, up to conjugacy, and then we determine the induced orbits and the orbit space. As a result of Theorem 3.1, we prove that the linear projection of the groups acting properly preserves either a two-dimensional definite subspace or a maximal totally isotropic subspace (Corollaries 3.2 and 3.4).

2. Groups acting with cohomogeneity one on $\mathbb{R}^{2,2}$

The $(p+q)$-dimensional pseudo-Euclidean space $\mathbb{R}^{p,q}$ is the $(p+q)$-dimensional real vector space $\mathbb{R}^{p+q}$ with the line element $ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$. Let $\mathrm{Iso}(\mathbb{R}^{p,q})$ denote the group of isometries of $\mathbb{R}^{p,q}$, that is, the group $O(p,q) \ltimes \mathbb{R}^{p,q}$. The multiplication and inversion on $\mathrm{Iso}(\mathbb{R}^{p,q})$ is given by $(V,v)(U,u) = (VU, v + V(u))$ and $(V,v)^{-1} = (V^{-1}, -V^{-1}(v))$ and the action of $\mathrm{Iso}(\mathbb{R}^{p,q})$ on $\mathbb{R}^{p,q}$ is given by $\mathrm{Iso}(\mathbb{R}^{p,q}) \times \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$, $((V,v),x) \mapsto V(x) + v$. The isometry group $\mathrm{Iso}(\mathbb{R}^{p,q})$ has four connected components. The identity component of $\mathrm{Iso}(\mathbb{R}^{p,q})$ is denoted by $\mathrm{Iso}_\circ(\mathbb{R}^{p,q}) = SO_\circ(p,q) \ltimes \mathbb{R}^{p,q}$.

To simplify computations in determining the Lie groups acting isometrically and with cohomogeneity one on $\mathbb{R}^{2,2}$, we consider the four-dimensional vector space of 2×2 real matrices $M(2,\mathbb{R})$ endowed with the line element defined by the determinant. Let E_{ij} denote the 2×2 matrix whose (i, j) entry is 1 and whose other entries are zero. Then $\{e'_1, e'_2, e'_3, e'_4\}$ is a basis for $M(2, \mathbb{R})$, where

$$e'_1 = E_{11} + E_{22}, \quad e'_2 = -E_{12} + E_{21}, \quad e'_3 = E_{12} + E_{21}, \quad e'_4 = E_{11} - E_{22}.$$

If $\{e_1, e_2, e_3, e_4\}$ is the standard basis of $\mathbb{R}^4$, then the linear function $\Psi : \mathbb{R}^{2,2} \to (M(2, \mathbb{R}), \det)$, defined by $e_i \mapsto e'_i$ for $1 \leq i \leq 4$, is a linear isometry. Hence we denote $M(2, \mathbb{R})$ endowed with the determinant line element by $M^{2,2}$. Let $\mathrm{Iso}_\circ(M^{2,2})$ denote the identity component of the isometry group of $M^{2,2}$. If $X \in M^{2,2}$, the translation T_X sending Y to $X + Y$ is an isometry. Then $T_X \circ T_Z = T_{X+Z} = T_Z \circ T_X$, and since T_0 is the identity, ${T_X}^{-1} = T_{-X}$. It follows that the set of all translations of M is a commutative subgroup of $\mathrm{Iso}_\circ(M^{2,2})$ and is isomorphic to $M(2, \mathbb{R})$, the additive Lie group of 2×2 real matrices, via $T_X \leftrightarrow X$. The following Lemma gives an explicit representation of $\mathrm{Iso}_\circ(M^{2,2})$.

Lemma 2.1. *Considering $M(2, \mathbb{R})$ as the commutative Lie group of all translations of $M^{2,2}$, we have*

$$\mathrm{Iso}_\circ(M^{2,2}) = \frac{SL(2, \mathbb{R}) \times SL(2, \mathbb{R})}{\mathbb{Z}_2} \ltimes_\varphi M(2, \mathbb{R}),$$

where $\varphi([\![A, B]\!])(T_X) = T_{AXB^{-1}}$.

Proof. First we determine the identity component of the Lie group of linear isometries of $M^{2,2}$, say $L_\circ(M^{2,2})$, that preserves the origin. Let $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. The homomorphism

$$\theta : G \to \mathrm{Iso}_\circ(M^{2,2})$$

with

$$\theta(A, B)(X) = AXB^{-1}, \quad X \in M^{2,2},$$

defines an isometric action of G on $M^{2,2}$, and

$$\ker\theta = \{(-I, -I), (I, I)\},$$

where I denotes the identity matrix. Then θ induces a faithful linear action of $G/\mathbb{Z}_2$ on $M^{2,2}$ given by $[\![A, B]\!]X = AXB^{-1}$. Hence $G/\mathbb{Z}_2 \subseteq L_\circ(M^{2,2})$. On the other hand, the isomorphism $SO_\circ(2, 2) \to L_\circ(M^{2,2})$ given by $g \mapsto \Psi \circ g \circ \Psi^{-1}$, implies that $\dim L_\circ(M^{2,2}) = 6$. Thus $(SL(2, \mathbb{R}) \times SL(2, \mathbb{R}))/\mathbb{Z}_2 = L_\circ(M^{2,2})$.

Now, if $f \in \mathrm{Iso}_\circ(M^{2,2})$, let $X = f(0) \in M^{2,2}$. Thus $T_{-X} \circ f(0) = 0$. Hence $T_{-X} \circ f \in L_\circ(M^{2,2})$, and so $f = T_X \circ [\![A, B]\!]$ for some $A, B \in SL(2, \mathbb{R})$. Clearly, this expression of f as $T_X \circ [\![A, B]\!]$ is unique.

For all $Z \in M^{2,2}$,

$$[\![A, B]\!] \circ T_X)(Z) = AXB^{-1} + AZB^{-1} = (T_{[\![A,B]\!]X} \circ [\![A, B]\!])(Z).$$

Hence, $[\![A, B]\!]T_X = T_{[\![A,B]\!]X} \circ [\![A, B]\!]$ that determines the multiplication rule on $\mathrm{Iso}_\circ(M^{2,2})$ as

$$([\![A, B]\!], T_X) \cdot ([\![C, D]\!], T_Y) = ([\![AC, BD]\!], T_{AYB^{-1}+X}).$$

This rule shows that the group of all translations of $M^{2,2}$ is a normal subgroup of $\mathrm{Iso}_\circ(M^{2,2})$. If we define $\varphi : G/\mathbb{Z}_2 \to \mathrm{Aut}(M(2,\mathbb{R}))$ by $\varphi([\![A,B]\!])(T_X) = T_{AXB^{-1}}$, then the function $([\![A,B]\!], T_X) \to T_X \circ [\![A,B]\!]$ is a Lie group isomorphism from $G/\mathbb{Z}_2 \ltimes_\varphi M(2,\mathbb{R})$ onto $\mathrm{Iso}_\circ(M^{2,2})$. □

By the proof of Lemma 2.1, we may consider any element of $\mathrm{Iso}_\circ(M^{2,2})$ of the from $([\![A,B]\!], X) \in (SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))/\mathbb{Z}_2 \ltimes_\varphi M(2,\mathbb{R})$. The action of $\mathrm{Iso}_\circ(M^{2,2})$ on $M^{2,2}$ is given by

$$([\![A,B]\!], X)Y = AYB^{-1} + X.$$

The Lie algebra of $\mathrm{Iso}_\circ(M^{2,2})$ is $\mathfrak{I} = (\oplus) \oplus_\phi \mathfrak{M}$, where $\mathfrak{M}$ denotes the space of 2×2 real matrices on which the bracket is identically zero, and $\phi(V,W)(x) = Vx - xW$. Thus the bracket on $\mathfrak{I}$ is

$$[(U,V) + x, (W,Z) + y] = (UW - WU, VZ - ZV) + \phi(U,V)(y) - \phi(W,Z)(x),$$

and hence the adjoint action of $\mathrm{Iso}_\circ(M^{2,2})$ on its Lie algebra is given as follows:

$$\mathrm{Ad}_{([\![A,B]\!],X)}((V,W)+y) = (AVA^{-1}, BWB^{-1}) - AVA^{-1}X + XBWB^{-1} + AyB^{-1}.$$

Therefore, to determine the Lie groups acting with cohomogeneity one on $\mathbb{R}^{2,2}$, we may specify the connected Lie subgroups of $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})) \ltimes_\varphi M(2,\mathbb{R})$ that act with cohomogeneity one on $M^{2,2}$.

It is known that any one-parameter Lie subgroup of $SL(2,\mathbb{R})$ is conjugate to one of the groups (see [3], p.436),

- $\mathbf{A} = \{A_t = e^t E_{11} + e^{-t} E_{22} \mid t \in \mathbb{R}\}$,
- $\mathbf{N} = \{N_t = E_{11} + E_{22} + tE_{12} \mid t \in \mathbb{R}\}$,
- $\mathbf{K} = \{K_t = (\cos t)(E_{11} + E_{22}) + (\sin t)(E_{21} - E_{12}) \mid t \in \mathbb{R}\}$,

The Lie groups **A**, **N** and **K** are one-parameter subgroups defined by

$$Y_\mathfrak{a} = E_{11} - E_{22}, \quad Y_\mathfrak{n} = E_{12}, \quad Y_\mathfrak{k} = E_{21} - E_{12},$$

respectively. The set $\{Y_\mathfrak{a}, Y_\mathfrak{n}, Y_\mathfrak{k}\}$ is a basis for $\mathfrak{sl}(2,\mathbb{R})$ and we fix this basis throughout the paper. The decomposition $SL(2,\mathbb{R}) = \mathbf{KAN}$ is known as the Iwasawa decomposition of the simple group $SL(2,\mathbb{R})$.

Each two-dimensional connected Lie subgroup of $SL(2,\mathbb{R})$ is conjugate to

$$\{A_{t,s} = e^t E_{11} + e^{-t} E_{22} + sE_{12} \mid t, s \in \mathbb{R}\},$$

which is isomorphic to $\mathrm{Aff}_\circ(\mathbb{R})$, the identity component of the group of affine transformations of the real line $\mathbb{R}$. Henceforth, we denote this group by $\mathrm{Aff}_\circ(\mathbb{R})$ in the paper.

The bracket rule on $\mathfrak{sl}(2,\mathbb{R})$ implies that

$$[Y_\mathfrak{k}, Y_\mathfrak{a}] = 2Y_\mathfrak{k} + 4Y_\mathfrak{n}, \quad [Y_\mathfrak{k}, Y_\mathfrak{n}] = -Y_\mathfrak{a}, \quad [Y_\mathfrak{a}, Y_\mathfrak{n}] = 2Y_\mathfrak{n}. \tag{2.1}$$

Let $(Y_{\mathfrak{k}},0)+u$, $(Y_{\mathfrak{a}},0)+v$, $(Y_{\mathfrak{n}},0)+w$, $(0,Y_{\mathfrak{k}})+x$, $(0,Y_{\mathfrak{a}})+y$ and $(0,Y_{\mathfrak{n}})+z$ belong to a subalgebra $\mathfrak{h}\subseteq\mathfrak{I}$, where $u,v,w,x,y,z\in\mathfrak{M}$. Then the following vectors should belong to $\mathfrak{h}$:

$$\begin{aligned}
[(Y_{\mathfrak{k}},0)+u,(Y_{\mathfrak{a}},0)+v] &= (2Y_{\mathfrak{k}}+4Y_{\mathfrak{n}},0)+(Y_{\mathfrak{k}}v-Y_{\mathfrak{a}}u),\\
[(Y_{\mathfrak{k}},0)+u,(Y_{\mathfrak{n}},0)+w] &= (-Y_{\mathfrak{a}},0)+(Y_{\mathfrak{k}}w-Y_{\mathfrak{n}}u),\\
[(Y_{\mathfrak{a}},0)+v,(Y_{\mathfrak{n}},0)+w] &= (2Y_{\mathfrak{n}},0)+(Y_{\mathfrak{a}}w-Y_{\mathfrak{n}}v),\\
[(0,Y_{\mathfrak{k}})+x,(0,Y_{\mathfrak{a}})+y] &= (0,2Y_{\mathfrak{k}}+4Y_{\mathfrak{n}})+(Y_{\mathfrak{k}}y-Y_{\mathfrak{a}}x),\\
[(0,Y_{\mathfrak{k}})+x,(0,Y_{\mathfrak{n}})+z] &= (0,-Y_{\mathfrak{a}})+(Y_{\mathfrak{k}}z-Y_{\mathfrak{n}}x),\\
[(0,Y_{\mathfrak{a}})+y,(0,Y_{\mathfrak{a}})+z] &= (0,2Y_{\mathfrak{n}})+(Y_{\mathfrak{a}}z-Y_{\mathfrak{a}}y).
\end{aligned}\tag{2.2}$$

By the fact that $\{(Y_{\mathfrak{k}},0),(Y_{\mathfrak{a}},0),(Y_{\mathfrak{n}},0),(0,Y_{\mathfrak{k}}),(0,Y_{\mathfrak{a}}),(0,Y_{\mathfrak{n}})\}$ is a basis for $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, we have

$$\begin{aligned}
&Y_{\mathfrak{k}}v-Y_{\mathfrak{a}}u=2u+4w, && Y_{\mathfrak{k}}w-Y_{\mathfrak{n}}u=-v, && Y_{\mathfrak{a}}w-Y_{\mathfrak{n}}v=2w,\\
&Y_{\mathfrak{k}}y-Y_{\mathfrak{a}}x=2x+4z, && Y_{\mathfrak{k}}z-Y_{\mathfrak{n}}x=-y, && Y_{\mathfrak{a}}z-Y_{\mathfrak{n}}y=2z
\end{aligned}\tag{2.3}$$

The relations (2) and (3) are key relations in determining the Lie algebras of Lie groups acting with cohomogeneity one on $M^{2,2}$ (see the proof of Theorem 2.5).

Finally, we define the following subspaces in $M^{2,2}$:

- the lightlike line $\ell=\mathbb{R}(e'_2-e'_3)$,
- the spacelike plane $M^{2,0}=\mathbb{R}e'_1\oplus\mathbb{R}e'_2$,
- the timelike plane $M^{0,2}=\mathbb{R}e'_3\oplus\mathbb{R}e'_4$,
- the Lorentz plane $M^{1,1}=\mathbb{R}e'_2\oplus\mathbb{R}e'_3$,
- the degenerate plane $\mathbb{W}^2=\mathbb{R}e'_1\oplus\ell$,
- the totally isotropic plane $\mathbb{V}^2=\mathbb{R}(e'_1-e'_4)\oplus\ell$.

The first step in classifying cohomogeneity one actions on $M^{2,2}$ is to determine Lie subgroups of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ acting linearly up to conjugacy. First, we define the indefinite unitary group of signature $(1,1)$ to be

$$U(1,1)=\{X\in M(2,\mathbb{C})\mid X^\star JX=J\},$$

where $J=E_{11}-E_{22}$. The special indefinite unitary group of signature $(1,1)$ is $SU(1,1)=U(1,1)\cap SL(2,\mathbb{C})$. Their Lie algebras are spanned by $\{X_1,X_2,X_3\}$ for SU(1, 1) and these three plus $\{X_4\}$ for $U(1,1)$, where

$$X_1=i(E_{11}-E_{22}),\quad X_2=E_{12}+E_{21},\quad X_3=i(E_{12}-E_{21}),\quad X_4=i(E_{11}+E_{22}).$$

Equivalently, $\mathfrak{u}(1,1)=\{X\mid X^\star J+JX=0\}$ and $\mathfrak{su}(1,1)=\{X\mid X^\star J+JX=0,\mathrm{Tr}(X)=0\}$. Thus $U(1,1)$ and $SU(1,1)$ are connected Lie groups of dimension 4 and 3, respectively.

Consider the linear map $\psi:\mathfrak{u}(1,1)\to\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{k}$ defined by

$$X_1\mapsto(Y_{\mathfrak{k}},0),\quad X_2\mapsto(Y_{\mathfrak{a}},0),\quad X_3\mapsto(Y_{\mathfrak{k}}+2Y_{\mathfrak{n}},0),\quad X_4\mapsto(0,Y_{\mathfrak{k}}).$$

It is easy to see that ψ is an injective Lie algebra homomorphism. This gives an explicit Lie algebra embedding of $\mathfrak{u}(1,1)$ into $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$. By this embedding, the following theorem from [11] is easily understood.

Theorem 2.2 ([11]). *Let $H \subset SO(2,n)$ be a connected Lie group that acts irreducibly on $\mathbb{R}^{2,n}$. Then H is conjugate to one of the following Lie groups:*

(1) *for arbitrary $n \geq 1$: $SO_\circ(2,n)$,*
(2) *for $n = 2p$: $U(1,p)$, $SU(1,p)$ or $S^1.SO_\circ(1,p)$ if $p > 1$,*
(3) *for $n = 3$: $SO_\circ(1,2) \subset SO(2,3)$.*

Lemma 2.3. *Let H be a connected Lie subgroup of $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ whose action is reducible on $M^{2,2}$. Then H is conjugate to some subgroup of one of the Lie groups $SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$ or $\mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$. Furthermore, if the action is of cohomogeneity one, then $H \subseteq SL(2,R) \times \mathrm{Aff}_\circ(\mathbb{R})$ up to conjugacy.*

Proof. By Theorem 2.2, if H acts irreducibly on $M^{2,2}$, then H is conjugate to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ or $U(1,1)$. Due to this theorem, every other connected proper Lie subgroup of $SO_\circ(2,2)$ preserves a nontrivial linear subspace of $M^{2,2}$. Every one-dimensional spacelike, timelike or lightlike subspace of $M^{2,2}$ is congruent by an element of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ to $\mathbb{R}e'_1$, $\mathbb{R}e'_4$ or ℓ, respectively. Let H be a Lie subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which acts on $M^{2,2}$ isometrically and preserves a one-dimensional linear subspace $\mathscr{L}$.

- If $\mathscr{L} = \mathbb{R}e'_1$, then $H \subseteq \mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$,
- If $\mathscr{L} = \mathbb{R}e'_4$, then $H \subseteq \{(P, e'_4 P e'^{-1}_4) \mid P \in SL(2,\mathbb{R})\}$,
- If $\mathscr{L} = \ell$, then $H \subseteq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

We notice that $\{(P, e'_4 P e'^{-1}_4) \mid P \in SL(2,\mathbb{R})\}$ is conjugate to $\mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$.

And any two-dimensional spacelike, timelike, Lorentzian, degenerate or totally isotropic subspace of $M^{2,2}$ is congruent to $M^{2,0}$, $M^{0,2}$, $M^{1,1}$, $\mathbb{W}^2$ or $\mathbb{V}^2$, respectively.

- If $\mathscr{L}$ is either $M^{2,0}$ or $M^{0,2}$, then $H \subseteq \mathbf{K} \times \mathbf{K}$,
- If $\mathscr{L}$ is $M^{1,1}$, then $H \subseteq \mathbf{A} \times \mathbf{A}$,
- If $\mathscr{L} = \mathbb{W}^2$, then $H \subseteq \{(A_{t,s}, A_{t,s'})|t, s, s' \in \mathbb{R}\}$,
- If $\mathscr{L} = \mathbb{V}^2$, then $H \subseteq SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

If H preserves a three-dimensional subspace, then it should preserve its orthogonal complement that is considered above.

To complete the proof, we show that the group $\mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))$ does not act with cohomogeneity one on $M^{2,2}$. Let $H = \mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$. Since H fixes $\mathbb{R}e'_1$, it preserves the subspace $\Pi = \mathbb{R}e'^{\perp}_1$ as well. The hyperplane $\Pi = \{T \in M^{2,2} \mid \mathrm{trace}(T) = 0\}$ is isometric to the Minkowski space $\mathbb{R}^{1,2}$ and the induced action of H on Π is linear and isometric, so it preserves the hyperquadrics contained in Π that are two- dimensional. Therefore the action of H on Π has no open orbit, i.e., its action on $M^{2,2}$ is not of cohomogeneity one. □

Now we are going to get the list of the groups acting linearly and with cohomogeneity one on $M^{2,2}$ up to conjugacy. Let $p_i : SL(2,\mathbb{R})\times SL(2,\mathbb{R}) \to SL(2,R)$ be the projection map on the first and second factor, where $i \in \{1,2\}$, that is a Lie group homomorphism. For any Lie subgroup H of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, we denote by p_i the restriction of p_i to H as well. Clearly, a product group $p_1(H) \times p_2(H)$ is conjugate to $p_2(H) \times p_1(H)$ by an element of $\mathrm{Iso}(M^{2,2})$ (considering H as a subgroup of $SO_\circ(2,2)$, they are conjugate by an element of $SO(2,2)$). Hence, to determine linear actions on $M^{2,2}$, we consider only the Lie subgroups H of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ with $\dim p_1(H) \geqslant \dim p_2(H)$.

Proposition 2.4. *Let H be a connected Lie subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ acting linearly and with cohomogeneity one on $M^{2,2}$. Then the following statements hold.*

(i) *If the action is irreducible, then H is conjugate to one of the groups $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ or $U(1,1)$.*

(ii) *If the action is reducible, then H is conjugate to one of the following Lie groups:*

$$\begin{array}{ccc} SL(2,R) \times \mathrm{Aff}_\circ(\mathbb{R}), & SL(2,\mathbb{R}) \times \{I\}, & SL(2,\mathbb{R}) \times \mathbf{A}, \\ SL(2,\mathbb{R}) \times \mathbf{N}, & SL(2,\mathbb{R}) \times \mathbf{K}, & \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R}), \\ \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}, & \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}, & \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{K}, \end{array}$$
$$\text{or} \quad \{(A_{t,s}, A_{\alpha t,s'}) \mid t,s,s' \in \mathbb{R}\},$$

where α is a fixed real number.

Proof. If the action of H is irreducible, then H is conjugate to either $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ or $U(1,1) \cong SL(2,\mathbb{R}) \times K$ by Theorem 2.2 . The actions of these groups are of cohomogeneity one since both of them act on the hyperquadric $\{x \in M^{2,2} \mid \det(x) = 1\}$ transitively.

Now assume that the action of H is reducible. By Lemma 2.3, it should be a subgroup of $SL(2,R) \times \mathrm{Aff}_\circ(\mathbb{R})$ up to conjugacy.

- Let $\dim p_1(H) = 3$, i.e., $p_1(H) = SL(2,\mathbb{R})$. Then $H = p_1(H) \times p_2(H)$ since $p_1(H)$ is simple and $p_2(H)$ is solvable. Hence, H is one of the groups:

$$SL(2,R) \times \mathrm{Aff}_\circ(\mathbb{R}), \quad SL(2,\mathbb{R}) \times \{I\}, \quad SL(2,\mathbb{R}) \times \mathbf{A}, \quad SL(2,\mathbb{R}) \times \mathbf{N},$$

  up to conjugacy.
- Let $\dim p_1(H) = 2$, i.e., $p_1(H) = \mathrm{Aff}_\circ(\mathbb{R})$. First, assume that $\dim p_2(H) = 1$. The action is of cohomogeneity one, so $\dim(\ker p_1) = 1$ and we have $\ker p_1 = p_2(H)$. Hence $H = p_1(H) \times p_2(H)$. By the well-known fact about one-dimensional subgroups of $SL(2,\mathbb{R})$, the Lie group H can be conjugate to one of the groups $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}$, $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}$ or $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{K}$.

  Now suppose that $\dim p_2(H) = 2$. If $\dim H = 3$, then $\ker p_1$ is a one-dimensional normal Lie subgroup of $\{I\} \times p_2(H)$, so

$$\ker p_1 = \{I\} \times \mathbf{N} = \exp(\{0\} \oplus \{tY_\mathfrak{n} \mid t \in \mathbb{R}\}). \tag{2.4}$$

  The Lie algebra $\mathfrak{h}$ is a three-dimensional subalgebra of

$$\mathfrak{aff}(\mathbb{R}) \oplus \mathfrak{aff}(\mathbb{R}) = \{(tY_\mathfrak{a} + sY_\mathfrak{n}, t'Y_\mathfrak{a} + s'Y_\mathfrak{n}) \mid t,t',s,s' \in \mathbb{R}\}.$$

Combining this with relation (2.4) implies that the projection $(tY_{\mathfrak{a}}+sY_{\mathfrak{n}}, t'Y_{\mathfrak{a}}+s'Y_{\mathfrak{n}}) \mapsto (t,s,s')$ is a linear isomorphism from $\mathfrak{h}$ onto $\mathbb{R}^3$. Thus $t'=t'(t,s)$ is a linear function $t':\mathbb{R}^2\to\mathbb{R}$. So there are fixed real numbers α and β such that $t'(t,s)=\alpha t+\beta s$. The closedness under the bracket of $\mathfrak{h}$ implies that $\beta=0$. Therefore, $\mathfrak{h}$ has the form $\{(tY_{\mathfrak{a}}+sY_{\mathfrak{n}}, \alpha tY_{\mathfrak{a}}+s'Y_{\mathfrak{n}}) \mid t,s,s'\in\mathbb{R}\}$, and thus $H=\{(A_{t,s}, A_{\alpha t,s'}) \mid t,s,s'\in\mathbb{R}\}$.

If $\dim H=4$, then $p_1(\mathfrak{h})=p_2(\mathfrak{h})=\mathrm{Aff}_\circ(\mathbb{R})$ and $\ker p_1$ is a two-dimensional normal subgroup of $\{I\}\times\mathrm{Aff}_\circ(\mathbb{R})$. So, $\ker p_1=\{0\}\times\mathrm{Aff}_\circ(\mathbb{R})$, and thus $H=\mathrm{Aff}_\circ(\mathbb{R})\times\mathrm{Aff}_\circ(\mathbb{R})$.

The theorem is proved. □

Let $L:(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))\ltimes M(2,\mathbb{R})\to SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ be the projection map, $((g_1,g_2),x)\mapsto(g_1,g_2)$, which is a Lie group homomorphism. Let H be a Lie subgroup of $(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))\ltimes M(2,\mathbb{R})$. Then $\ker L|_H$ is a normal subgroup of H that is called the *translation part* of H. The Lie subgroup $L(H)$ is called the *linear projection* of H in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. We use the same notation $L:(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))\oplus_\phi\mathfrak{M}\to(\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}))$, given by $((V,W),y)\mapsto(V,W)$, to denote the projection on the first factor as well. Let $\mathfrak{h}$ be the Lie algebra of H. Then $L(\mathfrak{h})$ is a subalgebra of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, $\ker L|_{\mathfrak{h}}$ is an ideal of $\mathfrak{h}$ and, obviously, $\ker L|_{\mathfrak{h}}=\mathfrak{h}\cap\mathfrak{M}$. Hence $L(\mathfrak{h})$ preserves $\mathfrak{h}\cap\mathfrak{M}$, i.e., $L(\mathfrak{h})(\mathfrak{h}\cap\mathfrak{M})\subseteq\mathfrak{h}\cap\mathfrak{M}$. By the *normalizer* of $\mathfrak{h}\cap\mathfrak{M}$ in $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$, we mean the maximal Lie subalgebra of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ preserving $\mathfrak{h}\cap\mathfrak{M}$. Now we are ready to state our main theorem.

Theorem 2.5. *Let H be a connected Lie subgroup of $\mathrm{Iso}(M^{2,2})$ acting on $M^{2,2}$ with cohomogeneity one. Then H is conjugate within $\mathrm{Iso}(M^{2,2})$ to one of the groups in Tables 2.1–2.6.*

We consider the proof of Theorem 2.5 according to the translation part of the acting group. The proof is a direct consequence of Lemmas 2.6–2.10. In any case, when the obtained group does not act with cohomogeneity one, we indicate it by determining the dimensions of the induced orbits. For all reminded cases we have $\dim H(q)=3$, where $q\in M^{2,2}$ with $q_{21}\neq 0$.

Consider the ordered basis $B=(E_{11},E_{12},E_{21},E_{22})$ for $\mathfrak{M}$. For any $q\in\mathfrak{M}$, let $q=(q_1,q_2,q_3,q_4)$ be its representation in this basis.

2.1. Subgroups with trivial translation part Here we assume that the translation part of the acting group H is trivial, and so $\mathfrak{M}\cap\mathfrak{h}=\{0\}$.

Lemma 2.6. *Let H be a connected Lie subgroup of $\mathrm{Iso}(M^{2,2})$ acting on $M^{2,2}$ with cohomogeneity one. If the translation part of H is trivial, then H is conjugate within $\mathrm{Iso}(M^{2,2})$ to one of the groups in Table 2.1.*

Proof. The translation part of H is trivial, so $\dim(\mathfrak{h}\cap\mathfrak{M})=0$. The cohomogeneity one assumption implies that $\dim L(\mathfrak{h})\geq 3$. By Proposition 2.4, $L(H)$

Subgroups with trivial translation part			
$SL(2,\mathbb{R})$ $\times SL(2,\mathbb{R})$	$U(1,1)$ $= SL(2,\mathbb{R}) \times \mathbf{K}$	$SL(2,R) \times \mathbf{A}$	$SL(2,\mathbb{R}) \times \mathbf{N}$
$\{(A_{t,s}, A_{\alpha t,s'}) \mid$ $t,s,s' \in \mathbb{R}\}$	$\mathrm{Aff}_\circ(\mathbb{R})$ $\times \mathrm{Aff}_\circ(\mathbb{R})$	$\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}$	$\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}$
$SU(1,1)$ $= SL(2,\mathbb{R}) \times \{I\}$	$\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{K}$		

Table 2.1: Groups acting linearly and with cohomogeneity one, up to conjugacy within $\mathrm{Iso}(M^{2,2})$, where $\alpha \in \mathbb{R}$.

can be one of the Lie subgroups in Table 2.1. We claim that if $L(H)$ is conjugate to one of these groups, then H is.

First, assume that $L(H) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. There exist $u, v, w, x, y, z \in \mathfrak{M}$ such that

$$\mathfrak{h} = \mathbb{R}((Y_\mathfrak{k},0)+u) + \mathbb{R}((Y_\mathfrak{a},0)+v) + \mathbb{R}((Y_\mathfrak{n},0)+w) \\ + \mathbb{R}((0,Y_\mathfrak{k})+x) + \mathbb{R}((0,Y_\mathfrak{a})+y) + \mathbb{R}((0,Y_\mathfrak{n})+z).$$

Then the obtained relations in (2.2) and (2.3) show that u, v, w, x, y and z have the following representations in the basis B:

$$u = (u_1,u_2,u_3,u_4), \quad v = (u_3,u_4,u_1,u_2), \quad w = (-u_1,-u_2,0,0), \\ x = (-u_4,u_3,u_2,-u_1), \quad y = (-u_3,u_4,u_1,-u_2), \quad z = (0,-u_3,0,u_1).$$

Let $p = (u_3,u_4,-u_1,-u_2)$. Then $\mathrm{Ad}((I,I),p)$ maps $(Y_\mathfrak{k},0)+u$, $(Y_\mathfrak{a},0)+v$, $(Y_\mathfrak{n},0)+w$, $(0,Y_\mathfrak{k})+x$, $(0,Y_\mathfrak{a})+y$ and $(0,Y_\mathfrak{n})+z$ to $(Y_\mathfrak{k},0)$, $(Y_\mathfrak{a},0)$, $(Y_\mathfrak{n},0)$, $(0,Y_\mathfrak{k})$, $(0,Y_\mathfrak{a})$ and $(0,Y_\mathfrak{n})$, respectively. Hence, $\mathrm{Ad}((I,I),p)(\mathfrak{h}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. Therefore H is conjugate to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

By a similar argument, one can see that if $L(H)$ is one of the remaining Lie groups, then H itself is conjugate to that group. The key point is in finding a suitable point p such that $\mathrm{Ad}((I,I),p)$ maps $\mathfrak{h}$ to $L(\mathfrak{h}) \oplus \{0\}$. □

Corollary 2.7. *Let H be a connected Lie subgroup of $\mathrm{Iso}(\mathbb{R}^{2,2})$ acting linearly and with cohomogeneity one on $\mathbb{R}^{2,2}$. If the action is reducible, then $H \subseteq \mathrm{Stab}(\mathbb{V}^2)$, where $\mathbb{V}^2$ is a maximal totally isotropic vector subspace of $\mathbb{R}^{2,2}$.*

Proof. Every group in Table 2.1, except the two groups $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and $U(1,1)$, acts reducibly and preserves one of the maximal totally isotropic subspace $\{tE_{12} + sE_{22} \mid t,s \in \mathbb{R}\}$ or $\{tE_{11} + sE_{12} \mid t,s \in \mathbb{R}\}$. □

2.2. Subgroups with a line as the translation part. Every one-dimensional spacelike, timelike or lightlike subspace of $M^{2,2}$ is congruent to $\mathbb{R}e'_1, \mathbb{R}e'_4$ or ℓ, respectively.

Lemma 2.8. *Let H be a connected Lie subgroup of $\mathrm{Iso}(M^{2,2})$ acting on $M^{2,2}$ with cohomogeneity one. If the translation part of H is a line, then H is conjugate within $\mathrm{Iso}(M^{2,2})$ to one of the groups in Table 2.2.*

Subgroups with a definite line as the translation part	
spacelike	**timelike**
$\mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))\times\mathbb{R}e'_1$	$\{(x,e'_4xe'^{-1}_4)\vert x\in SL(2,\mathbb{R})\}\times\mathbb{R}e'_4$
$\exp(\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+\lambda(E_{11}+E_{22}))$ $+\mathbb{R}(Y_\mathfrak{n},Y_\mathfrak{n}))\ltimes\mathbb{R}e'_1$	$\exp(\mathbb{R}((Y_\mathfrak{a},e'_4Y_\mathfrak{a}e'^{-1}_4)+\lambda(E_{11}-E_{22}))$ $+\mathbb{R}(Y_\mathfrak{n},e'_4Y_\mathfrak{n}e'^{-1}_4))\ltimes\mathbb{R}e'_4$

Subgroups with a lightlike line ℓ as the translation part
$\exp(\mathbb{R}((Y_\mathfrak{a},-Y_\mathfrak{a})+\lambda E_{12})+\mathbb{R}(Y_\mathfrak{n},0)+\mathbb{R}(0,Y_\mathfrak{n}))\ltimes\ell$
$\exp(\mathbb{R}(Y_\mathfrak{a},3Y_\mathfrak{a})+\mathbb{R}((Y_\mathfrak{n},0)+\lambda E_{22})+\mathbb{R}(0,Y_\mathfrak{n}))\ltimes\ell$
$\exp(\mathbb{R}(Y_\mathfrak{a},\frac{1}{3}Y_\mathfrak{a})+\mathbb{R}(Y_\mathfrak{n},0)+\mathbb{R}((0,Y_\mathfrak{n})+\lambda E_{11})\ltimes\ell$
$(\mathrm{Aff}_\circ(\mathbb{R})\times\mathbf{A})\ltimes\ell,\quad(\mathbf{A}\times\mathrm{Aff}_\circ(\mathbb{R}))\ltimes\ell$
$\exp(\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+\lambda(E_{11}+E_{22}))+\mathbb{R}(Y_\mathfrak{n},Y_\mathfrak{n}))\ltimes\ell$
$\exp(\mathbb{R}(Y_\mathfrak{a},aY_\mathfrak{a})+\mathbb{R}(Y_\mathfrak{n},0))\ltimes\ell$
$\exp(\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+\lambda E_{11})+\mathbb{R}(Y_\mathfrak{n},0))\ltimes\ell$
$\exp(\mathbb{R}(Y_\mathfrak{a},3Y_\mathfrak{a})+\mathbb{R}((Y_\mathfrak{n},0)+\lambda E_{22})\ltimes\ell$
$\exp(\mathbb{R}(Y_\mathfrak{a},bY_\mathfrak{n})+\mathbb{R}(Y_\mathfrak{n},0))\ltimes\ell$
$(\mathrm{Aff}_\circ(\mathbb{R})\times I)\ltimes\ell,\quad(I\times\mathrm{Aff}_\circ(\mathbb{R}))\ltimes\ell$
$\exp(\mathbb{R}(aY_\mathfrak{a},Y_\mathfrak{a})+\mathbb{R}(0,Y_\mathfrak{n}))\ltimes\ell$
$\exp(\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+\lambda E_{22})+\mathbb{R}(0,Y_\mathfrak{n}))\ltimes\ell$
$\exp(\mathbb{R}(3Y_\mathfrak{a},Y_\mathfrak{a})+\mathbb{R}((0,Y_\mathfrak{n})+\lambda E_{11})\ltimes\ell$
$\exp(\mathbb{R}(bY_\mathfrak{n},Y_\mathfrak{a})+\mathbb{R}(0,Y_\mathfrak{n}))\ltimes\ell$
$(\mathbf{A}\times\mathbf{A})\ltimes\ell,\quad(\mathbf{A}\times\mathbf{N})\ltimes\ell,\quad(\mathbf{N}\times\mathbf{A})\ltimes\ell$
$\exp(\mathbb{R}((Y_\mathfrak{n},0)+\lambda E_{22})+\mathbb{R}((0,Y_\mathfrak{n})+\mu E_{11}))\ltimes\ell$

Table 2.2: Here ℓ denotes the lightlike line $\mathbb{R}(e'_2-e'_3)\leq M^{2,2}$, λ,μ,a,b are fixed real numbers, where $a\in\mathbb{R}^+-\{1,3\}$ and $b\in\{\pm1\}$.

Proof. We may assume that $\mathfrak{h}\cap\mathfrak{M}$ is one of the one-dimensional subspaces $\mathbb{R}e'_1$, $\mathbb{R}e'_4$ or ℓ.

Case I: $\mathfrak{h}\cap\mathfrak{M}=\mathbb{R}e'_1$. The normalizer of $\mathbb{R}e'_1$ in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is $\mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))$, which implies that $L(H)\subseteq\mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))$. By the assumption $\mathfrak{h}\cap\mathfrak{M}=\mathbb{R}e'_1$, so $\dim L(H)\geqslant 2$.

- If $\dim L(H)=3$, then $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{k},Y_\mathfrak{k})+u)+\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+v)+\mathbb{R}((Y_\mathfrak{n},Y_\mathfrak{n})+w)\oplus\mathbb{R}e'_1$, where $u,v,w\in M^{2,2}$. Hence, by using (2.2) and (2.3), we have

$$u=(u_1,u_2,u_2,-u_1),\quad v=(0,v_3-2u_1,v_3,0),\quad w=\left(-\frac{1}{2}v_3,-u_2,0,\frac{1}{2}v_3\right).$$

  Let $p=\left(u_2,\frac{1}{2}v_3-u_1,-\frac{1}{2}v_3,0\right)$. Then $\mathrm{Ad}((I,I),p)$ maps $(Y_\mathfrak{k},Y_\mathfrak{k})+u,(Y_\mathfrak{a},Y_\mathfrak{a})+v$ and $(Y_\mathfrak{n},Y_\mathfrak{n})+w$ to $(Y_\mathfrak{k},Y_\mathfrak{k}),(Y_\mathfrak{a},Y_\mathfrak{a})$ and $(Y_\mathfrak{n},Y_\mathfrak{n})$ respectively. Hence, $\mathrm{Ad}((I,I),p)(\mathfrak{h})=\mathbb{R}(Y_\mathfrak{k},Y_\mathfrak{k})+\mathbb{R}(Y_\mathfrak{a},Y_\mathfrak{a})+\mathbb{R}(Y_\mathfrak{n},Y_\mathfrak{n})\oplus\mathbb{R}e'_1$. Therefore H is conjugate to $\mathrm{diag}(SL(2,\mathbb{R})\times SL(2,\mathbb{R}))\ltimes\mathbb{R}e'_1$.
- If $\dim L(H)=2$, then $L(H)=\mathrm{diag}(\mathrm{Aff}_\circ(\mathbb{R})\times\mathrm{Aff}_\circ(\mathbb{R}))$ up to conjugacy. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a},Y_\mathfrak{a})+u)+\mathbb{R}((Y_\mathfrak{n},Y_\mathfrak{n})+v)\oplus\mathbb{R}e'_1$, where $u,v\in\mathfrak{M}$. Then

(2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, u_1), \quad w = \left(-\frac{1}{2}u_3, v_2, 0, \frac{1}{2}u_3\right).$$

Let $p = \left(-v_2, \frac{1}{2}u_2, -\frac{1}{2}u_3, 0\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + u_1(E_{11} + E_{22})) + \mathbb{R}(Y_\mathfrak{n}, Y_\mathfrak{n}) \oplus \mathbb{R}e'_1$. Therefore H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda(E_{11} + E_{22})) + \mathbb{R}(Y_\mathfrak{n}, Y_\mathfrak{n})) \ltimes \mathbb{R}e'_1$, where λ is a fixed real number, (the action of this later group is orbit-equivalent to that of $\mathrm{diag}(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{R}e'_1$).

Case II: $\mathfrak{h} \cap \mathfrak{M} = \mathbb{R}e'_4$. The normalizer of $\mathbb{R}e'_4$ in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is $G = \{(x, e'_4 x e'^{-1}_4) \mid x \in SL(2, \mathbb{R})\}$, which implies that $L(H) \subseteq G$.

- If $\dim L(H) = 3$, then $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{k}, e'_4 Y_\mathfrak{k} e'^{-1}_4) + u) + \mathbb{R}((Y_\mathfrak{a}, e'_4 Y_\mathfrak{a} e'^{-1}_4) + v) + \mathbb{R}((Y_\mathfrak{n}, e'_4 Y_\mathfrak{n} e'^{-1}_4) + w) \oplus \mathbb{R}e'_4$, where $u, v, w \in M^{2,2}$. By using the relations obtained in (2.2) and (2.3), one gets that u, v and w have the following representations in the basis B:

$$u = (u_1, u_2, -u_2, u_1), \quad v = (0, 2(u_1 - u_2), 2u_2, 0), \quad w = (-u_2, -u_2, 0, -u_2).$$

  Let $p = (-u_2, u_1 - u_2, -u_2, 0)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_\mathfrak{k}, e'_4 Y_\mathfrak{k} e'^{-1}_4) + \mathbb{R}(Y_\mathfrak{a}, e'_4 Y_\mathfrak{a} e'^{-1}_4) + \mathbb{R}(Y_\mathfrak{n}, e'_4 Y_\mathfrak{n} e'^{-1}_4) \oplus \mathbb{R}e'_4$. Hence $H = \{(x, e'_4 x e'^{-1}_4) \mid x \in SL(2, \mathbb{R})\} \ltimes \mathbb{R}e'_4$ up to conjugacy.
- If $\dim L(H) = 2$, then $L(H) = \{(x, e'_4 x e'^{-1}_4) \mid x \in \mathrm{Aff}_\circ(\mathbb{R})\}$. Therefore, $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, e'_4 Y_\mathfrak{a} e'^{-1}_4) + u) + \mathbb{R}((Y_\mathfrak{n}, e'_4 Y_\mathfrak{n} e'^{-1}_4) + v) \oplus \mathbb{R}e'_4$ where $u, v \in \mathfrak{M}$. Then (2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, -u_1), \quad v = \left(-\frac{1}{2}u_3, v_2, 0, -\frac{1}{2}u_3\right).$$

  Let $p = \left(v_2, \frac{1}{2}u_2, -\frac{1}{2}u_3, 0\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, e'_4 Y_\mathfrak{a} e'^{-1}_4) + u_1(E_{11} - E_{22})) + \mathbb{R}(Y_\mathfrak{n}, e'_4 Y_\mathfrak{n} e'^{-1}_4) \oplus \mathbb{R}e'_4$ and so H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, e'_4 Y_\mathfrak{a} e'^{-1}_4) + \lambda(E_{11} - E_{22})) + \mathbb{R}(Y_\mathfrak{n}, e'_4 Y_\mathfrak{n} e'^{-1}_4)) \ltimes \mathbb{R}e'_4$, where λ is a fixed real number. The action of this group is orbit-equivalent to that of $\{(x, e'_4 x e'^{-1}_4) \mid x \in \mathrm{Aff}_\circ(\mathbb{R})\} \ltimes \mathbb{R}e'_4$.

Case III: Let $\mathfrak{h} \cap \mathfrak{M} = \ell$. The normalizer of ℓ in $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ is $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, which implies that $L(H) \subseteq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

If $\dim L(H) = 4$, then $L(H) = \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, 0) + u) + \mathbb{R}((Y_\mathfrak{n}, 0) + v) + \mathbb{R}((0, Y_\mathfrak{a}) + w) + \mathbb{R}((0, Y_\mathfrak{n}) + x) \oplus \ell$, where $u, v, w, x \in \mathfrak{M}$. Thus, the same argument as in the proof of Lemma 2.6 shows that H is conjugate to $(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \ell$. We claim that the action of this group is not of cohomogeneity one . In fact, for any $q \in M^{2,2}$, if $q_3 \neq 0$, then $\dim H(q) = 4$, if $q_3 = 0$, then $\dim H(q) \leqslant 2$. Thus, the case $\mathfrak{h} \cap \mathfrak{M} = \ell$, where $L(H) = \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, is excluded.

Subcase III-a: $\dim L(\mathfrak{h}) = 3$. Since $L(H) \subseteq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, then, up to conjugacy, the following cases may occur:

(i) $\dim p_1(L(H)) = 2$,

(ii) $\dim p_1(L(H)) = 1$.

In Case (i), if $\dim p_2(L(H)) = 2$, then $p_1(L(\mathfrak{h})) = \{tY_\mathfrak{a} + sY_\mathfrak{n} \mid t, s \in \mathbb{R}\}$ and $p_2(L(\mathfrak{h})) = \{uY_\mathfrak{a} + vY_\mathfrak{n} \mid u, v \in \mathbb{R}\}$. Since $\dim L(\mathfrak{h}) = 3$, one of the parameters t, s, u or v should be a linear function of the others. If $v = v(t, s, u)$ (respectively, $s = s(u, v, t)$), then the closedness of the bracket on $\mathfrak{h}$ implies that $v = 0$ (respectively, $s = 0$).

Let $u = at + bs + cv$, where a, b, c are fixed real numbers. The relation $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ implies that $b = c = 0$ and so $u = at$ where $a \in \mathbb{R}^*$. We claim that $a \in \{-1, \frac{1}{3}, 3\}$.

*Proof of the claim.*The Lie algebra $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, aY_\mathfrak{a}) + u) + \mathbb{R}((Y_\mathfrak{n}, 0) + v) + \mathbb{R}((0, Y_\mathfrak{n}) + w) \oplus \ell$ where $u, v, w \in \mathfrak{M}$. We consider the following different cases for a and in each case we use (2.2) and (2.3) to obtain a simple form of the group up to conjugacy.

If $a \notin \{\frac{1}{3},\ \pm 1,\ 3\}$, then

$$u = (u_1, u_2, u_3, u_4),$$
$$v = \left(-\frac{1}{a+1}u_3, \frac{1}{a-1}u_4, 0, 0\right), \quad w = \left(0, \frac{1}{a-1}u_1, 0, \frac{1}{a+1}u_3\right).$$

Let $p = \left(\frac{1}{1-a}u_1, \frac{1}{a+1}u_2, -\frac{1}{a+1}u_3, \frac{1}{a-1}u_4\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_\mathfrak{a}, aY_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n}) \oplus \ell$. Hence $H = \exp(\mathbb{R}(Y_\mathfrak{a}, aY_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \ell$ up to conjugacy. Let $q \in M^{2,2}$. If $q_3 \neq 0$ (respectively, $q_3 = 0$ and $q_1^2 + q_4^2 \neq 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 2$), and for any other point q we have $\dim H(q) = 1$. Hence H does not act with cohomogeneity one on $M^{2,2}$.

If $a = 1$, then

$$u = (0, u_2, u_3, 0), \quad v = \left(-\frac{1}{2}u_3, v_2, 0, 0\right), \quad w = \left(0, w_2, 0, \frac{1}{2}u_3\right).$$

Let $p = \left(-w_2, \frac{1}{2}u_2, -\frac{1}{2}u_3, v_2\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n}) \oplus \ell$ and so $H = \exp(\mathbb{R}(Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \ell$. For any $q \in M^{2,2}$, if $q_3 \neq 0$ (respectively, $q_3 = 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) \leqslant 2$). Hence H does not act with cohomogeneity one on $M^{2,2}$. Thus $a \neq 1$.

If $a = -1$, then a similar argument used for $a = 1$ shows that H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, -Y_\mathfrak{a}) + \lambda E_{12}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \ell$, where λ is a fixed real number (its action is orbit-equivalent to the action of $\exp(\mathbb{R}(Y_\mathfrak{a}, -Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \ell$). For any $q \in M^{2,2}$, if $\lambda q_3 \neq 0$, then $\dim H(q) = 3$, and thus H acts with cohomogeneity one on $M^{2,2}$.

If $a = 3$, then

$$u = (u_1, u_2, u_3, u_4), \quad v = \left(-\frac{1}{4}u_3, \frac{1}{2}u_4, 0, v_4\right), \quad w = \left(0, \frac{1}{2}u_1, 0, \frac{1}{4}u_3\right).$$

Let $p = \left(-\frac{1}{2}u_1, \frac{1}{4}u_2, -\frac{1}{4}u_3, \frac{1}{2}u_4\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_\mathfrak{a}, 3Y_\mathfrak{a}) + \mathbb{R}((Y_\mathfrak{n}, 0) + v_4E_{22}) + \mathbb{R}(0, Y_\mathfrak{n}) \oplus \ell$. Therefore H is conjugate to $\exp(\mathbb{R}(Y_\mathfrak{a}, 3Y_\mathfrak{a}) + \mathbb{R}((Y_\mathfrak{n}, 0) +$

$\lambda E_{22}) + \mathbb{R}(0, Y_{\mathfrak{n}})) \ltimes \ell$, where λ is a fixed real number. For any $q \in M^{2,2}$, if $q_3 \neq 0$, then $\dim H(q) = 4$, if $q_3 = 0$ and $q_1 q_4 \lambda \neq 0$, then $\dim H(q) = 3$. Hence H acts with cohomogeneity one on $M^{2,2}$ with at least one open orbit.

If $a = \frac{1}{3}$, then

$$u = (u_1, u_2, u_3, u_4), \quad v = \left(-\frac{3}{4}u_3, -\frac{3}{2}u_4, 0, 0\right), \quad w = \left(w_1, -\frac{3}{2}u_1, 0, \frac{3}{4}u_3\right).$$

Let $p = \left(\frac{3}{2}u_1, \frac{3}{4}u_2, -\frac{3}{4}u_3, -\frac{3}{2}u_4\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, \frac{1}{3}Y_{\mathfrak{a}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0) + \mathbb{R}((0, Y_{\mathfrak{n}}) + w_1 E_{11} \oplus \ell$, and so H is conjugate to $\exp(\mathbb{R}(3Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0) + \mathbb{R}((0, Y_{\mathfrak{n}}) + \lambda E_{11}) \ltimes \ell$, where λ is a fixed real number. For $q \in M^{2,2}$, if $q_3 \neq 0$, then $\dim H(q) = 4$, if $q_3 = 0$ and $q_1 q_4 \lambda \neq 0$, then $\dim H(q) = 3$. Hence H acts with cohomogeneity one on $M^{2,2}$ and there exists at least one open orbit. The claim is proved. □

In Case (ii), if $\dim p_2(L(H)) = 1$, then

$$L(H) \in \{\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}, \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}\}.$$

We claim that $L(H) \neq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}$, otherwise $\mathfrak{h}$ should be of the form $\mathbb{R}((Y_{\mathfrak{a}}, 0) + u) + \mathbb{R}((Y_{\mathfrak{n}}, 0) + v) + \mathbb{R}((0, Y_{\mathfrak{n}}) + w) \oplus \ell$, where $u, v, w \in \mathfrak{M}$. Then (2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, u_4), \quad v = (-u_3, -u_4, 0, 0), \quad w = (-u_1, 0, 0, u_3).$$

Let $p = (u_1, u_2, -u_3, -u_4)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, 0) + \mathbb{R}(Y_{\mathfrak{n}}, 0) + \mathbb{R}(0, Y_{\mathfrak{n}}) \oplus \ell$. Therefore $H = (\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}) \ltimes \ell$ up to conjugacy. Let $q \in M^{2,2}$. If $q_3 \neq 0$ (respectively, $q_3 = 0$ and $q_1^2 + q_4^2 \neq 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 2$), and for any other point q we have $\dim H(q) = 1$. Hence H does not act with cohomogeneity one on $M^{2,2}$. Thus $L(H) \neq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}$ as it is claimed.

Let $L(H) = \mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}$. Then a similar argument shows that H is conjugate to $(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}) \ltimes \ell$.

In Case (ii), $\dim L(H) = 3$ implies that $\dim p_2(L(H)) = 2$. Hence the same argument we used for the previous case shows that $H = (\mathbf{A} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \ell$ up to conjugacy.

Subcase III-b: $\dim L(H) = 2$. Considering various cases for $\dim p_1(H)$ and $\dim p_2(H)$, we obtain H up to conjugacy as follows.

⋆ If $\dim p_1(L(H)) = \dim p_2(L(H)) = 2$, then $p_1(L(H)) = p_2(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$ up to conjugacy. Since automorphisms of $\mathrm{Aff}_\circ(\mathbb{R})$ are conjugacies, for each automorphism $\rho \in \mathrm{Aut}(\mathrm{Aff}_\circ(\mathbb{R}))$ there is an $h \in \mathrm{Aff}_\circ(\mathbb{R})$ such that

$$\rho : \mathrm{Aff}_\circ(\mathbb{R}) \to \mathrm{Aff}_\circ(\mathbb{R}), \quad g \mapsto h^{-1}gh.$$

Hence $L(H) = \{(x, h^{-1}xh) \mid x \in \mathrm{Aff}_\circ(\mathbb{R})\}$ and so

$$\mathrm{Ad}(I, h)(L(H)) = \mathrm{diag}(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})).$$

The same argument we used for the second item of Case I shows that H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda(E_{11} + E_{22})) + \mathbb{R}(Y_{\mathfrak{n}}, Y_{\mathfrak{n}})) \ltimes \ell$, where λ is a fixed real number.

⋆ If $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 1$, then $p_1(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$ and $p_2(L(H)) \in \{\mathbf{A}, \mathbf{N}\}$ up to conjugacy.

- If $p_2(L(H)) = \mathbf{A}$, then

$$p_1(L(\mathfrak{h})) = \{tY_\mathfrak{a} + sY_\mathfrak{n} \mid t, s \in \mathbb{R}\}, \quad p_2(L(\mathfrak{h})) = \{rY_\mathfrak{a} | r \in \mathbb{R}\}.$$

Remind that $\dim L(\mathfrak{h}) = 2$. Hence $r = r(t, s)$ is a linear function $r : p_1(L(\mathfrak{h})) \to p_2(L(\mathfrak{h}))$, i.e., $r = at + bs$ for some $a, b \in \mathbb{R}$. But $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ implies that $b = 0$. Thus $L(\mathfrak{h}) = \{(tY_\mathfrak{a} + sY_\mathfrak{n}, atY_\mathfrak{a}) \mid t, s \in \mathbb{R}\}$ and $a \in \mathbb{R}^*$. If we denote this Lie algebra by $L(\mathfrak{h})^a$, then $(I, h)L^a(\mathfrak{h})(I, h)^{-1} = L^{-a}(\mathfrak{h})$, where $h = E_{12} - E_{21}$. Therefore, it is adequate to consider $L(\mathfrak{h})^a$ for $a \in \mathbb{R}_+$. Let us denote by $\mathfrak{h}^a$ the corresponding Lie algebra to $L(\mathfrak{h})^a$. Hence $\mathfrak{h}^a$ is of the form $\mathbb{R}((Y_\mathfrak{a}, aY_\mathfrak{a}) + u) + \mathbb{R}((Y_\mathfrak{n}, 0) + v) \oplus \ell$, where $u, v \in \mathfrak{M}$. Using (2.2) and (2.3), one gets the following.

If $a \in \mathbb{R}_+ - \{1, 3\}$, then

$$u = (u_1, u_2, u_3, 0), \quad v = \left(-\frac{1}{a+1}u_3, \frac{1}{a-1}u_4, 0, 0\right).$$

Let $p = \left(\frac{1}{1-a}u_1, \frac{1}{a+1}u_2, -\frac{1}{a+1}u_3, \frac{1}{a-1}u_4\right)$. Then we have $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, aY_\mathfrak{a})) + \mathbb{R}(Y_\mathfrak{n}, 0) \oplus \ell$. Therefore $H = \exp(\mathbb{R}(Y_\mathfrak{a}, aY_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0)) \ltimes \ell$ up to conjugacy.

If $a = 1$, then

$$u = (u_1, u_2, u_3, 0), \quad v = \left(-\frac{1}{2}u_3, v_2, 0, 0\right).$$

For $p = (0, \frac{1}{2}u_2, -\frac{1}{2}u_3, v_2)$ we have $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + u_1E_{11}) + \mathbb{R}(Y_\mathfrak{n}, 0) \oplus \ell$ and so H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11}) + \mathbb{R}(Y_\mathfrak{n}, 0)) \ltimes \ell$, where λ is a fixed real number.

If $a = 3$, then

$$u = (u_1, u_2, u_3, 0), \quad v = \left(-\frac{1}{4}u_3, \frac{1}{2}u_4, 0, v_4\right).$$

Let $p = \left(-\frac{1}{2}u_1, \frac{1}{4}u_2, -\frac{1}{4}u_3, \frac{1}{2}u_4\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, 3Y_\mathfrak{a})) + \mathbb{R}((Y_\mathfrak{n}, 0) + v_4E_{22}) \oplus \ell$ and so H is conjugate to $\exp(\mathbb{R}(Y_\mathfrak{a}, 3Y_\mathfrak{a}) + \mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) \ltimes \ell$, where λ is a fixed real number.

- If $p_2(L(H)) = \mathbf{N}$, then, by the same argument as above, one gets that $L(H) = \{(A_{t,s}, N_{bt}) \mid t, s \in \mathbb{R}\}$, where b is a fixed real nonzero number.

Let $b \in \mathbb{R}_+$. Then, for $h_b = \frac{1}{\sqrt{b}}E_{11} + \sqrt{b}E_{22} \in SL(2, \mathbb{R})$, $\mathrm{Ad}(I, h_b)$ maps bE_{12} to E_{12}. If $b \in \mathbb{R}_-$, then $\mathrm{Ad}(I, h_{-b})$ maps bE_{12} to $-E_{12}$. Hence we have the two cases $\{(A_{t,s}, N_t) \mid t, s \in \mathbb{R}\}$ and $\{(A_{t,s}, N_{-t}) \mid t, s \in \mathbb{R}\}$ for $L(H)$ up to conjugacy. This implies that $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, bY_\mathfrak{n}) + u) + \mathbb{R}((Y_\mathfrak{n}, 0) + v) \oplus \ell$, where $u, v \in \mathfrak{M}$ and $b \in \{\pm 1\}$. The relations in (2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, u_4), \quad v = (-u_3, bu_3 - u_4, 0, 0).$$

Let $p = (u_1, bu_1 + u_2, -u_3, bu_3 - u_4)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, bY_{\mathfrak{n}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0) \oplus \ell$. Therefore H is conjugate to $\exp(\mathbb{R}(Y_{\mathfrak{a}}, bY_{\mathfrak{n}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0)) \ltimes \ell$, where $b \in \{\pm 1\}$.

⋆ If $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 0$, then $L(H) = \mathrm{Aff}_\circ(\mathbb{R}) \times I$, where I denotes the trivial subgroup. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{a}}, 0) + u) + \mathbb{R}((Y_{\mathfrak{n}}, 0) + v) \oplus \ell$, where $u, v \in \mathfrak{M}$. Then (2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, u_4), \quad v = (-u_3, -u_4, 0, 0).$$

Let $p = (u_1, u_2, -u_3, -u_4)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, 0) + \mathbb{R}(Y_{\mathfrak{n}}, 0) \oplus \ell$, and so $H = (\mathrm{Aff}_\circ(\mathbb{R}) \times I) \ltimes \ell$ up to conjugacy. If $L(H) = I \times \mathrm{Aff}_\circ(\mathbb{R})$, then the same argument shows that $H = (I \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \ell$.

⋆ If $\dim p_1(L(H)) = 1$ and $\dim p_2(L(H)) = 2$, then, up to conjugacy, we have $p_1(L(H)) \in \{\mathbf{A}, \mathbf{N}\}$ and $p_2(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$. Thus, the same argument as that of the case, where $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 1$, of Subcase III-b shows that H is conjugate to one of the following groups:

- $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{22}) + \mathbb{R}(0, Y_{\mathfrak{n}})) \ltimes \ell$.
- $\exp(\mathbb{R}(3Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \mathbb{R}((0, Y_{\mathfrak{n}}) + \lambda E_{11}) \ltimes \ell$.
- $\exp(\mathbb{R}(aY_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \mathbb{R}(0, Y_{\mathfrak{n}})) \ltimes \ell$ if $a \in \mathbb{R}_+ - \{1, 3\}$.
- $\exp(\mathbb{R}(bY_{\mathfrak{n}}, Y_{\mathfrak{a}}) + \mathbb{R}(0, Y_{\mathfrak{n}})) \ltimes \ell$ and $b \in \{\pm 1\}$.

⋆ If $\dim p_1(L(H)) = \dim p_2(L(H)) = 1$, then

$$L(H) \in \{\mathbf{A} \times \mathbf{A}, \mathbf{A} \times \mathbf{N}, \mathbf{N} \times \mathbf{A}, \mathbf{N} \times \mathbf{N}\}.$$

- $L(H) = \mathbf{A} \times \mathbf{A}$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{a}}, 0) + u) + \mathbb{R}((0, Y_{\mathfrak{a}}) + v) \oplus \ell$, where $u, v \in \mathfrak{M}$. By using (2.2) and (2.3), one gets that

$$u = (u_1, u_2, u_3, u_4), \quad v = (-u_1, u_2, u_3, -u_4).$$

  Let $p = (u_1, u_2, -u_3, -u_4)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, 0) + \mathbb{R}(0, Y_{\mathfrak{a}}) \oplus \ell$ and so $H = (\mathbf{A} \times \mathbf{A}) \ltimes \ell$ up to conjugacy. A similar argument shows that if $L(H)$ is one of the groups $\mathbf{A} \times \mathbf{N}$ or $\mathbf{N} \times \mathbf{A}$, then H is conjugate to $(\mathbf{A} \times \mathbf{N}) \ltimes \ell$ or $(\mathbf{N} \times \mathbf{A}) \ltimes \ell$, respectively.
- $L(H) = \mathbf{N} \times \mathbf{N}$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{n}}, 0) + u) + \mathbb{R}((0, Y_{\mathfrak{n}}) + v) \oplus \ell$, where $u, v \in \mathfrak{M}$. Then (2.2) and (2.3) show that

$$u = (u_1, u_2, 0, u_4), \quad v = (v_1, v_2, 0, -u_1).$$

  Let $p = (-v_2, 0, u_1, u_2)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_{\mathfrak{n}}, 0) + u_4 E_{22}) + \mathbb{R}((0, Y_{\mathfrak{n}}) + v_1 E_{11}) \oplus \ell$, and so H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{n}}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_{\mathfrak{n}}) + \mu E_{11})) \ltimes \ell$, where λ and μ are fixed real numbers.

The theorem is proved. □

2.3. Subgroups with a plane as the translation part. Every two-dimensional spacelike, timelike, Lorentzian, degenerate or totally isotropic subspace of $M^{2,2}$ is congruent to $M^{2,0}$, $M^{0,2}$, $M^{1,1}$, $\mathbb{W}^2$ or $\mathbb{V}^2$, respectively.

Lemma 2.9. *Let H be a connected Lie subgroup of* $\mathrm{Iso}(M^{2,2})$ *acting on* $M^{2,2}$ *with cohomogeneity one. If the translation part of H is a plane, then H is conjugate within* $\mathrm{Iso}(M^{2,2})$ *to one of the groups in Tables 2.3, 2.4 or 2.5.*

Subgroups with a definite plane as the translation part	
spacelike	**timelike**
$(\mathbf{K}\times\mathbf{K})\ltimes M^{2,0}$	$(\mathbf{K}\times\mathbf{K})\ltimes M^{0,2}$
$\{(K_t,K_{at})\mid t\in\mathbb{R}\}\ltimes M^{2,0}$	$\{(K_t,K_{at})\mid t\in\mathbb{R}\}\ltimes M^{0,2}$
$\exp(\mathbb{R}((Y_{\mathfrak{k}},-Y_{\mathfrak{k}})+\lambda E_{21}+\mu E_{22}))\ltimes M^{2,0}$	$\exp(\mathbb{R}((Y_{\mathfrak{k}},-Y_{\mathfrak{k}})+\lambda E_{21}+\mu E_{22}))\ltimes M^{0,2}$
$(I\times\mathbf{K})\ltimes M^{2,0}$	$(I\times\mathbf{K})\ltimes M^{0,2}$

Table 2.3: Here $M^{2,0}$ and $M^{0,2}$ denote the subspaces $\mathbb{R}e'_1\oplus\mathbb{R}e'_2$ and $\mathbb{R}e'_3\oplus\mathbb{R}e'_4$, respectively, and $a,\lambda,\mu\in\mathbb{R}$ are fixed numbers and $a\neq -1$.

Subgroups with a Lorentzian plane as the translation part
$(\mathbf{A}\times\mathbf{A})\ltimes M^{1,1}$
$\{(A_t,A_{at})\mid t\in\mathbb{R}\}\ltimes M^{1,1}$
$\exp(\mathbb{R}((Y_{\mathfrak{a}},Y_{\mathfrak{a}})+\lambda E_{11}+\mu E_{22}))\ltimes M^{1,1}$
$\exp(\mathbb{R}((Y_{\mathfrak{a}},-Y_{\mathfrak{a}})+\lambda E_{12}+\mu E_{21}))\ltimes M^{1,1}$
$(I\times\mathbf{A})\ltimes M^{1,1}$

Table 2.4: Here $M^{1,1}$ denotes the subspace $\mathbb{R}e'_2\oplus\mathbb{R}e'_3$ and $a,\lambda,\mu\in\mathbb{R}$ are fixed numbers and $a\neq\pm1$.

Proof. We may assume that $\mathfrak{h}\cap\mathfrak{M}$ is equal to one of the two-dimensional subspaces $M^{2,0}$, $M^{0,2}$, $M^{1,1}$, $\mathbb{W}^2$ or $\mathbb{V}^2$.

Case I: $\mathfrak{h}\cap\mathfrak{M}=M^{2,0}$. The normalizer of $M^{2,0}$ in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is equal to $\mathbf{K}\times\mathbf{K}$, which implies that $L(H)\subseteq\mathbf{K}\times\mathbf{K}$.

- ⋆ If $\dim L(H)=2$, then $L(H)=\mathbf{K}\times\mathbf{K}$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{k}},0)+u)+\mathbb{R}((0,Y_{\mathfrak{k}})+v)\oplus M^{2,0}$, where $u,v\in\mathfrak{M}$. By a simple computation and using relations in (2.2) and (2.3), one can see that $\mathfrak{h}$ is conjugate to $\mathbb{R}(Y_{\mathfrak{k}},0)+\mathbb{R}(0,Y_{\mathfrak{k}})\oplus M^{2,0}$ and so $H=(\mathbf{K}\times\mathbf{K})\ltimes M^{2,0}$ up to conjugacy.
- ⋆ If $\dim L(H)=1$, then, up to conjugacy, $L(H)$ is one of the Lie groups $\{(K_t,K_{at})\mid t\in\mathbb{R}\}$ or $I\times\mathbf{K}$, where $a\in\mathbb{R}$. In the first case, $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{k}},aY_{\mathfrak{k}})+u)\oplus M^{2,0}$, where $u\in\mathfrak{M}$. Let $u=(u_1,u_2,u_3,u_4)$.

  If $a\neq\pm1$, then $\mathrm{Ad}((I,I),p)(\mathfrak{h})=\mathbb{R}(Y_{\mathfrak{k}},aY_{\mathfrak{k}})\oplus M^{2,0}$, where
  $$p=\left(\frac{au_2-u_3}{a^2-1},\frac{au_1+u_4}{1-a^2},\frac{au_4-u_1}{a^2-1},\frac{au_3-u_2}{1-a^2}\right).$$
  Hence $H=\{(K_t,K_{at})\mid t\in\mathbb{R}\}\ltimes M^{2,0}$ up to conjugacy.

  If $a=\pm1$, then, for $p=(-u_2,u_1,0,0)$, we have $\mathrm{Ad}((I,I),p)(\mathfrak{h})=\mathbb{R}((Y_{\mathfrak{k}},\mp Y_{\mathfrak{k}})+(u_3\pm u_2)E_{21}+(u_4\mp u_1)E_{22})\oplus M^{2,0}$. This implies that H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{k}},aY_{\mathfrak{k}})+\lambda E_{21}+\mu E_{22}))\ltimes M^{2,0}$, where λ and μ are fixed real numbers and $a\in\{\pm1\}$. However, for the case $a=+1$, we have $\mathrm{Ad}((I,I),p)(\mathbb{R}((Y_{\mathfrak{k}},Y_{\mathfrak{k}})+\lambda E_{21}+\mu E_{22})\oplus M^{2,0})=\mathbb{R}(Y_{\mathfrak{k}},Y_{\mathfrak{k}})\oplus M^{2,0}$, where $p=\frac{1}{2}(-\lambda,-\mu,\mu,\lambda)$.

Subgroups with a plane as the translation part	
degenerate	**totally degenerate**
$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + b\lambda E_{22}) + \mathbb{R}(Y_\mathfrak{n}, bY_\mathfrak{n})) \ltimes \mathbb{W}^2$	$\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R}) \ltimes \mathbb{V}^2$
$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{22}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{W}^2$	$\exp(\mathbb{R}(cY_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
$\exp(\mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_\mathfrak{n}) + \mu E_{11})) \ltimes \mathbb{W}^2$	$\exp(\mathbb{R}(3Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}((0, Y_\mathfrak{n}) + \lambda E_{11}) \ltimes \mathbb{V}^2$
$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes \mathbb{W}^2$	$(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}) \ltimes \mathbb{V}^2$, $(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}) \ltimes \mathbb{V}^2$
$\exp(\mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{W}^2$	$(\mathbf{A} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$, $(\mathbf{N} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$
$(I \times N) \ltimes \mathbb{W}^2$	$(\mathbf{K} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda(E_{11} + E_{22})) + \mathbb{R}(Y_\mathfrak{n}, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{a}, a'Y_\mathfrak{a}) + \mathbb{R}(Y_\mathfrak{n}, 0)) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11}) + \mathbb{R}(Y_\mathfrak{n}, 0)) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{a}, b'Y_\mathfrak{n}) + \mathbb{R}(Y_\mathfrak{n}, 0)) \ltimes \mathbb{V}^2$
	$(\mathrm{Aff}_\circ(\mathbb{R}) \times I) \ltimes \mathbb{V}^2$, $(I \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(dY_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(3Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}((0, Y_\mathfrak{n}) + \lambda E_{11}) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(b'Y_\mathfrak{n}, Y_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{k}, aY_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$(\mathbf{A} \times \mathbf{A}) \ltimes \mathbb{V}^2$, $(\mathbf{K} \times \mathbf{N}) \ltimes \mathbb{V}^2$, $(\mathbf{A} \times \mathbf{N}) \ltimes \mathbb{V}^2$
	$(\mathbf{N} \times \mathbf{A}) \ltimes \mathbb{V}^2$, $(\mathbf{K} \times \mathbf{A}) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_\mathfrak{n}) + \mu E_{11})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{k}, \lambda Y_\mathfrak{a})) \ltimes \mathbb{V}^2$, $\exp(\mathbb{R}(Y_\mathfrak{k}, \lambda Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{a}, \lambda Y_\mathfrak{a})) \ltimes \mathbb{V}^2$, $\exp(\mathbb{R}(Y_\mathfrak{a}, \lambda Y_\mathfrak{n})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}(Y_\mathfrak{n}, \lambda Y_\mathfrak{a})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes \mathbb{V}^2$
	$\exp(\mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{V}^2$
	$(\mathbf{A} \times I) \times \mathbb{V}^2$, $(\mathbf{N} \times I) \times \mathbb{V}^2$, $(I \times \mathbf{A}) \times \mathbb{V}^2$
	$(\mathbf{K} \times I) \times \mathbb{V}^2$, $(I \times \mathbf{N}) \times \mathbb{V}^2$

Table 2.5: Here $\mathbb{W}^2$ and $\mathbb{V}^2$ denote the degenerate plane $\mathbb{R}e_1' \oplus \ell$ and the totally isotropic plane $\mathbb{R}(e_1' - e_4') \oplus \ell$, respectively, and $a, b, c, a', b', c', \lambda, \mu \in \mathbb{R}$ are fixed numbers and $b \neq 1$, $c \neq 3$, $a' \in \mathbb{R}^+ - \{1\}$, $b' \in \{\pm 1\}$ and $c' \in \mathbb{R}^+ - \{3\}$.

Case II: $\mathfrak{h}\cap\mathfrak{M} = M^{0,2}$. Since the normalizer of $M^{0,2}$ in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is equal to $\mathbf{K}\times\mathbf{K}$, the same argument as in Case I shows that H is conjugate to one of the Lie groups $(\mathbf{K}\times\mathbf{K})\ltimes M^{0,2}$, $\{(K_t, K_{at}) \mid t\in\mathbb{R}\}\ltimes M^{0,2}$, $\exp(\mathbb{R}((Y_{\mathfrak{k}}, -Y_{\mathfrak{k}}) + \lambda E_{21} + \mu E_{22}))\ltimes M^{0,2}$ or $(I\times\mathbf{K})\ltimes M^{0,2}$.

Case III: $\mathfrak{h}\cap\mathfrak{M} = M^{1,1}$. The normalizer of $M^{1,1}$ in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is $\mathbf{A}\times\mathbf{A}$, which shows that $L(H)\subseteq\mathbf{A}\times\mathbf{A}$.

- ⋆ If $\dim L(H) = 2$, then $L(H) = \mathbf{A}\times\mathbf{A}$. Thus, using the same argument as in the fifth item of Subcase III-b of the proof of Lemma 2.8, one can see that $H = (\mathbf{A}\times\mathbf{A})\ltimes M^{1,1}$ up to conjugacy. Let $q\in M^{2,2}$. If $q_1\neq 0$, then $\dim H(q) = 3$, and thus H acts with cohomogeneity one on $M^{2,2}$.
- ⋆ If $\dim L(H) = 1$, then, up to conjugacy, $L(H)$ is one of the Lie group $I\times\mathbf{A}$ or $\{(A_t, A_{at}) \mid t\in\mathbb{R}\}$, where $a\in\mathbb{R}$. In the first, $H = (I\times\mathbf{A})\ltimes M^{1,1}$ and in the later, $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{a}}, aY_{\mathfrak{a}}) + u)\oplus M^{1,1}$, where $u\in\mathfrak{M}$. Let $u = (u_1, u_2, u_3, u_4)$.
  - For $a\neq\pm 1$, let $p = \left(\frac{1}{1-a}u_1, \frac{1}{a+1}u_2, -\frac{1}{a+1}u_3, \frac{1}{a-1}u_4\right)$. Then we have $\mathrm{Ad}((I,I),p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{a}}, aY_{\mathfrak{a}})\oplus M^{1,1}$, which implies that $H = \{(A_t, A_{at}) \mid t\in\mathbb{R}\}\ltimes M^{1,1}$ up to conjugacy. Let $q\in M^{2,2}$. If $q_1\neq 0$, then $\dim H(q) = 3$, and thus H acts with cohomogeneity one on $M^{2,2}$.
  - For $a = 1$, let $p = \left(0, \frac{1}{2}u_2, -\frac{1}{2}u_3, 0\right)$. Then $\mathrm{Ad}((I,I),p)(\mathfrak{h}) = \mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + u_1E_{11} + u_4E_{22})\oplus M^{1,1}$. Therefore H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11} + \mu E_{22}))\ltimes M^{1,1}$, where λ and μ are fixed real numbers. Let $q\in M^{2,2}$. If $q_1\neq 0$ and $\lambda^2+\mu^2\neq 0$, then $\dim H(q) = 3$.
  - For $a = -1$, let $p = \left(\frac{1}{2}u_1, 0, 0, -\frac{1}{2}u_4\right)$. Therefore $\mathrm{Ad}((I,I),p)(\mathfrak{h}) = \mathbb{R}((Y_{\mathfrak{a}}, -Y_{\mathfrak{a}}) + u_2E_{12} + u_3E_{21})\oplus M^{1,1}$. Hence H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{a}}, -Y_{\mathfrak{a}}) + \lambda E_{12} + \mu E_{21}))\ltimes M^{1,1}$, where λ and μ are fixed real numbers. Let $q\in M^{2,2}$. If $q_1\neq 0$ and $\lambda^2+\mu^2\neq 0$, then $\dim H(q) = 3$, and thus H acts with cohomogeneity one on $M^{2,2}$.

Case IV: $\mathfrak{h}\cap\mathfrak{M} = \mathbb{W}^2$. The identity component of the normalizer of $\mathbb{W}^2$ in $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ is

$$G = \{(A_{t,s}, A_{t,s'}) \mid t, s, s'\in\mathbb{R}\},$$

which implies that $L(H)\subseteq G$.

We claim that $\dim L(H)\leq 2$. Otherwise, if $\dim L(H) = 3$, then $L(H) = G$. An argument simimlar to that of the proof of Lemma 2.6 shows that $H = (A_{t,s}\times A_{t,s'})\ltimes\mathbb{W}^2$ up to conjugacy. Let $q\in M^{2,2}$. If $q_3\neq 0$ (respectively, $q_3 = 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 2$), which is in contradiction with the cohomogeneity one assumption.

Subcase IV-a: $\dim(L(H)) = 2$.

- If $\dim p_1(L(H)) = 2$, then $L(\mathfrak{h})\subseteq\{(tY_{\mathfrak{a}} + sY_{\mathfrak{n}}, tY_{\mathfrak{a}} + uY_{\mathfrak{n}}) \mid t, s, u\in\mathbb{R}\}$. Since $\dim L(H) = 2$, we may assume that $u = u(t,s)$ is a linear function $u:\mathbb{R}^2\to\mathbb{R}$, i.e., $u = at + bs$ for some $a, b\in\mathbb{R}$.

  Let $g = \left(1, \frac{1}{2}a, 0, 1\right)$. Then $\mathrm{Ad}((I,g))$ maps $(Y_{\mathfrak{a}}, Y_{\mathfrak{a}} + aY_{\mathfrak{n}})$ and $(Y_{\mathfrak{n}}, bY_{\mathfrak{n}})$ to $(Y_{\mathfrak{a}}, Y_{\mathfrak{a}})$ and $(Y_{\mathfrak{n}}, bY_{\mathfrak{n}})$, respectively. Thus, $L(\mathfrak{h}) = \{(tY_{\mathfrak{a}} + sY_{\mathfrak{n}}, tY_{\mathfrak{a}} + bsY_{\mathfrak{n}}) \mid$

$t, s \in \mathbb{R}\}$. Note that if $b \neq 0$ (respectively, $b = 0$), then $\dim p_2(L(H)) = 2$ (respectively, $\dim p_2(L(H)) = 1$). We claim that $b \neq 1$. Otherwise H is conjugate to $\mathrm{diag}(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{W}^2$. Then, for any $q \in M^{2,2}$, if $q_3 \neq 0$ (respectively $q_3 = 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 2$), which is a contradiction to the cohomogeneity one assumption. Thus $b \neq 1$ and so $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + u) + \mathbb{R}((Y_\mathfrak{n}, bY_\mathfrak{n}) + v) \oplus \mathbb{W}^2$, where $u, v \in \mathfrak{M}$. Using the relations (2.2) and (2.3), one gets

$$u = (u_1, u_2, u_3, bu_1), \quad v = \left(-\frac{1}{2}u_3, v_2, 0, \frac{1}{2}bu_3\right).$$

Let $p = \left(0, \frac{1}{2}u_2, -\frac{1}{2}u_3, v_2\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + u_1E_{11} + bu_1E_{22}) + \mathbb{R}(Y_\mathfrak{n}, bY_\mathfrak{n}) \oplus \mathbb{W}^2$. Henceforth H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + b\lambda E_{22}) + \mathbb{R}(Y_\mathfrak{n}, bY_\mathfrak{n})) \ltimes \mathbb{W}^2$, where λ is a fixed real number. Let $q \in M^{2,2}$. If $q_3 \neq 0$ and $b \neq -1$ (respectively, ($q_3 \neq 0$ and $b = -1$) or ($q_3 = 0$ and $\lambda \neq 0$)), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 3$) and so H acts with cohomogeneity one with at least one open orbit in $M^{2,2}$.

- Let $\dim p_1(L(H)) = 1$.
  - ⋆ If $\dim p_2(L(H)) = 2$, then the same argument as above shows that H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{22}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{W}^2$, where λ is a fixed real number. Here H also acts with cohomogeneity one on $M^{2,2}$ with at least one open orbit.
  - ⋆ If $\dim p_2(L(H)) = 1$, then, by the fact that $\dim L(H) = 2$, one gets that $L(\mathfrak{h}) = \{(sY_\mathfrak{n}, s'Y_\mathfrak{n}) \mid s, s' \in \mathbb{R}\}$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{n}, 0) + u) + \mathbb{R}((0, Y_\mathfrak{n}) + v) \oplus \mathbb{W}^2$, where $u, v \in \mathfrak{M}$. By using the same argument as in the last item of Subcase III-b of the proof of Lemma 2.8, it is seen that H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_\mathfrak{n}) + \mu E_{11})) \ltimes \mathbb{W}^2$, where λ and μ are fixed real numbers. Let $q \in M^{2,2}$. If $q_3 = 0$ and $\lambda^2 + \mu^2 \neq 0$, then $\dim H(q) = 3$, and thus H acts with cohomogeneity one on $M^{2,2}$.

Subcase IV-b: $\dim L(H) = 1$. When depending on the dimension of $p_1(L(H))$ and $p_2(L(H))$, the Lie group $L(H)$ is conjugate to one of the following groups:

(i) $\{(A_t, A_t) \mid t \in \mathbb{R}\}$,
(ii) $\{(N_t, N_{at}) \mid t \in \mathbb{R}\}$, $a \in \mathbb{R}$,
(iii) $I \times N$.

In Case (i), $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + u) \oplus \mathbb{W}^2$, where $u \in \mathfrak{M}$. By using the same argument as in the second item of Case III of the proof of Lemma 2.9, one can see that H is conjugate to $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes \mathbb{W}^2$, where λ and μ are fixed real numbers.

In case (ii), $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + u) \oplus \mathbb{W}^2$, where $u = (u_1, u_2, u_3, u_4) \in \mathfrak{M}$. We consider the cases $a \neq -1$ and $a = -1$ separately.

- For $a \neq -1$, let $p = \left(0, 0, \frac{1}{a+1}(u_1 - u_4), u_2\right)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + u_3E_{21} + (u_4 + au_1)E_{22}) \oplus \mathbb{W}^2$, and so H is conjugate to

$\exp(\mathbb{R}((Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{W}^2$, where λ and μ are fixed real numbers.

- For $a = -1$, let $p = (0, 0, u_1, u_2)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}((Y_{\mathfrak{n}}, -Y_{\mathfrak{n}}) + u_3 E_{21} + (u_4 - u_1)E_{22}) \oplus \mathbb{W}^2$. Therefore H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{n}}, -Y_{\mathfrak{n}}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{W}^2$, where λ and μ are fixed real numbers.

  Altogether, Case (ii) leads to the group $\exp(\mathbb{R}((Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{W}^2$, where a, λ and μ are fixed real numbers. This group acts with cohomogeneity one if $\lambda^2 + \mu^2 \neq 0$.

Finally, in Case (iii), it is easily seen that H is conjugate to $(I \times N) \ltimes \mathbb{W}^2$.

Case V: $\mathfrak{h} \cap \mathfrak{M} = \mathbb{V}^2$. The normalizer of $\mathbb{V}^2$ in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is equal to $SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, which implies that $L(H) \subseteq SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

If $L(H) = SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, then $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{k}}, 0) + u) + \mathbb{R}((Y_{\mathfrak{a}}, 0) + v) + \mathbb{R}((Y_{\mathfrak{n}}, 0) + w) + \mathbb{R}((0, Y_{\mathfrak{a}}) + x) + \mathbb{R}((0, Y_{\mathfrak{n}}) + y) \oplus \mathbb{V}^2$, where $u, v, w, x, y \in \mathfrak{M}$. An argument similar to that of the proof of Lemma 2.6 shows that $H = (SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$ up to conjugacy. Let $q \in M^{2,2}$. If $q_1 q_3 \neq 0$ (respectively, q_1 and q_3 are equal to zero), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 2$). Hence H does not act with cohomogeneity one on $M^{2,2}$. Therefore, $L(H) \subsetneq SL(2,\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

Subcase V-a: $\dim(L(H)) = 4$. We claim that $\dim p_1(L(H)) \neq 3$. Otherwise there should be a Lie group homomorphism $\phi : SL(2,\mathbb{R}) \to \mathrm{Aff}_\circ(\mathbb{R})$, with $\dim \ker(\phi) \neq 0$, which contradicts the fact that $SL(2,\mathbb{R})$ is simple. Thus $p_1(L(H))$ and $p_2(L(H))$ are two-dimensional, and so $L(H) = \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$. The same argument as that used in the proof of Lemma 2.6 shows that $H = \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R}) \ltimes \mathbb{V}^2$ up to conjugacy. Let $q \in M^{2,2}$. If $q_3 \neq 0$ (respectively, $q_3 = 0$ and $q_1 \neq 0$), then $\dim H(q) = 4$ (respectively, $\dim H(q) = 3$). Hence H acts with cohomogeneity one on $M^{2,2}$.

Subcase V-b: $\dim L(H) = 3$.

- If $p_1(L(H)) = SL(2,\mathbb{R})$, then $p_2(L(H)) = \{I\}$ since $SL(2,\mathbb{R})$ is simple. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{k}}, 0) + u) + \mathbb{R}((Y_{\mathfrak{a}}, 0) + v) + \mathbb{R}((Y_{\mathfrak{n}}, 0) + w) \oplus \mathbb{V}^2$, where $u, v, w \in \mathfrak{M}$. Using (2.2) and (2.3) show that

  $$u = (u_1, u_2, u_3, u_4), \quad v = (u_3, u_4, u_1, u_2), \quad w = (-u_1, -u_2, 0, 0).$$

  Let $p = (u_3, u_4, -u_1, -u_2)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{k}}, 0) + \mathbb{R}(Y_{\mathfrak{a}}, 0) + \mathbb{R}(Y_{\mathfrak{n}}, 0) \oplus \mathbb{V}^2$ and so $H = (SL(2,\mathbb{R}) \times I) \ltimes \mathbb{V}^2$. Let $q \in M^{2,2}$. If q_1 or $q_3 \neq 0$, then $\dim H(q) = 4$. If both q_1 and q_3 are equal to zero, then $\dim H(q) = 2$. Hence H does not act with cohomogeneity one on $M^{2,2}$ and the case $\mathfrak{h} \cap \mathfrak{M} = \mathbb{V}^2$, where $L(H) = SL(2,\mathbb{R}) \times I$, is excluded.
- If $\dim p_1(L(H)) = \dim p_2(L(H)) = 2$, then, up to conjugacy, $p_1(L(H)) = p_2(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$. Using the same argument as in the first item of Subcase III-b of the proof of Lemma 2.8 shows that H is conjugate to one of the following Lie subgroups:
  - ⋆ $\exp(\mathbb{R}(Y_{\mathfrak{a}}, cY_{\mathfrak{a}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0) + \mathbb{R}(0, Y_{\mathfrak{n}})) \ltimes \mathbb{V}^2$, if $c \in \mathbb{R} - \{\frac{1}{3}\}$.

- ⋆ $\exp(\mathbb{R}(Y_{\mathfrak{a}}, \frac{1}{3}Y_{\mathfrak{a}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0) + \mathbb{R}((0, Y_{\mathfrak{n}}) + \lambda E_{11}) \ltimes \mathbb{V}^2$, where λ is a fixed real number.

In all of the above cases, for any $q \in M^{2,2}$, if $q_3 \neq 0$, then $\dim H(q) = 4$, if $q_3 = 0$ and $q_1 \neq 0$, then $\dim H(q) = 3$. Hence H acts with cohomogeneity one on $M^{2,2}$ and there exists at least one open orbit.

- $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 1$. Then, up to cojugacy, $L(H)$ is one of the groups $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}$ or $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}$. By using the same argument as in the second item of Subcase III-a of the proof of Lemma 2.8, one can see that H is conjugate to one of the Lie subgroups $(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{A}) \ltimes \mathbb{V}^2$ or $(\mathrm{Aff}_\circ(\mathbb{R}) \times \mathbf{N}) \ltimes \mathbb{V}^2$. In both cases, if $q_3 \neq 0$, then $\dim H(q) = 4$, and if $q_3 = 0$ and $q_1 \neq 0$, then $\dim H(q) = 3$. Hence H acts with cohomogeneity one with at least one open orbit in $M^{2,2}$.
- $\dim p_1(L(H)) = 1$ and $\dim p_2(L(H)) = 2$. Then $p_2(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$ and $p_1(L(H))$ is conjugate to one the Lie groups $\mathbf{K}$, $\mathbf{A}$ or $\mathbf{N}$. The later two cases lead to $(\mathbf{A} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$ and $(\mathbf{N} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$, those were studied in the previous case.

  Now, let $L(H) = \mathbf{K} \times \mathrm{Aff}_\circ(\mathbb{R})$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_{\mathfrak{k}}, 0) + u) + \mathbb{R}((0, Y_{\mathfrak{a}}) + v) + \mathbb{R}((0, Y_{\mathfrak{n}}) + w) \oplus \mathbb{V}^2$, where $u, v, w \in \mathfrak{M}$. Then (2.2) and (2.3) show that
  $$u = (u_1, u_2, u_3, u_4), \quad v = (-u_3, u_4, u_1, -u_2), \quad w = (0, -u_3, 0, u_1).$$
  Let $p = (u_3, u_4, -u_1, -u_2)$. Then $\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_{\mathfrak{k}}, 0) + \mathbb{R}(0, Y_{\mathfrak{a}}) + \mathbb{R}(0, Y_{\mathfrak{n}}) \oplus \mathbb{V}^2$. Therefore H is conjugate to $(\mathbf{K} \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$.

Subcase V-c: $\dim L(H) = 2$. Here we consider various cases for dimensions $p_i(L(H))$, $i = 1, 2$.

- $\dim p_i(L(H)) = 2$, for $i = 1, 2$. The same argument as in the first item of Subcase III-b of the proof of Lemma 2.8 shows that H is conjugate to $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda(E_{11} + E_{22})) + \mathbb{R}(Y_{\mathfrak{n}}, Y_{\mathfrak{n}})) \ltimes \mathbb{V}^2$, where λ is a fixed real number (its action is orbit equivalent to the action of $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11}) + \mathbb{R}(Y_{\mathfrak{n}}, Y_{\mathfrak{n}})) \ltimes \mathbb{V}^2)$. Let $q \in M^{2,2}$. If $q_3 \neq 0$, then $\dim H(q) = 4$, if $q_3 = 0$ and $\lambda \neq 0$, then $\dim H(q) = 3$. Hence H acts with cohomogeneity one on $M^{2,2}$ with at least one open orbit in $M^{2,2}$.
- $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 1$. The same argument as in the second item of Subcase III-b of the proof of Lemma 2.8 shows that H is conjugate to one of the following Lie subgroups:
  - ⋆ $\exp(\mathbb{R}(Y_{\mathfrak{a}}, a'Y_{\mathfrak{a}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0)) \ltimes \mathbb{V}^2$ if $a' \in \mathbb{R}^+ - \{1\}$.
  - ⋆ $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11}) + \mathbb{R}(Y_{\mathfrak{n}}, 0)) \ltimes \mathbb{V}^2$, where λ is a fixed real number.
  - ⋆ $\exp(\mathbb{R}(Y_{\mathfrak{a}}, b'Y_{\mathfrak{n}}) + \mathbb{R}(Y_{\mathfrak{n}}, 0)) \ltimes \mathbb{V}^2$, where $b' \in \{\pm 1\}$.

  In all of the above cases for any $q \in M^{2,2}$, if $q_3 \neq 0$ (respectively, $q_3 = 0$ and $q_1 \neq 0$) then $\dim H(q) = 4$ (respectively, $\dim H(q) = 3$). Henceforth H acts with cohomogeneity one with at least one open orbit in $M^{2,2}$.
- $\dim p_1(L(H)) = 2$ and $\dim p_2(L(H)) = 0$. The same argument as in the third item of Subcase III-b of the proof of Lemma 2.8 shows that $H = (\mathrm{Aff}_\circ(\mathbb{R}) \times I) \ltimes \mathbb{V}^2$ up to conjugacy. This group acts with cohomogeneity one on $M^{2,2}$

since, for any $q \in M^{2,2}$, $\dim H(q) = 4$ if $q_3 \neq 0$, and $\dim H(q) = 3$ if $q_3 = 0$ and $q_1 \neq 0$.

- $\dim p_1(L(H)) = 1$ and $\dim p_2(L(H)) = 2$. Then, up to conjugacy, $p_1(L(H)) \in \{\mathbf{K}, \mathbf{A}, \mathbf{N}\}$ and $p_2(L(H)) = \mathrm{Aff}_\circ(\mathbb{R})$.

  If $p_1(L(H)) \in \{\mathbf{A}, \mathbf{N}\}$, then the same argument as in the fourth item of Subcase III-b of the proof of Lemma 2.8 shows that H is conjugate to one of the following Lie subgroups, those act with cohomogeneity one on $M^{2,2}$:

  ⋆ $\exp(\mathbb{R}(c'Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$ if $d \in \mathbb{R}_+$ and $c' \in \mathbb{R}^+ - \{3\}$,

  ⋆ $\exp(\mathbb{R}(3Y_\mathfrak{a}, Y_\mathfrak{a}) + \mathbb{R}((0, Y_\mathfrak{n}) + \lambda E_{11}) \ltimes \mathbb{V}^2$,

  ⋆ $\exp(\mathbb{R}(bY_\mathfrak{n}, Y_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$ and $b' \in \{\pm 1\}$.

  Now, let $p_1(L(H)) = \mathbf{K}$. Then $L(\mathfrak{h}) \subseteq \{(tY_\mathfrak{k}, rY_\mathfrak{a} + sY_\mathfrak{n}) \mid r, s, t \in \mathbb{R}\}$. Since $\dim L(H) = 2$, we may assume that $t = u(r, s)$ is a linear function. Hence $t = ar + bs$ for some fixed $a, b \in \mathbb{R}$, closedness under the bracket on $\mathfrak{h}$ shows that $b = 0$. Hence $\mathfrak{h}$ is of the form $\mathbb{R}((Y_\mathfrak{k}, aY_\mathfrak{a}) + u) + \mathbb{R}((0, Y_\mathfrak{n}) + v) \oplus \mathbb{V}^2$, where $u, v \in \mathfrak{M}$. Then (2.2) and (2.3) show that

$$u = (u_1, u_2, u_3, u_4), \quad v = \left(0, \frac{1}{a^2+1}(au_1 - u_3), 0, u_1 + au_3\right).$$

  Let $p = \frac{1}{a^2+1}(u_3 - au_1, u_4 + au_2, -u_1 - au_3, -u_2 + au_4))$. Then

$$\mathrm{Ad}((I, I), p)(\mathfrak{h}) = \mathbb{R}(Y_\mathfrak{k}, aY_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n}) \oplus \mathbb{V}^2.$$

  Therefore H is conjugate to $\exp(\mathbb{R}(Y_\mathfrak{k}, aY_\mathfrak{a}) + \mathbb{R}(0, Y_\mathfrak{n})) \ltimes \mathbb{V}^2$, where $a \in \mathbb{R}^*$.

- $\dim p_i(L(H)) = 1$, where $i = 1, 2$. Then, up to conjugacy,

$$L(H) \in \{\mathbf{K} \times \mathbf{A},\ \mathbf{K} \times \mathbf{N},\ \mathbf{A} \times \mathbf{A},\ \mathbf{A} \times \mathbf{N},\ \mathbf{N} \times \mathbf{A},\ \mathbf{N} \times \mathbf{N}\}.$$

  The same argument as in the fifth item of Subcase III-b of the proof of Lemma 2.8 shows that H is conjugate to one of the following Lie subgroups: $(\mathbf{A} \times \mathbf{A}) \ltimes \mathbb{V}^2$, $(\mathbf{K} \times \mathbf{N}) \ltimes \mathbb{V}^2$, $(\mathbf{A} \times \mathbf{N}) \ltimes \mathbb{V}^2$, $(\mathbf{N} \times \mathbf{A}) \ltimes \mathbb{V}^2$, $(\mathbf{K} \times \mathbf{A}) \ltimes \mathbb{V}^2$, and $\exp(\mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_\mathfrak{n}) + \mu E_{11})) \ltimes \mathbb{V}^2$, where λ and μ are fixed real numbers (its action is orbit equivalent to the action of $\exp(\mathbb{R}(Y_\mathfrak{n}, 0) + \mathbb{R}((0, Y_\mathfrak{n}) + \lambda E_{11})) \ltimes \mathbb{V}^2)$. In all of the mentioned groups we have $\dim H(q) = 3$ if $q_3 \neq 0$, except for the group $(\mathbf{K} \times \mathbf{A}) \ltimes \mathbb{V}^2$, where $\dim H(q) = 3$ if $q_1 - q_3 \neq 0$.

- $\dim p_1(L(H)) = 0$ and $\dim p_2(L(H)) = 2$, i.e., $H = (I \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$. Let $q \in M^{2,2}$. If $q_1 \neq 0$ or $q_3 \neq 0$, then $\dim H(q) = 3$. If both q_1 and q_3 are equal to zero, then $\dim H(q) = 2$. This shows that the actions of two groups $(I \times \mathrm{Aff}_\circ(\mathbb{R})) \ltimes \mathbb{V}^2$ and $(\mathrm{Aff}_\circ(\mathbb{R}) \times I) \ltimes \mathbb{V}^2$ are not orbit equivalent (compare this item with the third item of Subcase V-c).

Subcase V-d: $\dim L(H) = 1$. An argument similar to that used in Subcase V-c shows that if $\dim p_i(H) = 1$ for $i = 1, 2$, then H is conjugate to one of the following groups:

- $\exp(\mathbb{R}(Y_\mathfrak{k}, \lambda Y_\mathfrak{a})) \ltimes \mathbb{V}^2$,

- $\exp(\mathbb{R}(Y_{\mathfrak{k}}, \lambda Y_{\mathfrak{n}})) \ltimes \mathbb{V}^2$,
- $\exp(\mathbb{R}(Y_{\mathfrak{a}}, \lambda Y_{\mathfrak{a}})) \ltimes \mathbb{V}^2$,
- $\exp(\mathbb{R}(Y_{\mathfrak{a}}, \lambda Y_{\mathfrak{n}})) \ltimes \mathbb{V}^2$,
- $\exp(\mathbb{R}(Y_{\mathfrak{n}}, \lambda Y_{\mathfrak{a}})) \ltimes \mathbb{V}^2$,
- $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11} + \mu E_{22})) \ltimes \mathbb{V}^2$, (its action is orbit equivalent to the action of $\exp(\mathbb{R}((Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11})) \ltimes \mathbb{V}^2$),
- $\exp(\mathbb{R}((Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{V}^2$, (its action is orbit equivalent to the action of $\exp(\mathbb{R}((Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21})) \ltimes \mathbb{V}^2$).

In all of the above groups λ and μ are fixed real numbers.

If $\dim p_i(H) = 0$ for either $i = 1$ or $i = 2$, then H is conjugate to one of the groups $(\mathbf{K} \times I) \ltimes \mathbb{V}^2$, $(\mathbf{A} \times I) \ltimes \mathbb{V}^2$, $(\mathbf{N} \times I) \ltimes \mathbb{V}^2$, $(I \times \mathbf{A}) \ltimes \mathbb{V}^2$, or $(I \times \mathbf{N}) \ltimes \mathbb{V}^2$.

All of the above groups act with cohomogeneity one on $M^{2,2}$. In fact, for any of the mentioned groups we have $\dim H(q) = 3$ if $q_3 \neq 0$. □

2.4. Subgroups with a hyperplane as the translation part. Every three-dimensional subspace of $M^{2,2}$ is congruent by an element of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ to one of the subspaces $M^{2,1}$, $M^{1,2}$ or $\mathbb{W}^3$. For dimension reasons, it follows that the action of H is orbit equivalent to the action of one of the three pure translation subgroups $M^{2,1}$, $M^{1,2}$ or $\mathbb{W}^3$. Let Π be one of these three groups. The normalizer of Π in $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ should preserve Π and $\Pi^{\perp}$ as two subspaces of $M^{2,2}$. Therefore, using the proof of Lemma 2.3, one gets that

- if $\Pi = M^{1,2}$, then $L(H) \subseteq \mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$;
- if $\Pi = M^{2,1}$, then $L(H) \subseteq \{(P, e'_4 P e_4'^{-1}) | P \in SL(2,\mathbb{R})\}$;
- if $\Pi = \mathbb{W}^3$, then $L(H) \subseteq \mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$.

The above discussion determines the acting group H up to conjugacy. Thus, we get the following result.

Lemma 2.10. *Let H be a connected Lie subgroup of* $\mathrm{Iso}(M^{2,2})$ *acting on $M^{2,2}$ with cohomogeneity one. If the translation part of H is a hyperplane, then H is conjugate within* $\mathrm{Iso}(M^{2,2})$ *to one of the groups in Table 2.6.*

Subgroups with a hyperplane as the translation part		
Lorentzian	**anti-Lorentzian**	**degenerate**
$G_1 \ltimes M^{1,2}$	$G_2 \ltimes M^{2,1}$	$G_3 \ltimes \mathbb{W}^3$

Table 2.6: The groups G_1, G_2 and G_3 can be any linear subgroup of $\mathrm{diag}(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$, $\{(P, e'_4 P e_4'^{-1}) | P \in SL(2,\mathbb{R})\}$ and $\mathrm{Aff}_\circ(\mathbb{R}) \times \mathrm{Aff}_\circ(\mathbb{R})$, respectively.

3. Proper and nonproper actions

In this section, we determine proper and nonproper actions, which are induced by Lie groups in Tables 2.1 to 2.6. We recall that an action of a Lie group G on a manifold M is said to be *proper* if the mapping $\varphi : G \times M \to M \times M$, $(g, x) \mapsto (gx, x)$ is proper. Equivalently, for any sequences x_n in M and g_n in G, $g_n x_n \to y$ and $x_n \to x$ imply that g_n has a convergent subsequence. In particular, the stabilizer of any point is compact and the orbit space is Hausdorff.

The Lie groups in Table 2.1 act linearly on $M^{2,2}$, and so they fix the origin. Since none of them is compact, their actions are not proper.

Theorem 3.1. *Let G be a connected Lie subgroup of $\mathrm{Iso}(M^{2,2})$, which acts isometrically and with cohomogeneity one on $M^{2,2}$. Then the action is proper if and only if G is conjugate to one of the following Lie subgroups of $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})) \ltimes_\varphi M(2,\mathbb{R})$.*

(i) *Any group in Table 2.6, where its linear projection is either trivial or isomorphic to $SO(2)$.*

(ii) *All of the groups in Table 2.3.*

(iii) *The following Lie group of Table 2.4:*
  (1) $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes M^{1,1}$ *if* $\lambda^2 + \mu^2 \neq 0$.

(iv) *The following Lie groups of Table 2.5:*
  (2) $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes \mathbb{W}^2$ *if* $\lambda \neq \mu$,
  (3) $\exp(\mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + \lambda E_{21} + \mu E_{22})) \ltimes \mathbb{W}^2$ *if* $\lambda \neq 0$,
  (4) $\exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11})) \ltimes \mathbb{V}^2$ *if* $\lambda \neq 0$,
  (5) $\exp(\mathbb{R}((Y_\mathfrak{n}, aY_\mathfrak{n}) + \lambda E_{21})) \ltimes \mathbb{V}^2$ *if* $\lambda \neq \mu$,
  where $a \in \mathbb{R}$.

(v) *The following Lie group in Table 2.2:*
  (6) $\exp(\mathbb{R}((Y_\mathfrak{n}, 0) + \lambda E_{22}) + \mathbb{R}((0, Y_\mathfrak{n}) + \mu E_{11})) \ltimes \ell$ *if* $\lambda\mu > 0$,
  where a, λ and μ are fixed real numbers.

Proof. (i) If G is one of the groups in Table 2.6, then $L(G) = G_0$. The action is proper, so $L(G)$ should be compact. By the fact that G, as well as $L(G)$, is connected and the maximal compact subgroup of any of the groups in Table 2.6 is either trivial or isomorphic to $SO(2)$, one gets the result.

(ii) For the groups in Table 2.3, consider their representations in $\mathrm{Iso}(\mathbb{R}^{2,2})$ using the isometry $\Psi : \mathbb{R}^{2,2} \to M^{2,2}$, $e_i \mapsto e_i'$, where $1 \leqslant i \leqslant 4$, explained in Section 2. Then any group of Table 2.3 becomes a subgroup of $\mathrm{diag}[SO(2 \times SO(2)] \ltimes \mathbb{R}^{2,2}$. These groups preserve the Euclidean metric $dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, and so do Lie subgroups of $SO(4) \ltimes \mathbb{R}^4$, which act isometrically and with cohomogeneity one on the Euclidean space $\mathbb{E}^4$. Therefore their actions are proper.

(iii) In Table 2.4, we need to consider only the following two groups since any other group has a noncompact subgroup acting linearly, i.e., the stabilizer of the origin is noncompact:

$$\begin{cases} H_1 = \exp(\mathbb{R}((Y_\mathfrak{a}, -Y_\mathfrak{a}) + \lambda E_{12} + \mu E_{21})) \ltimes M^{1,1}, \\ H_2 = \exp(\mathbb{R}((Y_\mathfrak{a}, Y_\mathfrak{a}) + \lambda E_{11} + \mu E_{22})) \ltimes M^{1,1}, \end{cases}$$

where λ and μ are fixed real numbers. We claim that the action of H_1 is not proper. By a simple computation, one gets that

$$H_1 = \{g_{t,u,v} = ((e^t E_{11} + e^{-t}E_{22}, e^{-t}E_{11} + e^t E_{22}), (\lambda t + u)E_{12} + (\mu t + v)E_{21}) \mid t, u, v \in \mathbb{R})\}.$$

Then the noncompact subgroup $\{g_{t,-\lambda t,-\mu t} \in H_1 | t \in \mathbb{R}\}$ fixes the origin of $M^{2,2}$, which proves our claim.

Now we are going to consider the group H_2, where

$$H_2 = \left\{ h_{t,u,v} = \left(\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right), \begin{pmatrix} \lambda t & u \\ v & \mu t \end{pmatrix} \right) \middle| t, u, v \in \mathbb{R} \right\}.$$

Let $X_n = \Sigma_{i,j=1}^2 x_{ij}^n E_{ij}$ and h_{t_n,u_n,v_n} be two arbitrary sequences in $M^{2,2}$ and H_2, respectively, where $X_n \to \Sigma_{i,j=1}^2 x_{ij}E_{ij}$ and $h_{t_n,u_n,v_n}.X_n \to \Sigma_{i,j=1}^2 x'_{ij}E_{ij}$. This implies that

$$\begin{cases} \lambda t_n \to x'_{11} - x_{11}, \\ \mu t_n \to x'_{22} - x_{22}, \\ x_{12}^n e^{2t_n} + u_n \to x'_{12}, \\ x_{21}^n e^{-2t_n} + v_n \to x'_{21}. \end{cases}$$

Hence, if at least one of λ or μ is nonzero, then t_n, as well as u_n and v_n, are convergent. This shows that the action of H_2 on $M^{2,2}$ is proper if $\lambda^2 + \mu^2 \neq 0$. On the other hand, if $\lambda = \mu = 0$, then the stabilizer of the origin is the noncompact subgroup $\{h_{t,0,0} \in H_2 \mid t \in \mathbb{R}\}$, i.e., the action of H is nonproper. This completes the study of properness of the actions of groups in Table 2.4.

(iv) and (v). The argument is similar to that of (iii). □

To get a better visualization of the groups in Theorem 3.1, we give their representations in $\mathrm{Iso}(\mathbb{R}^{2,2})$ in the following corollary. We use the isometry $\Psi : \mathbb{R}^{2,2} \to M^{2,2}$, $e_i \mapsto e'_i$, where $1 \leqslant i \leqslant 4$, to compute these representations.

Corollary 3.2. *Let H be a connected Lie subgroup of $\mathrm{Iso}(\mathbb{R}^{2,2})$, which acts isometrically and with cohomogeneity one on $\mathbb{R}^{2,2}$. Then the action is proper if and only if H is conjugate to one of the following Lie groups in $SO(2,2) \ltimes \mathbb{R}^{2,2}$.*

(i) *The groups obtained from Table 2.6, those are the additive groups $\mathbb{R}^{1,2}$, $\mathbb{R}^{2,1}$ or $\mathbb{W}^3 = \mathbb{R}e_1 \oplus \mathbb{R}(e_2 - e_3) \oplus \mathbb{R}e_4$ up to conjugacy. The groups obtained from Table 2.3, their actions on $\mathbb{R}^{2,2}$ are orbit equivalent to one of the groups $\mathrm{diag}(I, SO(2)) \times (\mathbb{R}^2 \oplus \{0\})$ or $\mathrm{diag}(SO(2), I) \times (\{0\} \oplus \mathbb{R}^2)$.*

(ii) *One of the following Lie groups:*

$$(1) \left\{ \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh 2t & -\sinh 2t & 0 \\ 0 & -\sinh 2t & \cosh 2t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{\lambda+\mu}{2}t \\ u \\ v \\ \frac{\lambda-\mu}{2}t \end{pmatrix} \right) \middle| \begin{matrix} t, u, v \in \mathbb{R} \\ \text{and} \\ \lambda^2 + \mu^2 \neq 0 \end{matrix} \right\},$$

$$(2) \left\{ \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh 2t & -\sinh 2t & 0 \\ 0 & -\sinh 2t & \cosh 2t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{\lambda+\mu}{2}t + u \\ v \\ -v \\ \frac{\lambda-\mu}{2}t \end{pmatrix} \right) \middle| \begin{matrix} t, u, v \in \mathbb{R} \\ \text{and} \\ \lambda \neq \mu \end{matrix} \right\},$$

$$(3)\ \left\{\left(\begin{pmatrix} 1 & \frac{1-a}{2}t & \frac{1-a}{2}t & 0 \\ -\frac{1-a}{2}t & 1+\frac{a}{2}t^2 & \frac{a}{2}t^2 & \frac{1+a}{2}t \\ \frac{1-a}{2}t & -\frac{a}{2}t^2 & 1-\frac{a}{2}t^2 & -\frac{1+a}{2}t \\ 0 & \frac{1+a}{2}t & \frac{1+a}{2}t & 1 \end{pmatrix}, \begin{pmatrix} c\frac{1-a}{4}\lambda t^2+u \\ \frac{1}{3}\lambda t^3-\frac{1}{4}\mu t^2+\frac{1}{2}\lambda t+v \\ -\frac{1}{3}\lambda t^3+\frac{1}{4}\mu t^2+\frac{1}{2}\lambda t-v \\ \frac{1+a}{4}\lambda t^2-\frac{1}{2}\mu t \end{pmatrix}\right) \middle| \begin{matrix} t,u,v\in\mathbb{R} \\ \text{and} \\ \lambda\neq 0 \end{matrix}\right\},$$

$$(4)\ \left\{\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh 2t & -\sinh 2t & 0 \\ 0 & -\sinh 2t & \cosh 2t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{2}t+u \\ v \\ -v \\ \frac{\lambda}{2}t-u \end{pmatrix}\right) \middle| \begin{matrix} t,u,v\in\mathbb{R} \\ \text{and} \\ \lambda\neq 0 \end{matrix}\right\},$$

$$(5)\ \left\{\left(\begin{pmatrix} 1 & \frac{1-a}{2}t & \frac{1-a}{2}t & 0 \\ -\frac{1-a}{2}t & 1+\frac{a}{2}t^2 & \frac{a}{2}t^2 & \frac{1+a}{2}t \\ \frac{1-a}{2}t & -\frac{a}{2}t^2 & 1-\frac{a}{2}t^2 & -\frac{1+a}{2}t \\ 0 & \frac{1+a}{2}t & \frac{1+a}{2}t & 1 \end{pmatrix}, \begin{pmatrix} \frac{1-a}{4}\lambda t^2+u \\ \frac{1}{3}\lambda t^3+\frac{1}{2}\lambda t+v \\ -\frac{1}{3}\lambda t^3+\frac{1}{2}\lambda t-v \\ \frac{1+a}{4}\lambda t^2-u \end{pmatrix}\right) \middle| \begin{matrix} t,u,v\in\mathbb{R} \\ \text{and} \\ \lambda\neq 0 \end{matrix}\right\},$$

$$(6)\ \left\{\left(\begin{pmatrix} 1 & \frac{t-s}{2} & \frac{t-s}{2} & 0 \\ \frac{s-t}{2} & 1+\frac{st}{2} & \frac{st}{2} & \frac{s+t}{2} \\ \frac{t-s}{2} & -\frac{st}{2} & 1-\frac{st}{2} & -\frac{s+t}{2} \\ 0 & \frac{s+t}{2} & \frac{s+t}{2} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(\mu s+\lambda t) \\ \frac{1}{2}(\mu s^2-\lambda t^2)+u \\ -\frac{1}{2}(\mu s^2-\lambda t^2)-u \\ \frac{1}{2}(\mu s-\lambda t) \end{pmatrix}\right) \begin{matrix} t,s,u\in\mathbb{R} \\ \text{and} \\ \lambda\mu>0 \end{matrix}\right\},$$

where a, λ and μ are fixed real numbers.

Let S be a subspace of $\mathbb{R}^{2,2}$. We define the stabilizer of S as $\mathrm{Stab}(S) = \{g \in \mathrm{Iso}(\mathbb{R}^{2,2}) \mid gS \subseteq S\}$. The subspace $\mathbb{V}^2 = \mathbb{R}(e_1 - e_4) \oplus \mathbb{R}(e_2 - e_3)$ is the maximal totally isotropic subspace of $\mathbb{R}^{2,2}$ up to conjugacy. We refer the reader to [8] for the stabilizer of a maximal totally isotropic subspace in general case $\mathbb{R}^{p,q}$. As a consequence of Corollary 3.2, we get the following result.

Corollary 3.3. *Let H be a connected Lie subgroup of $\mathrm{Iso}(\mathbb{R}^{2,2})$, which acts properly, isometrically and with cohomogeneity one on $\mathbb{R}^{2,2}$. If there is no singular orbit, then $L(H) \subset \mathrm{Stab}(\mathbb{V}^2)$ up to conjugacy, where $L : O(2,2) \ltimes \mathbb{R}^{2,2} \to O(2,2)$ is the linear projection given by $(g, \tau) = g$.*

3.1. Orbits and orbit spaces of proper actions Here we are going to determine the orbits and the orbit spaces of the groups obtained in Corollary 3.2.

The orbits of the translation groups $\mathbb{R}^{1,2}$, $\mathbb{R}^{2,1}$ and $\mathbb{W}^3 = \mathbb{R}e_1 \oplus \mathbb{R}(e_2 - e_3) \oplus \mathbb{R}e_4$ are parallel affine hyperplanes, and so the orbit space is $\mathbb{R}$.

The set of the orbits of $\operatorname{diag}(I, SO(2)) \times (\mathbb{R}^2 \oplus \{0\})$ (respectively, $\operatorname{diag}(SO(2), I) \times (\{0\} \oplus \mathbb{R}^2)$) consists of a unique singular spacelike (respectively, timelike) orbit, congruent to $\mathbb{R}^{2,0}$ (respectively, $\mathbb{R}^{0,2}$), and the Lorentz cylinders $\mathbb{S}^1(r) \times \mathbb{R}^{2,0}$ (respectively, $\mathbb{S}^1(r) \times \mathbb{R}^{0,2}$) around the singular orbit, where $r > 0$. The orbit space is homeomorphic to $[0, +\infty)$.

The actions of the remaining groups from (1) to (6) of Corollary 3.2 are free since the groups are diffeomorphic to $\mathbb{R}^3$ and the actions are proper (stabilizer of any point is trivial). Hence the orbit space of each of these groups is $\mathbb{R}$.

The orbits of the group (1) of Corollary 3.2. Clearly, the action of this group is orbit equivalent to that of the translation group $T_{a,b} = \{(at, u, v, bt)^{tr} \mid t, u, v \in \mathbb{R}\}$, where a and b are fixed real numbers and $ab \neq 0$. Depending on the fixed numbers a and b, the causal character of the orbits may be Lorentzian, anti-Lorentzian or degenerate. The orbit space is $\mathbb{R}$.

The orbits of the group (2) of Corollary 3.2. Every orbit is a Lorentzian hypersurface. The orbit of the origin is a degenerate hyperplane and each other orbit is of the form $\mathbb{R} \times D$, where D is a Lorentzian generalized cylinder (see Theorem 4.1 in [7]).

The orbits of the groups (3), (4), (5), and (6) of Corollary 3.2. We claim that, for any of the mentioned groups, the tangent space of every induced orbit has a timelike, a spacelike and a lightlike tangent vector. To prove our claim, we use the representations of the groups in Theorem 3.1. Let $\mathfrak{g}_i$ denote their Lie algebras, where $3 \leqslant i \leqslant 6$. Let $p = \Sigma x_{ij} E_{ij} \in M^{2,2}$ and take the following vectors:

$$\begin{aligned}
X_3 &= (Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21} + \mu E_{22} + b(E_{11} + E_{22}) + cE_{12} \in \mathfrak{g}_3, \\
X_4 &= (Y_{\mathfrak{a}}, Y_{\mathfrak{a}}) + \lambda E_{11} + bE_{12} + cE_{22} \in \mathfrak{g}_4, \\
X_5 &= (Y_{\mathfrak{n}}, aY_{\mathfrak{n}}) + \lambda E_{21} + bE_{12} + cE_{22} \in \mathfrak{g}_5, \\
X_6 &= ((Y_{\mathfrak{n}}, 0) + \lambda E_{22}) + \alpha((0, Y_{\mathfrak{n}}) + \mu E_{11}) \in \mathfrak{g}_6,
\end{aligned}$$

where a, b, c and α are real numbers. Then each of the amounts

$$\begin{aligned}
\det\left(\frac{d}{dt}\bigg|_{t=0} (\exp(tX_3)p) \right) &= (x_{21} + b)(-ax_{21} + \mu + b) - \lambda(-ax_{11} + x_{22} + c), \\
\det\left(\frac{d}{dt}\bigg|_{t=0} (\exp(tX_4)p) \right) &= \lambda c + 2x_{21}(2x_{12} + b), \\
\det\left(\frac{d}{dt}\bigg|_{t=0} (\exp(tX_5)p) \right) &= x_{21}(c - ax_{21}) + \lambda(ax_{11} - x_{22} - b), \\
\det\left(\frac{d}{dt}\bigg|_{t=0} (\exp(tX_6)p) \right) &= -\mu\lambda\alpha^2 + (\mu\lambda - x_{21}^2)\alpha + \lambda x_{21}
\end{aligned}$$

may be positive, zero or negative for various choices of b, c and α. This proves our claim. The induced metrics on the obits of (4) and (5) are degenerate since the tangent space of any orbit contains the totally isotropic subspace $\mathbb{V}^2$. The induced metric on any orbit of the group (6) is also degenerate since the null vector $\frac{d}{dt}\big|_{t=0}\exp(tX)p$, where $X = e'_2 - e'_3$, is both tangent and normal to the orbit at p.

Corollary 3.4. *Let H be a Lie subgroup of* $\mathrm{Iso}(\mathbb{R}^{2,2})$ *acting properly, isometrically and with cohomogeneity one on* $\mathbb{R}^{2,2}$*. If there is a singular orbit, then it is a two-dimensional totally geodesic subspace on which the induced metric is definite.*

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Дії на чотиривимірному псевдоевклідовому просторі $\mathbb{R}^{2,2}$ з тривимірною орбітою

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У цій роботі ми класифікуємо зв’язні групи Лі з точністю до спряженості в $\mathrm{Iso}(R^{2,2})$, які діють ізометрично на чотиривимірному псевдоевклідовому просторі $\mathbb{R}^{2,2}$ таким чином, що є тривимірна індукована орбіта в $\mathbb{R}^{2,2}$. Потім ми надаємо перелік груп, що діють, у двох випадках: з власними та невласними діями. У випадку власної дії ми визначаємо явне представлення групи, що діє, в $SO(2,2) \ltimes \mathbb{R}^{2,2}$ і описуємо орбіти та простори орбіт.

Ключові слова: кооднорідність один, ізометрична дія, псевдоевклідів простір