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Multiplicity of Solutions to a p-q Fractional Laplacian System with Concave Singular Nonlinearities

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In this paper, we study the existence of multiple nontrivial nonnegative weak solutions to a coupled system of elliptic PDEs. The existence of solutions in the Nehari manifold is proved. The Lusternik–Schnirelman category is used to prove the existence of at least $\mathrm{cat}(\Omega)+1$ number of solutions, where Ω is a bounded domain in which the problem is considered.

Key words: Nehari manifold, Lusternik–Schnirelman category, singularity, multiplicity

Mathematical Subject Classification 2010: 35J35, 35J60

1. Introduction

In this section, we introduce the problem and discuss some developments in this direction in the literature. We consider the following problem:

$$
\begin{aligned}
-(\Delta_p)^s u - (-\Delta_q)^s u &= \lambda f(x) u^{r-1} + \nu \frac{1-\alpha}{2-\alpha-\beta} h(x) u^{-\alpha} v^{1-\beta} && \text{in } \Omega, && (1.1)\\
-(\Delta_p)^s v - (-\Delta_q)^s v &= \mu g(x) v^{r-1} + \nu \frac{1-\beta}{2-\alpha-\beta} h(x) u^{1-\alpha} v^{-\beta} && \text{in } \Omega, && (1.2)\\
u > 0,\ v &> 0 && \text{in } \Omega, && (1.3)\\
u = v &= 0 && \text{in } \mathbb{R}^N \setminus \Omega, && (1.4)
\end{aligned}
$$

where

(C): $\lambda, \mu, \nu > 0$, $0 < s, \alpha, \beta < 1$, $2-\alpha-\beta < q < \frac{N(p-1)}{N-s} < p < r \le p^*$.

We are mainly interested in positive solutions to (1.1)–(1.4). The functions $f, g, h > 0$ are measurable over Ω and are bounded almost everywhere in Ω, i.e., $f, g, h \in L^\infty(\Omega)$. The operator $(-\Delta_p)^s$ acting on a function, say U, is the fractional p-Laplacian operator which is defined as

$$(-\Delta_p)^s U(x) = C_{N,s}\, \mathrm{V.\,P.} \int_{\mathbb{R}^N} \frac{|U(x)-U(y)|^{p-2}(U(x)-U(y))}{|x-y|^{N+ps}} dy$$

for all $p \in [1, \infty)$, with $C_{N,s}$ being the normalizing constant. In a similar way, one can define $(-\Delta_q)^s$. Throughout the paper, we will assume $N \geq 2$, $sp < N$, $0 < s < 1$. A large amount of attention has been given of late to elliptic problems involving two Laplacian operators viz.

$$\begin{aligned} -(\Delta_p)u - (-\Delta_q)u &= \lambda |u|^{r-2}u + |u|^{p^*-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The main motivation for problems of this kind is the fundamental reaction-diffusion equation

$$\frac{\partial}{\partial t}u = \nabla \cdot [H(u)\nabla u] + c(x,u), \tag{1.5}$$

where $H(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. The problem is a model equation to phenomena in physics and other applied sciences such as in biophysics to model the cells, design of chemical reaction, plasma physics, etc. The reaction term has a polynomial form with respect to u. Of late, problem (1.5) with

$$H(u) = c(x,u)$$

was studied in [4, 6, 18, 19, 27, 29]. One may refer to Yin and Yang [31] who studied problem (1.5) when $p^2 < N$, $1 < q < p < r < p*$. The authors in [31] proved the existence of $\mathrm{cat}(\Omega)$ number of positive solutions using simple variational techniques. For $p = q$, $r = 2$, problem (1.5) reduces to the well-known *Brezis–Nirenberg problem* which was further studied for the case of critical growth in bounded and unbounded domains by many researchers (see [2, 3, 5, 25] and references therein). A common issue, which kept the interest to the problem, was to figure out a way for overcoming the lack of compactness in the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Two noteworthy contributions can be found in [10, 24].

At the same time, elliptic systems have also gained much attention, especially the system

$$-(\Delta_p)u = \lambda |u|^{r-2}u + \frac{2a}{a+b}|u|^{a-2}u|v|^b \qquad \text{in } \Omega, \tag{1.6}$$

$$-(\Delta_p)v = \mu |v|^{r-2}v + \frac{2b}{a+b}|u|^{a}|v|^{b-2}u \qquad \text{in } \Omega, \tag{1.7}$$

$$u = v = 0 \qquad \text{on } \partial\Omega, \tag{1.8}$$

where $a + b = p^*$. Ding and Xiao [12] studied (1.6)–(1.8) with the p-superlinear perturbation of $2 \leq p \leq r < p^*$ an extension of which can be found in the paper by Yin [30]. In both these works, in [12] and [30], the authors obtained the existence of $\mathrm{cat}(\Omega)$ number of solutions using the Lusternik–Schnirelman category. For the sublinear perturbation, Hsu [17] obtained the existence of two positive solutions for problem (1.6)–(1.8). Eight years ago, Fan [14] studied problem (1.6)–(1.8) for $p = 2$ and $1 < r < p$. Using the Nehari manifold and the

Lusternik–Schnirelman category, the author proved the admittance of at least $\mathrm{cat}(\Omega)+1$ positive solutions. When talking about the doubly nonlocal equation, we should refer to [16], where the following problem was considered:

$$\begin{aligned} (-\Delta_p)^{s_1}u + (-\Delta_q)^{s_2}u &= \lambda a(x)|u|^{\delta-2}u + b(x)|u|^{r-2}u && \text{in } \Omega \\ u &= 0 && \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $1 < \delta \le q \le p < r \le p^*_{s_1}$. Thereafter, in [7], the authors studied the problem

$$\begin{aligned} &(-\Delta_p)^s u = \lambda|u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} && \text{in } \Omega, \\ &(-\Delta_p)^s v = \mu|u|^{q-2}u + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v && \text{in } \Omega, \\ &u = v = 0 && \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

They guaranteed the multiplicity of solutions in a Nehari manifold. Further, Fu et al. [15] considered the following problem:

$$\begin{aligned} &(-\Delta_p)^s u = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^{\alpha-2}|v|^{\beta}v + \frac{\mu(x)}{\alpha\delta}|u|^{\gamma-2}|v|^{\delta}u && \text{in } \Omega, \\ &(-\Delta_p)^s v = \lambda c(x)|v|^{q-2}v + \lambda b(x)|u|^{\alpha}|v|^{\beta-2}v + \frac{\mu(x)}{\beta\delta}|u|^{\gamma}|v|^{\delta-2}v && \text{in } \Omega, \\ &u = v = 0 && \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

Here, $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, $1 < p < r$, $1 < q < \delta$, $\frac{\gamma}{p^*_s} + \frac{\delta}{q^*_s} < 1$, $\frac{1}{\alpha\delta} + \frac{1}{\beta\gamma} < 1$, p^*_s, q^*_s are fractional Sobolev critical exponents: $p^*_s = \frac{Np}{N-ps}$, $q^*_s = \frac{Nq}{N-qs}$.

Another noteworthy contribution was made by Zhen et al. [32], where the following critical system was studied:

$$\begin{aligned} &(-\Delta)^s u = \mu_1|u|^{2^*-2}u + \frac{\alpha\gamma}{2^*}|u|^{\alpha-2}|v|^{\beta}u && \text{in } \mathbb{R}^N, \\ &(-\Delta)^s v = \mu_2|v|^{2^*-2}v + \frac{\beta\gamma}{2^*}|u|^{\alpha}|v|^{\beta-2}v && \text{in } \mathbb{R}^N, \\ &u = v = 0 && \text{in } D_s(\mathbb{R}^N). \end{aligned}$$

Here $D_s(\mathbb{R}^N)$ is the completion of the space of compactly supported smooth functions with the norm

$$\|u\|^2_{D_s(\mathbb{R}^N)} = \frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

Motivated by the works of Li, Yang [21] and Choudhuri et al [9], we extend the results of the above problem with local operators and added singular nonlinearities. One may even consider this work to be a sequel to [9]. To our knowledge, there has not been any contribution in this direction and ours is entirely new. We now state the main result of this work.

Theorem 1.1. *Assume the condition (C) holds. Then there exists $\Lambda^* > 0$ such that if $\nu \in (0, \Lambda^*)$, problem (1.1)–(1.4) admits at least $\mathrm{cat}(\Omega)+1$ number of distinct solutions.*

2. Preliminaries

Let Y be a space that is defined as

$$Y = \left\{ u : \mathbb{R}^N \to \mathbb{R} \,\middle|\, u \text{ is measurable, } u|_\Omega \in L^p(Q), \text{ and } \frac{u(x)-u(y)}{|x-y|^{\frac{N+sp}{2}}} \in L^p(Q) \right\}$$

and is equipped with the Gagliardo norm

$$\|u\|_Y = |u|_p + \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dy\, dx \right)^{\frac{1}{p}},$$

where $\Omega \subset \mathbb{R}^N$, $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$. Here $|u|_p$ refers to the L^p-norm of u. We will frequently use the subspace Y_0 of Y which is defined as

$$Y_0 = \left\{ u \in X \,\middle|\, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

equipped with the norm

$$\|u\|_p = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dy\, dx \right)^{\frac{1}{p}}.$$

Remark 2.1. $Y \subset W^{s,p}(\Omega)$, $Y_0 \subset W^{s,p}(\Omega)$, where $W^{s,p}(\Omega)$ is the usual fractional order Sobolev space equipped with the norm (the Gagliardo norm)

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_p + \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dy\, dx \right)^{\frac{1}{p}}.$$

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then the space $(W_0^{s,p}(\Omega), \|\cdot\|_p)$ is defined by

$$W_0^{s,p}(\Omega) = \left\{ u \,\middle|\, \frac{u(x)-u(y)}{|x-y|^{\frac{N+sp}{p}}} \in L^p(\Omega\times\Omega),\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

equipped with the norm

$$\|u\|_p = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \right)^{\frac{1}{p}}.$$

We will refer to $|u|_r$ as the L^r-norm of u defined as $(\int_\Omega |u|^r dx)^{\frac{1}{r}}$ for $1 \le r < \infty$. Clearly, $W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$ is a reflexive Banach space [11]. We define the norm of any member of $W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$ as

$$\|(u,v)\|_p = (\|u\|_p + \|v\|_p)^{1/p}.$$

The best Sobolev constant is defined as

$$S = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{\|u\|_p^p}{\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p_s^*}}}, \tag{2.1}$$

and further we define

$$S' = \inf_{(u,v)\in X\setminus\{(0,0)\}} \frac{\|(u,v)\|_p^p}{(\int_\Omega |u|^{p^*} + |v|^{p^*}\,dx)^{\frac{p}{p^*}}}. \tag{2.2}$$

We also denote $M = \|h\|_\infty$, $M' = \max\{\|f\|_\infty, \|g\|_\infty\}$, where $\|\cdot\|_\infty$ denotes the essential supremum norm (or more commonly the L^∞-norm) of a function. We will seek for a solution in the *function* space $X = Z \times Z$, where $Z = W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$. The space X is equipped with the norm

$$\|(u,v)\| = \|(u,v)\|_p + \|(u,v)\|_q.$$

The space X is a reflexive Banach space. We now define the associated energy functional to problem (1.1)–(1.4) which is as follows:

$$\begin{aligned} J_{\alpha,\beta}(u,v) = \frac{1}{p}\|(u,v)\|_p^p + \frac{1}{q}\|(u,v)\|_q^q - \frac{1}{r}\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx \\ - \frac{\nu}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \end{aligned}$$

A function $(u,v) \in X$ is a weak solution to problem (1.1)–(1.4) if $u, v > 0$, $u^{-\alpha}\phi_1, v^{-\beta}\phi_2 \in L^1(\Omega)$, and

$$\begin{aligned} &\langle(-\Delta_p)^s u, \phi_1\rangle + \langle(-\Delta_p)^s v, \phi_2\rangle + \langle(-\Delta_q)^s u, \phi_1\rangle + \langle(-\Delta_q)^s v, \phi_2\rangle \\ &- \int_\Omega (\lambda f(x)u_+^{r-1}\phi_1 + \mu g(x)v_+^{r-1}\phi_2)\,dx - \nu\frac{1-\alpha}{2-\alpha-\beta}\int_\Omega h(x)u_+^{-\alpha}v_+^{1-\beta}\phi_1\,dx \\ &- \nu\frac{1-\beta}{2-\alpha-\beta}\int_\Omega h(x)u_+^{1-\alpha}v_+^{-\beta}\phi_2\,dx = 0. \end{aligned}$$

for each $\phi_2, \phi_2 \in X$. We have used the following notation,

$$\langle(-\Delta_{\bar r})^s w, \phi_i\rangle = \int_\Omega\int_\Omega \frac{|w(x)-w(y)|^{\bar r-2}(w(x)-w(y))(\phi_i(x)-\phi_i(y))}{|x-y|^{N+\bar r s}}\,dx\,dy$$

for $i = 1, 2$, $w = u$ or v, $\bar r = p$ or q. Observe that the nontrivial critical points of the functional $J_{\alpha,\beta}$ are the positive weak solutions of problem (1.1)–(1.4). Further, the reason that the functional $J_{\alpha,\beta}$ is not a C^1-functional does not allow us to apply classical variational methods. It is not difficult to verify that the energy functional $J_{\alpha,\beta}$ is not bounded below in X. However, we will show that $J_{\alpha,\beta}$ is bounded below on a *Nehari manifold* and we will extract solutions by minimizing the functional on suitable subsets.

We further define the Nehari manifold as follows:

$$\mathcal{M}_{\alpha,\beta} = \{(u,v) \in Z \setminus (0,0) \mid u, v > 0,\ \langle J'_{\alpha,\beta}(u,v), (u,v)\rangle = 0\}.$$

It is easy to see that a pair $(u,v) \in \mathcal{M}_{\alpha,\beta}$ if and only if

$$\|(u,v)\|_p^p + \|(u,v)\|_q^q - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx - \nu\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}\,dx = 0.$$

Just like for any problem, which has an involvement of a Nehari manifold, we also see here that

$$\begin{aligned}
J_{\alpha,\beta}(u,v) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q \\
&\quad + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&\geq \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p + \|(u,v)\|_q^q\right) \\
&\quad + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&\geq \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p - \nu\left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right)\|(u,v)\|_p^{2-\alpha-\beta}.
\end{aligned}$$

Since $2-\alpha-\beta<p$, we have that $J_{\alpha,\beta}$ is coercive and bounded below on $\mathcal{M}_{\alpha,\beta}$, and thus the functional is coercive and bounded below in $\mathcal{M}_{\alpha,\beta}$. Note that $J_{\alpha,\beta}(u,v)\geq 0$ for sufficiently small $\nu>0$ and for all $(u,v)\in\mathcal{M}_{\alpha,\beta}$. For $t\geq 0$, we define the fiber maps

$$\begin{aligned}
\Psi(t) = J_{\alpha,\beta}(tu,tv) &= \frac{t^p}{p}\|(u,v)\|_p^p + \frac{t^q}{q}\|(u,v)\|_q^q - \frac{t^r}{r}\int_\Omega\left(\lambda f(x)u^r + \mu g(x)v^r\right)dx \\
&\quad - \nu\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.
\end{aligned}$$

Then

$$\begin{aligned}
\Psi'(t) = t^{p-1}\|(u,v)\|_p^p + t^{q-1}\|(u,v)\|_q^q - t^{r-1}\int_\Omega\left(\lambda f(x)u^r + \mu g(x)v^r\right)dx \\
- \nu t^{1-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx
\end{aligned}$$

and

$$\begin{aligned}
\Psi''(t) &= (p-1)t^{p-2}\|(u,v)\|_p^p + (q-1)t^{q-2}\|(u,v)\|_q^q \\
&\quad - (r-1)t^{r-2}\int_\Omega\left(\lambda f(x)u^r + \mu g(x)v^r\right)dx \\
&\quad - \nu(1-\alpha-\beta)t^{-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.
\end{aligned}$$

Observe that $(u,v)\in\mathcal{M}_{\alpha,\beta}$ if and only if $\Psi'(1)=0$. In general, we have that $(u,v)\in\mathcal{M}_{\alpha,\beta}$ if and only if $\Psi'(1)=0$. Therefore, for $(u,v)\in\mathcal{M}_{\alpha,\beta}$, we have

$$\begin{aligned}
\Psi''(1) = (p-1)\|(u,v)\|_p^p + (q-1)\|(u,v)\|_q^q - (r-1)\int_\Omega\left(\lambda f(x)u^r + \mu g(x)v^r\right)dx \\
- \nu(1-\alpha-\beta)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx
\end{aligned}$$

$$
\begin{aligned}
&= (p-r)\|(u,v)\|_p^p + (q-r)\|(u,v)\|_q^q \\
&\quad + \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&= (p+\alpha+\beta-2)\|(u,v)\|_p^p + (q+\alpha+\beta-2)\|(u,v)\|_q^q \\
&\quad + (2-\alpha-\beta-r)\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\, dx.
\end{aligned}
$$

We thus split the Nehari manifold into three parts, namely,

$$
\begin{aligned}
\mathcal{M}_{\alpha,\beta}^{+} &= \{(u,v) \in \mathcal{M}_{\alpha,\beta} \mid \Psi''(1) > 0\}, \\
\mathcal{M}_{\alpha,\beta}^{-} &= \{(u,v) \in \mathcal{M}_{\alpha,\beta} \mid \Psi''(1) < 0\}, \\
\mathcal{M}_{\alpha,\beta}^{0} &= \{(u,v) \in \mathcal{M}_{\alpha,\beta} \mid \Psi''(1) = 0\},
\end{aligned}
$$

which corresponds to the collection of local minima, maxima and points of inflection, respectively. We now prove a lemma which follows the proof due to Hsu [17] (refer to Theorem 2.2).

Lemma 2.2. *For $(u,v) \in \mathcal{M}_{\alpha,\beta}$, there exists a positive constant A_0 that depends on $p, S, N, \alpha, \beta, |\Omega|$ such that*

$$
J_{\alpha,\beta}(u,v) \geq -\nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right].
$$

Proof. Applying the inequality

$$
\begin{aligned}
J_{\alpha,\beta}(u,v) \geq& \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) \\
&+ \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx
\end{aligned}
\tag{2.3}
$$

together with the Hölder inequality, the Young inequality, and the Sobolev embedding theorem [11] to (2.3), we have

$$
\begin{aligned}
J_{\alpha,\beta}(u,v) &\geq \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) - \nu\left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&\geq \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) - \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}}\left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right) \\
&\quad \times \int_\Omega \left(\frac{1-\alpha}{2-\alpha-\beta}|u|_{p^*}^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}|v|_{p^*}^{2-\alpha-\beta}\right)dx \\
&\geq \left(\frac{1}{p}-\frac{1}{r}\right)\left(\|(u,v)\|_p^p\right) - \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}} S^{\frac{\alpha+\beta-2}{p}}\left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right) \\
&\quad \times \left(\frac{1-\alpha}{2-\alpha-\beta}\|u\|_p^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}\|v\|_p^{2-\alpha-\beta}\right) \\
&\geq -\nu A_0(p,S,N,\alpha,\beta,|\Omega|)
\end{aligned}
$$

$$\times \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right].$$

The lemma is proved. □

Lemma 2.3. *There exists $\Lambda^* > 0$ such that if*

$$\nu \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right] \in (0, \Lambda^*), \tag{2.4}$$

then $\mathcal{M}^0_{\alpha,\beta} = \phi$.

Proof. Choose

$$\Lambda^* = \left((p-2+\alpha+\beta) \frac{1}{M'(\lambda+\mu)} \right)^{\frac{p}{r-p}} \frac{(r-p) S'^{\frac{rp}{N(r-p)} + \frac{2-\alpha-\beta}{p}}}{\nu M (r-2+\alpha+\beta)^{\frac{r}{r-p}} |\Omega|^{1-\frac{2-\alpha-\beta}{p^*}}}.$$

The proof follows by a contradiction. □

From Lemma (2.3), we have that if (2.4) holds, then $\mathcal{M}_{\alpha,\beta} = \mathcal{M}^+_{\alpha,\beta} \bigcup \mathcal{M}^-_{\alpha,\beta}$. We can define $i^+ = \inf_{(u,v)\in\mathcal{M}^+_{\alpha,\beta}} J_{\alpha,\beta}$ and $i^- = \inf_{(u,v)\in\mathcal{M}^-_{\alpha,\beta}} J_{\alpha,\beta}$ since the functional $J_{\alpha,\beta}$ is bounded below in $\mathcal{M}_{\alpha,\beta}$.

Remark 2.4. Henceforth, we will denote the norm convergence by $\to$, the weak convergence by $\rightharpoonup$ and Λ (or Λ^*) as any small parameter we will encounter or any complex representation in short form.

Lemma 2.5. *There exists $\Lambda^* > 0$ such that if (2.4) holds, then*

1. $i^+ < 0$,
2. $i^- \geq D_0$ *for some* $D_0 > 0$.

Proof. 1. Let $(u,v) \in \mathcal{M}^+_{\alpha,\beta} \subset \mathcal{M}_{\alpha,\beta}$. Then we have

$$\begin{aligned} 0 &< (r-p)\|(u,v)\|_p^p + (r-q)\|(u,v)\|_q^q \\ &< \nu(r+\alpha+\beta-2) \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx \end{aligned}$$

Further,

$$\begin{aligned} J_{\alpha,\beta}(u,v) &= \left(\frac{1}{p} - \frac{1}{r}\right) \|(u,v)\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|(u,v)\|_q^q \\ &\quad + \nu \left(\frac{1}{r} - \frac{1}{2-\alpha-\beta} \right) \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx \\ &< \left(\frac{1}{p} - \frac{1}{r}\right) \|(u,v)\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|(u,v)\|_q^q \\ &\quad - \frac{(r-p)}{r(2-\alpha-\beta)} \|(u,v)\|_p^p - \frac{(r-q)}{r(2-\alpha-\beta)} \|(u,v)\|_q^q \end{aligned}$$

$$
\begin{aligned}
&= \frac{(r-p)}{r}\left(\frac{1}{p}-\frac{1}{2-\alpha-\beta}\right)\|(u,v)\|_p^p \\
&\quad + \frac{(r-p)}{r}\left(\frac{1}{q}-\frac{1}{2-\alpha-\beta}\right)\|(u,v)\|_q^q < 0.
\end{aligned}
$$

Therefore, $i^+ = \inf_{(u,v)\in\mathcal{M}^+_{\alpha,\beta}} J_{\alpha,\beta}(u,v) < 0$.

2. Likewise, let us choose $(u,v)\in\mathcal{M}^-_{\alpha,\beta}$. We again appeal to the inequality

$$
\begin{aligned}
(p+\alpha+\beta-2)\|(u,v)\|_p^p &< (p+\alpha+\beta-2)\|(u,v)\|_p^p + (q+\alpha+\beta-2)\|(u,v)\|_q^q \\
&< (r+\alpha+\beta-2)\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx \\
&\le (r+\alpha+\beta-2)CM'(\lambda^{\frac{r}{r-p}} + \mu^{\frac{r}{r-p}})\|(u,v)\|_p^r
\end{aligned}
$$

by virtue of the fact that $(u,v)\in\mathcal{M}_{\alpha,\beta}$. Therefore,

$$
\|(u,v)\|_p \ge \left[\left(\frac{p+\alpha+\beta-2}{r+\alpha+\beta-2}\right)\frac{1}{CM'(\lambda^{\frac{r}{r-p}}+\mu^{\frac{r}{r-p}})}\right]^{\frac{1}{r-p}}.
$$

Let this constant be named as Λ. On proceeding further we have

$$
\begin{aligned}
J_{\alpha,\beta}(u,v) &= \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p + \left(\frac{1}{q}-\frac{1}{r}\right)\|(u,v)\|_q^q \\
&\quad + \nu\left(\frac{1}{r}-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\
&\ge \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p - \nu M|\Omega|^{1-\frac{2-\alpha-\beta}{p^*}} S^{\frac{\alpha+\beta-2}{p}}\left(\frac{1}{2-\alpha-\beta}-\frac{1}{r}\right) \\
&\quad \times\left(\frac{1-\alpha}{2-\alpha-\beta}\|u\|_p^{2-\alpha-\beta} + \frac{1-\beta}{2-\alpha-\beta}\|v\|_p^{2-\alpha-\beta}\right) \\
&\ge \left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^p - \nu A_0(p,s,N,\alpha,\beta,|\Omega|) \\
&\quad \times\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\|(u,v)\|_p^{2-\alpha-\beta} \\
&= \left[\left(\frac{1}{p}-\frac{1}{r}\right)\|(u,v)\|_p^{p+\alpha+\beta-2} - \nu A_0(p,s,N,\alpha,\beta,|\Omega|)\right. \\
&\quad \left.\times\left\{\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right\}\right]\|(u,v)\|_p^{2-\alpha-\beta} \\
&\ge \Lambda^{2-\alpha-\beta}\left[\left(\frac{1}{p}-\frac{1}{r}\right)\Lambda^{p+\alpha+\beta-2} - \nu A_0(p,s,N,\alpha,\beta,|\Omega|)\right. \\
&\quad \left.\times\left\{\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right\}\right].
\end{aligned}
$$

Then, for a sufficiently small $\Lambda^* > 0$ and $D_0 > 0$ such that (2.4) holds, we have $i^- \ge D_0 > 0$. $\square$

Remark 2.6. For better understanding the Nehari manifold and the fiber maps, we define the function

$$F_{u,v}(t) = t^{p-r}\|(u,v)\|_p^p + t^{q-r}\|(u,v)\|_q^q - \nu t^{2-\alpha-\beta-r}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

Then

$$\Psi'(t) = t^{r-1}[F_{u,v}(t) - \int_\Omega(\lambda f(x)u^r + \beta g(x)v^r)dx].$$

Observe that $\lim_{t\to\infty} F_{u,v}(t) = 0$ and $\lim_{t\to 0^+} F_{u,v}(t) = -\infty$. Further,

$$\begin{aligned} F'_{u,v}(t) &= (p-r)t^{p-r-1}\|(u,v)\|_p^p + (q-r)t^{q-r-1}\|(u,v)\|_q^q \\ &\quad - \nu(2-\alpha-\beta-r)t^{1-\alpha-\beta-r}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx \\ &= t^{1-\alpha-\beta-r}[(p-r)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (q-r)t^{q+\alpha+\beta}\|(u,v)\|_q^q \\ &\quad - \nu(2-\alpha-\beta-r)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx]. \end{aligned}$$

Let

$$\begin{aligned} \psi_{u,v}(t) = (p-r)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (q-r)t^{q+\alpha+\beta}\|(u,v)\|_q^q \\ - \nu(2-\alpha-\beta-r)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx. \end{aligned}$$

We also have

$$\lim_{t\to 0^+}\psi_{u,v}(t) = \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx, \quad \lim_{t\to\infty}\psi_{u,v}(t) = -\infty,$$

and

$$\begin{aligned} \psi'_{u,v}(t) &= (p-r)(p+\alpha+\beta)t^{p+\alpha+\beta-1}\|(u,v)\|_p^p \\ &\quad + (q-r)(q+\alpha+\beta)t^{q+\alpha+\beta-1}\|(u,v)\|_q^q < 0. \end{aligned}$$

Thus, for each $(u,v)\in X$ with $\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx > 0$, $F_{u,v}(t)$ attains its maximum at some $t_{\max} = t_{\max}(u,v)$. This unique $t_{\max}$ can be evaluated by solving for t from the equation

$$(r-p)t^{p+\alpha+\beta}\|(u,v)\|_p^p + (r-q)t^{q+\alpha+\beta}\|(u,v)\|_q^q = \nu(r+\alpha+\beta-2)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

A simple calculation yields

$$\begin{aligned} F_{u,v}(t_{\max}) &= t_{\max}^{p-r}\left(1 + \frac{r-p}{r+\alpha+\beta-2}t_{\max}^2\right)\|(u,v)\|_p^p \\ &\quad + t_{\max}^{q-r}\left(1 + \frac{r-q}{r+\alpha+\beta-2}t_{\max}^2\right)\|(u,v)\|_q^q > 0. \end{aligned}$$

Thus, for $t\in(0,t_{\max})$, we have $F'_{u,v}(t) > 0$ and $F'_{u,v}(t) < 0$ for $t\in(t_{\max},\infty)$.

We now have the following lemma as a consequence.

Lemma 2.7. *For every* $(u,v) \in X \setminus \{(0,0)\}$ *there exists a unique* $0 < t^+ < t_{max}$ *such that* $(t^+u, t^+v) \in \mathcal{M}^+_{\alpha,\beta}$ *and*

$$J_{\alpha,\beta}(t^+u, t^+v) = \inf_{t\geq 0} J_{\alpha,\beta}(tu, tv).$$

Furthermore, if

$$\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx > 0,$$

then there exists a unique $0 < t^+ < t_{\max} < t^-$ *such that* $(t^+u, t^+v) \in \mathcal{M}^+_{\alpha,\beta}$, $(t^-u, t^-v) \in \mathcal{M}^-_{\alpha,\beta}$ *and*

$$J_{\alpha,\beta}(t^+u, t^+v) = \inf_{0\leq t\leq t_{\max}} J_{\alpha,\beta}(tu, tv), \quad J_{\alpha,\beta}(t^-u, t^-v) = \sup_{t\geq 0} J_{\alpha,\beta}(tu, tv).$$

Proof. We only prove the case when

$$\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx > 0.$$

Thus the equation

$$F_{u,v}(t) = \int_\Omega (\lambda f(x)u^r + \beta g(x)v^r)\,dx$$

has only two solutions, namely $0 < t^+ < t_{\max} < t^-$ such that $I'_{\alpha,\beta}(t^+) > 0$ and $I'_{\alpha,\beta}(t^-) < 0$. Since

$$\Psi''(t^+) = (t^+)^{r-1}\left[F_{u,v}(t^+) - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx\right] > 0$$
$$\Psi''(t^-) = (t^-)^{r-1}\left[F_{u,v}(t^-) - \int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx\right] < 0,$$

therefore $(t^+u, t^+v) \in \mathcal{M}^+_{\alpha,\beta}$ and $(t^-u, t^-v) \in \mathcal{M}^-_{\alpha,\beta}$. Thus $\Psi(t)$ decreases in $(0, t^+)$, increases in (t^+, t^-) and decreases in (t^-, ∞). The lemma is proved. □

We now define the Palais–Smale (PS) sequence, the condition and the value in X for a functional $J_{\alpha,\beta}$.

Definition 2.8. Suppose, for $c \in \mathbb{R}$, a sequence $\{(u_n, v_n)\} \subset X$ is a $(PS)_c$-sequence for the functional $J_{\alpha,\beta}$ if $J_{\alpha,\beta}(u_n, v_n) \to c$ and $J'_{\alpha,\beta}(u_n, v_n) \to 0$ in X' as $n \to \infty$. Then

1. the number $c \in \mathbb{R}$ is a (PS)-value in X for the functional $J_{\alpha,\beta}$ if there exists a $(PS)_c$-sequence in X for $J_{\alpha,\beta}$;
2. the functional $J_{\alpha,\beta}$ satisfies the $(PS)_c$-condition in X for $J_{\alpha,\beta}$ if any $(PS)_c$-sequence admits a strongly convergent subsequence in X.

Remark 2.9. We will sometimes denote $\lim_{n\to\infty} x_n = 0$ as $x_n = o(1)$ for a sequence of real numbers (x_n).

Remark 2.10. X' will refer to the dual space of X.

Lemma 2.11. *For any* $0<\alpha,\beta<1$, *the functional* $J_{\alpha,\beta}$ *satisfies the* $(PS)_c$-*condition for*

$$c\in\left(-\infty,\frac{S'^{\frac{r}{r-p}}}{\Lambda}-\nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\right),$$

where $\Lambda=2M'(\lambda^{\frac{r}{r-p}}+\mu^{\frac{r}{r-p}})\}^{\frac{p}{r-p}}|\Omega|^{\frac{1}{r}}$. *Here,*

$$\frac{S'^{\frac{r}{r-p}}}{\Lambda}-\nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]>0$$

for a sufficiently small ν.

Proof. Suppose $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence in X for the functional $J_{\alpha,\beta}$ with

$$c\in\left(-\infty,\frac{S'^{\frac{r}{r-p}}}{\Lambda}-\nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]\right).$$

Then

$$J_{\alpha,\beta}(u_n,v_n)=c+o(1),\quad I'_{\alpha,\beta}(u_n,v_n)=o(1)\quad\text{as } n\to\infty. \tag{2.5}$$

We now claim that $\{(u_n,v_n)\}$ is bounded in X. We prove this claim by contradiction, i.e., say $\|(u_n,v_n)\|_p\to\infty$ as $n\to\infty$. Let

$$(\tilde{u}_n,\tilde{v}_n)=\left(\frac{u_n}{\|(u_n,v_n)\|_p},\frac{v_n}{\|(u_n,v_n\|_p)}\right).$$

Then $\|(\tilde{u}_n,\tilde{v}_n)\|_p=1$, which implies that $(\tilde{u}_n,\tilde{v}_n)$ is bounded in X. Therefore, due to the reflexivity of the space X, we have up to a subsequence

$$(\tilde{u}_n,\tilde{v}_n)\rightharpoonup(\tilde{u},\tilde{v})\quad\text{as } n\to\infty \text{ in } X.$$

This further implies that

$$\begin{aligned}
&\tilde{u}_n\rightharpoonup\tilde{u}, && \tilde{v}_n\rightharpoonup\tilde{v} && \text{in } W_0^{1,p}(\Omega),\\
&\tilde{u}_n\to\tilde{u}, && \tilde{v}_n\to\tilde{v} && \text{in } L^s(\Omega),\ 1\le s<p^*,
\end{aligned}$$
$$\int_\Omega\nu h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}dx\to\int_\Omega\nu h(x)u^{1-\alpha}v^{1-\beta}dx$$

as $n\to\infty$. The last convergence follows from Egoroff's theorem. From (2.5), we have

$$\frac{1}{p}\|(u_n,v_n)\|_p^p\|(\tilde{u_n},\tilde{v}_n)\|_p^p+\frac{1}{q}\|(u_n,v_n)\|_q^q\|(\tilde{u_n},\tilde{v}_n)\|_q^q$$

$$- \frac{1}{r}\|(u_n, v_n)\|_p^r \int_\Omega (\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)\, dx$$
$$- \frac{\nu}{2-\alpha-\beta}\|(u_n, v_n)\|_p^{2-\alpha-\beta} \int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta} dx = c + o(1)$$

and

$$\begin{aligned}\|(u_n, v_n)\|_p^p\|(\tilde{u}_n, \tilde{v}_n)\|_p^p + \|(u_n, v_n)\|_q^q\|(\tilde{u}_n, \tilde{v}_n)\|_q^q& \\ - \|(u_n, v_n)\|_p^r \int_\Omega (\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)\, dx& \\ - \nu\|(u_n, v_n)\|_p^{2-\alpha-\beta} \int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta}\, dx &= o(1)\end{aligned}$$

as $n \to \infty$. By the assumption we made, i.e., $\|(u_n, v_n)\|_p \to \infty$, we obtain

$$\begin{aligned}\frac{1}{p}\|(\tilde{u_n}, \tilde{v}_n)\|_p^p + \frac{1}{q}\|(u_n, v_n)\|_q^q \frac{\|(\tilde{u_n}, \tilde{v}_n)\|_q^q}{\|(u_n, v_n)\|_p^p}& \\ - \frac{1}{r}\|(u_n, v_n)\|_p^{r-p} \int_\Omega (\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)\, dx& \\ - \frac{\nu}{2-\alpha-\beta}\|(u_n, v_n)\|_p^{2-\alpha-\beta-p} \int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta} dx &= o(1)\end{aligned}$$

and

$$\begin{aligned}\|(\tilde{u}_n, \tilde{v}_n)\|_p^p + \|(u_n, v_n)\|_q^q \frac{\|(\tilde{u_n}, \tilde{v}_n)\|_q^q}{\|(u_n, v_n)\|_p^p}& \\ - \|(u_n, v_n)\|_p^{r-p} \int_\Omega (\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)\, dx& \\ - \nu\|(u_n, v_n)\|_p^{2-\alpha-\beta-p} \int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta} dx &= o(1)\end{aligned}$$

as $n \to \infty$. By using the above to the equations, we obtain

$$\begin{aligned}\left(1 - \frac{2-\alpha-\beta}{p}\right)\|(\tilde{u}_n, \tilde{v}_n)\|_p^p + \left(1 - \frac{2-\alpha-\beta}{q}\right)\|(u_n, v_n)\|_q^q \frac{\|(\tilde{u_n}, \tilde{v}_n)\|_q^q}{\|(u_n, v_n)\|_p^p}& \\ + \left(\frac{2-\alpha-\beta}{r} - 1\right)\|(u_n, v_n)\|_p^{r-p} \int_\Omega (\lambda f(x)\tilde{u}_n^r + \mu g(x)\tilde{v}_n^r)\, dx &= o(1)\end{aligned}$$

as $n \to \infty$. Therefore we have

$$\begin{aligned}\|(\tilde{u}_n, \tilde{v}_n)\|_p^p &= \frac{p(2-\alpha-\beta-q)}{q(p-2+\alpha+\beta)}\|(u_n, v_n)\|_q^q \frac{\|(\tilde{u_n}, \tilde{v}_n)\|_q^q}{\|(u_n, v_n)\|_p^p} \\ &\quad + \nu\frac{p(r-2+\alpha+\beta)}{r(p-2+\alpha+\beta)}\|(u_n, v_n)\|_p^{2-\alpha-\beta-p} \int_\Omega h(x)\tilde{u}_n^{1-\alpha}\tilde{v}_n^{1-\beta} dx + o(1)\end{aligned}$$

as $n \to \infty$. Thus we have $\|(\tilde{u}_n, \tilde{v}_n)\|_p^p \to \infty$, which is a contradiction to our assumption that $\|(\tilde{u}_n, \tilde{v}_n)\|_p = 1$. Therefore, the sequence $\{(u_n, v_n)\}$ is bounded in X.

We choose a subsequence to this bounded sequence, still denoted by $\{(u_n, v_n)\}$, such that

$$\begin{gathered}(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X,\\ u_n \to u, \quad v_n \to v \quad \text{in } L^{\mathbf{s}}(\Omega),\ 1 \le \mathbf{s} < p_s^*,\\ \int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r)\, dx \to \int_\Omega (\lambda f(x) u^r + \mu g(x) v^r)\, dx,\\ \nu \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx \to \nu \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx\end{gathered}$$

as $n \to \infty$.

By the Brezis–Lieb [20] theorem, we get

$$\begin{aligned}\|(u_n - u, v_n - v)\|_p^p &= \|(u_n, v_n)\|_p^p - \|(u, v)\|_p^p + o(1),\\ \int_\Omega (\lambda f(x)(u_n - u)^r + \mu g(x)(v_n - v)^r)\, dx &= \int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r)\, dx\\ &\quad - \int_\Omega (\lambda f(x) u^r + \mu g(x) v^r)\, dx + o(1)\\ \nu \int_\Omega h(x)(u_n - u)^{1-\alpha}(v_n - v)^{1-\beta} dx &= \nu \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx\\ &\quad - \nu \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx + o(1)\end{aligned}$$

as $n \to \infty$. Thus, for any $(\phi_2, \phi_2) \in X$, the following holds:

$$\lim_{n\to\infty} \langle I'_{\alpha,\beta}, (\phi_2, \phi_2)\rangle = \langle I'_{\alpha,\beta}(u, v), (\phi_1, \phi_2)\rangle = 0.$$

In other words, (u, v) is a critical point of $J_{\alpha,\beta}$. All we now need to show is that $(u_n, v_n) \to (u, v)$ in X. We use (2.5), the Brezis–Lieb lemma from [20] and some basic functional analyses to obtain

$$\begin{aligned}&\frac{1}{p}\|(u_n - u, v_n - v)\|_p^p + \frac{1}{q}\|(u_n - u, v_n - v)\|_q^q\\ &\quad - \frac{1}{r}\int_\Omega (\lambda f(x)(u_n - u)^r + \mu g(x)(v_n - v)^r)\, dx = c - J_{\alpha,\beta} + o(1)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}0 &= \langle I'_{\alpha,\beta}(u_n, v_n), (u_n - u, v_n - v)\rangle\\ &= \langle I'_{\alpha,\beta}(u_n, v_n) - I'_{\alpha,\beta}(u, v), (u_n - u, v_n - v)\rangle\\ &= \|(u_n - u, v_n - v)\|_p^p + \|(u_n - u, v_n - v)\|_q^q\\ &\quad - \int_\Omega (\lambda f(x)(u_n - u)^r + \mu g(x)(v_n - v)^r)\, dx + o(1)\end{aligned} \tag{2.7}$$

as $n \to \infty$. Without loss of generality, we let

$$\|(u_n - u, v_n - v)\|_p^p = c' + o(1), \quad \|(u_n - u, v_n - v)\|_q^q = d' + o(1)$$

and therefore

$$\int_{\Omega} (\lambda f(x)(u_n - u)^r + \mu g(x)(v_n - v)^r)\, dx = c' + d' + o(1)$$

as $n \to \infty$. Now, if $c' = 0$, the proof is immediate. On the contrary, we assume that $c' > 0$,

$$\begin{aligned}
\left(\frac{c'}{2}\right)^{\frac{p}{p^*}} \leq \left(\frac{c'+d'}{2}\right)^{\frac{p}{p^*}} &= \lim_{n\to\infty} \int_{\Omega} (\lambda f(x)(u_n - u)^r + \mu g(x)(v_n - v)^r)\, dx \\
&\leq M' \lim_{n\to\infty} \int_{\Omega} (\lambda |u_n - u|^r + \mu |v_n - v|^r)\, dx \\
&\leq M' \lim_{n\to\infty} |\Omega|^{\frac{1}{2-\alpha-\beta} - \frac{1}{r}} S'^{-\frac{r}{p}} \|(u_n - u, v_n - v)\|_p^r \\
&= M' |\Omega|^{\frac{1}{p} - \frac{1}{r}} S'^{-\frac{r}{p}} \left(\lambda^{\frac{r}{r-p}} + \mu^{\frac{r}{r-p}}\right) c'^{\frac{r}{p}}.
\end{aligned}$$

Thus,

$$c' \geq \frac{S'^{\frac{r}{r-p}}}{\{2M'(\lambda^{\frac{r}{r-p}} + \mu^{\frac{r}{r-p}})\}^{\frac{p}{r-p}} |\Omega|^{\frac{1}{r}}} = \frac{S'^{\frac{r}{r-p}}}{\Lambda}.$$

Therefore, from (2.6), (2.7) and the fact that $(u, v) \in \mathcal{M}_{\alpha,\beta} \bigcup \{(0,0)\}$, we have

$$\begin{aligned}
c' &= J_{\alpha,\beta}(u, v) + \frac{c'}{p} + \frac{d'}{q} - \frac{c'+d'}{r} \\
&\geq \frac{S'^{\frac{r}{r-p}}}{\Lambda} - \nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}},\right]
\end{aligned}$$

which contradicts

$$c' < \frac{S'^{\frac{r}{r-p}}}{\Lambda} - \nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right].$$

Thus $c' = 0$ and hence $(u_n, v_n) \to (u, v)$ as $n \to \infty$ in X. □

Remark 2.12. The functional $J_{\alpha,\beta}$ also satisfies the $(PS)_c$ condition for the case of $r = p_s^*$. In a way, we will try to find out the energy level c below which the functional satisfies the (PS) condition. We suppose that

$$\begin{aligned}
c < c_{**} = &\left(\frac{1}{R} - \frac{1}{p_s^*}\right) S^{\frac{N}{ps}} (f(x_i) + g(x_i)) \\
&- B \left(\frac{2-\alpha-\beta}{p_s^*} \frac{B}{A}\right)^{\frac{2-\alpha-\beta}{p_s^*-2+\alpha+\beta}} \left(\frac{p_s^* - 2 + \alpha + \beta}{p_s^*}\right).
\end{aligned}$$

The sequence (u_n, v_n) is bounded in X by the same argument. By the reflexivity of X, we have $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ as $n \to \infty$. Further, from the concentration-compactness result (refer to Theorem 2.5 of [23]), for these subsequences (still denoted by (u_n), (v_n)), we have that

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\, dy \rightharpoonup \mu \geq \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dy + \sum_{j\in I} \mu_j \delta_{x_j},$$

$$\int_{\mathbb{R}^N} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}\, dy \rightharpoonup \mu' \geq \int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dy + \sum_{j\in I'} \mu'_j \delta_{x'_j},$$

$$u_n^{p_s^*} \rightharpoonup d\nu = u^{p_s^*} + \sum_{j\in I} \nu_j \delta_{x_j},$$

$$v_n^{p_s^*} \rightharpoonup d\nu' = v^{p_s^*} + \sum_{j\in I'} \nu'_j \delta_{x'_j},$$

$$\mu_j \geq S\nu_j^{\frac{p}{p_s^*}}, \qquad \mu'_j \geq S{\nu'}_j^{\frac{p}{p_s^*}}$$

and therefore

$$\int_\Omega (\lambda f(x) u_n^{p_s^*} + \mu g(x) v_n^{p_s^*})\, dx \to \int_\Omega (\lambda f(x) u^r + \mu g(x) v^r)\, dx + \sum_{j\in I} \mu_j f(x_j) + \sum_{j\in I'} \nu'_j g(x'_j)$$

as $n \to \infty$. Further, by compact embedding results, we have

$$\nu \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx \to \nu \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx$$

as $n \to \infty$. Here, $\{x_j \mid j \in I\}$, $\{x'_j \mid j \in I'\}$ - I, I' are countable indexing sets, i.e., sets of distinct points in $\mathbb{R}^N$, $\{\nu_j \mid j \in I\} \in (0,\infty)$, $\{\mu_j \mid j \in I\} \in (0,\infty)$, $\{\nu'_j \mid j \in I\} \in (0,\infty)$, $\{\mu'_j \mid j \in I\} \in (0,\infty)$ and S is the best Sobolev constant defined earlier in this paper. Hence, if $J = I \cup I' = \varnothing$, then $u_n \to u$, $v_n \to v$ strongly in $L^{p_s^*}(\Omega)$. If not, we suppose $I \neq \varnothing$ and then choose $\zeta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \zeta \leq 1$, $\zeta(0) = 1$ with support in a unit ball of $\mathbb{R}^N$. Let us define, for any $\epsilon > 0$, the function $\zeta_{\epsilon,j}$ as $\zeta_{\epsilon,j} = \zeta(\frac{x-x_j}{\epsilon})$ for all $j \in J$. We have that $\langle J'_{\alpha,\beta}(u_n, v_n), \zeta_{\epsilon,j}(u_n, v_n)\rangle \to 0$ as $n \to \infty$. On testing wit $\zeta_{\epsilon,j}(u_n, v_n)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy + \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy \\
&\quad = \langle (-\Delta_p)^s u_n, \zeta_{\epsilon,j}\rangle + \langle (-\Delta_p)^s v_n, \zeta_{\epsilon,j}\rangle - \int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r) \zeta_{\epsilon,j}\, dx \\
&\qquad - \nu \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} \zeta_{\epsilon,j}^{2-\alpha-\beta} dx + o(1).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy \\
&\qquad \geq \lim_{\epsilon\to 0} \left[\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy + \mu_j \right] = \mu_j, \\
&\lim_{\epsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy \\
&\qquad \geq \lim_{\epsilon\to 0} \left[\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} \zeta_{\epsilon,j}\, dx\, dy + \mu'_j \right] = \mu'_j.
\end{aligned}$$

Furthermore, from the definition of $\zeta_{\epsilon,j}$, we also have

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\langle(-\Delta_p)^s u_n,\zeta_{\epsilon,j}\rangle=\lim_{\epsilon\to0}\lim_{n\to\infty}\langle(-\Delta_p)^s v_n,\zeta_{\epsilon,j}\rangle=0$$

and

$$\begin{aligned}&\lim_{\epsilon\to0}\lim_{n\to\infty}\int_\Omega(\lambda f(x)u_n^{p_s^*}+\mu g(x)v_n^{p_s^*})\zeta_{\epsilon,j}\,dx\\&=\lambda f(x_j)\nu_j+\mu g(x_j)\nu'_j+\lambda f(x_j)u(x_j)^{p_s^*}+\mu g(x_j)v(x_j)^{p_s^*}.\end{aligned}$$

From the compact embedding results in combination with the definition of $\zeta_{\epsilon,j}$, we have

$$\lim_{\epsilon\to0}\lim_{n\to\infty}\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}\zeta_{\epsilon,j}^{2-\alpha-\beta}dx=\lim_{\epsilon\to0}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}\zeta_{\epsilon,j}^{2-\alpha-\beta}dx=0.$$

From the above analysis, we get

$$\mu_j\le\mu_j+\mu'_j\le\lambda f(x_j)\nu_j,\quad \mu'_j\le\mu_j+\mu'_j\le\mu g(x_j)\nu'_j.$$

Therefore we have either $\nu_j=0$ or $S\le\lambda f(x_j)\nu_j^{\frac{ps}{N}}$ $\Big($and $\nu'_j=0$ or $S\le\mu g(x_j){\nu'}_j^{\frac{ps}{N}}\Big)$.

Consider

$$\begin{aligned}J_{\alpha,\beta}(u_n,v_n)&-\frac{1}{R}\langle J'_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle\\&=\left(\frac1p-\frac1R\right)\|(u_n,v_n)\|_p^p+\left(\frac1q-\frac1R\right)\|(u_n,v_n)\|_q^q\\&\quad+\left(\frac1R-\frac1{p_s^*}\right)\int_\Omega(\lambda f(x)u_n^{p_s^*}+\mu g(x)v_n^{p_s^*})\,dx\\&\quad+\nu\left(\frac1R-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx\\&\ge\left(\frac1R-\frac1{p_s^*}\right)\left(\int_\Omega(\lambda f(x)u^{p_s^*}+\mu g(x)v^{p_s^*})\,dx\right)\\&\quad+\nu\left(\frac1R-\frac{1}{2-\alpha-\beta}\right)\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx\\&\quad+\left(\frac1R-\frac1{p_s^*}\right)\left(\sum_{j\in I}\mu_{x_j}f(x_j)+\nu'_{x_j}g(x_j)\right),\end{aligned}$$

where $r<R<p_s^*$ and

$$J_{\alpha,\beta}(u_n,v_n)-\frac{1}{R}\langle J'_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle=c+o(1)\quad\text{as } n\to\infty.$$

Further, the function $m:x\mapsto At^{p_s^*}-Bt^{2-\alpha-\beta}$ for $A,B>0$, attains its minimum at some point, say, $t_0>0$, such that

$$m(x)\ge B\left(\frac{2-\alpha-\beta}{p_s^*}\frac{B}{A}\right)^{\frac{2-\alpha-\beta}{p_s^*-2+\alpha+\beta}}\left(\frac{2-\alpha-\beta-p_s^*}{p_s^*}\right).$$

Passing the limit $n \to \infty$, we obtain

$$c \geq \left(\frac{1}{R} - \frac{1}{p_s^*}\right) \left(\sum_{j\in I} \mu_{x_j} f(x_j) + \nu'_{x_j} g(x_j)\right) - B\left(\frac{2-\alpha-\beta}{p_s^*}\frac{B}{A}\right)^{\frac{2-\alpha-\beta}{p_s^*-2+\alpha+\beta}} \left(\frac{p_s^*-2+\alpha+\beta}{p_s^*}\right).$$

Thus we have

$$c \geq \left(\frac{1}{R} - \frac{1}{p_s^*}\right) S^{\frac{N}{ps}}(f(x_i)+g(x_i)) - B\left(\frac{2-\alpha-\beta}{p_s^*}\frac{B}{A}\right)^{\frac{2-\alpha-\beta}{p_s^*-2+\alpha+\beta}} \left(\frac{p_s^*-2+\alpha+\beta}{p_s^*}\right) = c_*,$$

which contradicts the assumption that $c < c_{**}$. As there are no extra terms appearing in the decomposition of the sequence (u_n, v_n), we have

$$\int_\Omega (\lambda f(x) u_n^{p_s^*} + \mu g(x) v_n^{p_s^*})\, dx \to \int_\Omega (\lambda f(x) u^{p_s^*} + \mu g(x) v^{p_s^*})\, dx$$

as $n \to \infty$. Finally, we have

$$\lim_{n\to\infty} \|u_n\|_p^p = \int_\Omega \lambda f(x) u^{p_s^*} dx + \nu \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx = \|u\|_p^p$$
$$\lim_{n\to\infty} \|v_n\|_p^p = \int_\Omega \mu g(x) v^{p_s^*} dx + \nu \int_\Omega h(x) u^{1-\alpha} v^{1-\beta} dx = \|v\|_p^p.$$

Thus, $\|u_n\|_p^p \to \|u\|_p^p$, $\|v_n\|_p^p \to \|v\|_p^p$ as $n \to \infty$. It is seen that the PS condition is satisfied by the functional even for the critical case of $r = p_s^*$.

Now we are to prove the existence of a local minimizer for $J_{\alpha,\beta}$ in $\mathcal{M}^+_{\alpha,\beta}$.

Lemma 2.13. *There exists $\Lambda^* > 0$ such that (2.4) holds, $J_{\alpha,\beta}$ has a minimizer $(u_\nu, v_\nu) \in \mathcal{M}^+_{\alpha,\beta}$, and it satisfies the conditions:*

(i) $J_{\alpha,\beta}(u_\nu, v_\nu) = i^+$ *is a weak solution to problem* (1.1)–(1.4),

(ii) $J_{\alpha,\beta}(u_\nu, v_\nu) \to 0$ *and* $\|(u_\nu, v_\nu)\|_p \to 0$, $\|(u_\nu, v_\nu)\|_q \to 0$ *as* $\nu \to 0$.

Proof. In order to prove (i), we follow Hsu [17, Theorem 4.2]. Since $i^+ = \inf_{(u,v)\in\mathcal{M}_{\alpha,\beta}}\{J_{\alpha,\beta}(u,v)\}$, there exists a sequence $(u_n, v_n) \in \mathcal{M}_{\alpha,\beta}$ such that $J_{\alpha,\beta}(u_n, v_n) \to i^+$ and $J'_{\alpha,\beta}(u_n, v_n) \to 0$ in X^* as $n \to \infty$. Since the functional $J_{\alpha,\beta}$ is coercive and therefore (u_n, v_n) is bounded in X. Thus, there exists a subsequence of (u_n, v_n), still denoted as (u_n, v_n), such that $((u_n, v_n)) \rightharpoonup (u, v) \in X$. So we have

$$u_n \rightharpoonup u, \quad v_n \rightharpoonup v,$$
$$u_n \to u, \quad v_n \to v \quad \text{a.e. in } \Omega,$$

$$u_n \to u, \quad v_n \to v \quad \text{in } L^s(\Omega) \text{ for } 1 \le s < p^*$$

as $n \to \infty$. This implies

$$\frac{2\nu}{2-\alpha-\beta}\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx \to \frac{2\nu}{2-\alpha-\beta}\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx$$

as $n \to \infty$. Clearly, (u, v) is a weak solution of (1.1)–(1.4). Also, since $(u_n, v_n) \in \mathcal{M}_{\alpha,\beta}$, we have

$$\begin{aligned}L^\nu_{\alpha,\beta}(u_n, v_n) = &\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n, v_n)\|_p^p\\ &+\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u_n, v_n)\|_q^q\\ &-\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}J_{\alpha,\beta}(u_n, v_n),\end{aligned}$$

where $L^\nu_{\alpha,\beta}(u_n, v_n) = \int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx$. Also,

$$\begin{aligned}L^\nu_{\alpha,\beta}(u_n, v_n) \ge &\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u, v)\|_p^p\\ &+\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u, v)\|_q^q-\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}i^+\\ \ge &-\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}i^+ > 0,\end{aligned}$$

where we have used the lower-semicontinuity of $\|\cdot\|_p$, $\|\cdot\|_q$ and $i^+ < 0$. Therefore $(u, v) \ne (0, 0)$, and thus we have a nontrivial weak solution.

Claim: We now claim that $(u_n, v_n) \to (u, v)$ in X and $J_{\alpha,\beta}(u, v) = i^+$. For any $(u_0, v_0) \in \mathcal{M}_{\alpha,\beta}$, we have

$$\begin{aligned}L^\nu_{\alpha,\beta}(u_0, v_0) = &\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_0, v_0)\|_p^p\\ &+\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}\left(\frac{1}{q}-\frac{1}{r}\right)\|(u_0, v_0)\|_q^q\\ &-\frac{r(2-\alpha-\beta)}{2\nu(r-2+\alpha+\beta)}J_{\alpha,\beta}(u_0, v_0).\end{aligned}$$

Thus,

$$\begin{aligned}i^+ &\le J_{\alpha,\beta}(u, v)\\ &\le \lim_{n\to\infty}\left[\left(\frac{1}{p}-\frac{1}{r}\right)\|(u_n, v_n)\|_p^p+\left(\frac{1}{q}-\frac{1}{r}\right)\|u_n, v_n\|_q^q-\frac{2\nu}{2-\alpha-\beta}L^\nu_{\alpha,\beta}(u_n, v_n)\right]\\ &= J_{\alpha,\beta}(u, v) = i^+.\end{aligned}$$

Then $J_{\alpha,\beta}(u,v) = i^+$. This also implies that $(u_n, v_n) \to (u,v)$ in X.

For the proof of (ii). let $(u_\nu, v_\nu) \in \mathcal{M}^+_{\alpha,\beta}$. From Lemmas 2.2, 2.3, we have that

$$0 > J_{\alpha,\beta}(u_\nu, v_\nu) \geq -\nu A_0 \left[\left(\frac{1-\alpha}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta} \right)^{\frac{p}{p+\alpha+\beta-2}} \right].$$

It is obvious that as $\nu \to 0$, we have $J_{\alpha,\beta}(u_\nu, v_\nu) \to 0$.

Further, we have

$$0 = \lim_{\nu\to 0} J_{\alpha,\beta}(u_\nu, v_\nu) = \lim_{\nu\to 0} \left[\left(\frac{1}{p} - \frac{1}{r} \right) \|(u_\nu, v_\nu)\|_p^p + \left(\frac{1}{q} - \frac{1}{r} \right) \|u_\nu, v_\nu\|_q^q \right.$$
$$\left. - \frac{2\nu}{2-\alpha-\beta} \int_\Omega h(x) u_\nu^{1-\alpha} v_\nu^{1-\beta} dx \right].$$

As it was seen earlier, the functional $J_{\alpha,\beta}$ is coercive over $\mathcal{M}^+_{\alpha,\beta}$, and therefore (u_ν, v_ν) is bounded. Also, using the fact $\lim\limits_{\nu\to 0} \frac{2\nu}{2-\alpha-\beta} \int_\Omega h(x) u_\nu^{1-\alpha} v_\nu^{1-\beta} dx = 0$, we clearly have

$$\lim_{\nu\to 0} \|(u_\nu, v_\nu)\|_p^p = 0 = \lim_{\nu\to 0} \|(u_\nu, v_\nu)\|_q^q. \qquad \square$$

Remark 2.14. For $\epsilon > 0$, let us define

$$u_\epsilon(x) = \frac{\eta(x)}{\left(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{N-sp}{p}}}, \quad v_\epsilon(x) = \frac{u_\epsilon(x)}{|u_\epsilon(x)|_{p_s^*}},$$

where $\eta(x) \in C_0^\infty(\Omega)$ is a radially symmetric function defined by

$$\eta(x) = \begin{cases} 1 & \text{if } |x| < \rho_0 \\ 0 & \text{if } |x| > 2\rho_0 \\ 0 \leq \eta(x) \leq 1 & \text{otherwise} \end{cases},$$

where ρ_0 is such that $B(0, 2\rho_0) \subset \Omega$ and $p_s^* = \frac{Np}{N-sp}$. Further, let

$$\int_\Omega \int_\Omega \frac{|\eta(x) - \eta(y)|^p}{|x-y|^{N+sp}} dx\, dy \leq C.$$

Then $\int_\Omega |v_\epsilon|^{p^*} dx = 1$ and we have the following estimates:

$$\int_\Omega |u_\epsilon|^t dx = \begin{cases} C_1 \epsilon^{\frac{N(p-1)-t(N-sp)}{p-1}} + O(1) & \text{if } t > \frac{N(p-1)}{N-sp} \\ C_1 |\ln \epsilon| + O(1) & \text{if } t = \frac{N(p-1)}{N-sp} \\ 0 \leq \eta(x) \leq 1 & \text{if } t < \frac{N(p-1)}{N-sp} \end{cases}$$

as $\epsilon \to 0$. In particular, we have

$$\int_\Omega \frac{|u_\epsilon(x) - u_\epsilon(y)|^p}{|x-y|^{N+sp}} dx\, dy = K_2 \epsilon^{\frac{sp-N}{p}} + O(1)$$

and
$$\left(\int_\Omega |u_\epsilon|^{p^*}dx\right)^{\frac{p}{p^*}} = K_3\epsilon^{\frac{sp-N}{p}} + O(1)$$
as $\epsilon \to 0$, where $K_1, K_2, K_3 > 0$ are independent of ϵ. There also exists ϵ_0 such that S, the best Sobolev constant, is close to $\frac{K_2}{K_3}$ for every $0 < \epsilon < \epsilon_0$. In other words, we will take $S \leq \frac{K_2}{K_3}$.

We now prove the following lemma which will be used for guaranteeing the multiplicity of solutions.

Lemma 2.15. *There exists ϵ_1, Λ^*, $\sigma(\epsilon) > 0$ such that for $\epsilon \in (0, \epsilon_1)$ and $\sigma \in (0, \sigma(\epsilon))$ under condition (2.4), we have*
$$\sup_{t\geq 0} J_{\alpha,\beta}(t_\epsilon\sqrt[p]{\nu}v_\epsilon, t_\epsilon\sqrt[p]{\nu}v_\epsilon) < c_{\alpha,\beta} - \sigma,$$
where
$$c_{\alpha,\beta} = \frac{r-p}{rp}S^{\frac{r}{r-p}} - \nu A_0\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right].$$

Proof. Define
$$a_\epsilon(t) = J_{\alpha,\beta}(t\sqrt[p]{\nu}v_\epsilon, t\sqrt[p]{\nu}v_\epsilon) = \frac{t^p}{p}\nu\|v_\epsilon\|_p^p + \frac{t^q}{q}(2\nu^{\frac{q}{p}})\int_\Omega |\nabla v_\epsilon|^q dx$$
$$- \frac{1}{r}\int_\Omega (\lambda f(x) + \mu g(x))(tv_\epsilon\nu^{\frac{1}{p}})^r dx - \frac{2\nu^{\frac{p-\alpha-\beta+2}{p}}t^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x)v_\epsilon^{2-\alpha-\beta}dx.$$
Clearly, $a_\epsilon(0) = 0$, $\lim_{t\to\infty} a_\epsilon(t) = -\infty$. Then there exists $t_\epsilon > 0$ such that
$$J_{\alpha,\beta}(t_\epsilon\sqrt[p]{\nu}v_\epsilon, t_\epsilon\sqrt[p]{b\nu}v_\epsilon) = \sup_{t\geq 0} I_{\lambda,\mu}(t\sqrt[p]{\nu}v_\epsilon, t\sqrt[p]{\nu}v_\epsilon).$$
This yields that
$$(2\nu)t_\epsilon^{p-1}\|v_\epsilon\|_p^p + (2\nu^{\frac{q}{p}})t_\epsilon^{q-1}\|v_\epsilon\|_q^q = t_\epsilon^{r-1}\int_\Omega (\lambda f(x) + \mu g(x))\left(\nu^{\frac{1}{p}}v_\epsilon\right)^r dx$$
$$+ 2\nu^{\frac{p-\alpha-\beta+2}{p}}t_\epsilon^{p_s^*-1}\int_\Omega h(x)v_\epsilon^{2-\alpha-\beta}dx. \tag{2.8}$$
From (2.8), we have the following:
$$t_\epsilon^{p+\alpha+\beta-2}\|v_\epsilon\|_p^p \leq t_\epsilon^{r+\alpha+\beta-2}\int_\Omega (\lambda f(x) + \mu g(x))\left(v_\epsilon\nu^{\frac{1}{p}}\right)^r dx$$
$$+ 2\nu^{\frac{p-\alpha-\beta+2}{p}}\int_\Omega h(x)v_\epsilon^{2-\alpha-\beta}dx \tag{2.9}$$
and
$$(2\nu)t_\epsilon^{p-q}\|v_\epsilon\|_p^p + 2\nu^{\frac{q}{p}}\|v_\epsilon\|_q^q \geq t_\epsilon^{r-q}\int_\Omega (\lambda f(x) + \mu g(x))\left(v_\epsilon\nu^{\frac{1}{p}}\right)^r dx. \tag{2.10}$$

From the estimates for u_ϵ, obtained in the Remark 2.14, i.e.,

$$\|v_\epsilon\|_p^p = CS + O\left(\epsilon^{\frac{N-sp}{p}}\right),$$
$$\int_\Omega |v_\epsilon|^r dx = O\left(\epsilon^{\frac{r(N-sp)}{p^2}}\right),$$
$$\int_\Omega |v_\epsilon|^{2-\alpha-\beta} dx = O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-sp)}{p^2}}\right)$$

as $\epsilon \to 0$, and from (2.8), it very easily follows now that

$$t_\epsilon^{p+\alpha+\beta-2}\left(CS + O\left(\epsilon^{\frac{N-sp}{p}}\right)\right) = CM' t_\epsilon^{r+\alpha+\beta-2} + 2M\nu^{\frac{p-\alpha-\beta+2}{p}} O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-sp)}{p^2}}\right)$$

as $\epsilon \to 0$, where we have used the estimate

$$\int_\Omega (\lambda f(x) + \mu g(x)) v_\epsilon^r dx \leq CM' \|v_\epsilon\|_{p^*}^r = CM'.$$

Thus, there exists $T_1 > 0$, $\epsilon_1 > 0$ such that for any $\epsilon \in (0, \epsilon_1)$, we have $t_\epsilon \geq T_1$. Likewise, we have

$$Ct_\epsilon^{2-\alpha-\beta-q} = (2\nu) t_\epsilon^{p-q}\left(S + O\left(\epsilon^{\frac{N-sp}{p}}\right)\right) + 2C\nu^{\frac{q}{p}} \tag{2.11}$$

as $\epsilon \to 0$. Then, there exists $T_2 > 0$, $\epsilon_2 > 0$ such that for any $\epsilon \in (0, \epsilon_2)$, we have $t_\epsilon \leq T_2$. Let $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$. Then, for any $\epsilon \in (0, \tilde{\epsilon})$, we have $T_1 \leq t_\epsilon \leq T_2$. Consider

$$b_\epsilon(t) = \frac{t^p}{p}\nu\|v_\epsilon\|_p^p - \frac{1}{r}\int_\Omega (\lambda f(x) + \mu g(x))\left(t v_\epsilon \nu^{\frac{1}{p}}\right)^r dx.$$

Then a simple calculation gives

$$\sup_{t\geq 0} b_\epsilon(t) = \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-sp}{p}}\right)$$

as $\epsilon \to 0$. Therefore, for any $\epsilon \in (0, \tilde{\epsilon})$, we have

$$\begin{aligned}
a_\epsilon(t_\epsilon) &= b_\epsilon(t_\epsilon) + \frac{t_\epsilon^q}{q}(\nu^{\frac{q}{p}})\|v_\epsilon\|_q^q - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} t_\epsilon^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&\leq b_\epsilon(t_\epsilon) + 2\nu^{\frac{q}{p}}\frac{t_\epsilon^q}{q}\|v_\epsilon\|_q^q - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} t_\epsilon^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&\leq b_\epsilon(t_\epsilon) + 2\nu^{\frac{q}{p}}\frac{T_2^q}{q}\|v_\epsilon\|_q^q - \frac{\nu^{\frac{p-\alpha-\beta+2}{p}} T_1^{2-\alpha-\beta}}{2-\alpha-\beta}\int_\Omega h(x) v_\epsilon^{2-\alpha-\beta} dx \\
&= \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{N-sp}{p}}\right) + O\left(\epsilon^{\frac{q(N-sp)}{p^2}}\right) - O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-sp)}{p^2}}\right) \\
&= \frac{r-p}{rp} S^{\frac{r}{r-p}} + O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-sp)}{p^2}}\right)
\end{aligned}$$

as $\epsilon \to 0$ because, according to the assumptions in problem (1.1)–(1.4), we have

$$0 < \frac{(2-\alpha-\beta)(N-sp)}{p^2} < \frac{q(N-sp)}{p^2} < \frac{N-sp}{p}.$$

Therefore, one can choose $\epsilon_1 > 0$, sufficiently small, Λ^*, $\sigma(\epsilon) > 0$ such that for $\epsilon \in (0, \epsilon_1)$, and $\sigma \in (0, \sigma(\epsilon))$ under condition (2.4), we obtain

$$-A_0\nu\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}} + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right] - \sigma = O\left(\epsilon^{\frac{(2-\alpha-\beta)(N-sp)}{p^2}}\right)$$

as $\epsilon \to 0$. □

3. Few useful lemmas

In this section, we recall and prove some important lemmas which are crucial for the proof of the main theorem. We first consider a submanifold of $\mathcal{M}^-_{\alpha,\beta}$ defined as follows:

$$\mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}) = \{(u,v) \in \mathcal{M}^-_{\alpha,\beta} \mid J_{\alpha,\beta}(u,v) \leq c_{\alpha,\beta}\}.$$

The main result we prove in this section is that problem (1.1)–(1.4) admits at least $\mathrm{cat}(\Omega)$ number of solutions in this set.

Definition 3.1.

(a) For a topological space X, we say that a non-empty closed subspace $Y \subset X$ is contractible to a point if and only if there exists a continuous mapping

$$\xi : [0,1] \times Y \to X$$

such that for some $x_0 \in X$, there hold

$$\xi(0,x) = x \quad \text{for all } x \in Y$$

and

$$\xi(1,x) = x_0 \quad \text{for all } x \in Y.$$

(b) If Y is a closed subset of a topological space X, then $\mathrm{cat}_X(Y)$ will denote the Lusternik–Schnirelman category of Y, i.e., the least number of closed and contractible sets in X which cover Y.

We now state an auxiliary lemma which can be found in the form of Theorem 1 in [1].

Lemma 3.2. *Suppose that M is a $C^{1,1}$ complete Riemanian manifold and $I \in C^1(M, \mathbb{R})$. Assume that for $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$:*

(i) *I satisfies the $(PS)_c$ condition for $c \leq c_0$,*

(ii) $\mathrm{cat}\,(u \in M : I(u) \leq c_0) \geq k$.

Then I has at least k critical points in $\{u \in M \mid I(u) \leq c_0\}$.

The following lemma is a standard one and can be proved if one works in the lines of the argument in [28].

Lemma 3.3. *Let $\{(u_n, v_n)\} \subset X$ be a nonnegative sequence of functions with*

$$\int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r)\, dx = 1$$

and $\|(u_n, v_n)\|_p^p \to S'$. Then there exists a sequence $\{(y_n, \theta_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$\omega_n(x) = (\omega_n^1(x), \omega_n^2(x)) = \theta_n^{\frac{N}{r}} (u_n(\theta_n x + y_n), v_n(\theta_n x + y_n))$$

contains a convergent subsequence, denoted again by $\{\omega_n\}$, such that

$$\omega_n \to \omega \quad \text{in } W^{1,p}\left(\mathbb{R}^N\right) \times W^{1,p}\left(\mathbb{R}^N\right),$$

where $\omega = (\omega^1, \omega^2) > 0$ in $\mathbb{R}^N$. Moreover, we have $\theta_n \to 0$ and $y_n \to y \in \overline{\Omega}$ as $n \to \infty$.

Up to translations, we assume that $0 \in \Omega$. Moreover, we choose $\delta > 0$ small enough such that $B_\delta = \left\{x \in \mathbb{R}^N \,\middle|\, \operatorname{dist}(x, \partial\Omega) < \delta\right\}$ and the sets

$$\Omega_\delta^+ = \left\{x \in \mathbb{R}^N \,\middle|\, \operatorname{dist}(x, \partial\Omega) < \delta\right\}, \quad \Omega_\delta^- = \left\{x \in \mathbb{R}^N \,\middle|\, \operatorname{dist}(x, \partial\Omega) > \delta\right\}$$

that are both homotopically equivalent to Ω. By using the idea of [14] or [22], we define a continuous mapping $\tau : \mathcal{M}_{\alpha,\beta}^- \to \mathbb{R}^N$ by setting

$$\tau(u, v) = \frac{\int_\Omega x(\lambda f u^r + \mu g v^r)\, dx}{\int_\Omega (\lambda f u^r + \mu g v^r)\, dx}.$$

Remark 3.4. As mentioned earlier in this paper that the functional $J_{\alpha,\beta}$ is not a C^1-functional, we might fail to apply some very useful techniques in variational techniques. For this reason, we will define a *cut-off* functional using a subsolution (refer to [13] for the definition) to the system in (1.1)–(1.4). Define

$$\overline{f}(x,t,s) = \begin{cases} f(x,t,s) & \text{if } t > \underline{u}, s > \underline{v}, \\ f(x,t,\underline{v}) & \text{if } t > \underline{u},\ \ s \leq \underline{v} \\ f(x,\underline{u},s) & \text{if } t \leq \underline{u},\ \ s > \underline{v} \\ f(x,\underline{u},\underline{v}) & \text{if } t \leq \underline{u},\ \ s \leq \underline{v} \end{cases},$$

where

$$\begin{aligned} f(x,t,s) = \lambda f(x) t^{r-1} + \mu g(x) s^{r-1} &+ \nu \frac{1-\alpha}{2-\alpha-\beta} h(x) t^{-\alpha} s^{1-\beta} \\ &+ \nu \frac{1-\beta}{2-\alpha-\beta} h(x) t^{1-\alpha} s^{-\beta} \end{aligned}$$

is a subsolution to (1.1)–(1.4) (the existence of such a solution can be guaranteed by the previous sections by taking $\lambda = \mu = 0$ in (1.1)–(1.4)). Let

$$\overline{F}(x,t,s) = \int_0^t \int_0^s \overline{f}(x,t,s)\, ds\, dt$$

and $(\underline{u}, \underline{v})$. Define a functional $\overline{I} : X \to \mathbb{R}$ as follows:

$$\overline{J}_{\alpha,\beta}(u,v) = \frac{1}{p}\|(u,v)\|_p^p + \frac{1}{q}\|(u,v)\|_q^q - \int_\Omega \overline{F}(x,u,v)\, dx. \tag{3.1}$$

The functional is C^1 (the proof follows the arguments of Lemma 6.4 in the Appendix of [26]) and it is weakly lower semicontinuous. Taking into account the way the functional was defined, it is not difficult to see that the critical points of the functional corresponding to problem (1.1)–(1.4) and that of the cut-off functional are the same.

Remark 3.5. We will continue to name the cut-off functional $\overline{J}_{\alpha,\beta}$ as $J_{\alpha,\beta}$.

We then have the following result.

Lemma 3.6. *There exists Λ^* such that if (2.4) holds, and $(u,v) \in \mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta})$, then $\tau(u,v) \in \Omega^+_\delta$.*

Proof. Let us assume that there exist sequences $\nu_n \to 0$ and $\{(u_n,v_n)\}$ such that $\tau(u_n,v_n) \notin \Omega^+_\delta$. By using the same tactics as in one of the previous lemmas (2.11), we conclude the boundedness of the sequence $\{(u_n,v_n)\}$ in X. Then we have

$$\nu_n \int_\Omega h(x) u_n^{1-\alpha} v_n^{1-\beta} dx \to 0 \quad \text{as } n \to \infty.$$

Therefore, we get

$$J_{\alpha,\beta}(u_n,v_n) = \left(\frac{1}{p} - \frac{1}{r}\right) \|(u_n,v_n)\|_p^p + \left(\frac{1}{q} - \frac{1}{r}\right) \|(u_n,v_n)\|_q^q + o(1) \le c_{\alpha,\beta}$$

and

$$\left(\frac{1}{p} - \frac{1}{r}\right) \|(u_n,v_n)\|_p^p \le c_{\alpha,\beta} \le \frac{S^{\frac{r}{r-p}}}{\Lambda}.$$

$$\|(u_n,v_n)\|_p^p \le \frac{rp}{r-p} \frac{S^{\frac{r}{r-p}}}{\Lambda}. \tag{3.2}$$

Since $\{(u_n,v_n)\} \subset \mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}) \subset \mathcal{M}^-_{\alpha,\beta}$, we have

$$\lim_{n\to\infty} \|(u_n,v_n)\|_p^p \lim_{n\to\infty} \le \int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r)\, dx \le \lim_{n\to\infty} M' |(u_n,v_n)|_{p^*}^r. \tag{3.3}$$

By (3.2) and (3.3), we get

$$S' \le \frac{\|(u_n,v_n)\|_p^p}{\{\int_\Omega (u_n^{p^*} + v_n^{p^*}) dx\}^{\frac{p}{p^*}}} \le C \|(u_n,v_n)\|_p^p = S' + o(1), \tag{3.4}$$

which implies that $\|(u_n, v_n)\|_p^p \to CSS'^{\frac{p}{r-p}}$ and

$$\int_\Omega (\lambda f(x) u_n^r + \mu g(x) v_n^r)\, dx \to C' S'^{\frac{p}{r-p}}$$

as $n \to \infty$.

Define

$$(\xi_n, \eta_n) = \left(\frac{u_n}{\left(\int_\Omega (\lambda f u_n^r + \mu g v_n^r)\, dx\right)^{1/r}}, \frac{v_n}{\left(\int_\Omega (\lambda f u_n^r + \mu g v_n^r)\, dx\right)^{1/r}} \right).$$

Clearly,

$$\int_\Omega (\lambda \xi_n^r + \mu \eta_n^r)\, dx = 1$$

and

$$\int_\Omega (|\nabla \xi_n|^p + |\eta_n|^p dx) \to S'^{\frac{p}{r-p}\frac{r-1}{r}} \quad \text{as } n \to \infty.$$

From Lemma 3.3, there exists a sequence $\{(y_n, \theta_n)\} \subset \mathbb{N} \times \mathbb{R}^+$ such that $\theta_n \to 0$, $y_n \to y \in \overline{\Omega}$ and

$$\omega(x) = (\omega_n^1(x), \omega_n^2(x)) = \theta_n^{\frac{N}{r}} (\xi_n(\theta_n x + y_n), \eta_n(\theta_n x + y_n)) \to (\omega_1, \omega_2)$$

with $\omega_1, \omega_2 > 0$ in $\mathbb{R}^N$ as $n \to \infty$.

Let $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi(x) = x$ in Ω. Then we guarantee that

$$\begin{aligned} \tau(u_n, v_n) &= \frac{\int_\Omega \chi(x)(\lambda f u_n^r + \mu g v_n^r)\, dx}{\int_\Omega (\lambda f u_n^r + \mu g v_n^r)\, dx} \\ &= \int_\Omega \theta_n^N \chi(\theta_n x + y_n)(\lambda \xi_n^r + \mu \eta_n^r)\, dx \\ &= \int_\Omega \chi(\theta_n x_n + y_n)(\lambda(\omega_n(x)^1)^r + \mu(\omega_n(x)^2)^r)\, dx. \end{aligned} \tag{3.5}$$

By the Lebesgue dominated convergence theorem, we have

$$\int_\Omega \chi(\theta_n x_n + y_n)(\lambda(\omega_n^1)^r + \mu(\omega_n^2)^r) dx \to y \in \overline{\Omega}$$

as $n \to \infty$. This implies that $\tau(x_n, y_n) \to y \in \overline{\Omega}$ as $n \to \infty$, which leads to a contradiction to our assumption. □

The analysis done till now tells us that $\inf_{M_\delta} u_{\alpha,\beta} > 0$ and $\inf_{M_\delta} v_{\alpha,\beta} > 0$, due to Lemma 2.13 and the definition of Ω_δ^-. Note that

$$M_\delta = \left\{ x \in \Omega \,\middle|\, \operatorname{dist}(x, \Omega_\delta^-) \le \frac{\delta}{2} \right\}$$

which is a compact set. Thus, by Lemma 2.15 and using the idea of Lemma 3.4 of [14], Lemma 3.3 of [8], we can obtain $\tilde{t}^- > 0$ such that

$$(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x - y), \tilde{t} \sqrt[p]{\nu} v_\epsilon(x - y)) \in \mathcal{M}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)$$

uniformly in $y \in \Omega_\delta^-$. Further, by Lemma 3.6, $\tau(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y), \tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y)) \in \Omega_\delta^-$. Thus we can define a map $\gamma : \Omega_\delta^- \to \mathcal{M}_{\alpha,\beta}(c_{\alpha,\beta} - \sigma)^-$ by

$$\gamma(y) = \begin{cases} (\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y), \tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y)) & \text{if } x \in B_\delta(y) \\ 0 & \text{otherwise} \end{cases}.$$

We will denote by $\tau_{\alpha,\beta}$ the restriction of τ over $\mathcal{M}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma)$. Observe that v_ϵ is a radial function, therefore for each $y \in \Omega_\delta^-$, we have

$$\begin{aligned}(\tau_{\alpha,\beta} \circ \gamma)(y) &= \frac{\int_\Omega x(\lambda f(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r + \mu g(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r)\, dx}{\int_\Omega (\lambda f(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r + \mu g(x)(\tilde{t}^- \sqrt[p]{\nu} v_\epsilon(x-y))^r)\, dx} \\ &= \frac{\int_\Omega (y+z)(\tilde{t}^-)^r \nu^{\frac{r}{p}} (\lambda f + \mu g) v_\epsilon^r dz}{\int_\Omega (\tilde{t}^-)^r \nu^{\frac{r}{p}} (\lambda f + \mu g) v_\epsilon^r dz} = y.\end{aligned}$$

From [14], we define the map $T_{\alpha,\beta} : [0,1] \times \mathcal{M}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma) \to \mathbb{R}^N$ by

$$T_{\alpha,\beta}(t,z) = t\tau_{\alpha,\beta}(z) + (1-t)\tau_{\alpha,\beta}(z).$$

We then have the following lemma.

Lemma 3.7. *To each $\epsilon \in (0, \epsilon_0)$, there exists $\Lambda^* > 0$ such that if (2.4) holds, we have $T_{\alpha,\beta}([0,1] \times \mathcal{M}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma)) \subset \Omega_\delta^-$.*

Proof. We prove by contradiction. Let there exist sequences $t_n \in [0,1]$, $\nu_n \to 0$ and $z_n = (u_n, v_n) \in \mathcal{M}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma)$ such that $T_{\alpha,\beta}(t_n, z_n) \notin \Omega_\delta^+$ for all n. We can assume that $t_n \to t \in [0,1]$. Thus, by Lemma 2.13 (ii) and an argument similar to that used in the proof of 3.6, we have

$$T_{\alpha,\beta}(t_n, z_n) \to y \in \overline{\Omega} \quad \text{as } n \to \infty,$$

which leads to a contradiction. □

We now prove the main result of this paper which roughly states that under certain assumptions on ν problem (1.1)–(1.4) admits at least $\operatorname{cat}(\Omega) + 1$ number of solutions.

Lemma 3.8. *If (u,v) is a critical point of $J_{\alpha,\beta}$ on $\mathcal{M}_{\alpha,\beta}^-$, then it is also a critical point of $J_{\alpha,\beta}$ in X.*

Proof. We follow the proof of Lemma 4.1 in [14] or Lemma 4.1 in [31]. Let (u,v) be a critical point of $J_{\alpha,\beta}$ in $\mathcal{M}_{\alpha,\beta}^-$. Then

$$\langle J'_{\alpha,\beta}(u,v), (u,v)\rangle = 0.$$

Define

$$\psi(u,v) = \langle J'_{\alpha,\beta}(u,v), (u,v)\rangle = \|(u,v)\|_p^p + \|(u,v)\|_q^q$$

$$-\int_\Omega (\lambda f(x)u^r + \mu g(x)v^r)\,dx - \nu\int_\Omega h(x)u^{1-\alpha}v^{1-\beta}dx.$$

Since we are now seeking to minimize $J_{\alpha,\beta}$ over the entire space X, the Lagrange multiplier method helps us in finding a $\theta(\neq 0)\in\mathbb{R}$ such that

$$J'_{\alpha,\beta}(u,v) = \theta\psi'(u,v), \tag{3.6}$$

where

$$\psi(u,v) = \langle J'_{\alpha,\beta}(u,v),(u,v)\rangle.$$

Since $(u,v)\in\mathcal{M}^-_{\alpha,\beta}$, by a simple computation, we have that $\psi'(u,v)<0$. Consequently, from (3.6), we have $J'_{\alpha,\beta}(u,v)=0$. □

Lemma 3.9. *There exists $\Lambda^*>0$ such that any sequence $\{(u_n,v_n)\}\subset\mathcal{M}^-_{\alpha,\beta}$ with $J_{\mathcal{M}^-_{\alpha,\beta}}(u_n,v_n)\to c\in(-\infty,c_{\alpha,\beta})$ and $J'_{\mathcal{M}^-_{\alpha,\beta}}(u_n,v_n)\to 0$ as $n\to\infty$ contains a convergent subsequence if (2.4) holds.*

Proof. By the Lagrange multiplier method, there exists a sequence $(a_n)\subset\mathbb{R}$ such that

$$\|I'_{\alpha,\beta}(u_n,v_n) - a_n\psi'_{\alpha,\beta}(u_n,v_n)\|_{X'}\to 0$$

as $n\to\infty$. Here,

$$\begin{aligned}\psi_{\alpha,\beta}(u_n,v_n) &= \langle I'_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle\\ &= \|(u_n,v_n)\|_p^p + \|(u_n,v_n)\|_q^q - \int_\Omega(\lambda f(x)u_n^r + \mu g(x)v_n^r)\,dx\\ &\qquad - \nu\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx.\end{aligned}$$

Then

$$I'_{\alpha,\beta}(u_n,v_n) = a_n\psi'_{\alpha,\beta}(u_n,v_n) + o(1)$$

as $n\to\infty$. Since $(u_n,v_n)\in\mathcal{M}^-_{\alpha,\beta}\subset\mathcal{M}_{\alpha,\beta}$, by a simple computation, we have

$$\langle\psi'_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle < 0.$$

Now suppose $\langle\psi'_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle\to 0$ as $n\to\infty$. Then we have

$$\begin{aligned}&\lim_{n\to\infty}(r-p)\|(u_n,v_n)\|_p^p + (r-q)\|(u_n,v_n)\|_q^q\\ &\qquad = \lim_{n\to\infty}\nu(1+\alpha+\beta)\int_\Omega h(x)u_n^{1-\alpha}v_n^{1-\beta}dx\\ &\qquad \le \nu(1+\alpha+\beta)M\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right.\\ &\qquad\quad \left. + \left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]^{\frac{p+\alpha+\beta-2}{p}}\lim_{n\to\infty}\|(u_n,v_n)\|_p^{2-\alpha-\beta}\end{aligned}$$

and

$$\lim_{n\to\infty}(p+\alpha+\beta-2)\|(u_n,v_n)\|_p^p+(q+\alpha+\beta-2)\|(u_n,v_n)\|_q^q$$
$$=\lim_{n\to\infty}(r+\alpha+\beta-2)\int_\Omega(\lambda f(x)u_n^r+\beta g(x)v_n^r)\,dx\leq\lim_{n\to\infty}M'\|(u_n,v_n)\|_p^{p^*},$$

where we have used the Hölder inequality and the Sobolev embedding. Then we have

$$\lim_{n\to\infty}\|(u_n,v_n)\|_p\leq(\nu C_1)^{\frac{1}{p}}\left[\left(\frac{1-\alpha}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}+\left(\frac{1-\beta}{2-\alpha-\beta}\right)^{\frac{p}{p+\alpha+\beta-2}}\right]^{\frac{1}{p}}$$

and

$$\|(u_n,v_n)\|_p=C_2^{\frac{1}{p^*-p}}+o(1)$$

as $n\to\infty$. Now, if we choose Λ^* small enough, this cannot hold. Therefore, let us assume that $\langle\psi_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle\to l<0$ as $n\to\infty$. Since $\langle J_{\alpha,\beta}(u_n,v_n),(u_n,v_n)\rangle=0$, we conclude that $a_n\to 0$, and thus $I'_{\alpha,\beta}(u_n,v_n)\to 0$ as $n\to\infty$. This gives us that

$$I_{\alpha,\beta}(u_n,v_n)=c<c_{\alpha,\beta}\quad\text{and}\quad I'_{\alpha,\beta}(u_n,v_n)\to 0\quad\text{as } n\to\infty.$$

Therefore, by Lemma 2.11, the proof is complete. □

Lemma 3.10. *Suppose that* (C) *and* (2.4) *hold. Then*

$$\operatorname{cat}(\mathcal{M}^-_{\lambda,\mu}(c_{\lambda,\mu}-\sigma))\geq\operatorname{cat}(\Omega).$$

Proof. Let $\operatorname{cat}(\mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma))=n$. Then, by the definition 3.1 of the category of a set in the sense of Lusternik–Schnirelman, we suppose that

$$\mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma)=A_1\cup A_2\cup\cdots\cup A_n,$$

where A_j, $j=1,2,\ldots,n$ are closed and contractible in $\mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma)$, i.e., there exists $h_j\in C([0,1]\times A_j,\mathcal{M}^-_{\alpha,\beta}(c_{\alpha,\beta}-\sigma))$ such that

$$h_j(0,z)=z,\ \ h_j(1,z)=\Theta\quad\text{for all } z\in A_j,$$

where $\Theta\in A_j$ is fixed. Consider $B_j=\gamma^{-1}(A_j)$, $j=1,2,\ldots,n$. Then the sets B_j are closed

$$\Omega_\delta^-=B_1\cup B_2\cup\cdots\cup B_n.$$

We now define the deformation $g_j:[0,1]\times B_j\to\Omega_\delta^+$ by setting

$$g_j(t,y)=T_{\alpha,\beta}\big(t,h_j(t,\gamma(y))\big)$$

under condition (2.4). Notice that

$$g_j(0,y)=T_{\alpha,\beta}(0,h_j(0,\gamma(y)))=(\tau_{\alpha,\beta}\circ\gamma)(y)=y\quad\text{for all } y\in B_j$$

and

$$g_j(1,y) = T_{\alpha,\beta}(0, h_j(1,\gamma(y))) = \tau_{\alpha,\beta}(\Theta) \in \Omega_\delta^+ \quad \text{for all } y \in B_j.$$

Thus the sets B_j, $j = 1, 2, \dots, n$ are contractible in Ω_δ^+. Therefore,

$$\mathrm{cat}(\mathcal{M}_{\alpha,\beta}^- - \sigma) \geq \mathrm{cat}_{\Omega_\delta^+}(\Omega_\delta^-) = \mathrm{cat}(\Omega).$$

The lemma is proved. □

Proof of Theorem 1.1. By Lemmas 2.11 and 3.9, the functional $I_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for $c \in (-\infty, c_{\alpha,\beta})$. Then, by Lemmas 3.2 and 3.10, we have that $I_{\alpha,\beta}$ has at least $\mathrm{cat}(\Omega)$ number of critical points in $\mathcal{M}_{\alpha,\beta}^-(c_{\alpha,\beta} - \sigma)$. By Lemma 3.8, we have that $I_{\alpha,\beta}$ has at least $\mathrm{cat}(\Omega)$ number of critical points in $\mathcal{M}_{\alpha,\beta}^-$. Further, since $\mathcal{M}_{\alpha,\beta}^+ \cap \mathcal{M}_{\alpha,\beta}^- = \phi$, the proof is now complete. □

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Множинність розв'язків систем з p-q дробовим лапласіаном з увігнутими сингулярними нелінійностями

Kamel Saoudi, Debajyoti Choudhuri, and Mouna Kratou

У цій роботі ми вивчаємо існування множинних нетривіальних невід'ємних слабких розв'язків сполученої системи еліптичних диференціальних рівнянь з частинними похідними. Доведено існування розв'язків на многовиді Негарі. Для доведення існування щонайменше $\mathrm{cat}(\Omega) + 1$ розв'язків використано категорію Люстерника–Шнірельмана, де Ω є обмеженою областю, в якій розглянуто цю задачу.

Ключові слова: многовид Негарі, категорія Люстерника–Шнірельмана, сингулярність, множинність