

Busemann Functions in Asymptotically Harmonic Finsler Manifolds

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In the present paper, we study Busemann functions in a general Finsler setting as well as in asymptotically harmonic Finsler manifolds. In particular, we show that Busemann functions are smooth on asymptotically harmonic Finsler manifolds.

Key words: Busemann function, asymptote, harmonic Finsler manifold, asymptotically harmonic Finsler manifold

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1. Introduction

Finsler geometry is a generalization of Riemannian geometry which is richer in content and much wider in scope. Working in the Finsler context may need different techniques that do not exist in the Riemannian framework. In [19], harmonic manifolds were introduced in the Finsler context. Recently, the study of harmonic and asymptotically harmonic Finsler manifolds of (α, β) -type has been discussed in [26]. It is known that Busemann functions play an important role in studying geometry of noncompact complete Riemannian manifolds with negative sectional curvature and harmonic manifolds (cf. [1, 16]). The convexity of Busemann functions is essential for the study of Hadamard Riemannian manifolds (cf. [22]). Furthermore, Busemann functions were used to study reversible Finsler manifolds of negative flag curvature [5], the splitting theorems for Finsler manifolds of non-negative Ricci curvature [12] and recently the relation between affine functions and Busemann functions on a complete Finsler manifold has been treated in [8]. Also, Finsler manifolds, whose Busemann functions are convex, were studied in [18]. The authors in [1, 9, 10, 23, 24] offered insightful discussions about Busemann functions in both complete Riemannian and Finslerian manifolds.

Our aim is to analyze Busemann functions in the context of Finsler geometry and then apply the obtained results for studying asymptotic harmonic Finsler (AHF) manifolds. For example, in a forward complete Finsler manifold we find the relation between Busemann functions associated with two asymptotic rays (in equation (3.5)). Also, we prove that any of two rays in an AHF-manifold are asymptotic if and only if the corresponding Busemann functions agree up to a

constant (see Theorem 4.1). Our results lead to the conclusion that Busemann functions are smooth in an AHF-space (see Theorem 4.7), which is a generalization of [16, Theorem 3.1] from the Riemannian to the Finsler context. Further, in Proposition 4.11, it is proved that if the horospheres of an AHF-manifold are minimal, then they have the bi-asymptotic property, that is, their asymptotic geodesics are bi-asymptotic.

The structure of the present paper is as follows. Section 2 is devoted to some preliminaries needed for better exposition of our work. Thereafter, in Section 3, we give some properties of the Busemann functions in a connected Finsler manifold without conjugate points. Then we study the relation of Busemann functions of asymptotic rays in a forward complete Finsler manifold. Finally, in Section 4, we conclude our work with the exploration of Busemann functions in the case of AHF-manifolds.

2. Preliminaries

We use the following notations: M denotes an n -dimensional, $n > 1$, orientable connected smooth manifold, (TM, π, M) , or simply TM , its tangent bundle and $TM_0 := TM \setminus \{0\}$ the tangent bundle with the null section removed. The tangent space at each $x \in M$ without the zero vector is denoted by T_xM_0 . The local coordinates (x^i) on M induce the local coordinates (x^i, y^i) on TM . Moreover, ∂_i and $\dot{\partial}_i$ denote partial differentiation with respect to x^i and y^i .

Definition 2.1 ([2]). A Finsler structure on a manifold M is a mapping $F : TM \rightarrow [0, \infty)$ such that F is C^∞ on TM_0 , positively homogeneous of degree one in y and the Hessian matrix $(g_{ij}(x, y))_{1 \leq i, j \leq n}$ is positive definite at each point y of TM_0 , where $g_{ij}(x, y) := \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, y)$.

We refer to [2,20] for further reading about Finsler geometry. A Finsler metric is Riemannian when $g_{ij}(x, y)$ are functions in x only. Further, a Finsler metric can be characterized in any tangent space T_xM by its unit vectors, which form a smooth strictly convex hypersurface I_xM called *indicatrix* at the point $x \in M$. When a Finsler metric is Riemannian, this hypersurface at each point of M is a Euclidean unit sphere. The indicatrix of F is $IM := \cup_{x \in M} I_xM$.

The distance d_F induced by F is defined in M by [20,27],

$$d_F(p, q) := \inf \left\{ \int_0^1 F(\dot{\eta}(t)) dt \mid \eta : [0, 1] \rightarrow M, C^1 \text{ curve joining } p \text{ to } q \right\}.$$

Remark 2.2. Note the following:

- (i) The Finsler distance is nonsymmetric, that is, $d_F(p, q) \neq d_F(q, p)$. In other words, the Finsler distance depends on the direction of the curve. Therefore the reverse of a general Finsler geodesic can not be a geodesic. The non-reversibility property is also reflected in the notion of Cauchy sequence and completeness [2, §6.2].

- (ii) Thus, being different from the Riemannian case, a positively (or forward) complete Finsler manifold (M, F) is not necessarily negatively (or backward) complete. The classical Hopf-Rinow theorem splits into forward and backward versions [2, §6.6]. A Finsler metric is *complete* if it is both forward and backward complete.
- (iii) Another main difference between Finsler and Riemannian geometries is that in a general Finsler manifold, the exponential map is only C^1 at the origin of $T_x M$ and it is C^∞ on $T_x M_0$.

A volume measure $d\mu$ (nondegenerate n -volume form) on M can be written in local coordinates as $d\mu = \sigma_\mu(x) dx^1 \wedge \cdots \wedge dx^n = \sigma_\mu(x) dx$, where $\sigma_\mu(x)$ is a positive smooth function on M . Unlike in Riemannian geometry, there are several non-equivalent definitions of volume forms used within Finsler geometry. The most well known among them are *Busemann–Hausdorff* $d\mu_{BH}$ and *Holmes–Thompson* $d\mu_{HT}$ volume forms [6]. Otherwise stated, we work with arbitrary but fixed volume form $d\mu$. That is, the forthcoming definitions and results hold for either Busemann–Hausdorff volume form or Holmes–Thompson volume form.

It is known that if F is a Finsler structure on M , then F induces a Minkowski norm on $T_x M$ at each point $x \in M$. Also, F^* , the dual structure of F , induces a Minkowski norm on $T_x^* M$. That is, $F^* : T^* M \rightarrow \mathbb{R}^+$ is defined, for all $(x, \alpha) \in T^* M$, by

$$F^*(x, \alpha) := \sup\{\alpha(\xi) \mid \xi \in I_x M\}.$$

The dual metric associated to F^* is given by $g_{ij}^*(x, \alpha) := \frac{1}{2} \frac{\partial^2 F^{*2}(x, \alpha)}{\partial \alpha^i \partial \alpha^j}$.

The *Legendre transformation* $J : TM \rightarrow T^* M$ associated with F is defined, for any point $x \in M$, by $J(x, y) = g_{ij}(x, y) y^i dx^j$, $\forall y \in T_x M_0$ and $J(0) = 0$. Let $J^* : T^* M \rightarrow TM$ be defined by

$$J^*(x, \alpha) = g_{ij}^*(x, \alpha) \alpha_i \partial_j, \quad \text{for all } \alpha \in T_x^* M_0 \text{ and } J^*(0) = 0,$$

where $g_{ij}^*(x, \alpha) := g^{ij}(J^*(\alpha))$.

Definition 2.3 ([20, §3.2]). The gradient of a differentiable function $f : M \rightarrow \mathbb{R}$ at a point $x \in M$, where $df(x) \neq 0$, is defined by

$$\nabla f(x) = J^*(x, df(x)) = g_{ij}^*(x, df(x)) \partial_i f(x) \partial_j. \quad (2.1)$$

Then $df(x)$ can be written as follows:

$$df(x, v) = g_{\nabla f(x)}(\nabla f(x), v), \quad v \in T_x M. \quad (2.2)$$

Remark 2.4. Unlike the Riemannian gradient, the gradient $\nabla f(x)$ is non-linear. It should be noted that when $df(x) = 0$, the gradient $\nabla f(x)$ is defined to be zero.

Definition 2.5 ([20]). A smooth function $f : M \rightarrow \mathbb{R}$ is called a Finsler distance if $F(\nabla f) = 1$.

A distance function r defined on an open subset Ω of (M, F) has some interesting geometric properties. Indeed, ∇r is a unit vector field on Ω and it induces a smooth Riemannian metric on Ω defined by

$$\hat{F}(x, v) := \sqrt{g_{\nabla r}(v, v)}, \quad v \in TM.$$

Furthermore, $\hat{F}(\hat{\nabla}r) = F(\nabla r) = 1$ by [20, Lemma 3.2.2].

Definition 2.6 ([20, §14.1]). Let $(M, F, d\mu)$ be a Finsler μ -space. For a C^2 function f , Shen’s Laplacian Δf of f is defined by $\Delta f = \operatorname{div}_\mu(\nabla f)$, that is,

$$\begin{aligned} \Delta f &= \frac{1}{\sigma_\mu(x)} \partial_k \left[\sigma_\mu(x) g^{kl}(x, \nabla f(x)) \partial_l f \right] \\ &= \left[g^{kl}(x, \nabla f(x)) \partial_k (\log(\sigma_\mu(x)) + \partial_k(g^{kl}(x, \nabla f(x)))) \right] \partial_l f \\ &\quad + g^{kl}(x, \nabla f(x)) \partial_l \partial_k f. \end{aligned} \tag{2.3}$$

Remark 2.7. Shen’s Laplacian is fully non-linear elliptic differential operator of the second order, cf. [4], which depends on the measure μ and it is defined on $U_f := \{x \in M \mid df(x) \neq 0\}$ by (2.3), and to be zero on $\{x \in M \mid df(x) = 0\}$.

Definition 2.8 ([20, §14.1]). For $u \in H^1_{loc}(M)$, the weak (or distributional) Laplacian of u is defined by

$$\int_M \phi \Delta u \, d\mu = - \int_M d\phi(\nabla u) \, d\mu \quad \text{for all } \phi \in C_c^\infty(M). \tag{2.4}$$

Definition 2.9 ([20, §14.3]). The Finsler mean curvature of the level hypersurface $r^{-1}(t)$ at $x \in M$ with respect to ∇r_x is defined by

$$\Pi_{\nabla r}(x) := \left. \frac{d}{dt} \log(\sigma_x(t, x^a)) \right|_{t=t_0} \quad \text{for some } t_0 \in \operatorname{Im}(r). \tag{2.5}$$

The Finsler Laplacian of a distance function r satisfies $\Delta r(x) = \Pi_{\nabla r}(x)$ [21].

Definition 2.10 ([19]). A forward complete Finsler manifold (M, F) endowed with a smooth volume measure $d\mu$ is (globally) harmonic if in polar coordinates the volume density function $\bar{\sigma}_p(r, y)$ is a radial function around (each) $p \in M$, where $\bar{\sigma}_p(r, y) := \frac{\sigma_p(r, y)}{\sqrt{\det(\dot{g}_p(p, y))}}$ and \dot{g}_p is the restriction of g on the indicatrix $I_p M$. That is, $\bar{\sigma}_p(r, y)$ is independent of $y \in I_p M$, and thus it can be written as $\bar{\sigma}_p(r)$.

Theorem 2.11 ([19]). *Let $(M, F, d\mu)$ be a forward complete Finsler μ -manifold. The following are equivalent:*

- (1) $(M, F, d\mu)$ is harmonic,
- (2) Shen’s Laplacian of a distance function is radial,
- (3) the Finsler mean curvature of all geodesic spheres of sufficiently small radii (all radii), expressed in polar coordinates, is a radial function.

Definition 2.12 ([19]). The Finsler mean curvature of horospheres Π_∞ is the mean curvature of the Finsler spheres of infinite radius and can be defined by

$$\Pi_\infty = \lim_{r \rightarrow \infty} \Pi_{\nabla r}(x).$$

Definition 2.13 ([19]). A forward complete, simply connected Finsler μ -manifold $(M, F, d\mu)$ without conjugate points is called asymptotically harmonic Finsler manifold (or shortly, AHF-manifold) if the Finsler mean curvature of horospheres is a real constant h .

Thus, a noncompact harmonic Finsler manifold with constant Finsler mean curvature of horospheres is an AHF-manifold. The examples of the AHF-manifold are given in [19]. Further, [7, 10, 28] offered perceptive discussions about harmonic and asymptotically Riemannian manifolds.

3. Analysis of Busemann functions with applications

An effective tool for studying various topics in differential geometry, such as the structure of harmonic spaces in Riemannian geometry, is Busemann functions. For more details about Busemann functions see [14–16, 24, 28] in the Riemannian context and [5, 9, 12, 17, 24] in the Finsler context.

Definition 3.1 ([12]). Let (M, F) be a forward complete Finsler manifold. A geodesic $\gamma : [0, \infty] \rightarrow M$ is called a forward ray if it is a globally minimizing unit speed Finslerian geodesic, that is, $d_F(\gamma(s), \gamma(t)) = t - s$ for all $s < t$ and $F(\dot{\gamma}) = 1$.

Now we recall the definition of *Busemann functions* in the context of Finsler geometry [12, 17, 23] and discuss some of their general properties.

Let (M, F) be a forward complete noncompact Finsler manifold without conjugate points. There always exists a forward ray $\gamma : [0, \infty) \rightarrow (M, F)$ emanating from each point $p := \gamma(0) \in M$ [17]. We define the following function associated to the ray γ :

$$b_{\gamma,t}(x) = d_F(x, \gamma(t)) - t, \quad \forall x \in M,$$

where d_F is the Finsler distance which is nonsymmetric.

Lemma 3.2. *For each $x \in M$, the function $b_{\gamma,t}(x)$ is monotonically decreasing with t . Moreover, $b_{\gamma,t}(x)$ is bounded below.*

Proof. Let $s, t \in [0, \infty)$ such that $s < t$. Using a triangle inequality for the nonsymmetric distance d , we have

$$t - s = d_F(\gamma(s), \gamma(t)) \leq d_F(\gamma(s), x) + d_F(x, \gamma(t)),$$

hence,

$$-s \leq d_F(\gamma(s), x) + d_F(x, \gamma(t)) - t = d_F(\gamma(s), x) + b_{\gamma,t}(x),$$

therefore,

$$-d_F(\gamma(0), x) \leq b_{\gamma,t}(x) \quad \text{at } s = 0.$$

Hence, $b_{\gamma,t}(x)$ is bounded below by $-d_F(\gamma(0), x)$.

To prove the decreasing of $b_{\gamma,t}$, we suppose $x \in M$ to be arbitrary but fixed and $s < t$. Using the triangle inequality, we have

$$d_F(x, \gamma(t)) \leq d_F(x, \gamma(s)) + d_F(\gamma(s), \gamma(t)) = d_F(x, \gamma(s)) + t - s,$$

therefore,

$$d_F(x, \gamma(t)) - t \leq d_F(x, \gamma(s)) - s \iff b_{\gamma,t}(x) \leq b_{\gamma,s}(x). \quad \square$$

In view of Lemma 3.2, the limit of the function $b_{\gamma,t}(x)$ as $t \rightarrow \infty$ exists. This justifies the following definition.

Definition 3.3. The limit $b_\gamma(x)$ of $b_{\gamma,t}(x)$ is called the Busemann function associated to the ray γ :

$$b_\gamma(x) = \lim_{t \rightarrow \infty} b_{\gamma,t}(x) = \lim_{t \rightarrow \infty} (d_F(x, \gamma(t)) - t). \quad (3.1)$$

We now give some properties of the Busemann functions.

Proposition 3.4. For a forward complete simply connected Finsler manifold without conjugate points, the following holds:

- (1) Along the ray $\gamma(t)$, we have $b_\gamma(\gamma(t)) = -t$ for all $t > 0$. Therefore, $b_\gamma(\gamma(0)) = b_\gamma(p) = 0$.
- (2) b_γ is 1-Lipschitz in the sense that

$$-d_F(y, x) \leq b_\gamma(y) - b_\gamma(x) \leq d_F(x, y), \quad x, y \in M. \quad (3.2)$$

Hence, b_γ is differentiable almost everywhere and it is uniformly continuous.

- (3) $b_{\gamma,t}$ converges to b_γ uniformly on each compact subset of M .

Proof. (1) follows directly from equation (3.1). (2) follows from the triangle inequality. (3) follows from Dini’s theorem. \square

It should be noted that Proposition 3.4 is proved in (cf. [12, 17]) under the condition that (M, F) is a noncompact forward complete Finsler manifold. In addition, for the Riemannian case, in [10, 11, 25] one can find the results similar to ours under different assumptions.

Proposition 3.5 ([21]). Busemann functions can be computed in a vector space equipped with a Finsler structure (V, F) as follows: for any vector $v \in V$, the Busemann function b_v associated to the ray $\eta_v(t) = tv$, $0 < t < \infty$, is given by

$$b_v(y) = -y^i \frac{\partial F(v)}{\partial y^i}. \quad (3.3)$$

Lemma 3.6. Let (M, F) be a forward complete Finsler manifold and let f be a Finsler distance on M . Then the following assertions hold:

- (1) The level sets of f have no critical points and viz. $f^{-1}(c)$ for any c are smooth hypersurfaces in M .
- (2) The integral curves of ∇f are unit speed Finslerian geodesics.
- (3) The level sets of f are parallel hypersurfaces along the direction ∇f . Consequently, f is linear along the integral curves of ∇f .

Proof. (1) A Finsler distance f on M by its definition means that $F(\nabla f) = 1$. Then f has no critical points and the rest of the proof follows directly.

(2) The proof follows from [2, Lemma 6.2.1].

(3) The proof follows from [27, §4]. That is, $d_F(f^{-1}(t), f^{-1}(s)) = s - t, t < s$. Consequently, f is linear along the integral curves of ∇f . Indeed, integrating both sides of $\dot{\eta}(t) = \nabla f \circ \eta$ yields $f(\eta(t)) = t + f(\eta(0))$. □

Remark 3.7. In the Finsler context, there is a slight difference in the definition of parallel hypersurfaces. This is due to the nonsymmetry of the distance d_F . Namely, if a hypersurface $f^{-1}(t)$ is parallel to $f^{-1}(s)$, it does not mean that $f^{-1}(s)$ is parallel to $f^{-1}(t)$, unless the Finsler metric is reversible, cf. [17].

Remark 3.8 ([14]). Let T be a distribution on a compact subset Ω of M . Then $\forall z \in \mathbb{Z}^+$ and we are to define the distributional derivative

$$\frac{d^z T}{dx^z}(\phi) = (-1)^z T(\phi^{(z)}), \quad \phi \in \mathcal{D}(\Omega).$$

That is,

$$\int_{\Omega} D^{(z)} T(\phi) d\mu = (-1)^z \int_{\Omega} T(D^{(z)} \phi) d\mu, \quad \phi \in \mathcal{D}(\Omega). \tag{3.4}$$

Lemma 3.9. We have $\Delta b_{\eta,t} \rightarrow \Delta b_{\eta}$ as $t \rightarrow \infty$ in the distributional sense.

Proof. Let $\Omega \subset M$ be a compact subset. Since $b_{\eta,t}$ is a continuous function, then it is locally integrable and therefore it is a distribution on Ω . Let ϕ be a test function, then $\Delta \phi$ is a test function as well. Thus,

$$\int_{\Omega} (\Delta b_{\eta,t})(\phi) d\mu = \int_{\Omega} b_{\eta,t}(\Delta \phi) d\mu.$$

Now, taking the limit as $t \rightarrow \infty$ yields

$$\int_{\Omega} b_{\eta}(\Delta \phi) d\mu = \int_{\Omega} (\Delta b_{\eta})(\phi) d\mu.$$

Hence, $\Delta b_{\eta,t} \rightarrow \Delta b_{\eta}$ in the distributional sense. □

For the corresponding result in Riemannian geometry, one can refer to [10,25].

Definition 3.10. Let (M, F) be a noncompact forward complete Finsler manifold. Let $\eta : [0, \infty) \rightarrow M$ be a ray. Another ray $\zeta : [0, \infty) \rightarrow M$ is said to be asymptotic to η if there exists a sequence $\{t_i\}_{i \in \mathbb{N}} \subset [0, \infty[$ and a sequence $\{\zeta_i\}_{i \in \mathbb{N}}$ such that

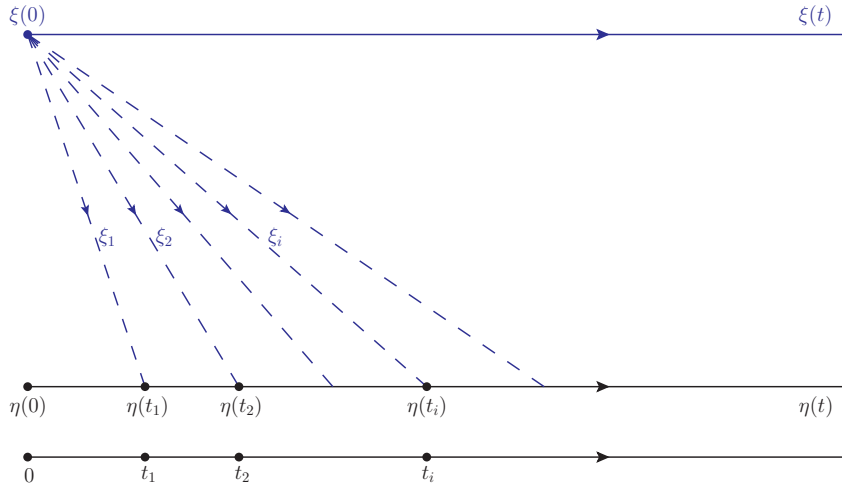
$$\lim_{i \rightarrow \infty} \zeta_i(t) = \zeta(t), \quad t \geq 0, \quad \lim_{i \rightarrow \infty} t_i = \infty,$$

and

$$\zeta_i : [0, d_F(\zeta(0), \eta(t_i))] \rightarrow M$$

is a sequence of minimal geodesics from $\zeta(0)$ to $\eta(t_i)$ (see [16] for the Riemannian case and [12, 17] for the Finsler one).

The following figure explains the definition:



The following result is a generalization of [15, Proposition 7.3.8] from the Riemannian to the Finsler context.

Proposition 3.11. *Let (M, F) be a forward complete Finsler manifold. If a ray ζ emanating from $p := \zeta(0)$ is asymptotic to η , then their Busemann functions are related by*

$$b_\eta(\zeta(t)) = b_\eta(p) + b_\zeta(\zeta(t)) = b_\eta(p) - t. \tag{3.5}$$

Consequently,

$$b_\eta(x) - b_\zeta(x) \leq b_\eta(p). \tag{3.6}$$

Proof. Let ζ be an asymptote to η from p . Then there exists a sequence $\{t_i\}_{i \in \mathbb{N}} \subset [0, \infty[$ and a sequence $\{\zeta_i\}_{i \in \mathbb{N}}$ of minimal geodesics from p to $\eta(t_i)$ such that

$$\begin{aligned} \lim_{i \rightarrow \infty} \zeta_i(t) &= \zeta(t), \quad t \geq 0, \quad \text{and} \quad \lim_{i \rightarrow \infty} t_i = \infty, \\ b_\eta(p) &:= \lim_{i \rightarrow \infty} (d_F(p, \eta(t_i)) - t_i) \\ &= \lim_{i \rightarrow \infty} (d_F(p, \zeta_i(s)) + d_F(\zeta_i(s), \eta(t_i)) - t_i) \\ &= d_F(p, \zeta(s)) + \lim_{i \rightarrow \infty} (d_F(\zeta(s), \eta(t_i)) - t_i) = s + b_\eta(\zeta(s)). \end{aligned}$$

That is, $b_\eta(q) - b_\eta(p) = -c$, where $\zeta(c) = q$, $c \geq 0$. Now we use equation (3.5) to prove (3.6) as indicated below. From the triangle inequality for the nonsymmetric distance d , we have

$$d_F(x, \eta(s)) - s \leq d_F(x, \zeta(t)) + d_F(\zeta(t), \eta(s)) - s$$

$$= d_F(x, \zeta(t)) - t + d_F(\zeta(0), \zeta(t)) + d_F(\zeta(t), \eta(s)) - s.$$

Now, let $s \rightarrow \infty$ in the above inequality. Then we obtain

$$b_\eta(x) \leq d_F(x, \zeta(t)) - t + d_F(p, \zeta(t)) + b_\eta(\zeta(t)).$$

Using (3.5), we get

$$b_\eta(x) \leq d_F(x, \zeta(t)) - t + d_F(p, \zeta(t)) + b_\eta(p) - t.$$

Therefore,

$$b_\eta(x) \leq d_F(x, \zeta(t)) - t + b_\eta(p).$$

Taking the limit $t \rightarrow \infty$ of both sides yields (3.6). \square

Actually, equation (3.5) represents a generalization of [5, Corollary 3.9].

Now we are to show that the asymptotes are unique. Consequently, the Busemann functions are distance functions, i.e., $F(\nabla b_\zeta) = 1$.

Corollary 3.12. *Let (M, F) be a forward complete Finsler manifold. Let $\eta : [0, \infty) \rightarrow M$ be a ray and $p \in M$. Then there exists a unique ray $\zeta(s) := \exp_p(sv)$ emanating from p that is asymptotic to η , where v is the initial velocity of ζ (cf. [12, 17]).*

Proof. Let ζ be an asymptote to η from p . From Proposition 3.4 (2), the Busemann function is differentiable almost everywhere, and hence one can differentiate both sides of equation (3.5) and get

$$\frac{d}{ds}(b_\eta(\zeta(s))) = -1.$$

Using (2.2), we have

$$\frac{d}{ds}(b_\eta(\zeta(s)))|_{s=0^+} = g(\nabla b_\eta(\zeta(s)), \dot{\zeta}(0)).$$

Hence,

$$\nabla b_\eta(p) = -\dot{\zeta}(0) = -v. \quad (3.7)$$

Therefore, there is only one asymptotic geodesic to η emanating from p , namely $\zeta(s) = \exp_p(s\nabla b_\eta(p))$. \square

4. Busemann functions in asymptotically harmonic manifolds

It should be noted that for a complete simply connected Riemannian manifold of nonpositive sectional curvature, the asymptotic relation between two rays is an equivalence relation [15]. However, imposing conditions on the flag curvature of a Finsler manifold does not suffice to make it an equivalence relation. This is because the asymptotic relation is neither transitive nor symmetric [24]. Even though, we prove the following.

Theorem 4.1. *In an AHF-manifold, two rays are asymptotic if and only if the corresponding Busemann functions agree up to a constant. Moreover, equation (3.6) represents an equivalence relation.*

Proof. Let ζ be a ray asymptotic to η and starting from p . Then it is clear from (3.6) that the function $(b_\eta - b_\zeta)$ attains its maximum at p . Let Ω be a bounded open subset of M containing p . Now, applying the “strong comparison principle” [6, Lemma 5.4], for $u = b_\zeta(x) + b_\eta(p)$, $v = b_\eta(x)$ and $\Lambda = 0$, yields that

$$b_\zeta(x) - b_\eta(x) = c, \tag{4.1}$$

where $c = b_\eta(p)$ is a constant and x lies in the component of Ω containing p , say, $x \in U$. This is because $u \geq v$ in Ω and $\Delta(b_\zeta(x) + b_\eta(p)) = \Delta b_\zeta(x) = h$, $\Delta b_\eta(x) = h$ as $(M, F, d\mu)$ is an AHF-manifold. Note that the set

$$A := \{z \in U \mid b_\zeta(z) - b_\eta(z) = c\}$$

is a non-void open bounded subset of Ω . Meanwhile, it is clear that A is a closed set. But our base manifold M is connected, therefore A is the whole M . One can easily show that the relation (4.1), $\eta \approx \zeta \Leftrightarrow b_\zeta(x) - b_\eta(x) = c$, is an equivalence relation. □

Corollary 4.2. *The level sets $b_\gamma^{-1}(t) := (b_\gamma(t))^{-1}$ of a Busemann function are smooth closed noncompact hypersurfaces of M and are called limit spheres or horospheres.*

Proof. Since b_γ is a distance function, then, by Lemma 3.6 (1), the level sets of b_γ have no critical points and viz. $b_\gamma^{-1}(c)$ for any c are smooth hypersurfaces in M . Moreover, when M is simply connected, the level sets $b_\gamma^{-1}(c)$ are noncompact hypersurfaces in M . □

In flat Riemannian manifolds [15, §7.3.2], horospheres are just affine hyperplanes, and in the case of constant negative sectional curvature, using the Poincare model, horospheres are Euclidean spheres internally tangent to the boundary sphere, minus the point of tangency. This may not be the case in Finsler manifolds (cf. [19]). As we have mentioned before, the Busemann function b_γ is a distance function and 1-Lipschitz. Consequently, we can define an AHF-manifold in the weak sense as follows.

Definition 4.3. A forward complete simply connected Finsler μ -manifold $(M, F, d\mu)$ without conjugate points is called an AHF-manifold in the weak sense if the weak Laplacian of every Busemann function is a real constant, that is, $\Delta b_\gamma = h$, where Δ is Shen’s Laplacian.

Another equivalent definition is the following.

Definition 4.4. A complete simply connected Finsler μ -manifold $(M, F, d\mu)$ without conjugate points is called an AHF-manifold in the weak sense if the weak forward and backward Laplacians of every Busemann function are real constants.

That is, $\overleftarrow{\Delta}b_{\bar{\eta}} = h$ and $\Delta b_{\eta} = h$, where $h \in \mathbb{R}$, Δ is Shen's Laplacian and $\overleftarrow{\Delta}$ is Shen's Laplacian associated with the reverse (backward) Finsler structure \overleftarrow{F} of F which is defined by $\overleftarrow{F}(v) := F(-v)$.

It is clear that an AHF-manifold is an AHF-manifold in the weak sense.

Remark 4.5. The constant h in the Riemannian case is non-negative. However, it may not be non-negative in Finsler setting.

Proposition 4.6. *For an AHF-manifold in the weak sense, the Finsler mean curvature of large geodesic spheres converges to the Finsler mean curvature of horospheres.*

Proof. The proof follows from Lemma 3.9. □

Theorem 4.7. *For any ray η in an AHF-manifold in the weak sense, the associated Busemann function b_{η} is smooth.*

Proof. Let b_{η} be the Busemann function associated to the ray η . We showed in Proposition 3.4 that b_{η} is 1-Lipschitz. Now, assume that $\Delta b_{\eta}(x) = h$, $h \in \mathbb{R}$ in the weak sense. That is, in view of (2.4),

$$h \int_M \phi \, d\mu = - \int_M d\phi(\nabla b_{\eta}) \, d\mu, \quad \text{for all } \phi \in C_c^{\infty}(M).$$

In a coordinate neighborhood Ω , Shen's Laplacian (2.3) of b_{η} can be written in the form

$$\Delta b_{\eta}(x) = A^{ij}(x, db_{\eta}(x)) \, \partial_i \partial_j b_{\eta}(x) + B^i(x, db_{\eta}(x)) \, \partial_i b_{\eta}(x),$$

where $A^{ij}(x, db_{\eta}(x)) := g^{ij}(x, \nabla b_{\eta}(x))$,

$$B^i(x, db_{\eta}(x)) := A^{ij}(x, db_{\eta}(x)) \, \partial_j (\log(\sigma_{\mu}(x)) + \partial_j(A^{ij}(x, db_{\eta}(x)))).$$

It is clear that the coefficients $A^{ij}(x, db_{\eta}(x))$, $B^i(x, db_{\eta}(x))$ are smooth functions for each $x \in M$ and $db_{\eta}(x) \in T_x M_0$ since $(M, F, d\mu)$ is a C^{∞} Finsler manifold equipped with a smooth volume form $d\mu$. Now, $\Delta b_{\eta} - h = 0$ can be written in the form $\mathbf{F}(d^2 b_{\eta}) = 0$ on a domain Ω . Thanks to [3, Theorem 41], we conclude that b_{η} is smooth on Ω . □

Remark 4.8. For a straight line $\zeta : \mathbb{R} \rightarrow M$ in a complete Finsler manifold, we have the two associated Busemann functions, b_{ζ} for the forward ray and $b_{\bar{\zeta}}$ for the backward ray $\bar{\zeta} := \zeta(-t)$, $t \geq 0$,

$$b_{\bar{\zeta}}(x) = \lim_{t \rightarrow \infty} d_F(\bar{\zeta}(t), x) - t.$$

Let us recall the definition of *bi-asymptote* in Finsler geometry [12, §4].

Definition 4.9. We say that a straight line $\zeta : \mathbb{R} \rightarrow M$ is bi-asymptotic to η if $\zeta|_{[0, \infty)}$ is asymptotic to $\eta|_{[0, \infty)}$ and $\bar{\zeta}(s) = \zeta(-s)$ is asymptotic to $\bar{\eta}$ with respect to \overleftarrow{F} .

Lemma 4.10. *For any straight line $\eta : \mathbb{R} \rightarrow M$ in an AHF-manifold with $h = 0$, the associated Busemann functions satisfy*

$$b_\eta + b_{\bar{\eta}} = 0. \tag{4.2}$$

Proof. The triangle inequality gives $b_\eta + b_{\bar{\eta}} \geq 0$, which means that $b_\eta \geq -b_{\bar{\eta}}$. By direct calculations, $b_\eta(\eta(s)) = -b_{\bar{\eta}}(\bar{\eta}(s))$. Let Ω be a bounded open set of M . It is easy to see that $b_\eta, -b_{\bar{\eta}} \in H^1(\Omega) \cap C(\Omega)$. Applying the “Strong comparison principle” [6, Lemma 5.4], by putting $u := b_\eta, v := -b_{\bar{\eta}}$ and $\Lambda := 0$, yields that $b_\eta + b_{\bar{\eta}} \leq 0$. Hence the result follows. \square

Proposition 4.11. *The bi-asymptotics are unique in any AHF-manifold whose Finsler mean curvature of all horospheres vanishes, i.e., $h = 0$.*

Proof. The proof follows from Lemma 4.10 by using the same technique as that of the proof of Corollary 3.12. \square

The next result is a special case of Theorem 4.7. However, the proof is different.

Proposition 4.12. *For any straight line η in an AHF-manifold (M, F) , the associated Busemann functions b_η and $b_{\bar{\eta}}$ are smooth functions on M in the case of $h = 0$.*

Proof. We have $\Delta b_\eta = 0, \overleftarrow{\Delta} b_{\bar{\eta}} = 0$. Since a harmonic function is a static solution to the heat equation and db_η does not vanish, then b_η is smooth by [12, Proposition 4.1] and [13, Theorem 4.9 and Remark 4.10]. Moreover, $b_\eta + b_{\bar{\eta}} = 0$ by Lemma 4.10, and thus $b_{\bar{\eta}} = -b_\eta$ is smooth. \square

Proposition 4.13. *Suppose f is a distance function on an AHF-manifold. Let η be the integral curve of ∇f starting from $p = \eta(0)$ such that $f(p) = 0$. If $\Delta f = h = \Delta b_\eta$, then $f = b_{\bar{\eta}}$.*

Proof. From the definition of the distance function, $F(\nabla f) = 1$. By Lemma 3.6 (3), it follows that the integral curve η of ∇f satisfies $f(\eta(t)) = t$ as $f(\eta(0)) = f(p) = 0$. Now, fix some $s > t$ and let $x \in f^{-1}(s)$,

$$d_F(\eta(t), x) \geq d_F(f^{-1}(t), f^{-1}(s)) = s - t,$$

thereby $d_F(\eta(t), x) + t \geq s = f(x)$ for all $x \in f^{-1}(s)$. Hence,

$$\lim_{t \rightarrow -\infty} (d_F(\eta(t), x) + t) \geq f(x).$$

Consequently, $b_{\bar{\eta}}(x) \geq f(x)$ for all $x \in f^{-1}(s)$ and $b_{\bar{\eta}}(p) = 0 = f(p)$. Now, applying [6, Lemma 5.4], we get $b_{\bar{\eta}}(x) = f(x), \forall x \in f^{-1}(s)$. That is, $f = b_{\bar{\eta}}$. \square

Let (M, F) be a complete simply connected Finsler manifold without conjugate points. Assume that at each $x \in M$ there exists a unique line ζ emanating from $x := \zeta(0)$ with $\dot{\zeta}(0) = v$. Under these conditions, each $v \in IM$ gives rise to a Busemann function b_ζ , where ζ is the line just defined above. This justifies the following definition of the total Busemann function [16, §5].

Definition 4.14. Let $A(M)$ be the set of differentiable functions from M to \mathbb{R} . The total Busemann function $B : IM \rightarrow A(M)$ is given by $(x, v) \mapsto b_{(x,v)} := b_\zeta$.

Proposition 4.15. Let $\{\gamma_n\}_{n \in \mathbb{N}}$ be a family of unit speed geodesic rays starting from a fixed point $p \in M$ with initial velocities $\{y_n\}_{n \in \mathbb{N}} \subset I_p M$. The sequence $\{b_{y_n}\}_{n \in \mathbb{N}}$ is uniformly bounded on each compact set.

Proof. Let Ω be a compact subset of M . Assume that $\{\gamma_n\}_{n \in \mathbb{N}}$ is a family of unit speed geodesic rays starting from a fixed point $p \in M$ with initial velocities $\{y_n\}_{n \in \mathbb{N}}$. Thus, we have a sequence $\{y_n\}_{n \in \mathbb{N}}$ of unit vectors in $I_p M$ and corresponding Busemann functions $\{b_{y_n}\}_{n \in \mathbb{N}}$. By (3.2), since each Busemann function b_{y_n} is 1-Lipschitz, we have

$$-d_F(x_o, p) \leq b_{y_n}(x_o) \leq d_F(p, x_o).$$

Taking the supremum over $x_o \in \Omega$, where $p \notin \Omega$, we get

$$-d_F(\Omega, p) = \sup_{x_o \in \Omega} -d_F(x_o, p) \leq \sup_{x_o \in \Omega} b_{y_n}(x_o) \leq \sup_{x_o \in \Omega} d_F(p, x_o) = d_F(p, \Omega).$$

Hence, $\{b_{y_n}\}_{n \in \mathbb{N}}$ is uniformly bounded. \square

Remark 4.16. Note the following:

- (i) Consider a sequence of unit vectors $\{y_n\}_{n \in \mathbb{N}}$ in $I_p M$ such that $y_n \rightarrow y$. Then $\{b_{y_n}\}_{n \in \mathbb{N}}$ is an equicontinuous family of Busemann functions which is pointwise bounded on each compact subset Ω of M (Proposition 4.15). Consequently, by *Ascoli–Arzela theorem*, $\{b_{y_n} : \Omega \rightarrow \mathbb{R}\}_{n \in \mathbb{N}}$ has a uniformly convergent subsequence, say, $\{b_{y_{n_k}}\}$, converging to some function f .
- (ii) The limit function f is differentiable almost everywhere. Therefore, the gradient ∇f is defined and the weak Laplacian Δf is defined.

Now, applying the definition of the distributional derivative (3.4), for $T = \nabla b_{v_{n_k}}$ and $z = 1$, we get

$$\begin{aligned} \int_{\Omega} (\Delta b_{v_{n_k}}) \phi \, d\mu &= - \int_{\Omega} d\phi(\nabla b_{v_{n_k}}) \, d\mu = \int_{\Omega} b_{v_{n_k}} \Delta \phi \, d\mu, \\ \int_{\Omega} (\Delta f) \phi \, d\mu &= - \int_{\Omega} d\phi(f) \, d\mu = \int_{\Omega} f \Delta \phi \, d\mu. \end{aligned}$$

Hence, in the distributional sense, we get

$$\nabla b_{y_{n_k}} \rightarrow \nabla f, \quad \Delta b_{y_{n_k}} \rightarrow \Delta f, \quad \text{as } n_k \rightarrow \infty.$$

- (iii) As $(M, F, d\mu)$ is an AHF-manifold, i.e., $\Delta b_{y_{n_k}} = h$, then $\Delta f = h$ in the distributional sense. Now, using the same technique of the proof as that of Theorem 4.7, we deduce that f is smooth.
- (iv) Both $f, b_{y_{n_k}} \in C^\infty(\Omega, \mathbb{R})$. Indeed, $\lim_{n_k \rightarrow \infty} b_{y_{n_k}} = f$.
- (v) Let η_v be the integral curve of ∇f starting from $p := \eta(0)$. Then, using Proposition 4.13, we conclude that $f = b_{\bar{\eta}}$ on Ω .

Table 4.1 summarizes the main differences between Riemannian and Finsler geometries used in the present work.

Geometric objects	Riemannian manifold (M, α)	Finsler manifold (M, F)
Metric	$\alpha_{ij}(x)$	$g_{ij}(x, y)$
Induced distance	$d_\alpha(p, q)$ is symmetric	$d_F(p, q)$ is non-symmetric
Exponential map \exp_x	C^∞ on $T_x M$	C^∞ on $T_x M_0$, C^1 at null section
Legendre transformation	linear	non-linear
Gradient of a function	linear	non-linear
Volume measure	canonically defined and unique	several non-equivalent (e.g., Holmes-Thompson)
Laplacian	unique (Laplace-Beltrami), linear elliptic operator	not unique (e.g., Shen's Laplacian) non-linear elliptic operator,
Asymptotic relation is an equivalence relation between two rays	for a simply connected complete (M, α) of non-positive sectional curvature [15](YES)	for a forward complete simply connected (M, F) of non-positive flag curvature (NO) [24]; but (YES) in case of Theorem 4.1
Parallel hypersurfaces: $f^{-1}(t)$ is parallel to $f^{-1}(s)$	means $f^{-1}(s)$ is parallel to $f^{-1}(t)$	does not mean $f^{-1}(s)$ is parallel to $f^{-1}(t)$; unless F is reversible
Busemann function	associated to a ray, 1-Lipschitz in the sense $ b_\gamma(q) - b_\gamma(p) \leq d_\alpha(p, q)$	associated to a forward ray, 1-Lipschitz in the sense of equation (3.2)
The mean curvature of horospheres	In AH-Riemannian space is always non-negative	In AH-Finsler space is a real constant

Table 4.1: Main differences between Riemannian and Finsler geometries.

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Функції Буземана в асимптотично гармонічних фінслерових многовидах

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У цій статті ми вивчаємо функції Буземана як в загальній фінслеровій поставі, так і для асимптотично гармонічних фінслерових многовидів. Зокрема, ми показуємо, що функції Буземана на асимптотично гармонічних фінслерових многовидах є гладкими.

Ключові слова: функція Буземана, асимптота, гармонічний фінслеровий многовид, асимптотично гармонічний фінслеровий многовид