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Existence and Asymptotic Behavior of Beam-Equation Solutions with Strong Damping and $p(x)$-Biharmonic Operator

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In this paper, we consider a nonlinear beam equation with a strong damping and the $p(x)$-biharmonic operator. The exponent $p(\cdot)$ of nonlinearity is a given function satisfying some condition to be specified. Applying Faedo–Galerkin's method, the existence of weak solutions is proved. Using Nakao's lemma, the asymptotic behavior of weak solutions is established with mild assumptions on the variable exponent $p(\cdot)$. We show the asymptotic behavior of the weak solution is exponentially and algebraically depending on the variable exponent. This work improves and extends many other results in the literature.

Key words: weak solutions, existence, asymptotic behavior, beam equation, $p(x)$-biharmonic operator, variable exponent

Mathematical Subject Classification 2010: 35A01, 35B40, 35D30, 35L25

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$. We consider the following problem

$$\begin{cases} u_{tt} + \Delta^2_{p(x)} u - \Delta u_t = f(x,t,u_t) & \text{in } Q_T, \\ u = 0, \qquad \Delta u = 0 & \text{on } \partial Q_T, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Delta^2_{p(x)}$ is the fourth-order operator called the $p(x)$-biharmonic operator which is defined by

$$\Delta^2_{p(x)} u = \Delta \left(|\Delta u|^{p(x)-2} \Delta u \right).$$

We introduce, for $0 < T < \infty$, $Q_T = \Omega \times (0,T)$, $\partial Q_T = \partial\Omega \times (0,T)$, and the functions $p(\cdot)$, $f(\cdot)$, $u_0(\cdot)$, and $u_1(\cdot)$ that satisfy the following conditions.

Let the variable exponent $p : \overline{\Omega} \to (1, \infty)$ satisfy the log-Hölder continuity condition

$$|p(x) - p(y)| \leq -\frac{c}{\log|x-y|} \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{1.2}$$

where $c > 0$ and $0 < \delta < 1$. The function $f \in C(\Omega \times [0, \infty) \times \mathbb{R})$ satisfies the following conditions with two positive constants c_1 and c_2 for all $(x, t, s) \in \Omega \times [0, \infty) \times \mathbb{R}$:

$$\begin{cases} f(x,t,s)\, s \geq -c_1 |s|^{q(x)}, \\ |f(x,t,s)| \leq c_2 |s|^{q(x)-1}, \end{cases} \tag{1.3}$$

where $q : \overline{\Omega} \to (1, \infty)$ is log-Hölder continuous.

The functions $p(\cdot)$ and $q(\cdot)$ satisfy, for all $x \in \overline{\Omega}$,

$$1 < p^- \leq p(x) \leq p^+ < \frac{N}{2}, \tag{1.4}$$

$$1 < q^- \leq q(x) \leq q^+ < \frac{Np(x)}{N - 2p(x)}, \tag{1.5}$$

where

$$p^- = \operatorname*{ess\,inf}_{x \in \overline{\Omega}} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x \in \overline{\Omega}} p(x),$$

$$q^- = \operatorname*{ess\,inf}_{x \in \overline{\Omega}} q(x), \qquad q^+ = \operatorname*{ess\,sup}_{x \in \overline{\Omega}} q(x).$$

Furthermore, we consider that

$$u_0 \in W^{2,p(x)}(\Omega) \cap W_0^{1,2}(\Omega) \tag{1.6}$$

and

$$u_1 \in L^2(\Omega). \tag{1.7}$$

Partial differential equations with variable exponents have many applications in mathematical physics, for example, problems of filtration processes in non-homogeneous porous media [8], wave equations [3, 4, 6], nonlinear beam equations [9], restoration and image processing [11,13,32], flow of electro-rheological or thermo-rheological fluids [1,6,7,14,31], plate equations with viscoelasticity, elasticity term or viscoelasticity term [2,4,5,15]. These equations are associated with operators called p-Laplacian, $p(x)$-Laplacian, $\vec{p}(x,t)$-Laplacian, p-biharmonic or $p(x)$-biharmonic.

To motivate our work, let us mention some results. In recent years, there has been a growing interest in studying elliptic problems involving a p-biharmonic operator of the type

$$\begin{cases} \Delta_p^2 u = f(x,u) & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Δ_p^2 is the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$. For example, Benedikt and Drábek [10] studied problem (1.8), taking $f(x,u) =$

$\lambda|u|^{p-2}u$, and provided estimates from below and from above for the principal eigenvalue of the p-biharmonic operator on a bounded domain with the Navier boundary conditions. They also applied these estimates for studying the asymptotic behavior of the principal eigenvalue when $p \to +\infty$. Liu, Chen, and Al-muaalemi [26] studied problem (1.8) with $f(x,u) = g(x,u) - V(x)|u|^{p-2}u$. For $V(x)$ and $g(x,u)$ satisfying specific conditions. They proved that the problem has Neharitype ground state solutions. Also, Bueno, Paes-Leme, and Rodrigues [12] studied problem (1.8) for $f(x,u) = \lambda g(x)|u|^{q-2}u + |u|^{p^*-2}u$, where p and q are such that $1 < p < \infty$, $N > 2p$, $1 < q < p$, and $p^* = \frac{Np}{N-2p}$. They proved the existence of infinitely many solutions for the problem in a bounded and smooth domain Ω with concave-convex nonlinearities depending on the parameter λ and a positive continuous function $g : \overline{\Omega} \to \mathbb{R}$. They simultaneously handled of the critical-case problems with both Navier and Dirichlet boundary conditions by applying the Ljusternik–Schnirelmann method. The multiplicity of solutions is obtained when λ is small enough. In the case of Navier boundary conditions, all solutions are positive and a regularity result is proved.

A natural generalization of problem (1.8) is obtained by replacing the classic p-biharmonic by the operator $p(x)$-biharmonic, that is,

$$\begin{cases} \Delta^2_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, \quad \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In this regard, Ge, Zhou, and Wu [17] studied problem (1.9) with the function $f(x,u) = \lambda V(x)|u|^{q(x)-2}u$, where λ is a positive real number, $p, q : \overline{\Omega} \to (1, +\infty)$ are continuous functions, and V is an indefinite weight function. Considering different situations concerning the growth rates involved in the quoted problem, they proved the existence of a continuous family of eigenvalues. The proofs of the main results are based on the mountain pass lemma and Ekeland's variational principle. Li and Tang [22] studied problem (1.9) with the Navier boundary condition and for $f(x,u) = \lambda|u|^{p(x)-2}u + g(x,u)$, where $\lambda \leq 0$ and $g(x,u)$ is a Carathéodory function. Using the mountain pass theorem, Fountain theorem, local linking theorem and symmetric mountain pass theorem, they established the existence of at least one solution and infinitely many solutions to this problem. Kong, in [20], considered problem (1.9), where $f(x,u) = \lambda b(x)|u|^{\gamma(x)-2}u - \lambda c(x)|u|^{\beta(x)-2}u - a(x)|u|^{p(x)-2}u$, with $\lambda > 0$ being a parameter, and $a, b, c, \beta, \gamma \in C(\overline{\Omega})$ being nonnegative functions. He proved the existence of weak solutions to the problem associated with the Navier boundary conditions.

For parabolic problems involving the p-biharmonic operator, we cite the paper of Liu and Guo [24], where the following initial boundary-value problem for a fourth-order degenerate parabolic equation was studied:

$$u_t + \Delta^2_p u + \lambda|u|^{p-2}u = 0,$$

where $\lambda > 0$ and $p > 2$. Under some assumptions on the initial value, the existence of weak solutions was proved by using the discrete-time method. The asymptotic behavior and the finite speed of propagation of perturbations of solutions were also discussed. Hao and Zhou in [19] investigated a class of p-biharmonic parabolic equation with nonlocal source in a bounded domain defined by

$$u_t + \Delta_p^2 u = |u|^q - \frac{1}{|\Omega|}\int_\Omega |u|^q dx,$$

where $\max\{1, \frac{2N}{N+4}\} < p \le 2$ and $q > 0$. The results on blow up, extinction and non-extinction of the solutions were demonstrated. Liu and Li [25] discussed the parabolic p-biharmonic equation with the logarithmic nonlinearity given by

$$u_t + \Delta_p^2 u = \lambda |u|^{q-2} u \log\left(|u|\right),$$

where $\lambda > 0$, $p > q > \frac{p+2}{2}$ and $p > \frac{N}{2}$. Basing on the difference and variation methods, they proved the existence of weak solutions for the initial boundary problem. In addition, they discussed the long-time behavior and the propagation of perturbations of solutions.

For parabolic problems involving the $p(x)$-biharmonic operator, we mention the work by Liu [27], where the initial-boundary-value problem for

$$u_t + \Delta_{p(x)}^2 u = |u|^{q(x)-2} u$$

was studied. He established the local existence of weak solutions and derived the finite-time blowup of solutions with nonpositive initial energy.

In this paper, our objective is to discuss the existence and asymptotic behavior of weak solutions for nonlinear hyperbolic problem (1.1) with strong damping Δu_t and $p(x)$-biharmonic operator. Our proofs are based on Faedo–Galerkin's method, Nakao's lemma, results of functional analysis, and the Lebesgue and Sobolev spaces with variable exponents. To the best of our knowledge, this is the first work dealing with problem (1.1) related to the existence and asymptotic behavior of beam equation solutions with strong damping and $p(x)$-biharmonic operator.

The paper is organized as follows. In Section 2, we introduce the functional spaces of Lebesgue and Sobolev with variable exponents and present a brief description of their main properties. In Section 3, we prove the existence of weak solutions for problem (1.1). In Section 4, we demonstrate the asymptotic behavior of weak solutions for problem (1.1). Finally, in Section 5, we give the conclusions of the paper.

2. Preliminaries

In this section, we present some results on the variable exponents in the Lebesgue and Sobolev spaces, $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$, respectively (see [6, 14]).

Let $p : \Omega \to (1, \infty)$ be a measurable function, where Ω is a domain of $\mathbb{R}^N$. We define the variable-exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is mensurable in } \Omega \text{ and } \rho(u) = \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

equipped with the following Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \,\middle|\, \rho\left(\frac{u}{\lambda}\right) \le 1\right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space (see [14]). If p^+ is finite, then the variable exponent $p(x)$ is bounded and, furthermore, we have the relations:

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\} \le \rho(u) \le \max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}\right\}. \tag{2.1}$$

The above inequalities (2.1) can be represented by

$$\min\left\{\rho(u)^{\frac{1}{p^-}}, \rho(u)^{\frac{1}{p^+}}\right\} \le \|u\|_{L^{p(\cdot)}(\Omega)} \le \max\left\{\rho(u)^{\frac{1}{p^-}}, \rho(u)^{\frac{1}{p^+}}\right\}. \tag{2.2}$$

Theorem 2.1 ([6, 14]). *If $p(x)$ and $q(x)$ are variable exponents such that $p(x) \ge q(x)$ for a.e. x in Ω, with Ω being bounded, then $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.*

Theorem 2.2 (Hölder's inequality [6]). *Suppose that $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{p'(\cdot)}(\Omega)$, $1 < p(x) < \infty$ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then*

$$\begin{aligned}\int_\Omega |uv|\,dx &\le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \\ &\le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.\end{aligned} \tag{2.3}$$

The variable-exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ is defined by

$$W^{m,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \,\middle|\, \forall\alpha \;\left(|\alpha| \le m \Rightarrow D^\alpha u \in L^{p(\cdot)}(\Omega)\right)\right\},$$

where m is a non-negative integer and D^α is the derivative in the sense of distributions. The variable-exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{m,p(\cdot)} = \|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} \|D^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions with a compact support contained in Ω. Throughout this paper, we denote by c_i various positive constants that may be different at different occurrences.

If X is a Banach space, then we denote by $L^p(0,T;X)$, with $1 \le p \le \infty$, the Banach space of measurable vector valued functions $u : (0,T) \to X$ such that $\|u(t)\|_X \in L^p(0,T)$, together with the norms

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\,dt\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\|u\|_{L^p(0,T;X)} = \operatorname*{ess\,sup}_{0\le t<T} \|u(t)\|_X, \quad p = \infty.$$

In addition, by $C^1(0,T;X)$, we denote the space of continuously differentiable functions on $[0,T]$ with values in X.

Theorem 2.3 ([14]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary. Suppose $p : \Omega \to (1, \infty)$ is a bounded function and log-Hölder continuous. If $q : \Omega \to (1, \infty)$, with $q^+ < N$, is a bounded and measurable function with*

$$q(x) \leq p^* = \frac{Np(x)}{N - 2p(x)}, \quad x \in \overline{\Omega},$$

then there is a continuous embedding $W^{2,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$.

Theorem 2.4 ([33]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary. Suppose $p : \Omega \to (1, \infty)$ is a bounded function and log-Hölder continuous in $\overline{\Omega}$. Then there is a constant c such that for each $u \in W_0^{2,p(\cdot)}(\Omega)$ we have*

$$\|u\|_{W_0^{2,p(\cdot)}(\Omega)} \leq c \|\Delta u\|_{L^{p(\cdot)}(\Omega)} .$$

Theorem 2.5 ([6]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\{\omega_i\}_{i=1}^{\infty}$ be an orthonormal base in $L^2(\Omega)$. Then, for any $\varepsilon > 0$, there is a constant $N_\varepsilon > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq \left(\sum_{i=1}^{N_\varepsilon} \left(\int_\Omega u\omega_i \, dx \right)^2 \right)^{\frac{1}{2}} + \varepsilon \|u\|_{W_0^{1,p(\cdot)}(\Omega)}$$

for all $u \in W_0^{1,p(\cdot)}(\Omega)$, where $2 \leq p < \infty$.

Next, we state Nakao's lemma, the proof of which can be found in [28]. With the help of this lemma, we demonstrate the asymptotic behavior of weak solutions for problem (1.1).

Lemma 2.6 (Nakao [28]). *Let $\Psi : (0, \infty) \to \mathbb{R}$ be a bounded nonnegative function. If there are two constants $\alpha > 0$ and $\beta \geq 0$ such that*

$$\sup_{t \leq s \leq t+1} \Psi^{1+\beta}(s) \leq \alpha \left(\Psi(t) - \Psi(t+1) \right), \quad t \geq 0,$$

then there are positive constants C and γ such that

$$\begin{cases} \Psi(t) \leq Ce^{-\gamma t} & \text{if } \beta = 0 \\ \Psi(t) \leq C(t+1)^{-\frac{1}{\beta}} & \text{if } \beta > 0 \end{cases}, \quad t \geq 0.$$

3. Existence of weak solutions

In this section, we prove the existence of weak solutions for problem (1.1), where the functions p, f, u_0 and u_1 satisfy the conditions given by (1.2)–(1.7).

Definition 3.1. The scalar function $u : Q_T \to \mathbb{R}$ is a weak solution for problem (1.1) if u satisfies simultaneously

$$u \in L^\infty \left(0, T; W_0^{2,p(\cdot)}(\Omega)\right) \cap C\left(0, T; W_0^{1,2}(\Omega)\right),$$

$$\frac{\partial u}{\partial t} \in L^\infty \left(0, T; L^2(\Omega)\right) \cap L^2 \left(0, T; W_0^{1,2}(\Omega)\right) \cap L^{q(\cdot)}(Q_T),$$

and

$$-\int_\Omega \frac{\partial u(x,0)}{\partial t}\varphi(x,0)\,dx - \int_{Q_T}\frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_{Q_T}|\Delta u|^{p(x)-2}\Delta u\Delta\varphi\,dx\,dt + \int_{Q_T}\nabla\left(\frac{\partial u}{\partial t}\right)\nabla\varphi\,dx\,dt = \int_{Q_T} f\left(x,t,\frac{\partial u}{\partial t}\right)\varphi\,dx\,dt$$

for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$ with $\varphi(x,T)=0$.

Theorem 3.2. *Let conditions (1.2), (1.4)–(1.7) hold, and let a function* $f \in C(\Omega\times[0,\infty)\times\mathbb{R})$ *satisfy the conditions*

$$\begin{cases} f(x,t,s)\,s \geq -c_3|s|^{q(x)} + c_4, \\ |f(x,t,s)| \leq c_5\left(|s|^{q(x)-1}+1\right), \end{cases} \tag{3.1}$$

with three positive constants c_3, c_4 *and* c_5 *for all* $(x,t,s)\in\Omega\times[0,\infty)\times\mathbb{R}$, *where* $q:\overline{\Omega}\to(1,\infty)$ *is log-Hölder continuous. Then problem (1.1) has a weak solution in the sense of Definition 3.1.*

Proof. We apply Faedo–Galerkin's method to problem (1.1) and show the existence of weak solutions. For that, as stated in [16, 21], we choose a sequence $\{\omega_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ such that $C_0^\infty(\Omega)\subset \overline{\bigcup_{n=1}^\infty V_n}^{C^2(\overline{\Omega})}$ and $\{\omega_j\}_{j=1}^\infty$ is a Hilbertian base in $L^2(\Omega)$, where $V_n = \langle \omega_1,\omega_2,\ldots,\omega_n\rangle$.

Due to the fact that $\bigcup_{n=1}^\infty V_n$ is dense in $C^2(\overline{\Omega})$, it is well known that if $u_0 \in W^{2,p(\cdot)}(\Omega)\cap W_0^{1,2}(\Omega)$ and $u_1\in L^2(\Omega)$, then there are $\psi_n,\phi_n\in V_n$ such that for $n\to\infty$,

$$\begin{cases} \psi_n \to u_0 & \text{in } W^{2,p(\cdot)}(\Omega)\cap W_0^{1,2}(\Omega), \\ \phi_n \to u_1 & \text{in } L^2(\Omega). \end{cases} \tag{3.2}$$

Faedo–Galerkin's method is used to find a sequence of solutions

$$u_n(x,t) = \sum_{j=1}^n \eta_{nj}(t)\omega_j(x) \in V_n \tag{3.3}$$

to the approximate problem

$$\int_\Omega \frac{\partial^2 u_n}{\partial t^2}v\,dx + \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta v\,dx + \int_\Omega \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla v\,dx = \int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)v\,dx, \quad v\in V_n. \tag{3.4}$$

Substituting (3.3) in (3.4) and taking $v=\omega_i$, with $1\leq i\leq n$, we obtain

$$\int_\Omega \sum_{j=1}^n \eta''_{nj}(t)\omega_j(x)\omega_i(x)\,dx$$

$$
\begin{aligned}
&+\int_\Omega \left|\sum_{j=1}^n \eta_{nj}(t)\Delta\omega_j(x)\right|^{p(x)-2}\left(\sum_{j=1}^n \eta_{nj}(t)\Delta\omega_j(x)\right)\Delta\omega_i(x)\,dx\\
&+\int_\Omega \left(\sum_{j=1}^n \eta'_{nj}(t)\nabla\omega_j(x)\right)\nabla\omega_i(x)\,dx\\
=&\int_\Omega f\left(x,t,\sum_{j=1}^n \eta'_{nj}(t)\omega_j(x)\right)\omega_i(x)\,dx,
\end{aligned}
$$

where $\eta'_{nj}(t)=\frac{\partial\eta_{nj}(t)}{\partial t}$ and $\eta''_{nj}(t)=\frac{\partial^2\eta_{nj}(t)}{\partial t^2}$. Defining the projection $P_{ni}(t,\mu,\nu):[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ as

$$
\begin{aligned}
P_{ni}(t,\mu,\nu)=&\int_\Omega \left|\sum_{j=1}^n \mu_{nj}(t)\Delta\omega_j(x)\right|^{p(x)-2}\left(\sum_{j=1}^n \mu_{nj}(t)\Delta\omega_j(x)\right)\Delta\omega_i(x)\,dx\\
&+\int_\Omega \left(\sum_{j=1}^n \nu_{nj}(t)\nabla\omega_j(x)\right)\nabla\omega_i(x)\,dx\\
&-\int_\Omega f\left(x,t,\sum_{j=1}^n \nu_{nj}(t)\omega_j(x)\right)\omega_i(x)\,dx,
\end{aligned}
\tag{3.5}
$$

where $1\le i\le n$, and using the fact that V_n is a Hilbertian base in $L^2(\Omega)$, we get

$$
\begin{cases}
\eta''_{n1}(t)+P_{n1}\left(t,\eta_{n1}(t),\eta'_{n1}(t)\right)=0,\\
\eta''_{n2}(t)+P_{n2}\left(t,\eta_{n2}(t),\eta'_{n2}(t)\right)=0,\\
\qquad\cdots\\
\eta''_{nn}(t)+P_{nn}\left(t,\eta_{nn}(t),\eta'_{nn}(t)\right)=0.
\end{cases}
\tag{3.6}
$$

Problem (3.6) can be rewritten as

$$
\begin{cases}
\eta''(t)+P_n\left(t,\eta(t),\eta'(t)\right)=0,\\
\eta(0)=U_{0n},\quad \eta'(0)=U_{1n},
\end{cases}
\tag{3.7}
$$

with

$$
\eta''(t)=\begin{bmatrix}\eta''_{n1}(t)\\ \eta''_{n2}(t)\\ \vdots\\ \eta''_{nn}(t)\end{bmatrix},\quad P_n\left(t,\eta(t),\eta'(t)\right)=\begin{bmatrix}P_{n1}\left(t,\eta_{n1}(t),\eta'_{n1}(t)\right)\\ P_{n2}\left(t,\eta_{n2}(t),\eta'_{n2}(t)\right)\\ \vdots\\ P_{nn}\left(t,\eta_{nn}(t),\eta'_{nn}(t)\right)\end{bmatrix}.
$$

We define

$$X(t)=\eta'(t), \tag{3.8}$$
$$Y(t)=(\eta(t),X(t)), \tag{3.9}$$
$$Z_n(t)=\left(X(t),-P_n(t,\eta(t))\right). \tag{3.10}$$

Thus, problem (3.7) becomes

$$\begin{cases} Y'(t) = Z_n(t, Y(t)), \\ Y(0) = (U_{0n}, U_{1n}). \end{cases} \tag{3.11}$$

From Carathéodory's theorem, it is easily seen that the system of nonlinear ordinary equations in the variable t has a local solution.

Remark 3.3. We have

$$P_n(t,\eta,\eta')\eta' \geq \frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx - c_6, \tag{3.12}$$

where $c_6 = c_4|\Omega| \geq 0$.

Proof. Using the projection (3.5), we get

$$P_n(t,\eta,\eta')\eta' = \int_\Omega |\Delta u_n|^{p(x)-2}\,\Delta u_n \sum_{i=1}^{n} \Delta\omega_i \eta'_{ni}\,dx + \int_\Omega \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\sum_{i=1}^{n}\omega_i\eta'_{ni}\,dx - \int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)\sum_{i=1}^{n}\omega_i\eta'_{ni}\,dx.$$

So, it follows that

$$P_n(t,\eta,\eta')\eta' = \int_\Omega |\Delta u_n|^{p(x)-2}\,\Delta u_n \Delta\left(\frac{\partial u_n}{\partial t}\right)dx + \int_\Omega \nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla\left(\frac{\partial u_n}{\partial t}\right)dx - \int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)\frac{\partial u_n}{\partial t}\,dx. \tag{3.13}$$

By (3.1), we have

$$-\int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)\frac{\partial u_n}{\partial t}dx \geq c_3\int_\Omega\left|\frac{\partial u_n}{\partial t}\right|^{q(x)}dx - c_6. \tag{3.14}$$

In addition, notice that

$$\int_\Omega |\Delta u_n|^{p(x)-2}\,\Delta u_n\Delta\left(\frac{\partial u_n}{\partial t}\right)dx = \frac{d}{dt}\int_\Omega\frac{|\Delta u_n|^{p(x)}}{p(x)}dx. \tag{3.15}$$

Replacing (3.14) and (3.15) in (3.13), we conclude (3.12). □

Now, returning to problem (3.11), we omit the arguments for simplicity. Composing problem (3.11) with Y, by applying the inner product and using (3.8), (3.9) and (3.10), we obtain that

$$Y'Y - \eta'\eta = -P_n\eta'. \tag{3.16}$$

Applying inequality (3.12) in (3.16), gives

$$Y'Y+\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx+\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx+c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \le \eta'\eta+c_6.$$

Since $\eta'\eta \le |\eta'|\,|\eta| \le \frac{1}{2}|\eta'|^2+\frac{1}{2}|\eta|^2$, we get

$$\begin{aligned}Y'Y+\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx+\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx+c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx&\\ \le \frac{1}{2}|\eta'|^2+\frac{1}{2}|\eta|^2+c_6.&\end{aligned}$$

By (3.9), we have

$$Y'Y+\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx+\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx+c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \le |Y|^2+c_6.$$

We know that $\frac{1}{2}\frac{d}{dt}|Y|^2 = Y'Y$. Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Y|^2+\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx+\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx+c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx.&\\ \le |Y|^2+c_6.&\end{aligned}\tag{3.17}$$

Integrating (3.17) from 0 to t, $t\le T$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt+\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx-\int_\Omega \frac{|\Delta u_n(x,0)|^{p(x)}}{p(x)}dx&\\ +\int_0^T\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx\,dt+c_3\int_0^T\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx\,dt&\\ \le \int_0^t |Y(s)|^2\,ds+c_6T.&\end{aligned}\tag{3.18}$$

By (3.2), $u_n(x,0)$ converges strongly in $W^{2,p(x)}(\Omega)\cap W_0^{1,2}(\Omega)$. Thus, $|\Delta u_n(x,0)|$ is bounded by a constant, that is,

$$|\Delta u_n(x,0)| \le c_7. \tag{3.19}$$

In addition, we have

$$\int_0^T\int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx\,dt = \left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|^2_{L^2(Q_T)} \le c_8, \tag{3.20}$$

$$\int_0^T\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx\,dt = \left\|\frac{\partial u_n}{\partial t}\right\|^{q(x)}_{L^{q(\cdot)}(Q_T)} \le c_9. \tag{3.21}$$

Replacing (3.19), (3.20) and (3.21) in (3.18), and defining $c_{10}(T) = c_6 T + c_7 - c_8 - c_3 c_9$, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \leq \int_0^t |Y(s)|^2\,ds + c_{10}(T).$$

Applying Gronwall's lemma, one has

$$\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}dx \leq c_{11}(T).$$

Since $p : \overline{\Omega} \to (1, \infty)$, it follows that

$$\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)}\,dx \leq \int_\Omega |\Delta u_n|^{p(x)}\,dx. \tag{3.22}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^t |Y|^2\,dt + \int_\Omega |\Delta u_n|^{p(x)}\,dx \leq c_{11}(T). \tag{3.23}$$

We know that $\int_\Omega |\Delta u_n|^{p(x)}dx \geq 0$, and thus

$$|Y(t) - Y(0)| \leq \sqrt{c(T)},$$

where $c(T) = 2c_{11}(T)$.

We denote

$$M_n = \max_{(t,Y)\,\in\,[0,T]\times\mathcal{B}\left(Y(0),\sqrt{c(T)}\right)} |Z_n(t,Y)| \quad \text{and} \quad \gamma_n \leq \min\left\{T, \frac{\sqrt{c(T)}}{M_n}\right\},$$

where $\mathcal{B}(Y(0), \sqrt{c(T)})$ is the ball with center $Y(0) \in \mathbb{R}^{2N}$ and radius $\sqrt{c(T)}$. By the definition, $Z_n(t,Y)$ is continuous with respect to (t,Y). Then, applying Peano's theorem (see [29]), it follows that problem (3.11) has a solution C^1 over the interval $[0,\gamma_n]$, which implies that over the same interval problem (3.7) has a solution C^2 denoted by $\eta_n^1(t)$.

Considering that $\eta(\gamma_n)$ and $\frac{\partial \eta(\gamma_n)}{\partial t}$ are the initial values of problem (3.7), we can repeat the previous process, and over the interval $[\gamma_n, 2\gamma_n]$ we obtain a solution C^2 denoted by $\eta_n^2(t)$.

We define

$$T = \left[\frac{T}{\gamma_n}\right]\gamma_n + \left(\frac{T}{\gamma_n}\right)\gamma_n \quad \text{with } 0 < \left(\frac{T}{\gamma_n}\right) < 1,$$

where $[\frac{T}{\gamma_n}]$ and $(\frac{T}{\gamma_n})$ are the integer part and the decimal part of $\frac{T}{\gamma_n}$. If we divide the interval $[0,T]$ in $[(i-1)\gamma_n, i\gamma_n]$, $i = 1, 2, \cdots, L$ and $[L\gamma_n, T]$, where $L = [\frac{T}{\gamma_n}]$, then there is a solution C^2 over the interval $[(i-1)\gamma_n, i\gamma_n]$ denoted by $\eta_n^i(t)$, and there is $\eta_n^{L+1}(t)$ over $[L\gamma_n, T]$. Therefore, we obtain a solution $\eta_n(t) \in C^2([0,T])$ as follows

$$\eta_n(t) = \begin{cases} ll\eta_n^1(t) & \text{if } t \in [0,\gamma_n] \\ \eta_n^2(t) & \text{if } t \in (\gamma_n, 2\gamma_n] \\ \cdots & \\ \eta_n^L(t) & \text{if } t \in ((L-1)\gamma_n, L\gamma_n] \\ \eta_n^{L+1}(t) & \text{if } t \in (L\gamma_n, T] \end{cases}.$$

Remark 3.4. The estimates

$$\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^2 dx + \int_\Omega |\Delta u_n|^{p(x)}\, dx + \int_\Omega |\nabla u_n|^2\, dx \le C_1, \quad (3.24)$$

$$\int_{Q_T} \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx\, dt + \int_{Q_T} |\Delta u_n|^{p(x)}\, dx\, dt + \int_{Q_T} \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx\, dt \le C_2 \quad (3.25)$$

are uniform with respect to n for all $t \in [0,T]$.

Proof. By equation (3.23), taking into account that $|Y(0)|^2$ is bounded, we obtain

$$|Y(t)|^2 + \int_\Omega |\Delta u_n|^{p(x)}\, dx \le c_{12}.$$

Using (3.3), (3.9), the fact that ω_j, $1 \le j \le n$, is a Hilbertian base and Poincaré's inequality, we obtain (3.24).

In addition, in (3.17), since $|Y|^2$ is bounded, then

$$\frac{d}{dt}\int_\Omega \frac{|\Delta u_n|^{p(x)}}{p(x)} dx + \int_\Omega \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx + c_3\int_\Omega \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx \le c_{13}.$$

Integrating from 0 to T and using $Q_T = \Omega \times [0,T]$, it follows that

$$\int_{Q_T} \frac{|\Delta u_n|^{p(x)}}{p(x)} dx\, dt + \int_{Q_T} \left|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right|^2 dx\, dt + c_3\int_{Q_T} \left|\frac{\partial u_n}{\partial t}\right|^{q(x)} dx\, dt \le c_{14}.$$

From (3.22), by hypothesis $c_3 \ge 0$, we conclude (3.25). □

Remark 3.5. The estimate

$$\||\Delta u_n|^{p(x)-2}\,\Delta u_n\|_{L^{p'(\cdot)}(Q_T)} + \left\|f\left(x,t,\frac{\partial u_n}{\partial t}\right)\right\|_{L^{q'(\cdot)}(Q_T)} \le C_3 \quad (3.26)$$

is uniform with respect to n for all $t \in [0,T]$.

Proof. By Remark 3.4, we obtain

$$\int_{Q_T} \left||\Delta u_n|^{p(x)-2}\nabla u_n\right|^{p'(x)} dx\, dt \le \int_{Q_T} |\nabla u_n|^{p(x)}\, dx\, dt \le c_{15}.$$

Thus,

$$\left\| |\Delta u_n|^{p(x)-2} \nabla u_n \right\|_{L^{p'(\cdot)}(Q_T)} \leq \max\left\{ \left(\int_\Omega |\Delta u_n|^{p(x)} dx \right)^{\frac{p^- -1}{p^-}}, \left(\int_\Omega |\Delta u_n|^{p(x)} dx \right)^{\frac{p^+ -1}{p^+}} \right\},$$

that is,

$$\left\| |\Delta u_n|^{p(x)-2} \nabla u_n \right\|_{L^{p'(\cdot)}(Q_T)} \leq c_{16}. \tag{3.27}$$

Using (3.1) and (3.21), it follows that

$$\left\| f\left(x, t, \frac{\partial u_n}{\partial t}\right) \right\|_{L^{q'(\cdot)}(Q_T)} \leq c_{17}. \tag{3.28}$$

From inequalities (3.27) and (3.28), we conclude (3.26). □

Using Remarks 3.4 and 3.5, there is a subsequence of u_n (still denoted by u_n) and u such that

$$\begin{aligned}
\frac{\partial u_n}{\partial t} &\overset{*}{\rightharpoonup} \frac{\partial u}{\partial t} && \text{in } L^\infty\big(0,T; L^2(\Omega)\big), \\
u_n &\overset{*}{\rightharpoonup} u && \text{in } L^\infty\big(0,T; W_0^{2,p(\cdot)}(\Omega)\big) \cap L^\infty\big(0,T; W_0^{1,2}(\Omega)\big), \\
\frac{\partial u_n}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } L^{q(\cdot)}(Q_T) \cap L^2\big(0,T; W_0^{1,2}(\Omega)\big), \\
|\Delta u_n|^{p(x)-2}\Delta u_n &\rightharpoonup \xi && \text{in } L^{p'(\cdot)}(Q_T), \\
f\left(x,t,\frac{\partial u_n}{\partial t}\right) &\rightharpoonup f\left(x,t,\frac{\partial u}{\partial t}\right) && \text{in } L^{q'(\cdot)}(Q_T).
\end{aligned}$$

Our next objective is to prove that there is a subsequence of u_n such that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^{q(\cdot)}(Q_T).$$

From (3.3) and since $\{\omega_j\}_{j=1}^n$ is a Hilbertian base, then

$$\int_\Omega \frac{\partial u_n}{\partial t} \omega_j \, dx = \eta'_{nj} \quad \text{and} \quad \int_\Omega \frac{\partial^2 u_n}{\partial t^2} \omega_j \, dx = \eta''_{nj}. \tag{3.29}$$

By Remark 3.4, it follows that $\eta'_{nj}(t)$ is uniformly bounded in $[0,T]$. Considering that $0 \leq t_1 < t_2 \leq T$, integrating (3.7) from t_1 to t_2, using (3.29) and defining $Q_{t_1}^{t_2} = \Omega \times [t_1, t_2]$, we get

$$\begin{aligned}
\int_\Omega \frac{\partial u_n(x,t_2)}{\partial t} \omega_j \, dx - \int_\Omega \frac{\partial u_n(x,t_1)}{\partial t} \omega_j \, dx + \int_{Q_{t_1}^{t_2}} |\Delta u_n|^{p(x)-2} \Delta u_n \Delta \omega_j \, dx \, dt \\
+ \int_{Q_{t_1}^{t_2}} \nabla \left(\frac{\partial u_n}{\partial t} \right) \nabla \omega_j \, dx \, dt = \int_{Q_{t_1}^{t_2}} f\left(x, t, \frac{\partial u_n}{\partial t}\right) \omega_j \, dx \, dt.
\end{aligned} \tag{3.30}$$

By (3.29), we obtain

$$\int_\Omega \frac{\partial u_n(x,t_2)}{\partial t}\omega_j\,dx - \int_\Omega \frac{\partial u_n(x,t_1)}{\partial t}\omega_j\,dx \le |\eta'_{nj}(t_2)-\eta'_{nj}(t_1)|. \tag{3.31}$$

Replace (3.31) in (3.30) and use Hölder's inequality (2.3) to get

$$\begin{aligned}|\eta'_{nj}(t_2)-\eta'_{nj}(t_1)| &\le 2\left\||\Delta u_n|^{p(x)-2}\Delta u_n\right\|_{L^{p'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}\\ &\quad + 2\left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|_{L^2\left(Q_{t_1}^{t_2}\right)}\|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}\\ &\quad + 2\left\|f\left(x,t,\frac{\partial u_n}{\partial t}\right)\right\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}.\end{aligned}$$

By Poincaré's inequality, it follows that

$$\begin{aligned}|\eta_{nj}(t_2)-\eta_{nj}(t_1)| &\le c_{18}\left\||\Delta u_n|^{p(x)-2}\Delta u_n\right\|_{L^{p'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)}\\ &\quad + c_{19}\left\|\nabla\left(\frac{\partial u_n}{\partial t}\right)\right\|_{L^2\left(Q_{t_1}^{t_2}\right)}\|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)}\\ &\quad + c_{20}\left\|f\left(x,t,\frac{\partial u_n}{\partial t}\right)\right\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}\|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}.\end{aligned}$$

By Remarks 3.4 and 3.5, $\||\Delta u_n|^{p(x)-2}\Delta u_n\|_{L^{p'(\cdot)}\left(Q_{t_1}^{t_2}\right)}$, $\|\nabla(\frac{\partial u_n}{\partial t})\|_{L^2\left(Q_{t_1}^{t_2}\right)}$ and $\|f(x,t,\frac{\partial u_n}{\partial t})\|_{L^{q'(\cdot)}\left(Q_{t_1}^{t_2}\right)}$ are bounded. Then

$$|\eta_{nj}(t_2)-\eta_{nj}(t_1)| \le c_{21}\left(\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)} + \|\nabla\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)} + \|\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}\right).$$

According to Theorems 2.1, 2.3 and 2.4, it follows that

$$|\eta_{nj}(t_2)-\eta_{nj}(t_1)| \le c_{22}\left(\|\Delta\omega_j\|_{L^{p(\cdot)}\left(Q_{t_1}^{t_2}\right)} + \|\Delta\omega_j\|_{L^2\left(Q_{t_1}^{t_2}\right)} + \|\Delta\omega_j\|_{L^{q(\cdot)}\left(Q_{t_1}^{t_2}\right)}\right).$$

Using (2.2), we get

$$\begin{aligned}&|\eta_{nj}(t_2)-\eta_{nj}(t_1)|\\ &\le c_{22}\Bigg[\max\left\{|t_2-t_1|^{\frac{1}{p^-}}\left(\int_\Omega|\Delta\omega_j|^{p(x)}\,dx\right)^{\frac{1}{p^-}}, |t_2-t_1|^{\frac{1}{p^+}}\left(\int_\Omega|\Delta\omega_j|^{p(x)}\,dx\right)^{\frac{1}{p^+}}\right\}\\ &\qquad + \max\left\{|t_2-t_1|^{\frac{1}{q^-}}\left(\int_\Omega|\Delta\omega_j|^{q(x)}\,dx\right)^{\frac{1}{q^-}}, |t_2-t_1|^{\frac{1}{q^+}}\left(\int_\Omega|\Delta\omega_j|^{q(x)}\,dx\right)^{\frac{1}{q^+}}\right\}\\ &\qquad + \max\left\{|t_2-t_1|^{\frac{1}{2}}\left(\int_\Omega|\Delta\omega_j|^2\,dx\right)^{\frac{1}{2}}\right\}\Bigg].\end{aligned}$$

Thus, the sequence $\eta_{nj}(t)$, with $1 \leq n < \infty$, is uniformly bounded and equicontinuous for fixed j and $n \geq j$ in $[0,T]$. Using Arzelà–Ascoli's theorem (see [18]), there is a subsequence such that $\eta_{nj}(t)$ converges uniformly in $[0,T]$ for some continuous function $\eta_j(t)$ for each fixed $j = 1, 2, \cdots$. We define

$$\bar{u}(x,t) = \sum_{j=1}^{\infty} \eta_j(t)\omega_j(x).$$

Then, for each $j \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} \int_\Omega \frac{\partial u_n}{\partial t}\omega_j\, dx = \int_\Omega \bar{u}\omega_j\, dx$$

uniformly in $[0,T]$. With the completeness of ω_j, we obtain that

$$\frac{\partial u_n}{\partial t} \rightharpoonup \bar{u} \text{ in } L^2(\Omega)$$

and uniformly in $[0,T]$ when $n \to \infty$. Furthermore, it turns out that $\bar{u} = \frac{\partial u}{\partial t}$. Using Remark 3.4 and Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_j\, dx \right)^2 dt = 0.$$

By Theorem 2.5, there is a positive number N_ε independent of n such that

$$\begin{aligned}\left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \leq{}& 2\sum_{j=1}^{N_\varepsilon} \int_0^T \left(\int_\Omega \left(\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right) \omega_j\, dx \right)^2 dt \\ &+ 2\varepsilon^2 \int_0^T \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|^2_{W_0^{1,2}(\Omega)} dt.\end{aligned}$$

Furthermore, by Remark 3.4, we have

$$\limsup_{n\to\infty} \left\| \frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t} \right\|_{L^2(Q_T)} \leq c_{23}\varepsilon^2.$$

The arbitrariness of ε implies that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2\left(Q_T\right).$$

Consequently, there is a subsequence of u_n such that

$$\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{a.e. in } Q_T.$$

For the continuity of f, we get

$$f\left(x, t, \frac{\partial u_n}{\partial t}\right) \to f\left(x, t, \frac{\partial u}{\partial t}\right) \quad \text{a.e. in } Q_T.$$

Our next objective is to prove that $u_n \to u$ in $L^{q(\cdot)}(Q_T)$. We know that $u_n \in W^{1,2}(Q_T)$. Thus, by Theorem 2.3, we can obtain a subsequence such that $u_n \to u$ in $L^2(Q_T)$ and a.e. in Q_T. By (1.4), Remark 3.4 and Theorem 2.3, we have

$$\int_\Omega |u_n|^{\frac{Np(x)}{N-2p(x)}}\,dx \le c_{24}, \quad t \in [0,T].$$

This implies that

$$\int_0^T \int_\Omega |u_n|^{\frac{Np(x)}{N-2p(x)}}\,dx\,dt \le c_{25}.$$

For any measurable subset $V \in Q_T$, if we use Hölder's inequality (2.3) and if $q(x) < p^* = \frac{Np(x)}{N-2p(x)}$, then

$$\int_V |u_n|^{q(x)}\,dx\,dt \le 2\,\||u_n|\|_{L^{\frac{p^*(\cdot)}{q(\cdot)}}(Q_T)} \|1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-q(\cdot)}}(V)} \le \|1\|_{L^{\frac{p^*(\cdot)}{p^*(\cdot)-q(\cdot)}}(V)}.$$

Thus, the sequence $|u_n|^{q(x)}$ with $1 \le n < \infty$ is equi-integrable in $L^1(Q_T)$. By Vitali's convergence theorem (see [30]),

$$\lim_{n\to\infty} \int_{Q_T} |u_n - u|^{q(x)}\,dx\,dt = 0.$$

Therefore, $u_n \to u$ in $L^{q(\cdot)}(Q_T)$.

Finally, our last objective here is to prove that $\xi = |\Delta u|^{p(x)-2}\Delta u$. We know that for all $\varphi \in C^1(0,T;C_0^\infty(\Omega))$, we can choose a sequence $\varphi_k \in C^1(0,T;V_k)$ such that $\varphi_k \to \varphi$ in $C^{1,2}(Q_T)$, where for any $u \in C^{1,2}(Q_T)$ its norm is given by

$$\|u\| = \sup_{\substack{|\alpha|\le 2\\ (x,t)\in\overline{Q_T}}} \left\{ |D^\alpha u|, \left|\frac{\partial u}{\partial t}\right| \right\}.$$

For all $\tau \in [0,T]$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2}\varphi_k\,dx\,dt \\
&\quad = \lim_{k\to\infty}\lim_{n\to\infty} \left(\int_\Omega \frac{\partial u_n(x,\tau)}{\partial t}\varphi_k(x,\tau)\,dx - \int_\Omega \frac{\partial u_n(x,0)}{\partial t}\varphi_k(x,0)\,dx \right) \\
&\qquad - \lim_{k\to\infty}\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial u_n}{\partial t}\frac{\partial \varphi_k}{\partial t}\,dx\,dt \\
&\quad = \lim_{k\to\infty} \left(\int_\Omega \bar{u}(x,\tau)\varphi_k(x,\tau)\,dx - \int_\Omega u_1\varphi_k(x,0)\,dx - \int_{Q_\tau} \frac{\partial u}{\partial t}\frac{\partial \varphi_k}{\partial t}\,dx\,dt \right) \\
&\quad = \int_\Omega \bar{u}(x,\tau)\varphi(x,\tau)\,dx - \int_\Omega u_1\varphi(x,0)\,dx - \int_{Q_\tau} \frac{\partial u}{\partial t}\frac{\partial \varphi}{\partial t}\,dx\,dt \\
&\quad = \lim_{n\to\infty} \int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2}\varphi\,dx\,dt,
\end{aligned}$$

where $Q_\tau = \Omega \times (0, \tau)$. By (3.4), it follows that

$$\int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2} \varphi_k \, dx \, dt + \int_{Q_\tau} \left(|\Delta u_n|^{p(x)-2} \Delta u_n \Delta \varphi_k + \nabla \left(\frac{\partial u_n}{\partial t} \right) \nabla \varphi_k \right) dx \, dt \\ = \int_{Q_\tau} f \left(x, t, \frac{\partial u_n}{\partial t} \right) \varphi_k \, dx \, dt.$$

Thus,

$$\lim_{n\to\infty} \int_{Q_\tau} \frac{\partial^2 u_n}{\partial t^2} \varphi \, dx \, dt \\ = \int_{Q_\tau} \left(f \left(x, t, \frac{\partial u}{\partial t} \right) \varphi - \xi \Delta \varphi - \nabla \left(\frac{\partial u}{\partial t} \right) \nabla \varphi \right) dx \, dt. \tag{3.32}$$

In addition, for any $\psi(x) \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega (\bar{u}(x,\tau) - u_1) \varphi \, dx &= \lim_{n\to\infty} \int_\Omega \left(\frac{\partial u_n(x,\tau)}{\partial t} - \frac{\partial u_n(x,0)}{\partial t} \right) \psi(x) \, dx \\ &= \lim_{n\to\infty} \int_0^\tau \int_\Omega \frac{\partial^2 u_n}{\partial t^2} \psi(x) \, dx \, dt \\ &= \int_{Q_\tau} \left(f \left(x, t, \frac{\partial u}{\partial t} \right) \varphi - \xi \Delta \varphi - \nabla \left(\frac{\partial u}{\partial t} \right) \nabla \varphi \right) dx \, dt \to 0, \end{aligned}$$

when $\tau \to 0$. Consequently, $\bar{u}(x,t)$ is weakly continuous in $L^2(\Omega)$, that is, we have $\bar{u}(x,t) \in C_w(0,T; L^2(\Omega))$. For all $\eta \in C^1([0,T])$ with $\eta(T) = 0$ and $\eta(0) = 1$, we get

$$\int_{Q_T} \frac{\partial u_n}{\partial t} \eta(t) \omega_i \, dx \, dt = - \int_\Omega u_n(x,0) \eta(0) \omega_i \, dx - \int_{Q_T} u_n(x,t) \eta'(t) \omega_i \, dx \, dt.$$

If $n \to \infty$, then

$$\int_\Omega (u(x,0) - u_0) \, \omega_i \, dx = 0 \quad \text{with } i = 1, 2, \ldots .$$

By the completeness of basis ω_i in $L^2(\Omega)$, we conclude that $u(x,0) = u_0$. Due to $\nabla u_n \overset{*}{\rightharpoonup} \nabla u$ in $L^\infty(0,T; L^2(\Omega))$ and $\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^2(0,T; W_0^{1,2}(\Omega))$, in the same way as in [23], we assume $u \in C(0,T; W_0^{1,2}(\Omega))$ and that there is a subsequence of u_n such that $\nabla u_n(x,T) \rightharpoonup \nabla u(x,T)$ in $(L^2(\Omega))^N$. Thus,

$$\int_\Omega |\nabla u(x,T)|^2 \, dx \leq \lim_{n\to\infty} \inf \int_\Omega |\nabla u_n(x,T)|^2 \, dx.$$

We take $\varphi = u_k$ in equation (3.32) and if $k \to \infty$, then

$$\int_\Omega \bar{u}(x,T) u(x,T) \, dx - \int_\Omega u_1 u_0 \, dx - \int_{Q_T} \left| \frac{\partial u}{\partial t} \right|^2 dx \, dt$$

$$+\int_{Q_T}\left(\xi\Delta u+\nabla\left(\frac{\partial u}{\partial t}\right)\nabla u\right)dx\,dt=\int_{Q_T}f\left(x,t,\frac{\partial u}{\partial t}\right)u\,dx\,dt. \tag{3.33}$$

Multiplying (3.7) by η_{nj}, adding j from 1 to n and integrating from 0 to T, we obtain

$$\int_0^T\int_\Omega\frac{\partial^2u_n}{\partial t^2}u_n\,dx\,dt+\int_0^T\int_\Omega\left(|\Delta u_n|^{p(x)}+\nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla u_n\right)dx\,dt$$
$$=\int_0^T\int_\Omega f\left(x,t,\frac{\partial u_n}{\partial t}\right)u_n\,dx\,dt.$$

Therefore,

$$\begin{aligned}0\le&\int_0^T\int_\Omega\left(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u)\,dx\,dt\\=&\int_0^T\int_\Omega\left(f\left(x,t,\frac{\partial u_n}{\partial t}\right)u_n-\nabla\left(\frac{\partial u_n}{\partial t}\right)\nabla u_n\right)dx\,dt\\&-\int_\Omega\frac{\partial u_n(x,T)}{\partial t}u_n(x,T)\,dx+\int_\Omega\frac{\partial u_n(x,0)}{\partial t}u_n(x,0)\,dx+\int_0^T\int_\Omega\left|\frac{\partial u_n}{\partial t}\right|^2dx\,dt\\&-\int_0^T\int_\Omega\left(|\Delta u_n|^{p(x)-2}\Delta u_n\Delta u+|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u)\ dx\,dt\end{aligned}$$

By equation (3.33), we have

$$\begin{aligned}\lim_{n\to\infty}\sup&\int_0^T\int_\Omega\left(|\Delta u_n|^{p(x)-2}\Delta u_n\Delta u+|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u)\,dx\,dt\\&\le\int_0^T\int_\Omega\left(f\left(x,t,\frac{\partial u}{\partial t}\right)u-\xi\Delta u\right)dx\,dt-\frac{1}{2}\int_\Omega|\nabla u(x,T)|^2\,dx\\&\quad+\frac{1}{2}\int_\Omega|\nabla u(x,0)|^2\,dx-\int_\Omega\bar{u}(x,T)u(x,T)\,dx\\&\quad+\int_\Omega u_1u_0\,dx+\int_0^T\int_\Omega\left|\frac{\partial u}{\partial t}\right|^2dx\,dt=0.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int_0^T\int_\Omega\left(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u)\,dxdt=0.$$

We define $Q_1=\{(x,t)\in Q_T\mid p(x)\ge 2\}$ and $Q_2=\{(x,t)\in Q_T\mid 1<p(x)<2\}$. When $n\to\infty$, we have

$$\begin{aligned}&\int_{Q_1}|\Delta u_n-\Delta u|^{p(x)}\,dx\,dt\\&\qquad\le c_{26}\int_{Q_1}\left(|\Delta u_n|^{p(x)-2}\Delta u_n-|\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n-\Delta u)\,dx\,dt\to 0.\end{aligned}$$

Moreover,

$$\int_{Q_2} |\Delta u_n - \Delta u|^{p(x)} dxdt$$
$$\leq c_{27}\left(\left\| \left[\left(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\right)(\Delta u_n - \Delta u)\right]^{\frac{p(x)}{2}} \right\|_{L^{\frac{2}{p(\cdot)}}(Q_2)} \left\| \left(|\Delta u_n|^{p(x)} + |\Delta u|^{p(x)}\right)^{\frac{2-p(x)}{2}} \right\|_{L^{\frac{2}{2-p(\cdot)}}(Q_2)}\right) \to 0.$$

Consequently, we obtain $\Delta u_n \to \Delta u$ in $L^{p(\cdot)}(Q_T)$. Then there is a subsequence of u_n such that $\Delta u_n \to \Delta u$ a.e. in Q_T. Moreover,

$$|\Delta u_n|^{p(x)-2}\Delta u_n \to |\Delta u|^{p(x)-2}\Delta u \quad \text{for a.a. } (x,t) \in Q_T.$$

To this end, we obtain that $\xi = |\Delta u|^{p(x)-2}\Delta u$, and thus the theorem on the existence of weak solutions to problem (1.1) is proved. □

4. Asymptotic behavior

In this section, we use Nakao's lemma to establish the asymptotic behavior of the weak solutions obtained by Theorem 3.2. Our main result is the following theorem.

Theorem 4.1. *If* $\max\{1, \frac{2N}{N+2}\} < p^-$, *then there are constants* $C > 0$ *and* $\gamma > 0$ *such that, for all* $t \geq 0$, *the weak solutions of problem* (1.1) *satisfy:*

$$\int_\Omega \left(\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + |\Delta u(x,t)|^{p(x)}\right) dx \leq \begin{cases} Ce^{-\gamma t} & \text{if } p^+ = 2 \\ C(t+1)^{-\frac{p^+}{p^+-2}} & \text{if } p^+ > 2 \end{cases}$$

if $q^- \geq 2$; *and*

$$\int_\Omega \left(\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + |\Delta u(x,t)|^{p(x)}\right) dx \leq \begin{cases} C(t+1)^{-\frac{p^+(q^- -1)}{p^+-q^-}} & \text{if } p^+ < q^- \\ Ce^{-\gamma t} & \text{if } p^+ \geq q^- \end{cases}$$

if $1 < q^- < 2$.

Proof. We define

$$I(t) = \frac{1}{2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)} dx.$$

By the definition, $I(t)$ is non-negative and uniformly bounded. Thus, there is $M > 0$ such that $I(t) \leq M$ for all $t \geq 0$, which implies $\frac{d}{dt}I(t) \leq 0$ and that $I(t)$ is non-increasing. In addition, notice that

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t} dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx \tag{4.1}$$

and

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \left(\frac{\partial u}{\partial t}\right) dx = \frac{d}{dt}\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}\, dx. \tag{4.2}$$

From the definition of $I(t)$ and by (4.1) and (4.2), we obtain

$$\frac{d}{dt}I(t) = \int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\, dx + \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \left(\frac{\partial u}{\partial t}\right) dx \le 0. \tag{4.3}$$

Using (3.4) and taking $v = \frac{\partial u}{\partial t}$, we have

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}\, dx + \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta \left(\frac{\partial u}{\partial t}\right) dx + \int_\Omega \nabla\left(\frac{\partial u}{\partial t}\right)\nabla\left(\frac{\partial u}{\partial t}\right) dx$$
$$= \int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial t}\, dx.$$

Replacing (4.3) in the equation above leads us to

$$\frac{d}{dt}I(t) + \int_\Omega \left(\nabla\left(\frac{\partial u}{\partial t}\right)\right)^2 dx = \int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial t}\, dx.$$

By using (1.3), it follows that

$$\frac{d}{dt}I(t) + \int_\Omega \left|\nabla\left(\frac{\partial u}{\partial t}\right)\right|^2 dx + c_1\int_\Omega \left|\frac{\partial u}{\partial t}\right|^{q(x)} dx \le 0. \tag{4.4}$$

Let us define

$$J^2(t) = I(t) - I(t+1). \tag{4.5}$$

Then, integrating (4.4) over $(t, t+1)$, we arrive at

$$J^2(t) \ge \int_t^{t+1}\int_\Omega \left(\left|\nabla\left(\frac{\partial u(x,s)}{\partial s}\right)\right|^2 + c_1\left|\frac{\partial u(x,s)}{\partial s}\right|^{q(x)}\right) dx\, ds.$$

Recall Poincaré's inequality

$$J^2(t) \ge c_{28}\int_t^{t+1}\int_\Omega \left|\frac{\partial u(x,s)}{\partial s}\right|^2 dx\, ds. \tag{4.6}$$

Using the mean value theorem and (4.6), we have $t_1 \in [t, t+\frac{1}{3}]$ and $t_2 \in [t+\frac{2}{3}, t+1]$ such that

$$\int_\Omega \left|\frac{\partial u(x,t_i)}{\partial t}\right|^2 dx \le \frac{1}{c_{28}}J^2(t), \quad i = 1, 2. \tag{4.7}$$

Multiplying equation (1.1) by a function u, integrating over Ω and using Green's formula, we get

$$\int_\Omega |\Delta u|^{p(x)}\,dx = -\int_\Omega \frac{\partial^2 u}{\partial t^2}u\,dx + \int_\Omega \nabla\left(\frac{\partial u}{\partial t}\right)\nabla u\,dx + \int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right)u\,dx.$$

Integrating over (t_1,t_2), we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega |\Delta u|^{p(x)}\,dx\,dt = &-\int_\Omega \frac{\partial u(x,t_2)}{\partial t}u(x,t_2)\,dx + \int_\Omega \frac{\partial u(x,t_1)}{\partial t}u(x,t_1)\,dx\\ &+\int_{t_1}^{t_2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx\,dt - \int_{t_1}^{t_2}\int_\Omega \nabla\left(\frac{\partial u}{\partial t}\right)\nabla u\,dx\,dt\\ &+\int_{t_1}^{t_2}\int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right)u\,dx\,dt. \qquad (4.8)\end{aligned}$$

Inequality (4.7), Hölder's inequality (2.3), the Sobolev embedding theorem 2.3 and the fact that $I(t)$ is non-increasing imply that

$$\begin{aligned}\left|\int_\Omega \frac{\partial u(x,t_i)}{\partial t}u(x,t_i)\,dx\right| &\le \|u(x,t_i)\|_{L^2(\Omega)}\left\|\frac{\partial u(x,t_i)}{\partial t}\right\|_{L^2(\Omega)}\\ &\le c_{29}\|\Delta u(x,t_i)\|_{L^{p(\cdot)}(\Omega)}J(t)\\ &\le c_{30}\left(\int_\Omega \frac{|\Delta u(x,t_i)|^{p(x)}}{p(x)}dx\right)^{\frac{1}{p^+}}J(t)\\ &\le c_{30}\,(I(t))^{\frac{1}{p^+}}J(t),\quad i=1,2, \qquad (4.9)\end{aligned}$$

where the third inequality in (4.9) is obtained using (2.2) as follows:

$$\begin{aligned}&\|\Delta u(x,t_i)\|_{L^{p(\cdot)}(\Omega)}\\ &\le (p^+)^{\frac{1}{p^-}}\max\left\{\left(\int_\Omega \frac{|\Delta u(x,t_i)|^{p(x)}}{p(x)}dx\right)^{\frac{1}{p^-}},\left(\int_\Omega \frac{|\Delta u(x,t_i)|^{p(x)}}{p(x)}dx\right)^{\frac{1}{p^+}}\right\}\\ &\le (p^+)^{\frac{1}{p^-}}\max\left\{M, M^{\frac{1}{p^-}-\frac{1}{p^+}}\right\}\left(\int_\Omega \frac{|\Delta u(x,t_i)|^{p(x)}}{p(x)}dx\right)^{\frac{1}{p^+}},\quad i=1,2.\end{aligned}$$

In a similar way, using the fact that $\nabla u\in (L^2(\Omega))^N$, $\frac{2N}{N+2}<p^-$ and Theorem 2.4, we have

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_\Omega \nabla\left(\frac{\partial u}{\partial t}\right)\nabla u\,dx\,dt\right| &\le \int_{t_1}^{t_2}\left\|\nabla\frac{\partial u(x,s)}{\partial s}\right\|_{L^2(\Omega)}\sup_{t\le s\le t+1}\|\nabla u(x,s)\|_{L^2(\Omega)}\,ds.\\ &\le c_{31}\,(I(t))^{\frac{1}{p^+}}J(t). \qquad (4.10)\end{aligned}$$

Using (1.3), Hölder's inequality (2.3), inequality (4.6) and the boundedness of $I(t)$ and $J(t)$, we get

$$\left|\int_{t_1}^{t_2}\int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right)u\,dx\,dt\right| \le 2c_2\int_{t_1}^{t_2}\left\|\left|\frac{\partial u}{\partial t}\right|^{q(x)-1}\right\|_{L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega)}\|u\|_{L^{q(\cdot)}(\Omega)}\,dt$$

$$\le c_{32}\left[\left(\int_{t_1}^{t_2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^{q(x)} dx\,dt\right)^{\frac{q^- -1}{q^-}} + \left(\int_{t_1}^{t_2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^{q(x)} dx\,dt\right)^{\frac{q^+ -1}{q^+}}\right](I(t))^{\frac{1}{p^+}}.$$

Thus,

$$\left|\int_{t_1}^{t_2}\int_\Omega f\left(x,t,\frac{\partial u}{\partial t}\right) u\,dx\,dt\right| \le c_{33}\,(J(t))^{\frac{2(q^- -1)}{q^-}}\,(I(t))^{\frac{1}{p^+}}. \tag{4.11}$$

From (4.8)–(4.11), we obtain

$$\int_{t_1}^{t_2}\int_\Omega |\Delta u|^{p(x)}\,dx\,dt$$
$$\le \frac{1}{c_{28}}J^2(t) + (2c_{30}+c_{31})J(t)\,(I(t))^{\frac{1}{p^+}} + c_{33}\,(J(t))^{\frac{2(q^- -1)}{q^-}}\,(I(t))^{\frac{1}{p^+}}.$$

Therefore,

$$\int_{t_1}^{t_2} I(t)\,dt \le c_{34}J^2(t) + c_{35}\left(J(t)\,(I(t))^{\frac{1}{p^+}} + (J(t))^{\frac{2(q^- -1)}{q^-}}\,(I(t))^{\frac{1}{p^+}}\right).$$

The monotonicity of $I(t)$ implies that $I(t+1) \le 3\int_{t_1}^{t_2} I(t)dt$, and by equation (4.5), we have $I(t+1) = I(t) - J^2(t)$. Furthermore, applying Yong's inequality, it follows that

$$I(t) \le c_{36}J^2(t) + c_{37}\left((J(t))^{\frac{2p^+(q^- -1)}{q^-(p^+ -1)}} + (J(t))^{\frac{p^+}{p^+ -1}}\right). \tag{4.12}$$

- If $q^- \ge 2$, then
$$\frac{2p^+(q^- -1)}{q^-(p^+ -1)} \ge \frac{p^+}{p^+ -1}.$$

Consequently, by the boundedness of $J(t)$ and by (4.12), it follows that

$$I(t) \le c_{36}J^2(t) + c_{38}\,(J(t))^{\frac{p^+}{p^+ -1}}.$$

Moreover, if $p^+ = 2$, then $I(t) \le c_{39}J^2(t)$. Since $I(t)$ is non-increasing, then, from Nakao's Lemma 2.6, there are two constants $C > 0$ and $\gamma > 0$ such that $I(t) \le Ce^{-\gamma t}$, $t > 0$, that is,

$$\int_\Omega \left(\left|\frac{\partial u}{\partial t}\right|^2 + |\Delta u|^{p(x)}\right) dx \le Ce^{-\gamma t}, \quad t > 0. \tag{4.13}$$

If $p^+ > 2$, then $I(t) \le c_{40}(J(t))^{\frac{p^+}{p^+ -1}}$. From Nakao's Lemma 2.6, there is a constant $C > 0$ such that $I(t) \le C(t+1)^{-\frac{p^+}{p^+ -2}}$, that is,

$$\int_\Omega \left(\left|\frac{\partial u}{\partial t}\right|^2 + |\Delta u|^{p(x)}\right) dx \le C(t+1)^{-\frac{p^+}{p^+ -2}}, \quad t > 0. \tag{4.14}$$

- If $1 < q^- < 2$, then
$$\frac{2p^+(q^- - 1)}{q^-(p^+ - 1)} < \frac{p^+}{p^+ - 1}.$$

Consequently, by the boundedness of $J(t)$ and by (4.12), we have

$$I(t) \leq c_{36} J^2(t) + c_{41} \left(J(t)\right)^{\frac{2p^+(q^- - 1)}{q^-(p^+ - 1)}}.$$

In addition, if $q^- \geq p^+$, then $I(t) \leq c_{42} J^2(t)$. Again, from Nakao's Lemma 2.6, there are two constants $C > 0$ and $\gamma > 0$ such that $I(t) \leq Ce^{-\gamma t}$, $t > 0$, that is,

$$\int_\Omega \left(\left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u|^{p(x)} \right) dx \leq Ce^{-\gamma t}, \quad t > 0. \tag{4.15}$$

Whereas, if $q^- < p^+$, then

$$I(t) \leq c_{43} \left(J(t)\right)^{\frac{2p^+(q^- - 1)}{q^-(p^+ - 1)}}.$$

Using Nakao's Lemma 2.6, it follows that $I(t) \leq C(t+1)^{-\frac{p^+(q^- - 1)}{p^+ - q^-}}$, $t > 0$, that is,

$$\int_\Omega \left(\left| \frac{\partial u}{\partial t} \right|^2 + |\Delta u|^{p(x)} \right) dx \leq C(t+1)^{-\frac{p^+(q^- - 1)}{p^+ - q^-}}, \quad t > 0. \tag{4.16}$$

Therefore, due to (4.13), (4.14), (4.15) and (4.16), we complete the proof of Theorem 4.1. □

5. Conclusion

In this paper, we established the existence and asymptotic behavior of weak solutions of a non-linear fourth-order beam equation with a strong dissipation and a lower order perturbation of the $p(x)$-biharmonic type over a bounded domain. This asymptotic behavior can be exponential and polynomial depending on the ranges of variable exponents. It was shown by basing on Nakao's lemma with classical functional analysis results and the Lebesgue and Sobolev spaces with variable exponents. In addition, using Faedo-Galerkin's method, the existence of weak solutions was proved.

The authors emphasize that the uniqueness of the weak solution to problem (1.1) has not been proven yet. Thus, as a future work, it is of great interest to study this uniqueness.

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Існування та асимптотичне поводження розв'язків рівняння балки із сильним демпфуванням і $p(x)$-бігармонічний оператор

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У цій роботі ми розглядаємо нелінійне рівняння балки із сильним демпфуванням та $p(x)$-бігармонічний оператор. Показник нелінійності $p(\cdot)$ є заданою функцією, яка задовольняє певні умови. Застосовуючи метод Фаедо–Ґалеркіна, ми довели існування слабких розв'язків. Застосовуючи лему Такао, ми встановили асимптотичне поводження слабких розв'язків за м'яких припущень щодо показника $p(\cdot)$. Ми доводимо, що асимптотичне поводження слабкого розв'язку є експоненційно і алгебраїчно залежним від змінного показника. Ця робота поліпшує та узагальнює багато інших результатів згаданих в літературі.

Ключові слова: слабки розв'язки, існування, асимптотичне поводження, рівняння балки, $p(x)$-бігармонічний оператор, змінний показник