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Solvability of Strongly Nonlinear Obstacle Parabolic Problems in Inhomogeneous Orlicz–Sobolev Spaces

Mohamed Bourahma, Jaouad Bennouna, Badr El Haji, and Abdelmoujib Benkirane

In this paper, we prove the existence result of solutions for the nonlinear unilateral problem associated to the parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) - \operatorname{div} \Phi(x,t,u) = \mu \quad \text{in } Q_T = \Omega \times (0,T),$$

where the lower order term Φ satisfies a generalized natural growth condition described by the appropriate Orlicz function Ψ, and the data μ is an integrable source term. No growth restrictions are assumed either on Ψ or on its complementary $\overline{\Psi}$. Therefore the solution is natural in this context.

Key words: unilateral parabolic problem, non-reflexive Orlicz spaces, natural growth

Mathematical Subject Classification 2010: 35K55, 35Q68, 35Q35

1. Introduction

In recent years, parabolic equations have got large applications that attract attention of many researchers in biology, image processing and electro-rheological fluids modeling.

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $Q_T = \Omega \times (0,T)$, where T is a positive real number and Ψ is an Orlicz function. Let $\mathrm{A} : D(\mathrm{A}) \subset W_0^{1,x} L_\Psi(Q_T) \to W^{-1,x} L_{\overline{\Psi}}(Q_T)$ be an operator of Leray–Lions type of the form

$$\mathrm{A}(u) := -\operatorname{div} a(x,t,u,\nabla u).$$

In this paper, we prove an existence theorem of entropy solutions in the setting of Orlicz spaces for the nonlinear unilateral parabolic problem associated to the following problem:

$$\frac{\partial u}{\partial t} + \mathrm{A}(u) - \operatorname{div} \Phi(x,t,u) = \mu \qquad \text{in } Q_T \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

where $u_0 \in L^1(\Omega)$, $\mu \in L^1(Q_T)$ and Φ satisfies the natural growth condition

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{\Psi}^{-1}(\Psi(|s|)) \tag{1.4}$$

with$\gamma \in E_{\overline{\Psi}}(Q_T)$.

In the classical Sobolev spaces, in some elliptic cases, Guibé et al. (see [8]) supposed on Φ the condition

$$|\Phi(x,s)| \leq c(x)\Big(1 + |s|\Big)^{p-1}. \tag{1.5}$$

In some parabolic cases (see [15]), they assumed the condition

$$|\Phi(x,t,s)| \leq c(x,t)\Big(1 + |s|^{\gamma}\Big) \tag{1.6}$$

with $\gamma = \frac{N+2}{N+p}(p-1)$ and $c \in L^r(Q_T)$ for $r > 0$.

Parabolic equations in Orlicz spaces have been widely studied since 2005 starting from the works of Meskine et al. (see [17, 18]). Later results were obtained, for instance, in the work of Moussa, Rhoudaf, and Mabdaoui (see [27]), where the existence of entropy solution for problem (1.1)–(1.3) was studied in the case $\mu \in L^1(Q_T)$ under the growth condition

$$|\Phi(x,t,s)| \leq \gamma(x,t) \cdot \overline{P}^{-1}(P(\delta|s|)), \tag{1.7}$$

where $\gamma \in L^\infty(Q_T)$ and $P \lll \Psi$.

For unilateral problems, see [5, 10, 25] and a later result by Rhoudaf et al. [31], where the existence of a solution for the unilateral problem associated to (1.1)–(1.3) was rigorously studied under the growth condition

$$|\Phi(x,t,s)| \leq \gamma(x,t) \cdot \overline{P}^{-1}(P(|s|)), \tag{1.8}$$

where $\gamma \in L^\infty(Q_T)$ and $P \lll \Psi$.

The main objective of this paper is how to deal with the existence of solutions for the obstacle problem associated to problem (1.1)–(1.3) in Orlicz spaces under a less restrictive assumption on the lower order term Φ, namely, where Φ verifies condition (1.4). We do not assume any restrictions either on the N-function Ψ or on its complementary $\overline{\Psi}$.

The imposed natural growth condition (1.4) on Φ leads to serious difficulties in proving the existence of approximate solutions and studying its convergence. These difficulties have been overcome by using the convexity of the N-function Ψ and Young's inequality on suitable quantities. Moreover, we use the very important observation that the norm convergence results from the modular convergence with every $\lambda > 0$ (see Lemma 2.3).

Let us briefly summarize the contents of this article. In Section 2, we collect some well-known preliminaries, results and properties of Orlicz–Sobolev spaces and inhomogeneous Orlicz–Sobolev spaces. Section 3 is devoted to basic assumptions, the problem setting and the proof of the main result.

2. Preliminaries

2.1. Orlicz–Sobolev spaces. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and convex function with

$$\Psi(t) > 0 \text{ for } t > 0, \quad \lim_{t\to 0} \frac{\Psi(t)}{t} = 0, \quad \text{and} \quad \lim_{t\to +\infty} \frac{\Psi(t)}{t} = +\infty.$$

The function Ψ is said to be an N-function or an Orlicz function. The N-function complementary to Ψ is defined as

$$\overline{\Psi}(t) = \sup \left\{ st - \Psi(s), s \geq 0 \right\}.$$

We recall that (see [1]),

$$\Psi(t) \leq t\overline{\Psi}^{-1}(\Psi(t)) \leq 2\Psi(t) \quad \text{for all } t \geq 0 \tag{2.1}$$

and the Young's inequality for all $s, t \geq 0$,

$$st \leq \overline{\Psi}(s) + \Psi(t).$$

We say that Ψ satisfies the Δ_2-condition if for some $k > 0$,

$$\Psi(2t) \leq k\Psi(t) \quad \text{for all } t \geq 0, \tag{2.2}$$

and if (2.2) holds only for $t \geq t_0$, then Ψ is said to satisfy the Δ_2-condition near infinity.

Let Ψ_1 and Ψ_2 be two N-functions. The notation $\Psi_1 \prec\prec \Psi_2$ means that Ψ_1 grows essentially less rapidly than Ψ_2, i.e.,

$$\forall \epsilon > 0 \quad \lim_{t\to\infty} \frac{\Psi_1(t)}{\Psi_2(\epsilon t)} = 0,$$

that is, the case if and only if

$$\lim_{t\to\infty} \frac{(\Psi_2)^{-1}(t)}{(\Psi_1)^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_\Psi(\Omega)$ (respectively, the Orlicz space $L_\Psi(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that

$$\int_\Omega \Psi(u(x))dx < \infty \quad \left(\text{respectively,} \int_\Omega \Psi\left(\frac{u(x)}{\lambda}\right)dx < \infty \text{ for some } \lambda > 0\right).$$

Endowed with the Luxemburg norm

$$\|u\|_\Psi = \inf \left\{ \lambda > 0 : \int_\Omega \Psi\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\},$$

and the so-called Orlicz norm, that is,

$$|||u|||_{\Psi,\Omega} = \sup_{\|v\|_{\overline{\Psi}} \leq 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

$L_\Psi(\Omega)$ is a Banach space and $K_\Psi(\Omega)$ is a convex subset of $L_\Psi(\Omega)$. The closure in $L_\Psi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\Psi(\Omega)$.

The Orlicz–Sobolev space $W^1L_\Psi(\Omega)$ (respectively, $W^1E_\Psi(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_\Psi(\Omega)$ (respectively, $E_\Psi(\Omega)$).

This is a Banach space under the norm

$$\|u\|_{1,\Psi} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_\Psi.$$

Thus, $W^1L_\Psi(\Omega)$ and $W^1E_\Psi(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_\Psi(\Omega)$. Denoting this product by ΠL_Ψ, we will use the weak topologies $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$ and $\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})$.

The space $W_0^1E_\Psi(\Omega)$ is defined as the norm closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1E_\Psi(\Omega)$ and the space $W_0^1L_\Psi(\Omega)$ as the $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_\Psi(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_\Psi(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega \Psi\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right) dx \to 0 \quad \text{for all } |\alpha| \le 1.$$

This implies the convergence for $\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})$.

If Ψ satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with the norm convergence. Recall that the norm $\|Du\|_\Psi$ defined on $W_0^1L_\Psi(\Omega)$ is equivalent to $\|u\|_{1,\Psi}$ (see [21]).

Let $W^{-1}L_{\overline{\Psi}}(\Omega)$ (respectively, $W^{-1}E_{\overline{\Psi}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{\Psi}}(\Omega)$ (respectively, $E_{\overline{\Psi}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open Ω has the segment property, then the space $\mathfrak{D}(\Omega)$ is dense in $W_0^1L_\Psi(\Omega)$ for the topology $\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})$ (see [21]). Consequently, the action of a distribution in $W^{-1}L_{\overline{\Psi}}(\Omega)$ on an element of $W_0^1L_\Psi(\Omega)$ is well defined. For more details one can see, for example, [1] or [26].

2.2. Inhomogeneous Orlicz–Sobolev spaces. Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q_T = \Omega \times (0,1)$. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \Omega$. The inhomogeneous Orlicz–Sobolev spaces are defined as follows:

$$W^{1,x}L_\Psi(Q_T) = \big\{u \in L_\Psi(Q_T) : D_x^\alpha u \in L_\Psi(Q_T) \quad \text{for all} \quad |\alpha| \le 1\big\}$$

and

$$W^{1,x}E_\Psi(Q_T) = \big\{u \in E_\Psi(Q_T) : D_x^\alpha u \in E_\Psi(Q_T) \quad \text{for all} \quad |\alpha| \le 1\big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\leq 1} \|D_x^\alpha u\|_{\Psi,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_\Psi(Q_T)$ which have as many copies as there are α-order derivatives, $|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$ and $\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}}))$. If $u \in W^{1,x}L_\Psi(Q_T)$, then the function : $t \mapsto u(t) = u(t,\cdot)$ is defined on $(0,T)$ with values in $W^1L_\Psi(\Omega)$. If, further, $u \in W^{1,x}E_\Psi(Q_T)$, then the concerned function is $W^1E_\Psi(\Omega)$-valued and strongly measurable. Furthermore, the following imbedding holds: $W^{1,x}E_\Psi(Q_T) \subset L^1(0,T;W^1E_\Psi(\Omega))$. The space $W^{1,x}L_\Psi(Q_T)$ is not in general separable. If $u \in W^{1,x}L_\Psi(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \|u(t)\|_{\Psi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\Psi(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_\Psi(Q_T)$ of $\mathfrak{D}(Q_T)$. It is proved that when Ω has the segment property, then each element u of the closure of $\mathfrak{D}(Q_T)$ with respect to the weak* topology $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$ is a limit, in $W^{1,x}L_\Psi(Q_T)$, of some subsequence $(u_n) \subset \mathfrak{D}(Q_T)$ for the modular convergence; i.e., if, for some $\lambda > 0$, such that for all $|\alpha| \leq 1$,

$$\int_{Q_T} \Psi\left(\frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\right) dx\,dt \to 0 \quad \text{as } n \to \infty.$$

This implies that (u_n) converges to u in $W^{1,x}L_\Psi(Q_T)$ for the weak topology $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$. Consequently,

$$\overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})} = \overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_\Psi, \Pi L_{\overline{\Psi}})}.$$

This space will be denoted by $W_0^{1,x}L_\Psi(Q_T)$. Furthermore,

$$W_0^{1,x}E_\Psi(Q_T) = W_0^{1,x}L_\Psi(Q_T) \cap \Pi E_\Psi.$$

We have then the following complementary system:

$$\left(W_0^{1,x}L_\Psi(Q_T), F, W_0^{1,x}E_\Psi(Q_T), F_0\right),$$

F being the dual space of $W_0^{1,x}E_\Psi(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{\Psi}}$ by the polar set $W_0^{1,x}E_\Psi(Q_T)^\perp$ denoted by $F = W^{-1,x}L_{\overline{\Psi}}(Q_T)$ and it is shown that

$$W^{-1,x}L_{\overline{\Psi}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{\Psi}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{\Psi},Q_T},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{\Psi}}(Q_T).$$

The space F_0 is then given by

$$W^{-1,x} L_{\overline{\Psi}}(Q_T) = \left\{ f = \sum_{|\alpha| \leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{\Psi}}(Q_T) \right\}$$

and is denoted by $F_0 = W^{-1,x} E_{\overline{\Psi}}(Q_T)$.

Lemma 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let Ψ, P and Q be N-functions such that $Q \lll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} \Psi(k_2 |s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from

$$P\left(E_\Psi, \frac{1}{k_2}\right) = \left\{ u \in L_\Psi(\Omega) : d(u, E_\Psi(\Omega)) < \frac{1}{k_2} \right\}$$

into $E_Q(\Omega)$.

Lemma 2.2 ([22])**.** *Let $u_k, u \in L_\Psi(\Omega)$. If $u_k \to u$ for the modular convergence, then $u_k \to u$ for $\sigma(L_\Psi, L_{\overline{\Psi}})$.*

Lemma 2.3. *If $u_n \to u$ for the modular convergence with every $\lambda > 0$ in $L_\Psi(\Omega)$, then $u_n \to u$ strongly in $L_\Psi(\Omega)$.*

Proof. We will use the Orlicz norm, for all $\lambda > 0$ we have

$$\int_\Omega \Psi\left(\frac{|u_k(x) - u(x)|}{\lambda} \right) dx \to 0 \quad \text{as } k \to \infty.$$

Thus $\Psi\left(\frac{|u_k(x)-u(x)|}{\lambda}\right)$ tends to 0 strongly in $L^1(\Omega)$ and so for a subsequence, still indexed by k, we can assume that $u_k \to u$ a.e. in Ω. For an arbitrary $v \in L_{\overline{\Psi}}(\Omega)$, there exists $\lambda_v > 0$ such that $\overline{\Psi}\left(\frac{v}{\lambda_v}\right) \in L^1(\Omega)$. By Young's inequality and the convexity of $\overline{\Psi}$, we can write

$$|(u_k(x) - u(x))v(x)| \leq \Psi\left(2\lambda_v |u_k(x) - u(x)|\right) + \frac{1}{2} \overline{\Psi}\left(\frac{v(x)}{\lambda_v} \right).$$

Applying Vitali's theorem, we obtain

$$\int_\Omega |(u_k(x) - u(x))v(x)|\, dx \to 0 \quad \text{for all } v \in L_{\overline{\Psi}}(\Omega)$$

and so

$$|||u_k - u|||_{\Psi,\Omega} = \sup_{\|v\|_{\overline{\Psi}} \leq 1} \int_\Omega |(u_k(x) - u(x))v(x)|\, dx \to 0 \text{ as } k \to \infty,$$

which yields the result. □

Lemma 2.4 ([21]). *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian, with $F(0)=0$. Let Ψ be an Orlicz function and let $u\in W^1L_\Psi(\Omega)$ (respectively, $W^1E_\Psi(\Omega)$). Then $F(u)\in W^1L_\Psi(\Omega)$ (respectively, $W^1E_\Psi(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(u)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x\in\Omega: u(x)\notin D\}\\ 0 & a.e.\ in\ \{x\in\Omega: u(x)\in D\}.\end{cases}$$

Lemma 2.5 ([21]). *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$, and let Ψ be an Orlicz function. We also assume that the set of discontinuity points D of F' is finite. Then the mapping $F: W^1L_\Psi(\Omega)\to W^1L_\Psi(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_\Psi,\Pi E_{\overline{\Psi}})$.*

Lemma 2.6 ([18]). *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, satisfying the segment property. Then*

$$\left\{u\in W_0^{1,x}L_\Psi(Q_T):\frac{\partial u}{\partial t}\in W^{-1,x}L_{\overline{\Psi}}(Q_T)+L^1(Q_T)\right\}\subset\mathcal{C}\big([0,T],L^1(\Omega)\big).$$

Lemma 2.7 (Integral Poincaré's type inequality in inhomogeneous Orlicz spaces [21]). *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let Ψ be an Orlicz function. Then there exist two positive constants $\delta,\lambda>0$ such that*

$$\int_{Q_T}\Psi(\delta|u(x,t)|)\,dx\,dt\leq\int_{Q_T}\lambda\Psi(|\nabla u(x,t)|)\,dx\,dt\quad \text{for all } u\in W_0^1L_\Psi(Q_T).$$

Lemma 2.8 ([24, Theorem 13.47]). *If $f_n\subset L^1(\Omega)$ with $f_n\to f\in L^1(\Omega)$ a.e. in Ω, $f_n,f\geq 0$ a. e. in Ω and $\int_\Omega f_n(x)\,dx\to\int_\Omega f(x)\,dx$, then $f_n\to f$ in $L^1(\Omega)$.*

Lemma 2.9 ([22]). *Suppose that Ω satisfies the segment property and let $u\in W_0^1L_\Psi(\Omega)$. Then there exists a sequence $(u_n)\subset\mathfrak{D}(\Omega)$ such that $u_n\to u$ for the modular convergence in $W_0^1L_\Psi(\Omega)$. Furthermore, if $u\in W_0^1L_\Psi(\Omega)\cap L^\infty(\Omega)$, then*

$$\|u_n\|_\infty\leq\ (N+1)\|u\|_\infty.$$

Lemma 2.10 (cf. [17]). *Let Ψ be an N-function. Let (u_n) be a sequence of $W^{1,x}L_\Psi(Q_T)$ such that $u_n\rightharpoonup u$ weakly in $W^{1,x}L_\Psi(Q_T)$ for $\sigma(\Pi L_\Psi,\Pi E_{\overline{\Psi}})$ and $\frac{\partial u_n}{\partial t}=h_n+k_n$ in $\mathfrak{D}'(Q_T)$ with h_n being bounded in $W^{-1,x}L_{\overline{\Psi}}(Q_T)$ and k_n being bounded in $L^1(Q_T)$. Then $u_n\to u$ strongly in $L^1_{Loc}(Q_T)$. If, further, $u_n\in W_0^{1,x}L_\Psi(Q_T)$, then $u_n\to u$ strongly in $L^1(Q_T)$.*

3. Basic assumptions and main result

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, satisfying the segment property, and let Ψ be an Orlicz function. Consider the following convex set:

$$\mathbf{K}_\psi=\left\{u\in W_0^{1,x}L_\Psi(Q_T):u\geq\psi\text{ a.e. in }Q_T\right\},\tag{3.1}$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Define the set

$$\mathcal{T}_0^{1,\Psi}(Q_T) := \Big\{ u : Q_T \to \mathbb{R} : u \text{ is measurable and } T_k(u) \in W_0^{1,x} L_\Psi(Q_T) \Big\}.$$

On the convex $\mathbf{K}_\psi$, we assume that

(C₁) $\psi^+ \in W_0^{1,x} L_\Psi(Q_T) \cap L^\infty(Q_T)$,

(C₂) for each $v \in \mathbf{K}_\psi \cap L^\infty(Q_T)$, there exists a sequence $\{v_j\} \subset \mathbf{K}_\psi \cap W_0^{1,x} E_\Psi(Q_T) \cap L^\infty(Q_T)$ such that $v_j \to v$ for the modular convergence,

(C₃) $\mathbf{K}_\psi \cap L^\infty(Q_T) \neq \varnothing$.

Let $\mathrm{A} : D(\mathrm{A}) \subset W_0^{1,x} L_\Psi(Q_T) \to W^{-1,x} L_{\overline{\Psi}}(Q_T)$ be an operator of Leray–Lions type of the form

$$\mathrm{A}(u) := -\operatorname{div} a(x,t,u,\nabla u).$$

This work aims to prove the existence of entropy solutions in the setting of Orlicz spaces for the nonlinear problem

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) - \operatorname{div} \Phi(x,t,u) = \mu \qquad \text{in } Q_T \tag{3.2}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega \tag{3.3}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{3.4}$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t) \in Q_T$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$, the following conditions:

(H₁) There exists a function $c(x,t) \in E_{\overline{\Psi}}(Q_T)$ and some positive constants k_1, k_2, k_3, ζ and an Orlicz function $P \prec\!\prec \Psi$ such that

$$|a(x,t,s,\xi)| \leq \zeta\big[c(x,t) + k_1 \overline{\Psi}^{-1}(P(k_2|s|)) + \overline{\Psi}^{-1}(\Psi(k_3|\xi|))\big].$$

(H₂) a is strictly monotone

$$\big(a(x,t,s,\xi) - a(x,t,s,\eta)\big) \cdot \big(\xi - \eta\big) > 0.$$

(H₃) a is coercive, there exists a constant $\beta > 0$ such that

$$a(x,t,s,\xi) \cdot \xi \geq \beta \Psi(|\xi|).$$

For the lower order term, we assume $\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ to be a Caratheodory function satisfying:

(H₄) For all $s \in \mathbb{R}$ and for almost every $x \in \Omega$,

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{\Psi}^{-1}(\Psi(|s|)),$$

where $\gamma \in E_{\overline{\Psi}}(Q_T)$.

(H₅) $\mu \in L^1(Q_T)$, u_0 is an element of $L^1(\Omega)$.

Lemma 3.1 ([27]). *Under assumptions* $(\mathbf{H_1})$–$(\mathbf{H_3})$*, let* (f_n) *be a sequence in* $W_0^{1,x}L_\Psi(Q_T)$ *such that*

$$f_n \rightharpoonup f \quad \text{in } W_0^{1,x}L_\Psi(Q_T) \text{ for } \sigma(\Pi L_\Psi(Q_T), \Pi E_{\overline{\Psi}}(Q_T)),$$
$$\Big(a(x,t,f_n,\nabla f_n)\Big)_n \quad \text{is bounded in } \Big(L_{\overline{\Psi}}(Q_T)\Big)^N,$$
$$\lim_{n,s\to\infty}\int_{Q_T}\Big(a(x,t,f_n,\nabla f_n)-a(x,t,f_n,\nabla f\chi_s)\Big)\cdot\Big(\nabla f_n-\nabla f\chi_s\Big)\,dx\,dt=0,$$

where χ_s *denotes the characteristic function of the set* $\Omega_s=\Big\{x\in\Omega:|\nabla f|\le s\Big\}$*. Then*

$$\nabla f_n \to \nabla f \quad \text{a.e. in } Q_T,$$
$$\lim_{n\to\infty}\int_{Q_T}a(x,t,f_n,\nabla f_n)\nabla f_n\,dx\,dt=\int_{Q_T}a(x,t,f,\nabla f)\nabla f\,dx\,dt,$$
$$\Psi(|\nabla f_n|)\to\Psi(|\nabla f|) \quad \text{in } L^1(Q_T).$$

In what follows, we will use the real function of a real variable, called the truncation at height $k>0$,

$$T_k(s)=\max\Big(-k,\min(k,s)\Big)=\begin{cases} s & \text{if } |s|\le k\\ k\frac{s}{|s|} & \text{if } |s|>k,\end{cases}$$

and its primitive is defined by

$$\widetilde{T}_k(s)=\int_0^s T_k(t)\,dt.$$

Note that $\widetilde{T}_k$ have the properties: $\widetilde{T}_k(s)\ge 0$ and $\widetilde{T}_k(s)\le k|s|$.

Definition 3.2. A measurable function u defined on Q_T is said to be a solution for the obstacle problem associated to (3.2)–(3.4) if $u\in\mathcal{T}_0^{1,\Psi}(Q_T)$ with $u\ge\psi$ a.e in Q_T and $\widetilde{T}_k(u(\cdot,t))\in L^1(\Omega)$ for every $t\in[0,T]$. Thus we have

$$\begin{aligned}\int_\Omega \widetilde{T}_k(u-v)\,dx+\Big\langle\frac{\partial v}{\partial t},T_k(u-v)\Big\rangle_{Q_\tau}+\int_{Q_\tau}a(x,t,u,\nabla u)\nabla T_k(u-v))\,dx\,dt\\ +\int_{Q_\tau}\Phi(x,t,u)\nabla T_k(u-v))\,dx\,dt\\ \le\int_{Q_\tau}\mu T_k(u-v))\,dx\,dt+\int_\Omega\widetilde{T}_k(u_0-v(0))\,dx,\end{aligned}\tag{3.5}$$

and

$$u(x,0)=u_0(x)\quad\text{for a.e } x\in\Omega,\tag{3.6}$$

for every $\tau\in[0,T]$, $k>0$ and for all $v\in W_0^{1,x}L_\Psi(Q_T)\cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t}\in W^{-1,x}L_{\overline{\Psi}}(Q_T)+L^1(Q_T)$, $\widetilde{T}_k(u(\cdot,t))\in L^1(\Omega)$ is the primitive function of the truncation function T_k defined above.

The main result of this paper is the following theorem.

Theorem 3.3. *Suppose that assumptions* **(C$_1$)**–**(C$_3$)** *and* **(H$_1$)**–**(H$_5$)** *hold true and $\mu \in L^1(Q_T)$. Then there exists at least one solution for problem* (3.2)–(3.4) *in the sense of definition* 3.2.

Proof. The proof of the above theorem is divided into four steps.

Step 1: Approximate problems. Let μ_n be a sequence of regular functions in $\mathcal{C}_0^\infty(Q_T)$ which converges strongly to μ in $L^1(Q_T)$ and such that $\|\mu_n\|_{L^1} \leq \|\mu\|_{L^1}$. For each $n \in \mathbb{N}^*$, put

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \text{ a.e } (x,t) \in Q_T, \quad s \in \mathbb{R},\ \xi \in \mathbb{R}^N,$$

and

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e } (x,t) \in Q_T, \forall s \in \mathbb{R}.$$

And let $u_{0n} \in \mathcal{C}_0^\infty(\Omega)$ such that

$$\|u_{0n}\|_{L^1} \leq \|u_0\|_{L^1} \quad \text{and} \quad u_{0n} \to u_0 \text{ in } L^1(\Omega).$$

Consider the following approximate problem:

$$u_n \in \mathbf{K}_\psi \tag{3.7}$$

$$\frac{\partial u_n}{\partial t} - \operatorname{div} a(x,t,u_n,\nabla u_n) - \operatorname{div} \Phi_n(x,t,u_n) = \mu_n \quad \text{in } Q_T \tag{3.8}$$

$$u_n(x,t=0) = u_{0n} \quad \text{in } \Omega \tag{3.9}$$

$$u_n = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{3.10}$$

Let $z_n(x,t,u_n,\nabla u_n) = a_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)$, which satisfies (A_1)–(A_4) of [23]. It remains to prove (A_4). For this end, we use Young's inequality technically as follows:

$$\begin{aligned}
|\Phi_n(x,t,u_n)\nabla u_n| &\leq |\gamma(x,t)||\nabla u_n| + \overline{\Psi}^{-1}(\Psi(|T_n(u_n)|))|\nabla u_n| \\
&= \frac{\beta^2}{\beta+2}\frac{\beta+2}{\beta^2}|\gamma(x,t)||\nabla u_n| \\
&\quad + \frac{\beta+1}{\beta}\overline{\Psi}^{-1}(\Psi(|T_n(u_n)|))\frac{\beta}{\beta+1}|\nabla u_n| \\
&\leq \frac{\beta^2}{\beta+2}\left(\overline{\Psi}\left(\frac{\beta+2}{\beta^2}|\gamma(x,t)|\right) + \Psi(|\nabla u_n|)\right) \\
&\quad + \overline{\Psi}\left(\frac{\beta+1}{\beta}\overline{\Psi}^{-1}(\Psi(|T_n(u_n)|))\right) + \Psi\left(\frac{\beta}{\beta+1}|\nabla u_n|\right).
\end{aligned}$$

While $\frac{\beta}{\beta+1} < 1$, using the convexity of Ψ and the fact that $\overline{\Psi}$ and $\overline{\Psi}^{-1} \circ \Psi$ are increasing functions, one has

$$|\Phi_n(x,t,u_n)\nabla u_n| \leq \frac{\beta^2}{\beta+2}\overline{\Psi}\left(\frac{\beta+2}{\beta^2}|\gamma(x,t)|\right) + \frac{\beta^2}{\beta+2}\Psi(|\nabla u_n|)$$

$$+\overline{\Psi}\left(\frac{\beta+1}{\beta}\overline{\Psi}^{-1}(\Psi(n))\right)+\frac{\beta}{\beta+1}\Psi\big(|\nabla u_n|\big).$$

Since $\gamma \in E_{\overline{\Psi}}(Q_T)$, $\overline{\Psi}\Big(\frac{\beta+2}{\beta^2}|\gamma(x,t)|\Big) \in L^1(\Omega)$, then we get

$$\Phi_n(x,t,u_n)\nabla u_n \geq -\Big(\frac{\beta^2}{\beta+2}+\frac{\beta}{\beta+1}\Big)\Psi\Big(|\nabla u_n|\Big)-C_n-F,$$

where F is a fixed L^1-function. Using this last inequality and $\mathbf{(H_3)}$, we obtain

$$\begin{aligned}z_n(x,t,u_n,\nabla u_n)\nabla u_n &\geq \Big(\beta-\frac{\beta^2}{\beta+2}-\frac{\beta}{\beta+1}\Big)\Psi\Big(|\nabla u_n|\Big)-C_n-F\\ &\geq \frac{\beta^2}{(\beta+1)(\beta+2)}\Psi\Big(|\nabla u_n|\Big)-F.\end{aligned}$$

Thus, from [18], the approximate problem (3.7)–(3.10) has at least one weak solution $u_n \in W_0^{1,x}L_\Psi(Q_T)$.

Step 2: A priori estimates. We prove some results which will be used later.

Proposition 3.4. *Suppose that assumptions* $\mathbf{(C_1)}$*–*$\mathbf{(C_3)}$ *and* $\mathbf{(H_1)}$*–*$\mathbf{(H_5)}$ *hold true and let* $(u_n)_n$ *be a solution of the approximate problem* (3.7)*–*(3.10)*. Then, for all* $k>0$*, there exists a constant* C_k*, not depending on* n*, such that*

$$\|T_k(u_n)\|_{W_0^{1,x}L_\Psi(Q_T)} \leq C_k \tag{3.11}$$

and

$$\lim_{k\to\infty}\text{meas}\big\{(x,t)\in Q_T : |u_n|>k\big\}=0. \tag{3.12}$$

Proof. First, by $\mathbf{(C_1)}$–$\mathbf{(C_3)}$, there exists $v_0 \in \mathbf{K}_\psi \cap L^\infty(Q_T)\cap W_0^{1,x}E_\Psi(Q_T)$. Testing the approximate problem (3.7)–(3.10) by $v=u_n-T_k(u_n-v_0)$, one has for every $\tau\in(0,T)$,

$$\begin{aligned}&\left\langle \frac{\partial u_n}{\partial t},(u_n-v_0)\right\rangle_{Q_\tau}+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx\,dt\\ &\quad+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla T_k(u_n-v_0)\,dx\,dt=\int_{Q_\tau}\mu_n T_k(u_n-v_0)\,dx\,dt.\end{aligned} \tag{3.13}$$

It follows that

$$\begin{aligned}\int_\Omega \widetilde{T}_k(u_n-v_0)(\tau)\,dx &+ \Big\langle\frac{\partial v_0}{\partial t},T_k(u_n-v_0)\Big\rangle_{Q_\tau}\\ &+\int_{Q_\tau}a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n-v_0))\,dx\,dt\\ &+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla T_k(u_n-v_0))\,dx\,dt\end{aligned}$$

$$\leq \int_{Q_\tau} \mu_n T_k(u_n - v_0))\,dx\,dt + \int_\Omega \widetilde{T}_k(u_{n0} - v_0(0))\,dx.$$

We have

$$\widetilde{T}_k(u_n - v_0)(\tau) \geq 0,$$

$$\int_\Omega \widetilde{T}_k(u_{n0} - v_0(0))\,dx \leq \int_\Omega k\mid(u_{n0} - v_0(0))\mid\,dx \leq kC_1,$$

$$\Big\langle \frac{\partial v_0}{\partial t}, T_k(u_n - v_0)\Big\rangle_{Q_\tau} \leq kC_2,$$

$$\int_{Q_T} \mu_n T_k(u_n - v_0)\,dx\,dt \leq k\|\mu\|_{L^1(Q_T)} \leq kC_3.$$

Seeing that $\Phi_n(x,t,u_n)\nabla T_k(u_n)$ is different from zero only on the set $\{|u_n| \leq k\}$, where $T_k(u_n) = u_n$, we have

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v_0)\,dx\,dt\\
&\qquad \leq \int_{\{|u_n - v_0| \leq k\}} |\Phi(x,t,T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\,dx\,dt\\
&\qquad\quad + \int_{\{|u_n - v_0| \leq k\}} |\Phi(x,t,T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\,dx\,dt + kC_4.
\end{aligned} \tag{3.14}$$

From $(\mathbf{H_4})$ and then Young's inequality for an arbitrary $\beta > 0$ (the constant of coercivity), using the convexity of Ψ with $\frac{\beta}{2(\beta+2)} < 1$, we have

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v_0)\,dx\,dt\\
&\leq \int_{\{|u_n - v_0| \leq k\}} \frac{2(\beta+2)}{\beta}\Big(\gamma(x,t) + \overline{\Psi}^{-1}(\Psi(|T_{k+\|v_0\|_\infty}(u_n)|))\Big)\frac{\beta}{2(\beta+2)}|\nabla u_n|\,dx\,dt\\
&\quad + \int_{\{|u_n - v_0| \leq k\}} \Big(\gamma(x,t) + \overline{\Psi}^{-1}(\Psi(|T_{k+\|v_0\|_\infty}(u_n)|))\Big)|\nabla v_0|\,dx\,dt + kC_4\\
&\leq \frac{\beta}{2(\beta+2)}\int_{\{|u_n - v_0| \leq k\}} \Psi(|\nabla u_n|)\,dx\,dt + C_5(k,\beta)
\end{aligned} \tag{3.15}$$

since $\gamma \in E_{\overline{\Psi}}(Q_T)$, $(\nabla v_0) \in (L_\Psi(\Omega))^N$. Furthermore, we can write

$$\begin{aligned}
&\int_{\{|u_n - v_0| \leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\qquad \leq \frac{\beta}{\beta+1}\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\frac{\beta+1}{\beta}\nabla v_0\,dx\,dt\\
&\qquad\quad + \frac{\beta}{2(\beta+2)}\int_{\{|u_n - v_0| \leq k\}} \Psi(|\nabla u_n|)\,dx\,dt + C_6(k,\beta).
\end{aligned} \tag{3.16}$$

Use now $(\mathbf{H_2})$ to evaluate the second term in (3.16),

$$\frac{\beta}{\beta+1}\int_{\{|u_n - v_0| \leq k\}} a(x,t,u_n,\nabla u_n)\frac{\beta+1}{\beta}\nabla v_0\,dx\,dt$$

$$\leq \frac{\beta}{\beta+1}\left(\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\right.$$
$$\left.-\int_{\{|u_n-v_0|\leq k\}} a\left(x,t,u_n,\frac{\beta+1}{\beta}\nabla v_0\right)\left(\nabla u_n-\frac{\beta+1}{\beta}\nabla v_0\right)dx\,dt\right). \quad (3.17)$$

Hence, (3.16) becomes

$$\begin{aligned}\left(1-\frac{\beta}{\beta+1}\right)&\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &\leq \int_{\{|u_n-v_0|\leq k\}}\left|a\left(x,t,u_n,\frac{\beta+1}{\beta}\nabla v_0\right)\right|\left|\frac{\beta+1}{\beta}\nabla v_0\right|dx\,dt\\ &\quad+\int_{\{|u_n-v_0|\leq k\}}\left|a\left(x,t,u_n,\frac{\beta+1}{\beta}\nabla v_0\right)\right||\nabla u_n|\,dx\,dt\\ &\quad+\frac{\beta}{2(\beta+2)}\int_{\{|u_n-v_0|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt+C_7(k,\beta). \qquad (3.18)\end{aligned}$$

Using again Young's inequality as in (3.16) for the third term of (3.18) and using $(\mathbf{H_1})$, we get

$$\begin{aligned}\left(1-\frac{\beta}{\beta+1}\right)&\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\ &\leq \frac{\beta}{2(\beta+2)}\int_{\{|u_n-v_0|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\\ &\quad+\frac{\beta}{2(\beta+2)}\int_{\{|u_n-v_0|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt+C_8(k,\beta). \qquad (3.19)\end{aligned}$$

Thanks to $(\mathbf{H_3})$, it follows that

$$\left(\beta(1-\frac{\beta}{\beta+1})-\frac{\beta}{\beta+2}\right)\int_{\{|u_n-v_0|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\leq C_9(k,\beta). \qquad (3.20)$$

Since $\left(\beta(1-\frac{\beta}{\beta+1})-\frac{\beta}{\beta+2}\right)=\frac{\beta}{\beta+1}-\frac{\beta}{\beta+2}>0$, we have

$$\int_{\{|u_n-v_0|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\leq C(k,\beta). \qquad (3.21)$$

Finally, since $\{|u_n|\leq k\}\subset\{|u_n-v_0|\leq k+\|v_0\|_\infty\}$, one has

$$\begin{aligned}\int_{Q_T}\Psi\big(|\nabla T_k(u_n)|\big)\,dx\,dt&\leq\int_{\{|u_n|\leq k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\\ &\leq\int_{\{|u_n-v_0|\leq k+\|v_0\|_\infty\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\leq C(k,\beta). \qquad (3.22)\end{aligned}$$

To prove (3.12), from (3.22), we have

$$\int_{Q_T}\Psi\big(|\nabla T_k(u_n)|\big)\,dx\,dt\leq C(k,\beta).$$

If $C(k,\beta)\le 1$, by Poincaré's inequality, there exists $\lambda>0$ and δ such that

$$\int_{Q_T}\Psi\big(\delta|T_k(u_n)|\big)\,dx\,dt\le\lambda\int_{Q_T}\Psi\big(|\nabla T_k(u_n)|\big)\,dx\,dt,$$

then for all $n,k>0$, we obtain

$$\begin{aligned}\text{meas}\big\{|u_n|>k\big\}&=\frac{1}{\Psi(\delta k)}\int_{\{|u_n|>k\}}\Psi\big(\delta|T_k(u_n)|\big)\,dx\,dt\\&\le\frac{1}{\Psi(\delta k)}\int_{Q_T}\Psi\big(\delta|T_k(u_n)|\big)\,dx\,dt\\&\le\frac{\lambda}{\Psi(\delta k)}\int_{Q_T}\Psi\big(|\nabla T_k(u_n)|\big)\,dx\,dt\\&\le\frac{\lambda}{\Psi(\delta k)}\to 0\quad\text{as } k\to\infty.\end{aligned}\tag{3.23}$$

If $C(k,\beta)\ge 1$ and $\frac{1}{C(k,\beta)}\le 1$, using $P\lll\Psi$ appearing in assumption ($\mathbf{H_1}$), which implies that for all $\epsilon>0$, there exists a constant d_ϵ such that $P(t)\le\Psi(\epsilon t)+d_\epsilon$. Using again Poincaré's inequality, we obtain for $\epsilon<\frac{1}{C(k,\beta)}\le 1$ and for all $n,k>0$,

$$\begin{aligned}\text{meas}\big\{|u_n|>k\big\}&=\frac{1}{P(\delta k)}\int_{\{|u_n|>k\}}P\big(\delta|T_k(u_n)|\big)\,dx\,dt\\&\le\frac{1}{P(\delta k)}\int_{Q_T}\big(\Psi\big(\epsilon\delta|T_k(u_n)|\big)+d_\epsilon\big)\,dx\,dt\\&\le\frac{1}{P(\delta k)}\left(\frac{1}{C(k,\beta)}\int_{Q_T}\Psi\big(\delta|T_k(u_n)|\big)\,dx\,dt+d_\epsilon|Q_T|\right)\\&\le\frac{\lambda}{P(\delta k)}\left(\frac{1}{C(k,\beta)}\int_{Q_T}\Psi\big(|\nabla T_k(u_n)|\big)\,dx\,dt+d_\epsilon|Q_T|\right)\\&\le\frac{\lambda(1+d_\epsilon|Q_T|)}{P(\delta k)}\to 0\quad\text{as } k\to\infty.\end{aligned}\tag{3.24}$$

The proposition is proved. □

Lemma 3.5. *Let u_n be a solution of the approximate problem (3.7)–(3.10). Then:*

(i) $u_n\to u$ *a.e. in* Q_T,

(ii) $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ *is bounded in* $(L_{\overline{\Psi}}(Q_T))^N$.

Proof. To prove **(i)**, we proceed as in [27,30]. Taking a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that

$$\Gamma_k(s)=\begin{cases}s & \text{for } |s|\le\frac{k}{2}\\ k & \text{for } |s|\ge k\end{cases}$$

and multiplying the approximate problem (3.7)–(3.10) by $\Gamma_k'(u_n)$, we obtain

$$\frac{\partial\Gamma_k(u_n)}{\partial t}-\text{div}\Big(a(x,t,u_n,\nabla u_n)\Gamma_k'(u_n)\Big)+a(x,t,u_n,\nabla u_n)\Gamma_k''(u_n)\nabla u_n$$

$$-\operatorname{div}\Big(\Gamma_k'(u_n)\Phi_n(x,t,u_n)\Big)+\Gamma_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n=\mu_n\Gamma_k'(u_n).$$

Remarking that $\overline{\Psi}^{-1}\circ\Psi$ is an increasing function, $\gamma\in E_{\overline{\Psi}}(Q_T)$, $\operatorname{supp}(\Gamma_k')$, $\operatorname{supp}(\Gamma_k'')\subset[-k,k]$, by using Young's inequality, we get

$$\begin{aligned}
&\left|\int_{Q_T}\Gamma_k'\Phi_n(x,t,u_n)\,dx\,dt\right|\\
&\le\|\Gamma_k'\|_{L^\infty}\left(\int_{Q_T}|\gamma(x,t)|\,dx\,dt+\int_{Q_T}\overline{\Psi}^{-1}\big(\Psi(|T_k(u_n)|)\big)\,dx\,dt\right)\\
&\le\|\Gamma_k'\|_{L^\infty}\left(\int_{Q_T}\big(\overline{\Psi}(|\gamma(x,t)|)+\Psi(1)\big)\,dx\,dt+\int_{Q_T}\overline{\Psi}^{-1}\big(\Psi(k)\big)\,dx\,dt\right)<C_{1,k},
\end{aligned}$$

and (here, we use also (3.22))

$$\begin{aligned}
&\left|\int_{Q_T}\Gamma_k''\Phi_n(x,t,u_n)\nabla u_n\,dx\,dt\right|\\
&\le\|\Gamma_k''\|_{L^\infty}\left(\int_{Q_T}|\gamma(x,t)|\,dx\,dt+\int_{Q_T}\overline{\Psi}^{-1}(\Psi(|T_k(u_n)|))|\nabla T_k(u_n)|\,dx\,dt\right)\\
&\le\|\Gamma_k''\|_{L^\infty}\left(\int_{Q_T}\big(\overline{\Psi}(|\gamma(x,t)|)+\Psi(1)\big)\,dx\,dt+\int_{Q_T}\Psi(k)\,dx\,dt\right.\\
&\qquad\left.+\int_{Q_T}\Psi(|\nabla T_k(u_n)|)\,dx\,dt\right)<C_{2,k},
\end{aligned}\tag{3.25}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Then all above implies that

$$\frac{\partial\Gamma_k(u_n)}{\partial t}\quad\text{is bounded in } L^1(Q_T)+W^{-1,x}L_{\overline{\Psi}}(Q_T).\tag{3.26}$$

Hence, by Lemma 2.10 and using the same techniques as in [29], we can deduce that there exists a measurable function $u\in L^\infty(0,T;L^1(\Omega))$ such that

$$u_n\to u\quad\text{a.e. in } Q_T,$$

and for every $k>0$,

$$T_k(u_n)\rightharpoonup T_k(u)\quad\text{weakly in } W^{1,x}L_\Psi(Q_T)\text{ for }\sigma(\Pi L_\Psi,\Pi E_{\overline{\Psi}})\tag{3.27}$$

and

$$T_k(u_n)\to T_k(u)\quad\text{strongly in } L^1(Q_T)\text{ and a.e. in } Q_T.\tag{3.28}$$

For **(ii)**, we use the Banach–Steinhaus theorem. Let $\phi\in(E_\Psi(Q_T))^N$ be an arbitrary function. From (H_2), we can write

$$\big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\phi)\big)\big(\nabla T_k(u_n)-\phi\big)\ge0$$

which gives

$$\int_{Q_T}a(x,t,T_k(u_n),\nabla T_k(u_n))\phi\,dx\,dt$$

$$\begin{aligned}&\le \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt\\&\quad+\int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx\,dt.\end{aligned}\tag{3.29}$$

Let us denote by J_1 and J_2 the first and the second integrals in the right-hand side of (3.29) so that

$$J_1=\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt.$$

Going back to (3.19), it is seen that

$$\begin{aligned}&\left(1-\frac{\beta}{\beta+1}\right)\int_{\{|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\&\qquad\le \frac{\beta}{2(\beta+2)}\int_{\{|u_n-v_0|\le k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt\\&\qquad\quad+\frac{\beta}{2(\beta+2)}\int_{\{|u_n-v_0|\le k\}}\Psi\big(|\nabla u_n|\big)\,dx\,dt+C_8(k,\beta).\end{aligned}$$

By (3.22), there exists a positive constant C_{J_1} independent of n such that

$$J_1\le C_{J_1}.\tag{3.30}$$

Now we estimate the integral J_2. To this end, remark that

$$\begin{aligned}J_2&=\int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx\,dt\\&\le\int_{Q_T}|a(x,t,T_k(u_n),\phi)||\phi|\,dx\,dt+\int_{Q_T}|a(x,t,T_k(u_n),\phi)||\nabla T_k(u_n)|\,dx\,dt.\end{aligned}$$

In addition, let η be large enough. From **($\mathbf{H_1}$)** and the convexity of $\overline{\Psi}$, we get

$$\begin{aligned}&\int_{Q_T}\overline{\Psi}\left(\frac{|a(x,t,T_k(u_n),\phi)|}{\eta}\right)dx\,dt\\&\quad\le\int_{Q_T}\overline{\Psi}\left(\frac{\zeta\left(c(x,t)+k_1\overline{\Psi}^{-1}\big(P(k_2|T_k(u_n)|\big)+\overline{\Psi}^{-1}\big(\Psi(k_3|\phi|)\big)\right)}{\eta}\right)dx\,dt\\&\quad\le\frac{\zeta}{\eta}\int_{Q_T}\overline{\Psi}(c(x,t))\,dx\,dt+\frac{\zeta k_1}{\eta}\int_{Q_T}\overline{\Psi}\big(\overline{\Psi}^{-1}(P(k_2|T_k(u_n)|))\big)\,dx\,dt\\&\qquad+\frac{\zeta}{\eta}\int_{Q_T}\overline{\Psi}\big(\overline{\Psi}^{-1}(\Psi(k_3|\phi|))\big)\,dx\,dt\\&\quad\le\frac{\zeta}{\eta}\int_{Q_T}\overline{\Psi}(c(x,t))\,dx\,dt+\frac{\zeta k_1}{\eta}\int_{Q_T}P(k_2k)\,dx\,dt+\frac{\zeta}{\eta}\int_{Q_T}\Psi(k_3|\phi|)\,dx\,dt.\end{aligned}$$

Since $\phi\in(E_\Psi(Q_T))^N$, $c(x,t)\in E_{\overline{\Psi}}(Q_T)$, we deduce that $\{a(x,t,T_k(u_n),\phi)\}$ is bounded in $(L_{\overline{\Psi}}(Q_T))^N$ and we have that $\{\nabla T_k(u_n)\}$ is bounded in $(L_\Psi(Q_T))^N$.

Consequently, $J_2 \leq C_{J_2}$, where C_{J_2} is a positive constant not depending on n. And then we obtain

$$\int_{Q_T} a(x, T_k(u_n), \nabla T_k(u_n))\phi \, dx \, dt \leq C_{J_1} + C_{J_2} \quad \text{for all } \phi \in (E_\Psi(Q_T))^N.$$

Finally, $\{a(x,t,T_k(u_n), \nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{\Psi}}(Q_T))^N$. □

Step 3: Almost everywhere convergence of the gradients. In this step, most parts of the proof of the proposition below are the same as in [27,31]. Thus we give only those which are different.

Proposition 3.6. *Let u_n be a solution of the approximate problem (3.7)–(3.10). Then, for all $k \geq 0$, we have (for a subsequence still denoted by u_n), as $n \to +\infty$:*

(i) $\nabla u_n \to \nabla u$ *a.e. in* Q_T*;*

(ii) $a(x,t,T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u), \nabla T_k(u))$ *weakly in* $(L_{\overline{\Psi}}(Q_T))^N$*;*

(iii) $\Psi(|\nabla T_k(u_n)|) \to \Psi(|\nabla T_k(u)|)$ *strongly in* $L^1(Q_T)$*.*

Proof. Let $\theta_j \in \mathfrak{D}(Q_T)$ be a sequence such that $\theta_j \to u$ in $W_0^{1,x}L_\Psi(Q_T)$ for the modular convergence and let $\psi_i \in \mathfrak{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Put $Z_{i,j}^l = T_k(\theta_j)_l + \mathrm{e}^{-lt}\, T_k(\psi_i)$, where $T_k(\theta_j)_l$ is the mollification with respect to time of $T_k(\theta_j)$. Notice that $Z_{i,j}^l$ is a smooth function having the following properties:

$$\begin{gathered}\frac{\partial Z_{i,j}^l}{\partial t} = l(T_k(\theta_j) - Z_{i,j}^l), \quad Z_{i,j}^l(0) = T_k(\psi_i), \quad \text{and } |Z_{i,j}^l| \leq k,\\ Z_{i,j}^l \to T_k(u)_l + \mathrm{e}^{-lt}\, T_k(\psi_i) \quad \text{in } W_0^{1,x}L_\Psi(Q_T) \text{ modularly as } j \to \infty,\\ T_k(u)_l + \mathrm{e}^{-lt}\, T_k(\psi_i) \to T_k(u) \quad \text{in } W_0^{1,x}L_\Psi(Q_T) \text{ modularly as } l \to \infty.\end{gathered}$$

Let h_m be the function defined on $\mathbb{R}$ for any $m \geq k$ by

$$h_m(r) = \begin{cases} 1 & \text{if } |r| \leq m \\ -|r| + m + 1 & \text{if } m \leq |r| \leq m+1 \\ 0 & \text{if } |r| \geq m+1 \end{cases}.$$

Put $E_m = \{(x,t) \in Q_T : m \leq |u_n| \leq m+1\}$ and define $\varphi_{n,j,m}^{l,i} = (T_k(u_n) - Z_{i,j}^l)h_m(u_n)$. Testing the approximate problem (3.7)–(3.10) by the test function $u_n - \varphi_{n,j,m}^{l,i}$, we get

$$\begin{aligned}\left\langle \frac{\partial u_n}{\partial t}, \varphi_{n,j,m}^{l,i} \right\rangle &+ \int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla Z_{i,j}^l)h_m(u_n)\, dx\, dt\\ &+ \int_{Q_T} a(x,t,u_n,\nabla u_n)(T_k(u_n) - Z_{i,j}^l)\nabla u_n h_m'(u_n)\, dx\, dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{E_m} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-Z^l_{i,j})\,dx\,dt\\
&+\int_{Q_T} \Phi_n(x,t,u_n)\nabla u_n h_m(u_n)(\nabla T_k(u_n)-\nabla Z^l_{i,j})\,dx\,dt\\
&=\int_{Q_T} \mu_n \varphi^{l,i}_{n,j,m}\,dx\,dt.
\end{aligned}
$$

For simplicity, we will denote by $\epsilon(n,j,l,i)$ and $\epsilon(n,j,l)$ any quantities such that

$$
\lim_{i\to+\infty}\lim_{l\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,l,i)=0, \quad \lim_{l\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,l)=0.
$$

We have the following lemma which can be found in [27,31].

Lemma 3.7 (cf. [31]). *Let $\varphi^{l,i}_{n,j,m}=(T_k(u_n)-Z^l_{i,j})h_m(u_n)$. Then, for any $k\geq 0$, we have*

$$
\left\langle \frac{\partial u_n}{\partial t}, \varphi^{l,i}_{n,j,m}\right\rangle \geq \epsilon(n,j,l,i),
$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(Q_T)+W^{-1,x}L_{\overline{\Psi}}(Q_T)$ and $L^\infty(Q_T)\cap W^{1,x}_0 L_\Psi(Q_T)$.

To complete the proof of Proposition 3.6, we establish the results below. For any fixed $k\geq 0$, we have:

($\mathbf{r_1}$) $\displaystyle\int_{Q_T} \mu_n\varphi^{l,i}_{n,j,m}\,dx\,dt=\epsilon(n,j,l)$;

($\mathbf{r_2}$) $\displaystyle\int_{Q_T} \Phi_n(x,t,u_n)\nabla u_n h_m(u_n)(\nabla T_k(u_n)-\nabla Z^l_{i,j})\,dx\,dt=\epsilon(n,j,l)$;

($\mathbf{r_3}$) $\displaystyle\int_{E_m} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n)-Z^l_{i,j})\,dx\,dt=\epsilon(n,j,l)$;

($\mathbf{r_4}$) $\displaystyle\int_{Q_T} a(x,t,u_n,\nabla u_n)(T_k(u_n)-Z^l_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt\leq\epsilon(n,j,l,m)$;

($\mathbf{r_5}$) $\displaystyle\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)]$
$$\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\,dt\leq\epsilon(n,j,l,m,s).$$

The proofs of ($\mathbf{r_1}$) and ($\mathbf{r_3}$)–($\mathbf{r_5}$) are the same as in [27,30,31]. To prove ($\mathbf{r_2}$), to this end, for $n\geq m+1$, we have

$$
\Phi_n(x,t,u_n)h_m(u_n)=\Phi(x,t,T_{m+1}(u_n))h_m(T_{m+1}(u_n)) \text{ a.e in } Q_T.
$$

Put

$$
P_n=\overline{\Psi}\left(\frac{|\Phi(x,t,T_{m+1}(u_n))-\Phi(x,t,T_{m+1}(u))|}{\eta}\right).
$$

Since Φ is continuous with respect to its third argument and $u_n\to u$ a.e in Q_T, then $\Phi(x,t,T_{m+1}(u_n))\to\Phi(x,t,T_{m+1}(u))$ a.e in Ω as n goes to infinity, besides $\overline{\Psi}(0)=0$, it follows that

$$
P_n\to 0 \quad \text{a.e in } \Omega \text{ as } n\to\infty. \tag{3.31}
$$

Using now the convexity of $\overline{\Psi}$ and **(H₄)**, for every $\eta > 0$ and $n \geq m+1$, we have

$$\begin{aligned}
P_n &= \overline{\Psi}\left(\frac{|\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))|}{\eta}\right) \\
&\leq \overline{\Psi}\left(\frac{2\gamma(x,t) + \overline{\Psi}^{-1}\big(\Psi(|T_{m+1}(u_n)|)\big) + \overline{\Psi}^{-1}\big(\Psi(|T_{m+1}(u)|)\big)}{\eta}\right) \\
&\leq \overline{\Psi}\left(\frac{2}{\eta}|\gamma(x,t)| + \frac{2}{\eta}\overline{\Psi}^{-1}\big(\Psi(m+1)\big)\right) \\
&= \overline{\Psi}\left(\frac{1}{2}\frac{4}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{4}{\eta}\overline{\Psi}^{-1}\big(\Psi(m+1)\big)\right) \\
&\leq \frac{1}{2}\overline{\Psi}\left(\frac{4}{\eta}|\gamma(x,t)|\right) + \frac{1}{2}\overline{\Psi}\left(\frac{4}{\eta}\overline{\Psi}^{-1}\big(\Psi(m+1)\big)\right). \qquad (3.32)
\end{aligned}$$

We put $C_m^\eta(x,t) = \frac{1}{2}\overline{\Psi}\left(\frac{4}{\eta}|\gamma(x,t)|\right) + \frac{1}{2}\overline{\Psi}\left(\frac{4}{\eta}\overline{\Psi}^{-1}(\Psi(m+1))\right)$. Since $\gamma \in E_{\overline{\Psi}}(Q_T)$, we have $C_m^\eta \in L^1(Q_T)$, Then, by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_{Q_T} P_n\,dx\,dt = \int_{Q_T}\lim_{n\to\infty} P_n\,dx\,dt = 0.$$

This implies that $\{\Phi(x,t,T_{m+1}(u_n))\}$ converges modularly to $\Phi(x,t,T_{m+1}(u))$ as $n \to \infty$ in $(L_{\overline{\Psi}}(Q_T))^N$. Moreover, $\Phi(x,t,T_{m+1}(u_n))$, $\Phi(x,t,T_{m+1}(u))$ lie in $(E_{\overline{\Psi}}(Q_T))^N$. Indeed, from **(H₄)**, for every $\eta > 0$, we have

$$\begin{aligned}
&\int_{Q_T}\overline{\Psi}\left(\frac{|\Phi(x,t,T_{m+1}(u_n))|}{\eta}\right)dx\,dt \\
&\quad\leq \int_{Q_T}\overline{\Psi}\left(\frac{1}{\eta}|\gamma(x,t)| + \frac{1}{\eta}\overline{\Psi}^{-1}\big(\Psi(|T_{m+1}(u_n)|)\big)\right)dx\,dt \\
&\quad\leq \int_{Q_T}\overline{\Psi}\left(\frac{1}{2}\frac{2}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{2}{\eta}\overline{\Psi}^{-1}\big(\Psi(m+1)\big)\right)dx\,dt \\
&\quad\leq \int_{Q_T}\frac{1}{2}\overline{\Psi}\left(\frac{2}{\eta}|\gamma(x,t)|\right)dx\,dt + \int_{Q_T}\frac{1}{2}\overline{\Psi}\left(\frac{2}{\eta}\overline{\Psi}^{-1}\big(\Psi(m+1)\big)\right)dx\,dt < \infty
\end{aligned}$$

since $\gamma \in E_{\overline{\Psi}}(Q_T)$ and Ω is bounded, the same for $\Phi(x,t,T_{m+1}(u))$. Due to Lemma 2.3, we can deduce that $\Phi(x,t,T_{m+1}(u_n)) \to \Phi(x,t,T_{m+1}(u))$ strongly in $(E_{\overline{\Psi}}(Q_T))^N$. Furthermore, $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_\Psi(Q_T))^N$ as n goes to infinity and it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T}\Phi(x,t,u_n)h_m(u_n)\big(\nabla T_k(u_n) - \nabla Z_{i,j}^l\big)\,dx\,dt \\
&\quad= \int_{Q_T}\Phi(x,t,u)h_m(u)\big(\nabla T_k(u) - \nabla Z_{i,j}^l\big)\,dx\,dt.
\end{aligned}$$

Using the modular convergence of $Z_{i,j}^l$ as $j \to \infty$ and then $l \to \infty$, we get (r_2). As a consequence of Lemma 3.1, the results of Proposition 3.6 follow. □

Step 4: Passing to the limit. Now we will pass to the limit. Let $v \in W^{1,x}L_\Psi(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{\Psi}}(Q_T) + L^1(Q_T)$. From [17, Lemma 5, Theorem 3], there exists a prolongation $v_p = v$ on Q_T, $v_p \in W^{1,x}L_\Psi(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R})$ and

$$\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{\Psi}}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}).$$

There also exists a sequence $(\omega_j) \subset \mathfrak{D}(\Omega \times \mathbb{R})$ such that

$$\omega_j \to v_p \text{ in } W_0^{1,x}L_\Psi(\Omega \times \mathbb{R}) \text{ and } \frac{\partial \omega_j}{\partial t} \to \frac{\partial v_p}{\partial t} \text{ in } W^{-1,x}L_{\overline{\Psi}}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})$$

for the modular convergence, and $\|\omega_j\|_{\infty,Q_T} \leq (N+2)\|v\|_{\infty,Q_T}$.

Testing the approximate problem (3.7)–(3.10) by $v = u_n - T_k(u_n - \omega_j)\chi_{(0,\tau)}$ with $\tau \in [0,T]$, we get

$$\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau} + \int_{Q_\tau} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n))\nabla T_k(u_n - \omega_j)\,dx\,dt$$
$$+ \int_{Q_\tau} \Phi(x,t,T_{k_0}(u_n))\nabla T_k(u_n - \omega_j)\,dx\,dt = \int_{Q_\tau} \mu_n T_k(u_n - \omega_j)\,dx\,dt, \quad (3.33)$$

where $k_0 = k + (N+2)\|v\|_{\infty,Q_T}$. This implies, with

$$E_{n,j} := Q_\tau \cap \{|u_n - \omega_j| \leq k\},$$

that

$$\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau} + \int_{E_{n,j}} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n))\nabla u_n\,dx\,dt$$
$$- \int_{E_{n,j}} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n))\nabla \omega_j\,dx\,dt$$
$$+ \int_{Q_\tau} \Phi(x,t,T_{k_0}(u_n))\nabla T_k(u_n - \omega_j)\,dx\,dt$$
$$= \int_{Q_\tau} \mu_n T_k(u_n - \omega_j)\,dx\,dt. \quad (3.34)$$

Our aim here is to pass to the limit in each term in (3.34). Let us start by the terms of the left-hand side.

The limit of the first term $\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j)\rangle_{Q_\tau}$ is as follows:

$$\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau} = \Big\langle \frac{\partial u_n}{\partial t} - \frac{\partial \omega_j}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau}$$
$$+ \Big\langle \frac{\partial \omega_j}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau}$$
$$= \int_\Omega \widetilde{T}_k(u_n - \omega_j)\,dx + \Big\langle \frac{\partial \omega_j}{\partial t}, T_k(u_n - \omega_j)\Big\rangle_{Q_\tau}$$

$$- \int_\Omega \widetilde{T}_k(u_{0n} - \omega_j(0))\,dx. \tag{3.35}$$

Since $u_n \to u$ in $C([0,T], L^1(\Omega))$ (see [17]), by Lebesgue's theorem, we have

$$\int_\Omega \widetilde{T}_k(u_n - \omega_j)\,dx \to \int_\Omega \widetilde{T}_k(u - \omega_j)\,dx \quad \text{as } n \to \infty.$$

Passing to the limit in (3.35), we get

$$\begin{aligned} \lim_{n\to\infty} \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n - \omega_j) \right\rangle_{Q_\tau} &= \int_\Omega \widetilde{T}_k(u - \omega_j)\,dx + \left\langle \frac{\partial \omega_j}{\partial t}, T_k(u - \omega_j) \right\rangle_{Q_\tau} \\ &\quad - \int_\Omega \widetilde{T}_k(u_0 - \omega_j(0))\,dx. \end{aligned}$$

For the second and the third terms of (3.34), we have from **(ii)** of Proposition 3.6,

$$a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \rightharpoonup a(x,t,T_{k_0}(u), \nabla T_{k_0}(u)) \quad \text{weakly in } (L_{\overline{\Psi}}(Q_T))^N.$$

Thus Fatou's lemma allows us to get

$$\begin{aligned} \liminf_{n\to\infty} \Bigg(\int_{E_{n,j}} & a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \nabla u_n\,dx\,dt \\ & - \int_{E_{n,j}} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \nabla \omega_j\,dx\,dt \Bigg) \\ \geq & \int_{E_{n,j}} a(x,t,T_{k_0}(u), \nabla T_{k_0}(u)) \nabla u\,dx\,dt \\ & - \int_{E_{n,j}} a(x,t,T_{k_0}(u), \nabla T_{k_0}(u)) \nabla \omega_j\,dx\,dt. \end{aligned} \tag{3.36}$$

Concerning the fourth term of the left-hand side of (3.34), we proceed as in (3.32) to get

$$\Phi(x,t,T_{k_0}(u_n)) \to \Phi(x,t,T_{k_0}(u)) \quad \text{as } n \to \infty.$$

And since

$$\nabla T_k(u_n - \omega_j) \rightharpoonup \nabla T_k(u - \omega_j) \quad \text{in } L_\Psi(Q_T) \text{ as } n \to \infty,$$

we can deduce

$$\int_{Q_\tau} \Phi(x,t,T_{k_0}(u_n)) \nabla T_k(u_n - \omega_j)\,dx\,dt \to \int_{Q_\tau} \Phi(x,t,T_{k_0}(u)) \nabla T_k(u - \omega_j)\,dx\,dt.$$

Finally, we turn to the right-hand side of (3.34). Since

$$T_k(u_n - \omega_j) \to T_k(u - \omega_j) \quad \text{weakly* in } L^\infty \text{ as } n \to \infty,$$

we obtain

$$\int_{Q_\tau} \mu_n T_k(u_n - \omega_j)\,dx\,dt \to \int_{Q_\tau} \mu T_k(u - \omega_j)\,dx\,dt.$$

Now we are ready to pass to the limit as $n \to \infty$ in each term of (3.34) to conclude that

$$
\begin{aligned}
\int_\Omega \widetilde{T}_k(u-\omega_j)\,dx &+ \left\langle \frac{\partial \omega_j}{\partial t}, T_K(u-\omega_j)\right\rangle_{Q_\tau} \\
&+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-\omega_j)\,dx\,dt \\
&+ \int_{Q_\tau} \Phi(x,t,u)\nabla T_k(u_n-\omega_j)\,dx\,dt \\
&\le \int_\Omega \widetilde{T}_k(u_0-\omega_j(0))\,dx + \int_{Q_\tau} \mu T_k(u-\omega_j)\,dx\,dt. \qquad (3.37)
\end{aligned}
$$

Passing to the limit in (3.37) as $j \to \infty$, we obtain

$$
\begin{aligned}
\int_\Omega \widetilde{T}_k(u-v)\,dx &+ \left\langle \frac{\partial v}{\partial t}, T_k(u-v)\right\rangle_{Q_\tau} \\
&+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v)\,dx\,dt \\
&+ \int_{Q_\tau} \Phi(x,t,u)\nabla T_k(u_n-v)\,dx\,dt \\
&\le \int_\Omega \widetilde{T}_k(u_0-v(0))\,dx + \int_{Q_\tau} \mu T_k(u-v)\,dx\,dt. \qquad (3.38)
\end{aligned}
$$

It remains to show that u satisfies the initial condition of (3.7)–(3.10). To do this, recall that $\frac{\partial u_n}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\overline{\Psi}}(Q_T)$. As a consequence, Aubin's type Lemma (cf [32] , Corollary 4 and Lemma 2.6 imply that u_n lies in a compact set of $C^0([0,T]; L^1(\Omega))$. It follows that $u_n(x,t=0) = u_{0n}$ converges to $u(x,t=0)$ strongly in $L^1(\Omega)$. Thus we conclude that

$$
u(x,t=0) = u_0(x) \quad \text{in } \Omega.
$$

The proof of the main result is completed. □

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Mohamed Bourahma,

Laboratory LAMA, Department of mathematics, Faculty of Sciences Dhar el Mahraz, Sidi Mohamed Ben Abdellah University, PB 1796 Atlas, Fez, Morocco,
E-mail: mohamedbourahma@gmail.com

Jaouad Bennouna,

Laboratory LAMA, Department of mathematics, Faculty of Sciences Dhar el Mahraz, Sidi Mohamed Ben Abdellah University, PB 1796 Atlas, Fez, Morocco,
E-mail: jbennouna@hotmail.com

Badr El Haji,

AFNLA Team, Department of Mathematics, Faculty of Sciences Tetouan, Abdelmalek Essaadi University BP 2121, Tetouan, Morocco,
E-mail: badr.elhaji@gmail.com

Abdelmoujib Benkirane,

Laboratory LAMA, Department of mathematics, Faculty of Sciences Dhar el Mahraz, Sidi Mohamed Ben Abdellah University, PB 1796 Atlas, Fez, Morocco,
E-mail: abd.benkirane@gmail.com

Розв'язність сильно нелінійних параболічних проблем з перешкодами в неоднорідних просторах Орлича–Соболєва

Mohamed Bourahma, Jaouad Bennouna, Badr El Haji, and Abdelmoujib Benkirane

У цій роботі ми доводимо існування розв'язків для нелінійної однобічної задачі, пов'язаної з параболічним рівнянням

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) - \operatorname{div} \Phi(x,t,u) = \mu \quad \text{in } Q_T = \Omega \times (0,T),$$

де член нижчого порядку Φ задовольняє узагальнену природну умову зростання, описану певною функцією Орлича Ψ, і функція μ є інтегровним членом витоку. Жодних обмежень зростання не накладається ані на Ψ, ані на його спряжене $\overline{\Psi}$. Отже, розв'язок є природним у цьому контексті.

Ключові слова: однобічна параболічна задача, нерефлексивний простір Орлича, природне зростання