Journal of Mathematical Physics, Analysis, Geometry
2022, Vol. 18, No. 3, pp. 382–405
doi: https://doi.org/10.15407/mag18.03.382

Asymptotic Absence of Poles of Ihara Zeta Function of Large Erdős–Rényi Random Graphs

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Using recent results on the concentration of the largest eigenvalue and maximal vertex degree of large random graphs, we show that the infinite sequence of Erdős–Rényi random graphs $G(n, \rho_n/n)$ such that $\rho_n/\log n$ infinitely increases as $n \to \infty$ verifies a version of the graph theory Riemann Hypothesis.

Key words: random graphs, random matrices, Ihara zeta function, graph theory Riemann hypothesis.

Mathematical Subject Classification 2010: 05C80,11M50,15B52,60B20

1. Ihara zeta function and graph theory Riemann hypothesis

Given a finite connected non-oriented graph $\Gamma = (V, E)$ with the vertex set $V = (\alpha_1, \ldots, \alpha_n)$ and the edge set E, the Ihara zeta function (IZF) $Z_\Gamma(u)$ is determined for sufficiently small $|u|$ by the equality

$$Z_\Gamma(u) = \prod_{[C]} \left(1 - u^{\nu(C)}\right)^{-1}, \tag{1.1}$$

where $[C]$ denotes the equivalence class of closed primitive backtrackless tailless paths C and $\nu(C) = k - 1$, k being the length of C [19]. The k-step path over the graph $C = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{k-1}}, \alpha_{i_k})$, $\{\alpha_{i_l}, \alpha_{i_{l+1}}\} \in E$ is closed when $\alpha_{i_k} = \alpha_{i_1}$. The path C is backtrackless if $\alpha_{i_{l-1}} \neq \alpha_{i_{l+1}}$ for all $l = 2, \ldots, k-1$. The path C is tailless if $\alpha_{i_2} \neq \alpha_{i_{k-1}}$. The equivalence class $[C]$ includes C and all paths obtained from C with the help of all cyclic permutation of its elements. The closed path C is primitive if there is no smaller path $\tilde{C}$ such that $C = \tilde{C}^k$.

Zeta function (1.1) was introduced by Y. Ihara in the algebraic context [19]. Ihara's theorem says that the IZF (1.1) is the reciprocal of a polynomial and that for sufficiently small $|u|$,

$$Z_\Gamma(u)^{-1} = (1 - u^2)^{r-1} \det\left(I + u^2(B - I) - uA\right), \quad u \in \mathbb{C}, \tag{1.2}$$

where $A = (a_{ij})_{i,j=1,\ldots,N}$ is the adjacency matrix of Γ, $B = \operatorname{diag}\left(\sum_{j=1}^{N} a_{ij}\right)$ and

$$r - 1 = \operatorname{Tr}(B - 2I)/2. \tag{1.3}$$

Let us note that the Ihara zeta function can also be determined as the exponential expression [30],

$$Z_\Gamma(u) = \exp\left\{\sum_{k\geq 1} \frac{\mathcal{N}_k}{k} u^k\right\}, \tag{1.4}$$

where $\mathcal{N}_k$ is the number of classes of closed backtrackless tailless primitive paths of the length k over the edges of Γ. The Ihara's theorem was proved initially for $q+1$-regular graphs then it was generalized by Bass to the cases of possibly irregular graphs [2] (see also [12]).

There exists an analog of the Riemann hypothesis formulated for $q+1$-regular graphs $\Gamma = X^{(q+1)}$ with the help of the Ihara zeta function. According to the definition by Stark and Terras [28], a graph $X^{(q+1)}$ verifies the *graph theory Riemann hypothesis* (GTRH) iff its Ihara zeta function is such that

$$\operatorname{Re} s \in (0,1) \text{ and } \left(Z_{X^{(q+1)}}\left(q^{-s}\right)\right)^{-1} = 0 \quad \text{imply} \quad \operatorname{Re} s = \frac{1}{2}. \tag{1.5}$$

This relation means that the graph $X^{(q+1)}$ is such that there is no poles of the Ihara zeta function $Z_X(u)$ in the disk $1/q < |u| < 1$ excepting those situated on the circle $|u| = 1/\sqrt{q}$. The following statement is formulated by Stark and Terras [28] as a corollary of the formula (1.2).

Lemma 1.1. *A finite $(q+1)$-regular graph $X^{(q+1)}$ satisfies the Riemann hypothesis iff for every eigenvalue λ of its adjacency matrix A_X, we have*

$$|\lambda| \neq q+1 \quad \textit{implies} \quad |\lambda| \leq 2\sqrt{q}. \tag{1.6}$$

We reproduce the proof of this lemma in Section 4 of the present paper. It is fairly simple and uses an elementary observation that in the case of $q+1$-regular graphs, relations (1.2) and (1.5) reduce the problem to the study of zeroes of the quadratic equation

$$1 + qu^2 - \lambda u = 0,$$

whose discriminant is negative if and only if $|\lambda| < 2\sqrt{q}$.

Relation (1.5) can be reformulated in a more convenient for us form as follows: for any complex $v \in D_q$,

$$D_q = \left\{z \in \mathbb{C} : \frac{1}{\sqrt{q}} < |z| < \sqrt{q}\right\}, \tag{1.7}$$

the statement

$$\left(Z_{X^{(q+1)}}\left(\frac{v}{\sqrt{q}}\right)\right)^{-1} = 0 \quad \text{implies} \quad |v| = 1 \tag{1.8}$$

is true. This means that the regular graph $X^{(q+1)}$ verifies the GTRH if and only if the function

$$\Phi^{(q+1)}(v) = Z_{X^{(q+1)}}\left(\frac{v}{\sqrt{q}}\right) \tag{1.9}$$

has no poles in the region $v \in D_1^{(q)} \cup D_2^{(q)}$, where

$$D_1^{(q)} = \{v \in \mathbb{C} : 1/\sqrt{q} < |v| < 1\} \quad \text{and} \quad D_2^{(q)} = \{v \in \mathbb{C} : 1 < |v| < \sqrt{q}\}\,. \tag{1.10}$$

Relation (1.6) represents a widely known in graph theory and applications the second eigenvalue conjecture (see [1] and references therein). This condition means that the distance between the maximal and the second maximal in absolute value eigenvalues of a $q+1$-regular graph is greater than $q+1-2\sqrt{q}$. The $(q+1)$-regular graphs that satisfy condition (1.6) are determined by Lubotzky, Phillips and Sarnak as the *Ramanujan graphs* [23] (see also the review [26] and references therein). The Ramanujan graphs are known to be good expanders that make good communication networks (see, e.g., [11, 26]). This property can be explained by the observation that the diameter of a $(q+1)$-regular graph is minimized by minimizing the second maximal eigenvalue because the maximal eigenvalue is always equal to $q+1$ [8, 26, 27]. It is proved by Friedman [13, 14] that the proportion of $q+1$-regular graphs with n vertices such that (1.6) is true with $2\sqrt{q}$ replaced by $2\sqrt{q}+\varepsilon$ goes to 1 as $n \to \infty$ for any $\varepsilon > 0$.

The notion of the Ramanujan graphs mostly concerns regular graphs. Several extensions of this notion were considered in [24, 28]. A definition of the Ramanujan graphs in the case of non-regular graphs was proposed by Lubotzky [24]. It is given in terms of the eigenvalues of the adjacency matrix of a graph Γ, its spectral radius and the spectral radius of the adjacency operator on the universal covering tree of Γ.

The case when Γ is chosen at random from the set of all possible graphs of n vertices was considered in [20]. More precisely, the eigenvalue distribution of the matrix $I + H(u) = I + u^2(B - I) - uA$ was studied, where A is the adjacency matrix of the Erdős–Rényi random graphs $G(n, \rho_n/n)$ (see the next section for the rigorous definition of the ensemble $G(n,p)$). It is shown that in the limit $n \to \infty$ and $\rho_n/\log n \to \infty$ the limiting eigenvalue distribution of $H(u)$ with properly normalized parameter $u = v/\sqrt{\rho_n}$ exists and its density is given by a shift of the Wigner semi-circle distribution. Then one can show that the limit of the mean value of $\frac{1}{n}\log Z_\Gamma(v/\sqrt{\rho_n})$, if it exists, satisfies the version of (1.5).

Another approach allowing to include the case of non-regular random graphs into consideration is based on the following representation of zeta function (1.1):

$$Z_\Gamma(u)^{-1} = \det(I - uW_\Gamma), \tag{1.11}$$

where W_Γ is the *non-backtracking matrix* of Γ (see [16, 28] and references therein). Representation (1.11) is known as the Ihara–Bass formula and can serve as the basis of the proof of the Ihara theorem (1.2) for connected graphs.

Regarding representation (1.11), Stark and Terras defined Γ to satisfy the graph theory Riemann hypothesis if the matrix W_Γ has no eigenvalues with absolute values inside the interval $(\sqrt{r_{W_\Gamma}}, r_{W_\Gamma})$, where r_{W_Γ} is the Perron–Frobenius eigenvalue of W_Γ. In the paper [5], the spectral properties of the non-backtracking matrix W_Γ of the the Erdős–Rényi random graphs $\Gamma \in G(n, p_n)$ were studied. It is shown that the graphs $\Gamma \in G(n, \alpha/n)$ verify a weak Ramanujan property in the sense that in the limit $n \to \infty$ they satisfy with high probability the graph theory Riemann hypothesis formulated above [18]. More precisely, it is proved that $r_{W_\Gamma} \sim \alpha$ and all other eigenvalues λ of W_Γ verify $|\lambda| \le \sqrt{\alpha} + o(1)$ with high probability as $n \to \infty$. It is stated that in this sense, the Erdős–Rényi random graphs $G(n, \alpha/n)$ asymptotically satisfy the graph theory Riemann hypothesis [5]. In this work, the term "with high probability" is used to indicate the situation when one or another statement is true with probability $1 + o(1)$, $n \to \infty$, i.e., tending to 1 as n infinitely increases. This shows that the results of [5] are much in the spirit of Friedman's statements cited above. It should also be noted that the results of [5] as well as those of [13, 14] are obtained with the help of the study of moments of non-backtracking matrix W_Γ.

The limiting eigenvalue distribution of non-backtracking matrices W_Γ of Erdős–Rényi random graphs $G(n, p)$ were also studied in the asymptotic regimes when either $p \approx \text{const}$ or $p = o(1)$ as $n \to \infty$ [31]. More fine spectral characteristics of W_Γ such as the presence of isolated eigenvalues inside and outside of the bulk of the spectrum of W_Γ were considered in [9] for a generalization of the Erdős–Rényi random graphs $G(n, \alpha/n)$ in the case when $\alpha/\log n \to \infty$ as $n \to \infty$. It is shown, in particular, that with probability $1 - o(1)$ all eigenvalues of W_Γ are located on the distance $o(\sqrt{\alpha})$ of a circle of radius $\sqrt{\alpha - 1}$ excepting two ones that are close to 1 and α as $n \to \infty$. Since eigenvalues of W_Γ determine uniquely the poles of $Z_\Gamma(u)$ (1.11), the results of [9] can be interpreted in the sense that the proportion of Erdős–Rényi random graphs that verify the graph theory Riemann hypothesis (1.5) tends to one as $n \to \infty$. This formulation put [9] in line with the works [13, 14] and [5]. The results of [9] are obtained on the base of the known facts from spectral properties of the adjacency matrices A_Γ of random graphs combined with the concentration results on the elements of B and perturbation theorems by Bauer and Fike allowing one to study the spectrum of the non-backtracking matrix W_Γ on the base of the knowledge of that of the adjacency matrix A_Γ (see also [5]).

In this paper, we follow the approach of [20] based on the study of the spectrum of $I + u^2(B - I) - uA$ of the right-hand side of (1.2). This method seems to be more simple and transparent than that using the non-backtracking matrix W_Γ. Using classical perturbation theorems of the Weyl type for singular values of matrices, concentration properties of B [21] and recent results on the concentration of the maximal eigenvalue of A_Γ [25], we prove a statement that can be regarded as an improvement of that of [9]. Namely, we show that the infinite series of probabilities of the events that the normalized zeta function $Z_\Gamma(v/\sqrt{\rho_n})$ of the Erdős–Rényi random graphs $G(n, \rho_n/n)$ in the asymptotic regime when $\rho_n \gg \log n$ has a pole in any domain close to $D = \{v \in \mathbb{C} : |v| \ne 1\}$ converges. The result obtained in the present paper can be regarded as a one more confirmation of the conjecture that almost all Erdős–Rényi random graphs $\{G(n, p_n)\}_{n \ge 1}$ satisfy, in the limit $n \to \infty$, $np_n/\log n \to \infty$, a version of the graph theory Riemann hypothesis.

2. Ihara zeta function of Erdős–Rényi random graphs

Let us consider a family of jointly independent random variables

$$\mathcal{A}_{n,\rho} = \{a_{ij}^{(n,\rho)},\ 1 \le i < j \le n\}$$

that have the probability distribution

$$a_{ij}^{(n,\rho)} = \begin{cases} 1 & \text{with probability} p_n = \rho/n \\ 0 & \text{with probability } 1-\rho/n \end{cases}, \quad 0 < \rho < n.$$

We assume that the family $\mathcal{A}_{n,\rho}$ is determined on a probability space $\Omega_{n,\rho}$ and denote by $\mathbb{E} = \mathbb{E}_{n,\rho}$ the mathematical expectation with respect to the probability measure $P = P_{n,\rho}$ generated by $\mathcal{A}_{n,\rho}$.

The ensemble of real symmetric random matrices $A^{(n,\rho)}$ with elements

$$\left(A^{(n,\rho)}\right)_{ij} = \begin{cases} a_{ij}^{(n,\rho)} & \text{if } i < j \\ a_{ji}^{(n,\rho)} & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}, \quad i,j \in \{1,\dots,n\}, \tag{2.1}$$

can be regarded as the adjacency matrix of a non-oriented random graph $\Gamma^{(n,\rho)}$. The family of such random graphs $\{\Gamma^{(n,\rho)}\}$ is usually denoted by $G(n,p_n)$, where we have taken $p_n = \rho/n$. This family is equivalent in many aspects to the ensemble of random Erdős–Rényi graphs [10] and is often referred to simply as the Erdős–Rényi random graphs (see monograph [4]).

Given $\Gamma^{(n,\rho)}$, we consider the corresponding right-hand side of (1.2) an say that it determines the Ihara zeta function of $\Gamma^{(n,\rho)}$ despite of the fact that this graph can be disconnected,

$$(1-u^2)^{r-1} \det\left(I + u^2(B^{(n,\rho)} - I) - uA^{(n,\rho)}\right) = \left(Z_{\Gamma^{(n,\rho)}}(u)\right)^{-1}, \quad u \in \mathbb{C}, \tag{2.2}$$

where

$$\left(B^{(n,\rho)}\right)_{ij} = \delta_{ij} \sum_{k=1}^{n} a_{ik}^{(n,\rho)}. \tag{2.3}$$

According to (1.3), we denote in (2.2)

$$r = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}^{(n,\rho)} - n + 1. \tag{2.4}$$

We study IZF (2.2) in the limiting transition $n \to \infty$ when the average value of the vertex degree of $\Gamma^{(n,\rho_n)}$ given by $(B^{(n,\rho_n)})_{ii}$ (2.3) goes to infinity more rapidly than $\log n$. This means that

$$n \to \infty, \quad \rho_n/\log n = \chi_n \to \infty. \tag{2.5}$$

We denote this limiting transition by $(\rho,\chi)_n \to \infty$. Writing (2.5), we assume that an infinite sequence $(\chi_n)_{n\ge1}$ is determined and ρ_n is given by the relation

$$\rho_n = \chi_n \log n, \quad n \ge 1.$$

In what follows, we omit the subscript in ρ_n when no confusion can arise.

In the paper [20], it is shown that in the limit (2.5), it is natural to consider (2.4) with the spectral parameter u normalized by the square root of $\rho = \rho_n$. Thereby we introduce a normalized version of Ihara zeta function (2.2) with the spectral parameter $u = v/\sqrt{\rho}$,

$$\tilde{Z}^{(n,\rho)}(v,\omega) = Z_{\Gamma^{(n,\rho)}(\omega)}\left(\frac{v}{\sqrt{\rho}}\right), \quad \omega \in \Omega_{n,\rho_n}, \quad v \in \mathbb{C},$$

where

$$\left(Z_{\Gamma^{(n,\rho)}}\left(\frac{v}{\sqrt{\rho}}\right)\right)^{-1} = \left(1 - \frac{v^2}{\rho}\right)^{r-1} \det\left(-vH^{(n,\rho)}\right), \tag{2.6}$$

where

$$H^{(n,\rho)}(v) = I + \frac{v^2}{\rho}\left(B^{(n,\rho)} - I\right) - \frac{1}{\sqrt{\rho}} A^{(n,\rho)}.$$

Our main result is given by the following statement.

Theorem 2.1. *Let $D_{\varepsilon,\varepsilon'}$ with $\varepsilon, \varepsilon' > 0$ be a union of two complex domains*

$$D^{(1)}_{\varepsilon,\varepsilon'} = \{z \in \mathbb{C} : \varepsilon' < |z| < 1 - \varepsilon\} \text{ and } D^{(2)}_{\varepsilon,\varepsilon'} = \{z \in \mathbb{C} : 1 + \varepsilon < |z| < 1/\varepsilon'\}. \tag{2.7}$$

We consider two subsets $\Phi^{(1)}_{n,\rho}(\varepsilon,\varepsilon')$ and $\Phi^{(2)}_{n,\rho}(\varepsilon,\varepsilon')$ determined by the relation

$$\Phi^{(i)}_{n,\rho}(\varepsilon,\varepsilon') = \left\{\omega \in \Omega : \text{ there exists } v \in D^{(i)}_{\varepsilon,\varepsilon'} \text{ such that } \left(\tilde{Z}^{(n,\rho)}(v,\omega)\right)^{-1} = 0\right\}, \tag{2.8}$$

for $i = 1$ and $i = 2$. With the choice of

$$\varepsilon_n = \frac{2}{(\chi_n)^{1/8}} \quad \text{and} \quad \varepsilon'_n = \frac{1}{\sqrt{\rho_n}(1-\kappa)}, \quad \kappa > 0, \tag{2.9}$$

the following series of probabilities converge,

$$\sum_{n \ge 1} P_{n,\rho_n}\left(\Phi^{(i)}_{n,\rho_n}(\varepsilon_n, \varepsilon'_n)\right) < \infty, \quad i = 1, 2 \tag{2.10}$$

under asymptotic condition (2.5).

We prove Theorem 2.1 in Section 3 below. Let us note that we can prove slightly more powerful statement by adding to the complex domains $D^{(i)}_{\varepsilon,\varepsilon'}$ (2.7) real intervals

$$I^{(1)}_{\hat{\varepsilon},\varepsilon'} = \{v \in \mathbb{R} : \varepsilon' < v < 1 - \hat{\varepsilon}\} \quad \text{and} \quad I^{(2)}_{\hat{\varepsilon},\varepsilon'} = \{v \in \mathbb{R} : 1 + \hat{\varepsilon} < v < 1/\varepsilon'\},$$

where $\varepsilon' = \varepsilon'_n$ is given by (2.9) and

$$\hat{\varepsilon}_n = \frac{\hat{h}}{(\chi_n)^{1/4}} \tag{2.11}$$

with sufficiently large $\hat{h} > 0$. We concentrate ourselves on the main case of complex domains, the case of real v is briefly discussed at the end of Section 3.

Let us formulate a corollary of Theorem 2.1 that characterizes the points of the complex plane in the form close to (1.10).

Corollary 2.2. *For any given*

$$v \in D^{(1)} \cup D^{(2)} = \{z : z \in \mathbb{C}, 0 < |z| < 1\} \cup \{z : z \in \mathbb{C}, 1 < |z|\},$$

the following series of probabilities converges,

$$\sum_{n \geq 1} P_{n,\rho_n} \left(\left\{ \omega \in \Omega_{n,\rho_n} : \left(\tilde{Z}^{(n,\rho_n)}(v,\omega) \right)^{-1} = 0 \right\} \right) < \infty, \tag{2.12}$$

where $\rho_n = \chi_n \log n$, $\chi_n \to \infty$ *as* $n \to \infty$.

The proof of this statement follows immediately from the proof of Theorem 2.1.

Let us note that the statement of Corollary 2.2 can be formulated in its equivalent form,

$$\sum_{n \geq 1} P_{n,\rho_n} \left(\left\{ \omega : \left(Z_{\Gamma^{(n,\rho_n)}} \left(\frac{v}{\sqrt{\rho}_n} \right) \right)^{-1} = 0 \right\} \right) = +\infty \quad \text{implies} \quad |v| = 1. \tag{2.13}$$

This shows that Corollary 2.2 can be regarded as a direct analog of the statement (1.8) in the case of large random graphs.

Let us outline the proof of Theorem 2.1. Definition (2.6) shows that if $v^2 \neq \rho$, then the function $Z_{\Gamma^{(n,\rho)}}$ has a pole at v if and only if

$$v\left(1 - \frac{1}{\rho}\right) + v - \lambda_j \left(\frac{1}{\sqrt{\rho}} A + v \left(\frac{1}{\rho} B - I \right) \right) = 0 \tag{2.14}$$

for some j, where $\lambda_j(M)$ denotes the j-th eigenvalue of M. If one accepts that the expression in last braces asymptotically vanishes

$$\left\| \frac{1}{\rho} B - I \right\| = o(1), \tag{2.15}$$

then it can be regarded as a small perturbation of the eigenvalues of $\tilde{A} = A/\sqrt{\rho}$. Neglecting the corresponding term in the right-hand side of (2.14) as well as the vanishing term $-1/\rho$ in the first braces of (2.14), one could say that (2.14) is equivalent to the condition that

$$\frac{1}{v} + v - \lambda_j(\tilde{A}) = 0 \tag{2.16}$$

for some $j \in \{1, \ldots, n\}$. It is known from [15, 22] that the normalized adjacency matrices of the Erdős–Rényi random graphs $G(n, \rho/n)$ have all eigenvalues, excepting the maximal one $\lambda_1(\tilde{A})$, asymptotically bounded in absolute value by $2 + \delta_n$ in the limit $n \to \infty$, $\rho \gg \log n$, where δ_n tends to zero. Therefore the probability of the event (2.16) rapidly decays for all $j \in \{2, \ldots, n\}$ as $n \to \infty$ for any v verifying $|v| \neq 1$. It is worthy to note that the important part of this proposition is that it can be proved uniformly with respect to complex v belonging to growing domains $D^{(1)}$ and $D^{(2)}$ (2.7). To do this, we use an important property of the ellipsoidal curves of the form

$$\mathcal{E}_r = \left\{ w(z) = z + \frac{1}{z}, \quad z = re^{i\varphi}, \quad 0 \le \varphi < 2\pi \right\} \tag{2.17}$$

and the distance between the points of real axis and $\mathcal{E}_r$.

To establish the convergence of the series (2.10), we use recent results on the concentration properties of eigenvalues of adjacency matrices of random graphs [3, 25]. Relation (2.15) reflects the concentration property of diagonal elements of $B - I$,

$$\max_{1\le i\le n} \left| \frac{1}{\rho} \left(B^{(n,\rho)} \right)_{ii} - 1 \right| = o(1), \quad n, \rho \to \infty, \tag{2.18}$$

that is also well known in the literature (see, for example, the monograph [4] and the paper [21]). The convergence (2.18) can be interpreted as an asymptotic regularity of the Erdős–Rényi random graphs when the average vertex degree goes to infinity. Taking into account this observation, one can say that the proof of Theorem 2.1 is equivalent in certain sense to the proof of a stochastic version of Lemma 1.1 by Stark and Terras.

3. Proof of Theorem 2.1

Let us rewrite definition (2.7) in the form

$$H^{(n,\rho_n)}(v) = \tilde{A}^{(n,\rho_n)} - \gamma^{(n,\rho_n)}(v) I - v\hat{B}^{(n,\rho_n)}, \tag{3.1}$$

where $\hat{B} = \tilde{B} - I(n-1)/n$,

$$\left(\tilde{B}^{(n,\rho)} \right)_{ij} = \frac{1}{\rho} \left(B^{(n,\rho)} \right)_{ij}$$

and

$$\left(\tilde{A}^{(n,\rho)} \right)_{ij} = \frac{1}{\sqrt{\rho}} a_{ij}^{(n,\rho)}, \quad i, j \in \{1, \dots, n\}.$$

We also denote

$$\gamma^{(n,\rho_n)}(v) = \frac{1 - v^2/\rho}{v} + \frac{v(n-1)}{n} = v + \frac{1}{v} - v\left(\frac{1}{n} + \frac{1}{\rho_n} \right).$$

The poles of $\tilde{Z}^{(n,\rho)}(v, \omega)$ (2.6) with $v^2 \ne \rho$ correspond to zeroes of the determinant of $H^{(n,\rho)}$. Introducing the subset

$$\Phi_{n,\rho}(v) = \{\omega :\ \det(H^{(n,\rho)}(v)) = 0\}, \tag{3.2}$$

we can write that the subsets $\Phi^{(i)}$ of (2.8) are given by the relations

$$\Phi_{n,\rho}^{(i)}(\varepsilon_n, \varepsilon_n') = \bigcup_{v \in D^{(i)}_{\varepsilon_n, \varepsilon_n'}} \Phi_{n,\rho}(v), \quad i = 1, 2.$$

It should be noted that $\Phi^{(1)}$ and $\Phi^{(2)}$ are determined as uncountable unions of measurable events; since the probability space $\Omega_{n,\rho}$ generated by (2.1) can be viewed as a discrete set, we conclude that $\Phi^{(i)}$ are both measurable. In what follows, we replace denotations of events $\{\omega : \Upsilon(\omega)\}$ simply by $\{\Upsilon\}$. We will omit the superscripts n and ρ in $H^{(n,\rho)}$ and everywhere below, when no confusion can arise. We prove relation (2.10) with $i = 2$ in the full extent. The proof of (2.10) for the case of $i = 1$ is given in less detail.

3.1. Proof of Theorem 2.1 for the case $i = 2$. The following elementary statement shows that the study of $\det(H(v))$ (3.1) can be reduced to the study of the product of singular values of $H(v)$. We denote these singular values by

$$\sigma_1(H(v)) \le \sigma_2(H(v)) \le \cdots \le \sigma_n(H(v)). \tag{3.3}$$

Here and below we omit the superscripts n and ρ_n everywhere when no confusion can arise.

Lemma 3.1. *For any $v \in \mathbb{C}$, the equivalence*

$$\det(H(v)) \neq 0 \iff \prod_{k=1}^{n} \sigma_k(H(v)) \neq 0 \tag{3.4}$$

is true.

Proof. Since $\tilde{A}$ is a real symmetric matrix and $\hat{B}$ is a real diagonal one, then we can write for hermitian conjugate that

$$\big(H(v)\big)^*_{ij} = \overline{\big(H(v)\big)_{ji}} = (1-\delta_{ji})\tilde{A}_{ji} - \bar{v}\delta_{ji}\hat{B}_{ii} - \overline{\gamma(v)} = \big(H(\bar{v})\big)_{ij},$$

and therefore

$$H^*(v) = H(\bar{v}) \quad \text{and} \quad \overline{H(v)} = H(\bar{v}).$$

It is easy to see that $\lambda(v)$ is an eigenvalue of $H(v)$ if and only if $\overline{\lambda(v)}$ is the eigenvalue of $H(\bar{v})$. Then

$$\det(H(v)) \neq 0 \iff \det(H(\bar{v})) \neq 0 \iff \det(H^*(v)H(v)) \neq 0.$$

The last statement is equal to that of the right-hand side of (3.4). Let us note that the last condition of (3.4) is equivalent to $\sigma_1(H(v)) \neq 0$ because of (3.3) and due to positivity of $\sigma_i(H(v))$, $1 \le i \le n$. Lemma 3.1 is proved. □

The study of singular values of $H(v)$ (3.3) can be reduced to the study of singular values of $\tilde{A} - \gamma(v)I$ (3.1) due to the concentration property (2.15) of the diagonal matrix $\hat{B}$. Using the Weyl's inequality for singular values of n-dimensional matrices X and Y (see [7] and [29], Exercice 22),

$$|\sigma_i(X+Y) - \sigma_i(X)| \le \|Y\|, \quad i = 1, \ldots, n, \tag{3.5}$$

where $\|Y\|$ is the operator norm of Y, we obtain that

$$|\sigma_i(H^{(n,\rho)}(v)) - \sigma_i(\tilde{A}^{(n,\rho)} - \gamma^{(n,\rho)}(v)I)| \le \|v\hat{B}\| \\ = |v| \max_{i=1,\dots,n} |\hat{\Delta}_i^{(n,\rho)}|, \quad i = 1,\dots,n, \tag{3.6}$$

where

$$\hat{\Delta}_i^{(n,\rho)} = \tilde{b}_{ii}^{(n,\rho)} - \frac{n-1}{n} = \frac{1}{\rho} \sum_{\substack{j\in\{1,\dots,n\},\\ j\neq i}} \left(a_{ij}^{(n,\rho)} - \frac{\rho}{n} \right).$$

We denote

$$\hat{\Delta}_{\max}^{(n,\rho)} = \max_{i=1,\dots,n} |\hat{\Delta}_i^{(n,\rho)}|. \tag{3.7}$$

We have seen above that $\Phi(v) = \{\sigma_1(H(v)) = 0\}$ and that

$$\{\sigma_1(H(v)) = 0\} \subseteq \left\{ \sigma_1(\tilde{A} - \gamma^{(n,\rho)}(v)I) \le |v|\hat{\Delta}_{\max}^{(n,\rho)} \right\}. \tag{3.8}$$

Elementary calculation shows that

$$\sigma_i(\tilde{A} - \gamma^{(n,\rho)}(v)I) = \lambda_i\big((\tilde{A} - \gamma^{(n,\rho)}(v)I)(\tilde{A} - \gamma^{(n,\rho)}(v)I)^*\big) \\ = \lambda_i\big((\tilde{A} - \alpha(v)I)^2\big) + \beta(v)^2, \tag{3.9}$$

where $\alpha(v)$ and $\beta(v)$ are the real and the imaginary parts of $\gamma(v)$. Diagonalizing $\tilde{A} - \alpha(v)I$, we deduce from (3.9) that

$$\sigma_i(\tilde{A} - \gamma^{(n,\rho)}(v)I) = \big(\lambda_i(\tilde{A}) - \alpha(v)\big)^2 + \beta(v)^2 = |\lambda_i(\tilde{A}) - \gamma(v)|^2. \tag{3.10}$$

It follows from (3.8) and (3.10) that for any $v \in D^{(2)}_{\varepsilon,\varepsilon'}$,

$$\Phi_{n,\rho}(v) \subseteq \left\{ \min_{i=1,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma^{(n,\rho)}(v)|^2}{|v|} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\} \\ \subseteq \left\{ \inf_{v\in D^{(2)}_{\varepsilon,\varepsilon'}} \min_{i=1,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma^{(n,\rho)}(v)|^2}{|v|} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\} = R^{(n,\rho)}(\varepsilon,\varepsilon'). \tag{3.11}$$

Then

$$\Phi^{(2)}_{n,\rho}(\varepsilon,\varepsilon') = \cup_{v\in D^{(2)}_{\varepsilon,\varepsilon'}} \{\sigma_1(H(v)) = 0\} \subseteq R^{(n,\rho)}(\varepsilon,\varepsilon'), \tag{3.12}$$

where we omitted ε and ε' in R. Let us note that the minimums of (3.11) can be taken in arbitrary order, in particular, such that

$$\inf_{v\in D^{(2)}_{\varepsilon,\varepsilon'}} = \inf_{1+\varepsilon<r<1/\varepsilon'} \inf_{0\le\varphi<2\pi}.$$

We denote these minimums by $M_r^{(2)}$ and M_φ, respectively.

In (3.11), the term $|\lambda_i(\tilde{A}) - \gamma^{(n,\rho)}(v)|^2$ is a squared distance between the real $\lambda_i(\tilde{A})$ and the complex number

$$\gamma^{(n,\rho)}(v) = v(1-\tau) + \frac{1}{v}, \quad \tau = \frac{1}{n} + \frac{1}{\rho}. \tag{3.13}$$

Let us note that the application $\gamma^{(n,\rho)}(\cdot) : \mathbb{C} \to \mathbb{C}$ can be viewed as a slightly modified version of the Zhukovsky transform $w(z)$ (2.17). For given r, the points

$$\mathcal{E}(\dot{a}, \dot{b}) = \left\{ \gamma^{(n,\rho)}(re^{i\varphi}), \quad 0 \le \varphi < 2\pi \right\}$$

form an ellipsoid in the complex plane $\mathbb{R}^2 = \{(x,y) : x = \operatorname{Re} w,\ y = \operatorname{Im} w\}$. An important property of the minimal distance from an ellipsoid to a given real point λ is studied in Section 4.

We write that

$$R^{(n,\rho)}(\varepsilon, \varepsilon') \subseteq R_1^{(n,\rho)} \cup R_2^{(n,\rho)}, \tag{3.14}$$

where

$$R_1^{(n,\rho)} = \left\{ M_r^{(2)} M_\varphi \frac{|\lambda_1(\tilde{A}) - \gamma(re^{i\varphi})|^2}{r} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\}$$

and

$$R_2^{(n,\rho)} = \left\{ M_r^{(2)}\, M_\varphi \min_{i=2,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma(re^{i\varphi})|^2}{r} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\}.$$

Let us study first the term $R_2^{(n,\rho)}$. To do this, we consider the matrix

$$\breve{A}^{(n,\rho)} = \frac{1-\delta_{ij}}{\sqrt{\rho}} \left(a_{ij}^{(n,\rho)} - \frac{\rho}{n} \right). \tag{3.15}$$

It is known [15] that

$$\lambda_1(\breve{A}) \ge \lambda_2(\tilde{A}) \ge \cdots \ge \lambda_n(\tilde{A}) \ge \lambda_n(\breve{A}). \tag{3.16}$$

For completeness, we reproduce the proof of (3.16) by Füredi and Komlós [15] in Section 4. It follows from (3.16) that

$$\min_{i=2,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma(re^{i\varphi})|^2}{r} \ge \min_{i=1,\dots,n} \frac{|\lambda_i(\breve{A}) - \gamma(re^{i\varphi})|^2}{r}.$$

Let us introduce the set

$$\Upsilon_\delta = \{\omega : \lambda_{\max}(\breve{A}) \le 2 + \delta\}, \quad \delta > 0.$$

Then we can write an obvious inclusion

$$R_2^{(n,\rho)} \subseteq (R_2^{(n,\rho)} \cap \Upsilon_\delta) \cup \bar{\Upsilon}_\delta. \tag{3.17}$$

We choose δ and ε such that the point $2+\delta$ lies inside the minimal ellipsoid,

$$2 + \delta < \min_{1+\varepsilon<r} \gamma(r),$$

(see relation (4.9) of Section 4 for more details). In Section 4, we show that (see Lemma 4.2)

$$M_\varphi|\lambda_i(\breve{A}) - \gamma(re^{i\varphi})|^2 I_{\Upsilon_\delta} \ge M_\varphi|\lambda_{\max}(\breve{A}) - \gamma(re^{i\varphi})|^2 I_{\Upsilon_\delta},$$

where I_υ is the indicator function of Υ. It is also proved in Lemma 4.2 that the minimal distance between $\lambda_{\max}(\breve{A})$ and the ellipsoid $\mathcal{E}(\dot{a}, \dot{b})$ is attained at the right extremum of $\mathcal{E}(\dot{a}, \dot{b})$ given by $\dot{a} = \gamma(r)$ (see (4.7)),

$$M_\varphi|\lambda_{\max}(\breve{A}) - \gamma(re^{i\varphi})|^2 I_{\Upsilon_\delta} = |\lambda_{\max}(\breve{A}) - \gamma(r)|^2 I_{\Upsilon_\delta} \ge |(2+\delta) - \gamma(r)|^2.$$

Finally, the minimal value of $|(2+\delta) - \gamma(r)|^2/r$ with respect to $r \in (1+\varepsilon, 1/\varepsilon')$ is given by the value

$$\min_{1+\varepsilon<r<1/\varepsilon'} \frac{|(2+\delta) - \gamma(r)|^2}{r} = F^{(1)}(2+\delta, \varepsilon, \varepsilon', \tau) = \frac{(\varepsilon^2 - \delta(1+\varepsilon) - \tau(1+\varepsilon)^2)^2}{(1+\varepsilon)^3},$$

(see Lemma 4.2). Taking into account obvious inclusion,

$$R_2^{(n,\rho)} \cap \Upsilon_\delta \subseteq \left\{ M_r^{(2)}\, M_\varphi \frac{|\lambda_{\max}(\breve{A}) - \gamma(re^{i\varphi})|^2}{r} I_{\Upsilon_\delta} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\},$$

we can write that

$$P\left(R_2^{(n,\rho)} \cap \Upsilon_\delta\right) \le P\left(\hat{\Delta}_{\max}^{(n,\rho)} \ge F^{(1)}(2+\delta, \varepsilon, \varepsilon', \tau)\right). \tag{3.18}$$

It is proved in [21] that if $\rho_n = \chi_n \log n$ with $\chi_n \to \infty$ as $n \to \infty$, then for any positive ν the following upper bound holds:

$$P\left(\hat{\Delta}_{\max}^{(n,\rho_n)} > \nu\right) \le \frac{1}{n^{\log(\nu\sqrt{\chi_n}(1+o(1)))}}, \quad n \to \infty. \tag{3.19}$$

We see that in the limit (2.5), when χ_n infinitely increases, we can consider (3.19) with vanishing $\nu_n \to 0$ such that $\nu_n \ge 3/\sqrt{\chi_n}$. This allows us to choose ε in the right-hand side of (3.18) such that $\varepsilon \to 0$ as $n \to \infty$. In this case, $F^{(1)}(2+\delta, \varepsilon, \varepsilon', \tau) = \varepsilon^4(1+o(1))$ for vanishing δ such that $\delta = o(\varepsilon^2)$ (see relation (4.13) of Section 4). Then the choice of $\varepsilon_n = 2\chi_n^{-1/8}$ (2.9) is sufficient to conclude that

$$\sum_{n=1}^{\infty} P(R_2^{(n,\rho_n)} \cap \Upsilon_{\delta_n}) < \infty, \quad \delta_n = o(\varepsilon_n^2), \tag{3.20}$$

where ρ_n satisfies conditions (2.5).

Let us show that

$$\sum_{n\ge1} P(\bar{\Upsilon}_{\delta_n}) < \infty \quad \text{with} \quad \delta_n = \frac{1}{\chi_n^{3/8}}. \tag{3.21}$$

It is easy to see that starting from some n_0, the random matrix $\breve{A}^{(n,\rho_n)}$ (3.15) satisfies the conditions of Theorem 2.7 of [3]. According to this theorem, there exists a constant $C > 0$ such that

$$\mathbb{E}\|\breve{A}^{(n,\rho)}\| < 2 + \frac{C}{\sqrt{\chi_n}}, \quad (\rho, \chi)_n \to \infty. \tag{3.22}$$

Adding to this result the general concentration inequality

$$P\left(\left|\|\breve{A}^{(n,\rho)}\| - \mathbb{E}\|\breve{A}^{(n,\rho)}\|\right| \geq t\right) \leq \frac{2}{n^{c\chi_n t^2}}, \quad t > 0, \tag{3.23}$$

with some $c > 0$ [6], we get the following asymptotic upper bound:

$$P\left(\|\breve{A}\| \geq 2 + \delta\right) \leq P\left(\left|\|\breve{A}\| - \mathbb{E}\|\breve{A}\|\right| \geq \delta - \frac{C}{\sqrt{\chi_n}}\right) \leq \frac{2}{n^{c\chi_n(\delta/2)^2}}. \tag{3.24}$$

It follows from (3.22) that

$$\bar{\Upsilon}_\delta = \{\lambda_{\max}(\breve{A}) > 2 + \delta\} \subseteq \{\lambda_{\max}(\breve{A}) - \mathbb{E}\lambda_{\max}(\breve{A}) > \delta - C/\sqrt{\chi_n}\}.$$

Then, using (3.23), we get

$$P(\bar{\Upsilon}_\delta) \leq \left(|\lambda_{\max}(\breve{A}) - \mathbb{E}\lambda_{\max}(\breve{A})| > \delta - C/\sqrt{\chi_n}\right) \leq \frac{2n^{2\delta c C\sqrt{\chi_n}}}{n^{c\chi_n\delta^2} \cdot n^{cC^2}}. \tag{3.25}$$

Relation (3.25) shows that (3.21) is true as well as (3.20). Then it follows from (3.17) that

$$\sum_{n\geq 1} P\left(R_2^{(n,\rho_n)}\right) < \infty. \tag{3.26}$$

Let us study the subset $R_1^{(n,\rho)}$ (3.14). We denote

$$\Psi_\kappa = \left\{\lambda_1(\tilde{A}) \geq \sqrt{\rho}(1-\kappa)\right\}$$

and observe that

$$R_1^{(n,\rho)} \subseteq (R_1^{(n,\rho)} \cap \Psi_\kappa) \cup \bar{\Psi}_\kappa. \tag{3.27}$$

Regarding the subset $R_1^{(n,\rho)} \cap \Psi_\kappa$, we can repeat the previous reasoning based on the properties of an ellipsoid and write that

$$R_1^{(n,\rho)} \cap \Psi_\kappa \subseteq \left\{\min_{1+\varepsilon<r<1/\varepsilon'} \frac{|\sqrt{\rho}(1-\kappa) - \gamma(r)|^2}{r} \leq \hat{\Delta}_{\max}^{(n,\rho)}\right\}. \tag{3.28}$$

With the help of the second part of Lemma 4.2, we deduce from (3.28) that

$$P(R_1^{(n,\rho)} \cap \Psi_\kappa) \leq P\left(\hat{\Delta}_{\max}^{(n,\rho)} \geq F^{(2)}(\sqrt{\rho}(1-\kappa), \varepsilon, \varepsilon', \tau)\right), \tag{3.29}$$

where

$$F^{(2)}\left(\sqrt{\rho}(1-\kappa), \varepsilon, \varepsilon', \tau\right) = \varepsilon'\left((\varepsilon')^2(1-\tau) - q\varepsilon' + 1\right)^2,$$

under condition

$$1/\varepsilon' < \frac{q + \sqrt{q^2 - 4(1-\tau)}}{2}, \quad q = \sqrt{\rho}(1-\kappa). \tag{3.30}$$

If $\varepsilon' = (\sqrt{\rho}(1-h))^{-1}$ and $\kappa < h$, then (3.30) is verified and

$$F^{(2)}\left(\sqrt{\rho}(1-\kappa), \varepsilon, \varepsilon', \tau)\right), = \sqrt{\rho}(h-\kappa)^2(1+o(1)).$$

Substituting this relation into the right-hand side of (3.29), we obtain the following upper bound:

$$P(R_1^{(n,\rho)} \cap \Psi_\kappa) \le P\left(\hat{\Delta}_{\max}^{(n,\rho)} \ge \sqrt{\rho}(h-\kappa)^2\right).$$

It follows from the last estimate and (3.19) that for any $h > \kappa$ we have

$$P(R_1^{(n,\rho)} \cap \Psi_\kappa) \le \frac{1}{n^{\log(\sqrt{\chi_n \rho_n}(h-\kappa)^2(1+o(1))}}.$$

Therefore, we can write that

$$\sum_{n\ge 1} P(R_1^{(n,\rho)} \cap \Psi_\kappa) < \infty. \tag{3.31}$$

Let us estimate the probability of $\bar{\Psi}_\kappa$. The maximal eigenvalue of the adjacency matrix $A^{(n,\rho_n)}$ (2.1) of the Erdős–Rényi random graphs $G(n, p_n)$ was studied by Krivelevich and Sudakov [22]. It follows from the results of [22] that with large probability, the maximal eigenvalue is greater than $(1 + o(1)) \max\left(\sqrt{D_{\max}}, np_n\right)$, where $D_{\max}$ is the maximal vertex degree of the graph $\Gamma^{(n,\rho_n)}$, that is,

$$P\left\{\omega :\ \lambda_1\left(A^{(n,\rho_n)}(\omega)\right) \ge \max\left(\sqrt{\max_{i=1,\dots,n} b_{ii}^{(n,\rho_n)}}, \rho_n\right)(1+o(1))\right\} \to 1$$

in the limit $(\rho, \chi)_n \to \infty$ (2.5). Then we can write that

$$\mathbb{E}\lambda_1(\tilde{A}^{(n,\rho_n)}) \ge (1+o(1))\sqrt{\rho_n}, \quad (\rho,\chi)_n \to \infty. \tag{3.32}$$

In the paper [25], the inequality

$$P\left(\sup_{\rho_n > C\log n}\left|\lambda_1(\tilde{A}^{(n,\rho_n)}) - \mathbb{E}\lambda_1(\tilde{A}^{(n,\rho_n)})\right| > \frac{t}{\sqrt{\rho_n}}\right) \le 4e^{-t^2/32} \tag{3.33}$$

is proved for all $t > C$. Taking into account that $\rho_n = \chi_n \log n$ and choosing $t^2 = s^2\rho_n$, we deduce from (3.33) that

$$P\left(\left|\lambda_1(\tilde{A}^{(n,\rho_n)}) - \mathbb{E}\lambda_1(\tilde{A}^{(n,\rho_n)})\right| > s\right) \le \frac{4}{n^{\chi_n s^2/32}}. \tag{3.34}$$

It follows from (3.32) that

$$\bar{\Psi}_\kappa \subseteq \left\{\lambda_1(\tilde{A}) \le \sqrt{\rho}(1-\kappa)\right\} \subseteq \left\{\mathbb{E}\lambda_1(\tilde{A}) - \lambda_1(\tilde{A}) \le \sqrt{\rho}\kappa(1+o(1))\right\}.$$

Then (3.34) implies the following upper bound:

$$P\left(\bar{\Psi}_\kappa\right) \le P\left(|\mathbb{E}\lambda_1(\tilde{A}) - \lambda_1(\tilde{A})| \ge \sqrt{\rho}\kappa(1+o(1))\right) \le \frac{4}{n^{\chi_n \rho_n \kappa^2(1+o(1))/32}}.$$

Then, clearly.

$$\sum_{n\ge 1} P(\bar{\Psi}_\kappa) < \infty$$

for any $\kappa > 0$. Taking into account this convergence, as well as (3.27), (3.31), and (3.34), we conclude that

$$\sum_{n\ge 1} P\left(R_1^{(n,\rho_n)}\right) < \infty.$$

Combining this relation with (3.26) and remembering (3.12), we conclude that relation (2.10) is true for the case $i = 2$.

3.2. Proof of Theorem 2.1 for the case $i = 1$. In analogy with (3.11), we can write that

$$\Phi(v) \subseteq \left\{ \inf_{v\in D^{(1)}_{\varepsilon,\varepsilon'}} \min_{i=1,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma^{(n,\rho)}(v)|^2}{|v|} \le \hat{\Delta}^{(n,\rho)}_{\max} \right\} = S^{(n,\rho)}(\varepsilon,\varepsilon'). \qquad (3.35)$$

We introduce the subsets $S_1^{(n,\rho)}$ and $S_2^{(n,\rho)}$ similarly to (3.14) and (3.15), where the minimum $M_r^{(2)}$ is replaced by $M_r^{(1)} = \inf_{\varepsilon'<r<1-\varepsilon}$. Then we can write that

$$S_2^{(n,\rho)} \cap \Upsilon_\delta \subseteq \left\{ M_r^{(1)} M_\varphi \frac{|\lambda_{\max}(\breve{A}) - \gamma(re^{i\varphi})|^2}{r} I_{\Upsilon_\delta} \le \hat{\Delta}^{(n,\rho)}_{\max} \right\},$$

and therefore

$$P(S_2^{(n,\rho)} \cap \Upsilon_\delta) \le P(\hat{\Delta}^{(n,\rho)}_{\max} \ge G^{(1)}(2+\delta,\varepsilon,\varepsilon',\tau)). \qquad (3.36)$$

Relation (4.19) shows that $G^{(1)}(2+\delta,\varepsilon,\varepsilon',\tau) = \varepsilon^4(1+o(1))$ as $\varepsilon \to 0$ and then

$$\sum_{n=1}^{\infty} P(S_2^{(n,\rho)} \cap \Upsilon_\delta) < \infty \qquad (3.37)$$

with the same choice of $\varepsilon_n = 2(\chi_n)^{-1/8}$ (2.9) and $\delta_n = (\chi_n)^{-3/8}$ as in the previous subsection.

Let us study $S_1^{(n,\rho)}$. Repeating the arguments of the previous subsection, we can write that

$$S_1^{(n,\rho)} \cap \Psi_\kappa \subseteq \left\{ \min_{\varepsilon'<r<1-\varepsilon} \frac{|\sqrt{\rho}(1-\kappa) - \gamma(r)|^2}{r} \le \hat{\Delta}^{(n,\rho)}_{\max} \right\}$$

and, by Lemma 4.3, we have

$$P(S_1^{(n,\rho)} \cap \Psi_\kappa) \le P\left(\hat{\Delta}_{\max}^{(n,\rho)} \ge G^{(2)}(\sqrt{\rho}(1-\kappa), \varepsilon, \varepsilon', \tau)\right). \tag{3.38}$$

Choosing $1/\varepsilon' = \sqrt{\rho}(1-h)$, we obtain with the help of (4.20) an upper estimate,

$$P(S_1^{(n,\rho)} \cap \Psi_\kappa) \le P\left(\hat{\Delta}_{\max}^{(n,\rho)} \ge \rho^{3/2}(1-h)(h-\kappa)^2\right).$$

Then (3.19) implies the convergence

$$\sum_{n\ge 1} P(S_1^{(n,\rho)} \cap \Psi_\kappa) < \infty. \tag{3.39}$$

The upper bounds for the probabilities $P(\bar{\Upsilon}_\delta)$ and $P(\bar{\Psi}_\kappa)$ for the case $i = 1$ are the same as for the case $i = 2$. Relations (3.37), (3.39) together with (3.21) and (3.34) show that convergence (2.10) is true for the case $i = 1$. Theorem 2.1 is proved.

3.3. The case of real v. Regarding the particular case $v \in \mathbb{R}$, we can use the following inequalities instead of (3.6):

$$|\lambda_i(H^{(n,\rho)}(v)) - \lambda_i(\tilde{A}^{(n,\rho)} - \gamma^{(n,\rho)}(v)I)| \le |v|\hat{\Delta}_{\max}^{(n,\rho)}, \quad i = 1, \dots, n, \tag{3.40}$$

where eigenvalues of $H^{(n,\rho)}$ and $\tilde{A}^{(n,\rho)}$ are ordered in, say, decreasing order. Relation (3.40) is a corollary of more precise Weyl's inequality for hermitian matrices (see, e.g., [17]). Using (3.40), we can write that

$$\{\omega : \det(H(v)) = 0\} = \bigcup_{i=1}^{n} \{\omega : \lambda_i(H(v)) = 0\}$$

and

$$\{\omega : \lambda_i(H(v)) = 0\} \subseteq \left\{ \frac{|\lambda_i(\tilde{A}) - \gamma(v)|}{|v|} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\}.$$

Then, in complete analogy with (3.2), (3.11) and (3.12), we get the inclusions

$$\hat{\Phi}_{n,\rho}^{(2)}(\hat{\varepsilon}, \varepsilon') = \cup_{1+\hat{\varepsilon}<v<1/\varepsilon'} \{\omega : \det(H(v)) = 0\} \subseteq \hat{R}_1^{(n,\rho)} \cup R_2^{(n,\rho)},$$

where

$$\hat{R}_1^{(n,\rho)} = \left\{ M_v^{(2)} \frac{|\lambda_1(\tilde{A}) - \gamma(v)|}{|v|} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\}$$

and

$$\hat{R}_2^{(n,\rho)} = \left\{ M_v^{(2)} \min_{i=2,\dots,n} \frac{|\lambda_i(\tilde{A}) - \gamma(v)|}{|v|} \le \hat{\Delta}_{\max}^{(n,\rho)} \right\}$$

with $M_v^{(2)} = \min_{1+\hat\varepsilon<v<1/\varepsilon'}$. Regarding the last event, we can repeat all computations of the previous subsection with $F^{(1)}(2+\delta,\varepsilon,\varepsilon',\tau)$ replaced by

$$\hat F^{(1)}(2+\delta,\hat\varepsilon,\varepsilon',\tau) = \frac{\varepsilon^2-\delta(1+\varepsilon)-\tau(1+\varepsilon)^2}{(1+\varepsilon)^3} = O(\hat\varepsilon^2), \quad \hat\varepsilon\to 0,$$

in the asymptotic regime when $\delta=\hat\varepsilon^2/h$ with sufficiently large h. In this case, relations (3.18) and (3.19) take the form

$$P(\hat R_2^{(n,\rho)}\cap\Upsilon_\delta) \le P\left(\hat\Delta_{\max}^{(n,\rho)}\ge\hat\varepsilon^2\right) \le \frac{1}{n^{\log(\hat\varepsilon^2\sqrt{\chi_n}(1+o(1)))}}\,, \quad n\to\infty. \tag{3.41}$$

It is easy to see that (2.11) is sufficient for the convergence of the series (3.41),

$$\sum_{n\ge1} P\left(\hat R_2^{(n,\rho_n)}\cap\Upsilon_{\delta_n}\right)<\infty, \quad \delta_n=\hat\varepsilon_n^2/h. \tag{3.42}$$

It follows from the upper bound (3.25) that to have the series of $P(\bar\Upsilon_{\delta_n})$ convergent, we need to make the value $\chi_n\delta_n^2$ sufficiently large. This observation together with the last condition of (3.42) shows that the rate (2.11) is close to the optimal one from the technical point of view.

4. Auxiliary results and statements

4.1. Proof of Lemma 1.1 by Stark and Terras. We reproduce here the proof of Lemma 1.1 given by Stark and Terras [28]. It is based on the observation that for a finite $(q+1)$-regular graph, relation (1.3) takes the form

$$(Z_{X^{(q+1)}}(u))^{-1} = (1-u^2)^{r-1}\prod_{j=1}^{n}\left(1-u\lambda_j+qu^2\right), \tag{4.1}$$

where $\lambda_1\le\cdots\le\lambda_n$ are eigenvalues of A. Then the poles of $Z_{X^{(q+1)}}(u)$ are given by zeros of quadratic polynomials $1-u\lambda_j+qu^2$, $1\le j\le n$.

One can write

$$1-u\lambda_j+qu^2=(1-\alpha_j u)(1-\beta_j u)$$

with $\alpha_j\beta_j=q$ and $\alpha_j+\beta_j=\lambda_j$. Then α_j and β_j are given by the roots of the quadratic equation

$$x^2-\lambda_j x+q=0,$$

and therefore

$$\alpha_j,\beta_j=\frac{\lambda_j+\sqrt{\lambda_j^2-4q}}{2}. \tag{4.2}$$

Thus, the values α_j and β_j are complex conjugate if and only if

$$|\lambda_j|\le 2\sqrt{q}, \tag{4.3}$$

and in this case,

$$|\alpha_j|^2=|\beta_j|^2=q.$$

The last equalities mean that if $s = \sigma + i\tau$ is such that

$$q^s = \alpha_j \quad \text{or} \quad q^s = \beta_j, \tag{4.4}$$

then $\mathrm{Re}(s) = \sigma = 1/2$. If $|\lambda_j| = q + 1$, then it follows from (4.2) and (4.3) that $\mathrm{Re}(s) = 0$ or 1. Finally, if $|\lambda_j| \neq q + 1$, then $|\lambda_j| < q + 1$ that therefore (4.4) implies inequalities $0 < \mathrm{Re}(s) < 1$. This argument completes the proof of equivalence between (1.5) and (1.6).

4.2. Distance property of ellipsoid. Let $\mathcal{E}(a, b)$ denote the family of points on $\mathbb{R}^2$,

$$\mathcal{E}(a,b) = \left\{ (s,t) : s,t \in \mathbb{R},\ \left(\frac{s}{a}\right)^2 + \left(\frac{t}{b}\right)^2 = 1 \right\}.$$

We assume that $b < a$. Given $x \in [0, a]$, we determine the distance $\mathcal{D}(x)$ between the point $(x, 0)$ and the ellipse $\mathcal{E}(a, b)$ by the formula

$$\mathcal{D}(x)^2 = \min_{(s,t)\in\mathcal{E}(a,b)} ((s-x)^2 + t^2) = \min_{s\in[-a,a]} \phi(x,s),$$

where

$$\phi(x, s) = (s - x)^2 + b^2 - b^2 s^2/a^2.$$

Lemma 4.1. *There exists a critical point $x_0 = a(1 - b^2/a^2)$ such that*

$$\mathcal{D}(x)^2 = \begin{cases} b^2\left(1 - \dfrac{x^2}{a^2-b^2}\right) & \text{if } 0 \le x \le x_0 \\ (a-x)^2 & \text{if } x_0 \le x \le a \end{cases}. \tag{4.5}$$

Proof. The proof of Lemma 4.1 is based on the elementary analysis of the derivative

$$\frac{\partial}{\partial s}\phi(x,s) = 2(s-x) - 2b^2 s/a^2.$$

If $x \in [0, x_0]$, then this derivative equals zero at the point $\tilde{s} = \tilde{s}(x) = x/(1 - b^2/a^2)$ and

$$\mathcal{D}(x)^2 = \phi(x, \tilde{s}(x)).$$

If $x \in [x_0, a]$, then the derivative $\phi'_s(x, s)$ has no zero on the interval $s \in [0, a]$, and therefore

$$\mathcal{D}(x)^2 = \phi(x, a) = (a - x)^2.$$

This means that if $x < x_0$, then the corresponding point $\tilde{l} = (\tilde{s}, \tilde{t})$ is such that $\tilde{t} > 0$; if $x \ge x_0$, then the point $\tilde{s}$ coincides with the right extremity of the ellipsoid, $\tilde{s} = a$ for all $x \ge x_0$. The derivative of $\mathcal{D}(x)^2$ is a discontinuous function and the distance $\mathcal{D}(x)^2$ shows a kind of "phase transition" of the first order at $x = x_0$. □

Let us point out that the distance $\mathcal{D}(x)$ (5.5) is a decreasing function for all $x \in [0, a]$ and for any $s \in [0, a]$,

$$\min_{x\in[0,s]} \mathcal{D}(x) = \mathcal{D}(s). \tag{4.6}$$

Now we turn to the case of $\gamma^{(n,\rho)}(v)$ (3.13). Denoting $s = \mathrm{Re}(\gamma^{(n,\rho)}(v))$ and $t = \mathrm{Im}(\gamma^{(n,\rho)}(v))$, we observe that the family of points

$$\dot{\mathcal{E}} = \mathcal{E}(\dot{a}, \dot{b}) = \left\{\gamma^{(n,\rho)}(re^{i\varphi}), \quad 0 \le \varphi < 2\pi\right\}$$

is given by an ellipsoid with the half-axes

$$\dot{a} = r(1-\tau) + \frac{1}{r}, \quad \dot{b} = r(1-\tau) - \frac{1}{r}.$$

Regarding the difference between $\lambda \in \mathbb{R}$ and $\gamma^{(n,\rho)}$, we see that its absolute value is bounded from below by the distance (4.5),

$$|\lambda - \gamma^{(n,\rho)}|^2 \ge \mathcal{D}(\lambda)^2$$

with a and b replaced by $\dot{a}$ and $\dot{b}$. Taking into account that

$$\dot{x}_0 = \frac{\dot{a}^2 - \dot{b}^2}{\dot{a}} = \frac{4(1-\tau)}{r(1-\tau) + 1/r},$$

one can easily see that if

$$\varepsilon < 1 \quad \text{and} \quad \tau < 1,$$

then $\dot{x}_0 < 2 + \delta$. Therefore we can write a version of (4.6),

$$\min_{\lambda\in[0,2+\delta]} \min_{\varphi\in[0,2\pi)} |\lambda - \gamma^{(n,\rho)}|^2 = \left(r(1-\tau) + \frac{1}{r} - 2 - \delta\right)^2. \tag{4.7}$$

Elementary analysis of the function

$$\gamma(x) = x(1-\tau) + \frac{1}{x}$$

shows that if

$$\varepsilon > \frac{1}{\sqrt{1-\tau}} - 1 \quad \text{and} \quad \varepsilon > \varepsilon_0 = \frac{(2\tau + \delta) + \sqrt{4\delta + \delta^2 + 4\tau}}{2(1-\tau)}, \tag{4.8}$$

then

$$\inf_{1+\varepsilon<x} \gamma(x) > 2 + \delta. \tag{4.9}$$

Regarding (4.7), we see that it remains to study the minimal value of the function

$$f(x) = \frac{1}{x}\left(x(1-\tau) + \frac{1}{x} - q\right)^2 = \frac{\left(x^2(1-\tau) - qx + 1\right)^2}{x^3}$$

over the interval $x \in (1+\varepsilon, 1/\varepsilon')$ in the case when $q = 2+\delta$. Elementary analysis shows that the derivative $f'(x)$ has four zeroes,

$$x_{1,2} = \frac{q \mp \sqrt{q^2 - 4(1-\tau)}}{2(1-\tau)} \quad \text{and} \quad x_{3,4} = \frac{-q \mp \sqrt{q^2 + 12(1-\tau)}}{2(1-\tau)},$$

such that $x_3 < 0 < x_1 < x_4 < 1$ and

$$1 < x_2 = \frac{q + \sqrt{q^2 - 4(1-\tau)}}{2(1-\tau)}.$$

We will also need to minimize $f(x)$ in the case when $q = \sqrt{\rho}(1-\kappa)$.

Lemma 4.2. *Let positive ε and ε' verify the inequality $1+\varepsilon < 1/\varepsilon'$. If q is greater than $2-\tau$ and such that $x_2 < 1+\varepsilon$, then*

$$F^{(1)}(q, \varepsilon, \varepsilon', \tau) = \inf_{1+\varepsilon<x<1/\varepsilon'} f(x) = f(1+\varepsilon) = \frac{\big((1+\varepsilon)^2(1-\tau) - q(1+\varepsilon) + 1\big)^2}{(1+\varepsilon)^3}. \tag{4.10}$$

If q is greater than $2+\tau$ and such that $1/\varepsilon' < x_2$, then

$$F^{(2)}(q, \varepsilon, \varepsilon', \tau) = \inf_{1+\varepsilon<x<1/\varepsilon'} f(x) = f(1/\varepsilon') = \frac{1}{\varepsilon'}\big((1-\tau) - q\varepsilon' + (\varepsilon')^2\big)^2. \tag{4.11}$$

Proof. The proof of Lemma 4.2 is based on the observation that $f(x)$ has a local minimum at x_2 and is strictly decreasing on the interval $[1, x_2)$ and strictly increasing on the interval $(x_2, +\infty)$. Simple computations show that (4.10) and (4.11) are true. □

Regarding our main asymptotic regime when

$$\varepsilon, \delta, \tau \to 0, \quad \delta = o(\varepsilon^2), \quad \text{and} \quad \tau = o(\delta), \tag{4.12}$$

we conclude that condition (4.8) and the conditions of Lemma 4.2 are satisfied and deduce from (4.10) that the asymptotic relation

$$F^{(1)}(2+\delta, \varepsilon, \varepsilon', \tau) = \varepsilon^4(1+o(1)), \quad \varepsilon \to 0, \tag{4.13}$$

is true.

Regarding (4.11), in the case when $q = \sqrt{\rho}(1-\kappa)$ and $1/\varepsilon' = \sqrt{\rho}(1-h)$, we conclude that in the limit of infinite ρ, the condition $\kappa < h$ is sufficient for the inequality $1/\varepsilon' < x_2$ to hold asymptotically. A simple computation shows that in this case

$$F^{(2)}(\sqrt{\rho}(1-\kappa), \varepsilon, \varepsilon', \tau) = \sqrt{\rho}\,(h-\kappa)^2(1+o(1)), \quad \rho \to \infty. \tag{4.14}$$

Let us study the minimum of $\gamma(x)$ over the interval $r \in (\varepsilon', 1-\varepsilon)$. The first observation is that if

$$\varepsilon > \frac{-(2\tau+\delta) + \sqrt{\delta^2 + 4\delta + 4\tau}}{2(1-\tau)}, \tag{4.15}$$

then the following analogue of (4.9) is verified:

$$\min_{0<x<1-\varepsilon} \gamma(x) > 2+\delta. \tag{4.16}$$

Lemma 4.3. *Let positive ε and ε' verify the inequality $\varepsilon' < 1-\varepsilon$. If q is greater than $2+\tau$ and such that $1-\varepsilon < x_1$, then*

$$G^{(1)}(q,\varepsilon,\varepsilon',\tau) = \inf_{\varepsilon'<x<1-\varepsilon} f(x) = f(1-\varepsilon) = \frac{\left(\varepsilon^2 - \delta(1+\varepsilon) - \tau(1+\varepsilon)^2\right)^2}{(1-\varepsilon)^3}. \tag{4.17}$$

If q is such that $x_1 < \varepsilon'$, then

$$G^{(2)}(q,\varepsilon,\varepsilon',\tau) = \inf_{\varepsilon'<x<1-\varepsilon} f(x) = f(\varepsilon') = \frac{\left((\varepsilon')^2(1-\tau) - q\varepsilon' + 1\right)^2}{(\varepsilon')^3}. \tag{4.18}$$

Proof. The proof of Lemma 4.3 is based on elementary computations that we do not present here. □

Regarding the asymptotic regime (4.12), we see that condition (4.15) and the conditions of Lemma 4.3 are verified. Then we conclude that

$$G^{(1)}(2+\delta,\varepsilon,\varepsilon',\tau) = \varepsilon^4(1+o(1)), \quad \varepsilon \to 0. \tag{4.19}$$

Regarding (4.18), in the case when $q = \sqrt{\rho}(1-\kappa)$ and $1/\varepsilon' = \sqrt{\rho}(1-h)$, we conclude that in the limit of infinite ρ the condition $\kappa < h$ is sufficient for the inequality $\varepsilon' > x_1$ to hold asymptotically. Then the relation

$$G^{(2)}\left(\sqrt{\rho}(1-\kappa),\varepsilon,\varepsilon',\tau\right) = \rho^{3/2}(1-h)(h-\kappa)^2(1+o(1)) \tag{4.20}$$

is true in the asymptotic regime (4.12).

4.3. Proof of Füredi–Komlós inequalities. For completeness, we reproduce the proof of inequalities (3.16) resulting from two lemmas below [15].

Lemma 4.4. *If $\tilde{A} = (a_{ij})$ is an $n \times n$ real symmetric matrix and $\breve{A} = \tilde{A} - tJ$ (where J is the matrix with all 1 entries), then*

$$\lambda_2(\tilde{A}) \le \lambda_1(\breve{A}).$$

Proof. The relation $\lambda_1(\tilde{A}) = \max_{\|\mathbf{x}\|=1} \mathbf{x}\tilde{A}\mathbf{x}$ and the Courant-Fischer theorem imply that

$$\lambda_2(\tilde{A}) = \min_{\mathbf{v}} \max_{(\mathbf{x},\mathbf{v})=0,\, \|\mathbf{x}\|=1} \mathbf{x}\tilde{A}\mathbf{x},$$

and therefore

$$\lambda_2(\tilde{A}) \le \max_{(\mathbf{x},\mathbf{1})=0,\, \|\mathbf{x}\|=1} \mathbf{x}\tilde{A}\mathbf{x} = \max_{(\mathbf{x},\mathbf{1})=0,\, \|\mathbf{x}\|=1} \mathbf{x}(\breve{A}+tJ)\mathbf{x}$$

$$= \max_{(\mathbf{x},\mathbf{1})=0,\, \|\mathbf{x}\|=1} \mathbf{x}\breve{A}\mathbf{x} \le \lambda_1(\breve{A})$$

since $(\mathbf{x},\mathbf{1}) = 0$ implies $J\mathbf{x} = \mathbf{0}$. Lemma 4.4 is proved. □

Lemma 4.5. *If $\tilde{A} = (a_{ij})$ is a real symmetric matrix and $\breve{A} = \tilde{A} - tJ$, $t > 0$, then*

$$\lambda_{-\infty}(\tilde{A}) \geq \lambda_{-\infty}(\breve{A}).$$

Proof. For $t > 0$, the matrix tJ is positive definite (i.e., $\mathbf{x}tJ\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbf{R}^n$), and hence

$$\lambda_{-\infty}(\tilde{A}) = \min_{\|\mathbf{x}\|=1} \mathbf{x}\tilde{A}\mathbf{x} \geq \min_{\|\mathbf{x}\|=1} \mathbf{x}\breve{A}\mathbf{x} + \min_{\|\mathbf{x}\|=1} \mathbf{x}tJ\mathbf{x}$$

$$\geq \min_{\|\mathbf{x}\|=1} \mathbf{x}\breve{A}\mathbf{x} = \lambda_{-\infty}(\breve{A}).$$

Lemma 4.5 is proved. □

Acknowledgments. The author would like to thank the anonymous referee for valuable remarks and discussion that helped me much to improve the paper.

References

[1] N. Alon, *Eigenvalues and expanders*, Combinatorica **6** (1986), 83–96.

[2] H. Bass, *The Ihara-Selberg zeta function of a tree lattice*, Internat. J. Math. **3** (1992), 717–797.

[3] F. Benaych-Georges, C. Bordenave, and A. Knowles, *Spectral radii of sparse random matrices*, Ann. Inst. Henri Poincaré Probab. Stat. **56** (2020), no. 3, 2141–2161.

[4] B. Bollobaś, *Random graphs*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2001.

[5] C. Bordenave, M. Lelarge,and L. Massoulié, *Nonbacktracking spectrum of random graphs: community detection and nonregular ramanujan graphs*, Ann. Probab. **46** (2018), 1–71.

[6] S. Boucheron, G. Lugosi, and P. Massart, *Concentration inequalities: a nonasymptotic theory of independence*, Oxford University Press, Oxford, 2013.

[7] D. Chafai, *Singular values of random matrices*, available from: https://djalil.chafai.net/docs/sing.pdf

[8] F.R.K. Chung, *Diameters and eigenvalues*, J. Amer. Math. Soc. **2** (1989), 187–196.

[9] S. Coste and Y. Zhu, *Eigenvalues of the non-backtracking operator detached from the bulk*, Random Matrices Theory Appl. **10** (2021), 2150028.

[10] P. Erdős and A. Rényi, *On random graphs. I*, Publ. Math. Debrecen **6** (1959), 290–297.

[11] *Expanding graphs* (Ed. J. Friedman), DIMACS Series on Discrete Math. and Theor. Comput. Sci., **10**, Amer. Math. Soc., Providence, RI, 1993.

[12] D. Foata and D. Zeilberger, *A combinatorial proof of Bass's evaluations of the Ihara-Selberg zeta function for graphs*, Trans. Amer. Math. Soc. **351** (1999), 2257–2274.

[13] J. Friedman, *On the second eigenvalue and random walks in random d-regular graphs*, Combinatorica **11** (1991), 331–362.

[14] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Mem. Amer. Math. Soc. **195** (2008), 910.

[15] Z. Füredi and J. Komlós, *The eigenvalues of random symmetric matrices*, Combinatorica **1** (1981), 233–241.

[16] K. Hashimoto, *Zeta functions of finite graphs and representation of p-adic groups*, Adv. Stud. Pure Math. **15** (1989), 211–280.

[17] R.A. Horn and C.R. Johnson, *Matrix analysis*, Cambridge University Press, 1985.

[18] M.D. Horton, H.M. Stark, and A.A. Terras, *What are zeta functions of graphs and what are they good for?* Contemporary Mathematics **415** (2006), 173–190.

[19] Y. Ihara, *On discrete subgroups of the two by two projective linear group over $\mathfrak{p}$-adic fields*, J. Math. Soc. Japan **18** (1966) 219–235.

[20] O. Khorunzhiy, *On eigenvalue distribution of random matrices of Ihara zeta function of large random graphs*, J. Math. Phys. Anal. Geom. **13** (2017), 268–282.

[21] O. Khorunzhiy, *On asymptotic properties of Bell polynomials and concentration of vertex degree of large random graphs*, J. Theor. Probab. **35** (2022), 20–51.

[22] M. Krivelevich and B. Sudakov, *The largest eigenvalue of sparse random graphs*, Combin. Probab. Comput. **12** (2003), 61–72.

[23] A. Lubotzky, R. Phillips and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), 261–277.

[24] A. Lubotzky, *Cayley graphs: eigenvalues, expanders and random walks*, Surveys in combinatorics, London Math. Soc. Lecture Notes **18** (1995), 155–189.

[25] G. Lugosi, S. Mendelson, and N. Zhivotovsky, *Concentration of the spectral norm of Erdős–Rényi random graphs*, Bernoulli, **26** (2020), 2253–2274.

[26] M. Ram Murty, *Ramanujan graphs*, J. Ramanujan Math. Soc. **18** (2003), 1–20.

[27] G. Quenell, *Spectral diameter estimates for k-regular graphs*, Advances in Math. **106** (1994) 122–148.

[28] H.M. Stark and A.A. Terras, *Zeta Functions of Finite Graphs and Coverings*, Adv. Math. **121** (1996), 124–165.

[29] T. Tao, *Eigenvalues and sums of Hermitian matrices*, 254A, Notes 3a, available from: https://terrytao.wordpress.com/tag/weyl-inequalities/.

[30] A. Terras, *Zeta functions of graphs: a stroll through the garden*, Cambridge Studies in Advanced Mathematics, **128**, Cambridge University Press, Cambridge, 2011.

[31] K. Wang and P.M. Wood, *Limiting empirical spectral distribution for the non-backtracking matrix of an Erdős–Rényi graph*, preprint, https://arxiv.org/abs/1710.11015

Received June 29, 2021, revised February 16, 2022.

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Асимптотична відсутність полюсів дзета функції Іхари для великих випадкових графів Ердоша–Реньї

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Скориставшись результатами про концентрацію найбільшого власного значення і максимального степеня вершини великих випадкових графів, ми доводимо, що нескінченна послідовність випадкових графів Ердоша–Реньї $G(n, \rho_n/n)$ така, що $\rho_n/\log n$ нескінченно зростає, коли $n \to \infty$, задовольняє версію гіпотези Рімана для теорії графів.

Ключові слова: випадкові графи, випадкові матриці, дзета функція Іхари, гіпотеза Рімана для теорії графів