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Simple Closed Geodesics on Regular Tetrahedra in Spaces of Constant Curvature

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In the current survey, the results on a behavior of simple closed geodesics on regular tetrahedra in three-dimensional spaces of constant curvature are presented.

Key words: closed geodesics, regular tetrahedron, hyperbolic space, spherical space

Mathematical Subject Classification 2010: 53C22, 52B10

Contents

1. Introduction

A closed geodesic is called simple if this geodesic is not self-intersecting and does not go along itself. At the end of the 19th century, while working on the three-body problem, H. Poincare [39] stated a problem of the existence of geodesic lines on smooth convex two-dimensional surfaces. Since then, the methods for finding closed geodesics on regular surfaces of positive and negative curvature have been developed. In 1927, G.D. Birkhoff [6] proved that there exists at least one simple closed geodesic on an n-dimensional Riemannian manifold homeomorphic to a sphere. In contrast to this, there are non-smooth convex closed surfaces in Euclidean space that are free from simple closed geodesics. From the generalization of the Gauss–Bonnet theorem for polyhedra, there follows a necessary condition for the existence of a simple closed geodesic on a convex polyhedron in $\mathbb{E}^3$. This condition does not hold for most convex polyhedra, but it holds for regular polyhedra, in particular for regular tetrahedra.

In the current survey, we present the results on the behavior of simple closed geodesics on regular tetrahedra in three-dimensional spaces of constant curvature. D. Fuchs and E. Fuchs supplemented and systematized the results on closed geodesics on regular polyhedra in $\mathbb{E}^3$ (see [16,18]). V.Yu. Protasov [41] obtained a condition for the existence of simple closed geodesics on an arbitrary tetrahedron in Euclidean space.

A.A. Borisenko and D.D. Sukhorebska studied simple closed geodesics on regular tetrahedra in three-dimensional hyperbolic and spherical spaces (see [7,9, 10]). In Euclidean space, the faces of a tetrahedron have zero Gaussian curvature, and the curvature of a tetrahedron is concentrated only on its vertices. In the hyperbolic or spherical space, the Gaussian curvature of faces is $k = -1$ or 1, and the curvature of a tetrahedron is determined not only by its vertices, but also by its faces. In the hyperbolic space, the planar angle α of a face of a regular tetrahedron satisfies $0 < \alpha < \pi/3$. In the spherical space, the planar angle α satisfies $\pi/3 < \alpha \leq 2\pi/3$. In both cases the intrinsic geometry of a tetrahedron depends on the planar angle. The behavior of closed geodesics on a regular tetrahedron in three-dimensional spaces of constant curvature k depends on the sign of k.

2. Historical notes and main results

In [39], Henri Poincare studied properties of the solutions of the three-body problem, in particular, periodical and asymptotic solutions. He found that the key difficulty of this problem could be formulated as an independent problem of describing geodesics lines on a convex surface. In [40], H. Poincare showed the existence of a simple closed geodesic on a convex smooth surface S that is an embedding of the two-dimensional sphere into Euclidean space $\mathbb{E}^3$ with induced metric. He considered the shortest simple closed curve dividing S into two pieces of equal total Gaussian curvature. Moreover, H. Poincare stated a conjecture on the existence of at least three simple closed geodesics on a smooth closed convex two-dimensional surface in $\mathbb{E}^3$. Later, in 1927, G.D. Birkhoff proved that there exists at least one simple closed geodesic on an n-dimensional Riemannian manifold homeomorphic to a sphere [6].

In 1929, L.A. Lusternik and L.G. Schnirelmann [30,31] published the proof of Poincare's conjecture. However, their proof contained some gaps which were filled in by W. Ballmann in 1978 [4] and independently by I. Taimanov in 1992 [47]. In 1951–1952, L.A. Lusternik and A.I. Fet [14, 29] proved the existence of a closed geodesic on an n-dimensional regular closed manifold.

Using the ideas of G.D. Birkhoff, it was proved that every Riemannian metric on a two-dimensional sphere carries infinitely many geometrically distinct closed geodesics, cf. J. Franks [15] and V. Bangert [5]. The methods of the proof were restricted to surfaces. The condition of the existence of infinitely many closed geodesics on a compact simply-connected manifold of arbitrary dimension is more complicated. In 1969, D. Gromoll and W. Meyer [20] showed that there always exist infinitely many distinct periodic geodesics on an arbitrary compact manifold M, provided some weak topological condition holds: if the sequence of Betti numbers of the free loop space LM of M is unbounded. W. Ziller [50] proved that this condition on the free loop space holds for symmetric spaces of rank > 1. H.B. Rademacher [42] showed that for a C^4-regular metric on a compact Riemannian manifold with finite fundamental group there are infinitely many geometrically distinct closed geodesics.

In 1898, J. Hadamard [23] showed that on a closed surface of negative curvature any closed curve, that is not homotopic to zero, can be deformed into the closed curve of minimal length within its free homotopy group. This minimal curve is unique and it is a closed geodesic. Then it is interesting to estimate the number of closed geodesics, depending on the length of these geodesics, on a compact manifold of negative curvature. H. Huber [25,26] proved that on a complete closed two-dimensional manifold of constant curvature -1 the number of closed geodesics of length at most L has the order of growth e^L/L as $L \to \infty$. For compact n-dimensional manifolds of negative curvature this result was generalized by Ya.G. Sinai [46], G.A. Margulis [32], M. Gromov [21], and others.

In I. Rivin's work [43], and later in M. Mirzakhani's work [34], it was proved that on a complete hyperbolic (constant negative curvature) Riemannian surface of genus g and with n cusps the number of simple closed geodesics of length at most L is asymptotic to (positive) constant times $L^{6g-6+2n}$ as $L \to \infty$. One can also refer to [13,44] for details.

Theorems about geodesic lines on convex two-dimensional surfaces play an important role in geometry "in the large" of convex surfaces in spaces of constant curvature. Important results on this subject were obtained by S. Cohn-Vossen [11], A.D. Alexandrov [2], and A.V. Pogorelov [36]. In one of his earliest works, A.V. Pogorelov proved that on a closed convex surface of Gaussian curvature $\leq k$, $k > 0$, each geodesic of length $< \pi/\sqrt{k}$ is the shortest path between its endpoints [37]. V.A. Toponogov [48] proved that on a C^2-regular closed surface of curvature $\geq k > 0$ the length of a simple closed geodesic is at most $2\pi/\sqrt{k}$. V.A. Vaigant and O.Yu. Matukevich [49] proved that on this surface a geodesic of length $\geq 3\pi/\sqrt{k}$ has the point of self-intersection.

Geodesics have also been studied on non-smooth surfaces, including convex polyhedra in $\mathbb{E}^3$. Since a geodesic is the locally shortest curve, it can not pass through any point for which the full angle is less than 2π (see [2]). P. Gruber [22] showed that in the sense of Baire categories [27] most convex surfaces (no regularity required) do not contain a closed geodesic. A.V. Pogorelov [38] generalized L.A. Lusternik and L.G. Schnirelmann's result showing that on any closed convex surface there are at least three closed quasi-geodesics. Whereas a geodesic has exactly π surface angle to either side at each point, a quasi-geodesic has at most π surface angle to either side at each point. Unlike geodesics, quasi-geodesics can pass through the vertices with the full angle $< 2\pi$ on the surface [3].

On a convex polyhedron a geodesic has the following properties:

1) it consists of line segments on faces of a polyhedron;
2) it forms equal angles with edges on adjacent faces;

3) a geodesic cannot pass through a vertex of a convex polyhedron [2].

G. Galperin [19] presented a necessary condition for the existence of a simple closed geodesic on a convex polyhedron in $\mathbb{E}^3$. It is based on a generalization of the Gauss–Bonnet theorem for polyhedra. The curvature of a convex polyhedron in $\mathbb{E}^3$ is concentrated on its vertices. Let $\theta_1, \ldots, \theta_n$ be the full angles around the vertices $A_1, \ldots, A_n$ of a convex polyhedron. The curvature of the vertex A_i is $\omega_i = 2\pi - \theta_i$, $i = 1, \ldots, n$. If there is a simple closed geodesic on a convex polyhedron, then there should necessarily be a subset $I \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} \omega_i = 2\pi.$$

This condition does not hold for most polyhedra, but it holds for regular polyhedra. D. Fuchs and E. Fuchs supplemented and systematized the results on closed geodesics on regular polyhedra in the three-dimensional Euclidean space (see [16, 18]). K. Lawson and others [28] obtained a complete classification of simple closed geodesics on the eight-convex polyhedra (deltahedra) whose faces are all equilateral triangles.

In [41], V.Yu. Protasov obtained a condition for the existence of simple closed geodesics on an arbitrary tetrahedron in Euclidean space and evaluated from above the number of these geodesics in terms of the difference from π the sum of the angles at a vertex of the tetrahedron. In particular, it is proved that a simplex has infinitely many different simple closed geodesics if and only if all the faces are equal triangles. A. Akopyan and A. Petrunin [1] showed that if a closed convex surface M in $\mathbb{E}^3$ contains arbitrarily long simple closed geodesic, then M is a tetrahedron whose faces are equal triangles.

Definition 2.1. A simple closed geodesic on a tetrahedron has *type* (p, q) if it has p vertices on each of two opposite edges of the tetrahedron, q vertices on each of other two opposite edges, and $(p + q)$ vertices on each of the remaining two opposite edges.

On a regular tetrahedron in Euclidean space, for each ordered pair of coprime integers (p, q) there exists a whole class of simple closed geodesics of type (p, q), up to the isometry of the tetrahedron. On the development of the tetrahedron, geodesics in each class are parallel to each other. Furthermore, in each class there is a simple closed geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron [9].

J. O'Rourke and C. Vilcu [35] considered simple closed quasi-geodesics on tetrahedra in $\mathbb{E}^3$.

In [12], D. Davis and others considered geodesics which begin and end at vertices (and do not touch other vertices) on a regular tetrahedron and cube. It was proved that a geodesic as above never begins and ends at the same vertex and computed the probabilities with which a geodesic starting from a given vertex ends at every other vertex. D. Fuchs [17] obtained similar results for a regular octahedron and icosahedron (in particular, such a geodesic never ends at the point it begins).

Denote a simply-connected complete Riemannian n-dimensional manifold of constant curvature $k \in \{-1, 0, 1\}$ by M_k^n. A polyhedron in M_k^3 is a surface obtained by gluing finitely many geodesic polygons from M_k^2. In particular, a regular tetrahedron in M_k^3 is a closed convex polyhedron whose all faces are regular geodesic triangles from M_k^2 and all vertices are regular trihedral angles. From Alexandrov's gluing theorem [3], it follows that the polyhedron in M_k^3 with the induced metric is a compact Alexandrov surface $A(k)$ with the curvature bounded below by k. Notice that in $\mathbb{E}^3(M_0^3)$ the curvature of a tetrahedron is concentrated only on its vertices. In the hyperbolic or spherical space, the Gaussian curvature of faces is $k = -1$ or 1, respectively, and the curvature of a tetrahedron is determined not only by its vertices, but also by its faces.

In [45], J. Rouyer and C. Vilcu studied the existence or non-existence of simple closed geodesics on most (in the sense of Baire category [27]) Alexandrov surfaces. In particular, it was proved that most surfaces in $A(-1)$ have infinitely many, pairwise disjoint, simple closed geodesics, and most surfaces in $A(1)$ have no simple closed geodesics.

As we have said before, on a regular tetrahedron in Euclidean space $\mathbb{E}^3$, for each ordered pair of coprime integers (p, q) there exist infinitely many simple closed geodesics of type (p, q) that are parallel to each other on the development of the tetrahedron. It follows from the fact that the development of a tetrahedron along the geodesic is contained in the standard triangular tiling of the plane. Moreover, the vertices of the tiling can be labeled in such a way that for any development the labeling of vertices of the tetrahedron matches the labeling of vertices of the tiling. This is something that holds only for regular tetrahedra and only in $\mathbb{E}^3$ [18].

In the spherical space $\mathbb{S}^3$, the planar angle α of the faces of a tetrahedron satisfies $\pi/3 < \alpha \leq 2\pi/3$. The intrinsic geometry of the tetrahedron depends on α. If the planar angle $\alpha = 2\pi/3$, then the tetrahedron is a unit two-dimensional sphere. Hence, there are infinitely many simple closed geodesics on it and they are great circles of the sphere. In the following, we consider α such that $\pi/3 < \alpha < 2\pi/3$. In [10], A.A. Borisenko and D.D. Sukhorebska proved that on a regular tetrahedron in spherical space there exists the finite number of simple closed geodesics. The length of all these geodesics is less than 2π.

It was found that for any coprime integer (p, q) there exist the numbers α_1 and α_2, depending on p, q and satisfying the inequalities $\pi/3 < \alpha_1 < \alpha_2 < 2\pi/3$, such that

1) if $\pi/3 < \alpha < \alpha_1$, then on a regular tetrahedron in spherical space with the planar angle α there exists a unique simple closed geodesic of type (p, q), up to the rigid motion of this tetrahedron, and it passes through the midpoints of two pairs of opposite edges of the tetrahedron;
2) if $\alpha_2 < \alpha < 2\pi/3$, then on a regular tetrahedron with the planar angle α there is no simple closed geodesic of type (p, q).

In [7], A.A. Borisenko gave the necessary and sufficient condition for the existence of a simple closed geodesic on a regular tetrahedron in $\mathbb{S}^3$. We will consider it in details in Section 4.

Unlike in $\mathbb{S}^3$, on a regular tetrahedron in hyperbolic space $\mathbb{H}^3$ there are infinitely many simple closed geodesics. Recall that the planar angle α of a regular tetrahedron in $\mathbb{H}^3$ satisfies $0 < \alpha < \pi/3$. In [9], A.A. Borisenko and D.D. Sukhorebska proved that on a regular tetrahedron in hyperbolic space for any coprime integers (p, q), $0 \le p < q$, there exists a unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p, q), and it passes through the midpoints of two pairs of opposite edges of the tetrahedron. These geodesics exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space. As a part of the proof, there was found a constant $d(\alpha) > 0$ for $\alpha \in (0, \pi/3)$ such that the distances from the vertices of the regular tetrahedron to any simple closed geodesic is greater than $d(\alpha)$. It should be noticed that this property holds only for simple closed geodesics on regular tetrahedra in $\mathbb{H}^3$. In $\mathbb{E}^3$ or $\mathbb{S}^3$, for any $\varepsilon > 0$, there is a simple closed geodesic γ such that the distance from a tetrahedron vertex to γ is $< \varepsilon$.

Furthermore, in [9], it was proved that the number of simple closed geodesics of length bounded by L is asymptotic to $c(\alpha)L^2$ when $L \to \infty$. If $\alpha \to 0$, then $c(\alpha) \to c_0 > 0$. If the planar angle α of a regular tetrahedron in hyperbolic space is zero, then the vertices of the tetrahedron become cusps. Then the limiting tetrahedron is a noncompact surface homeomorphic to a sphere with four cusps with a complete regular Riemannian metric of constant negative curvature. The genus of this surface is zero. In [43], Rivin showed that the number of simple closed geodesics on this surface has order of growth L^2.

In [7], A.A. Borisenko proved that if the planar angles of any tetrahedron in hyperbolic space are at most $\pi/4$, then for any pair of coprime integers (p, q) there exists a simple closed geodesic of type (p, q). This situation differs from Euclidean space, where there are no simple closed geodesics on a generic tetrahedron [19].

3. Closed geodesics on a regular tetrahedron in $\mathbb{E}^3$

Consider a regular tetrahedron $A_1A_2A_3A_4$ with the edge of length 1 in Euclidean space.

Fix a point of a geodesic on the edge of the tetrahedron and roll the tetrahedron along the plane in such a way that the geodesic always touches the plane. The traces of the faces form the *development* of the tetrahedron on a plane and the geodesic is a line segment inside the development.

A development of a regular tetrahedron in $\mathbb{E}^3$ is a part of the standard triangulation of Euclidean plane. Denote the vertices of the triangulation in accordance with the vertices of the tetrahedron (see Fig. 3.1). We introduce a rectangular Cartesian coordinate system with the origin at A_1 and the x-axis along the edge A_1A_2 containing X. Then the vertices A_1 and A_2 have the coordinates $\big(l, k\sqrt{3}\big)$, and the coordinates of A_3 and A_4 are $\big(l + 1/2, (2k + 1)\sqrt{3}/2\big)$, where k, l are integers.

Choose two identically oriented edges A_1A_2 of the triangulation that do not belong to the same line. Take two points $X(\mu, 0)$ and $X'(\mu + q + 2p, q\sqrt{3})$ on them, where $0 < \mu < 1$ such that the segment XX' does not contain any vertex of the triangulation. The segment XX' corresponds to the simple closed geodesic γ of type (p, q) on a regular tetrahedron in Euclidean space. If (p, q) are coprime integers, then γ does not repeat itself. On a tetrahedron, γ has p vertices on each of two opposite edges of the tetrahedron, q vertices on each of other two opposite edges, and $(p+q)$ vertices on each of the remaining two opposite edges, and thus γ has type (p, q).

The length of γ is equal to

$$L = 2\sqrt{p^2 + pq + q^2}. \tag{3.1}$$

Notice that the segments of a geodesic lying on the same face of the tetrahedron are parallel to each other. It follows that a closed geodesic on a regular tetrahedron in Euclidean space does not have points of self-intersection.

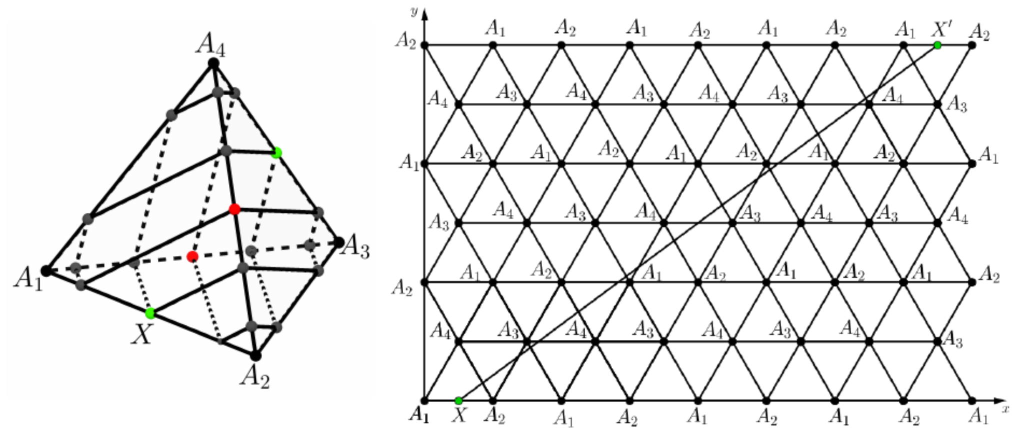

Fig. 3.1

If $q = 0$ and $p = 1$, then the geodesic consists of four segments that consecutively intersect four edges of the tetrahedron and the geodesic does not intersect a pair of opposite edges.

Theorem 3.1.

1. *On a regular tetrahedron in Euclidean space, for each ordered pair of coprime integers (p, q) there exists the whole class of simple closed geodesics of type (p, q), up to the isometry of the tetrahedron. On the development of the tetrahedron, geodesics in each class are parallel to each other* [18].
2. *In every class there is a simple closed geodesic passing through the midpoints of two pairs of opposite edges of the tetrahedron* [9].

Proof. For each pair of coprime integers (p, q), construct a segment connecting the points $X(\mu_0, 0)$ and $X'(\mu_0 + q + 2p, q\sqrt{3})$. Chose $\mu_0 \in (0, 1)$ such that XX' does not contain any vertex of the triangulation. Then XX' corresponds to the simple closed geodesic γ of type (p, q) on a regular tetrahedron in Euclidean space.

Consider the segments parallel to XX'. They are characterized by the equation

$$y = \frac{q\sqrt{3}}{q+2p}(x-\mu).$$

We can change μ until the line touches a vertex of the tiling. Then for each pair (p,q) there are $\mu_1, \mu_2 \in (0,1)$ such that $\mu_1 \le \mu_0 \le \mu_2$ and for all $\mu \in (\mu_1, \mu_2)$, the segment joining $X(\mu, 0)$ and $X'(\mu + q + 2p, q\sqrt{3})$ corresponds to the simple closed geodesic of type (p,q) on a regular tetrahedron. Therefore, the first part of the theorem is proved.

To prove the second part, consider the lines

$$\gamma_i : y = \frac{q\sqrt{3}}{q+2p}(x-\mu_i), \quad i = 1,2, \tag{3.2}$$

passing through the vertices of the tiling. It means that there exist the integer numbers c_1 and c_2 such that the points $P_1\left(c_1(q+2p)/2q + \mu_1, c_1\sqrt{3}/2\right)$ and $P_2\left(c_2(q+2p)/2q + \mu_2, c_2\sqrt{3}/2\right)$ are the vertices of the tiling and γ_1 passes through P_1 and γ_2 passes through P_2.

Consider the closed geodesic γ_0 parallel to γ such that the equation of γ_0 is

$$y = \frac{q\sqrt{3}}{q+2p}\left(x - \frac{\mu_1+\mu_2}{2}\right).$$

It passes through the point

$$P_0\left(\frac{c_1+c_2}{2}\,\frac{q+2p}{2q} + \frac{\mu_1+\mu_2}{2},\ \frac{c_1+c_2}{2}\,\frac{\sqrt{3}}{2}\right).$$

Consider three cases:

1) the points P_1 and P_2 belong to the line A_1A_2;
2) the points P_1, P_2 belong to the line A_3A_4;
3) the point P_1 belongs to the line A_1A_2 and the point P_2 belongs to the line A_3A_4.

In each of this cases it is easy to show that P_0 is a midpoint of some edge of the tiling.

Then, let us prove that if a geodesic passes through the midpoint of one edge, then it passes through the midpoints of two pairs of opposite edges. Assume that a closed geodesic γ_0 passes through the midpoint of the edge A_1A_2. Then the equation of γ_0 is

$$y = \frac{q\sqrt{3}}{q+2p}\left(x - \frac{1}{2}\right). \tag{3.3}$$

The vertices A_3 and A_4 belong to the line $y_v = (2k+1)\sqrt{3}/2$, and their first coordinate is $x_v = l + 1/2$ $(k, l \in \mathbb{Z})$. Substituting the coordinates of the points A_3 and A_4 to equation (3.3), we get

$$q(2l - 2k - 1) = 2p(2k+1). \tag{3.4}$$

If q is even, then there exist k and l satisfying equation (3.4). It follows that γ_0 passes through the vertex of the tiling. It contradicts the properties of γ_0, therefore q is an odd integer.

The points $X_1\,(1/2, 0)$ and $X_1'\left(q+2p+1/2, q\sqrt{3}\right)$ satisfy equation (3.3). These points are the midpoint of the edge A_1A_2 on the tetrahedron. Suppose that the point X_2 is the midpoint of X_1X_1'. Then the coordinates of X_2 are $\left(q/2+p+1/2, q\sqrt{3}/2\right)$. Substituting $q = 2k+1$, we obtain $X_2\left(k+p+1, (k+1/2)\sqrt{3}\right)$. Since the second coordinate of X_2 is $(k+1/2)\sqrt{3}$, where k is an integer, the point X_2 belongs to the line that contains the vertices A_3 and A_4. It follows that X_2 is the midpoint of the edge A_3A_4 because the first coordinate of X_2 is an integer.

Let $Y_1\left(q/4+p/2+1/2,\ q\sqrt{3}/4\right)$ be the midpoint of X_1X_2. Substituting $q = 2k+1$, we obtain $Y_1\left((k+p+1)/2+1/4,\ (k/2+1/4)\sqrt{3}\right)$. From the second coordinate we have that Y_1 belongs to the line passing in the middle of the horizontal lines $y = k\sqrt{3}/2$ and $y = (k+1)\sqrt{3}/2$. Looking at the first coordinate of Y_1, which has $1/4$ added, we can see that Y_1 is the center of A_1A_3, or A_3A_2, or A_2A_4, or A_4A_1.

In a similar way, consider the midpoint $Y_2\left(3q/4+3p/2+1/2,\ 3q\sqrt{3}/4\right)$ of X_2X_1' Then Y_2 is the midpoint of the edge that is opposite to the edge with Y_1. □

Corollary 3.2. *The development of the tetrahedron obtained by unrolling along a closed geodesic consists of four equal polygons. Two adjacent polygons can be transformed into each other by rotating them through an angle π around the midpoint of their common edge.*

Proof. For any closed geodesic γ, we get the equivalent closed geodesic γ_0 that passes through the midpoints of two pairs of the opposite edges on the tetrahedron. Let the points X_1, X_2 and Y_1, Y_2 on γ_0 be the midpoints of the edges A_1A_2, A_4A_3 and A_1A_3, A_2A_4, respectively.

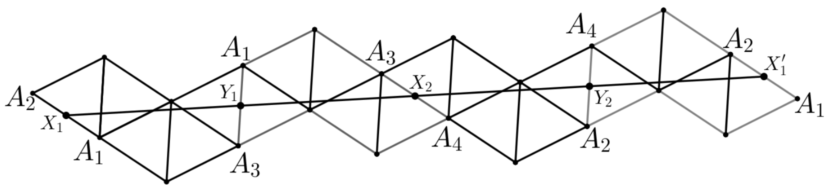

Fig. 3.2

Consider the rotation of the regular tetrahedron through π around the line passing through the points X_1 and X_2. This rotation is the isometry of the regular tetrahedron. The points Y_1 and Y_2 are swapped. Furthermore, the segment of γ_0 that starts at X_1 on the face $A_1A_2A_4$ is mapped to the segment of γ_0 that starts from the point X_1 on $A_1A_2A_3$. It follows that the segments X_1Y_1 and X_1Y_2 are swapped. For the same reason, after the rotation the segments X_2Y_1 and X_2Y_2 of γ_0 are also swapped.

From this rotation, we get that the development of the tetrahedron along the segment $Y_1X_1Y_2$ of the geodesic is central symmetric with respect to the point X_1. And the development along $Y_1X_2Y_2$ is central symmetric with respect to X_2.

Now, consider the rotation of the regular tetrahedron through π around the line passing through the points Y_1 and Y_2. By the same argument as above, we obtain that the development of the tetrahedron along the segment $X_1Y_1X_2$ of the geodesic is central symmetric with respect to Y_1, and the development along the segment $X_2Y_2X_1$ is central symmetric with respect to Y_2 (see Fig. 3.2). □

Lemma 3.3. *Let γ be a simple closed geodesic of type (p, q) on a regular tetrahedron in Euclidean space such that γ intersects the midpoints of two pairs of opposite edges. Then the distance h from the vertices of the tetrahedron to γ satisfies the inequality*

$$h \geq \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}. \tag{3.5}$$

Proof. Suppose γ intersects the edge A_1A_2 at the midpoint X. Then geodesic γ is unrolled into the segment XX' lying at the line

$$y = \frac{q\sqrt{3}}{q + 2p}\left(x - \frac{1}{2}\right).$$

The segment XX' intersects the edges A_1A_2 at the points

$$(x_b, y_b) = \left(\frac{2(q + 2p)k + q}{2q}, k\sqrt{3}\right),$$

where $k \leq q$. Since XX' does not pass through the vertices of tiling, x_b can not be an integer. Hence, on the edge A_1A_2, the distance from the vertices to the points of γ is not less than $1/2q$.

Analogously, on the edge A_3A_2, the distance from the vertices of the tetrahedron to the points of γ is not less than $1/2p$.

Develop the faces $A_1A_2A_4$ and $A_2A_4A_3$ to the plane. Choose the points B_1 at the edge A_2A_1 and B_2 at the edge A_2A_3 such that the length A_2B_1 is $1/2q$ and the length A_2B_2 is $1/2p$. Let A_2H be the height of the triangle $B_1A_2B_2$. Then

$$|A_2H| = \frac{\sqrt{3}}{4\sqrt{p^2 + pq + q^2}}.$$

The distance h from the vertex A_2 to γ is not less than $|A_2H|$. □

The pair of coprime integers (p, q) determines the combinatorical structure of a simple closed geodesic and hence the order of intersections with the edges of the tetrahedron.

In [41], the generalization of simple closed geodesics on a polyhedron was proposed. A polyline on a tetrahedron is a curve consisting of line segments which connect the points consecutively on the edges of this tetrahedron. An *abstract geodesic* on a tetrahedron is a closed polyline with the following properties:

1) it does not have points of self-intersection, and adjacent segments of it lie on different faces;
2) it crosses more than three edges and does not pass through the vertices of the tetrahedron.

For any two tetrahedra, we can fix a one-to-one correspondence between their vertices and label the corresponding vertices of the tetrahedra identically. Then two closed geodesics on these tetrahedra are called *equivalent* if they intersect the identical labelled edges in the same order.

Proposition 3.4 ([41]). *For every abstract geodesic $\tilde{\gamma}$ on a tetrahedron in Euclidean space there exists an equivalent simple closed geodesic γ on a regular tetrahedron in Euclidean space.*

A vertex of a geodesic γ is called a *link node* if it and two neighboring vertices of γ lie on the edges of the same vertex A_i of the tetrahedron, and these three vertices are the vertices of the geodesic that are closest to A_i.

Proposition 3.5 ([41]). *Let γ_1^1 and γ_1^2 be the segments of a simple closed geodesic γ starting at a link node on a regular tetrahedron, and let γ_2^1 and γ_2^2 be the next segments and so on. Then, for each $i = 2, \dots, 2p+2q-1$, the segments γ_i^1 and γ_i^2 lie on the same face of the tetrahedron, and there are no other geodesic points between them. The segments γ_{2p+2q}^1 and γ_{2p+2q}^2 meet at the second link node of the geodesic.*

4. Simple closed geodesics on regular tetrahedra in $\mathbb{S}^3$

4.1. The main definition and examples. A *spherical triangle* is a convex polygon on a unit sphere bounded by the three shortest lines. A *regular tetrahedron* $A_1A_2A_3A_4$ in three-dimensional spherical space $\mathbb{S}^3$ is a closed convex polyhedron such that all its faces are regular spherical triangles and all its vertices are regular trihedral angles. A planar angle α of a regular tetrahedron in $\mathbb{S}^3$ satisfies the conditions $\pi/3 < \alpha \leq 2\pi/3$. Notice that then there exists a unique (up to the rigid motion) tetrahedron in spherical space with the given planar angle. The length of the edges is equal to

$$a = \arccos\left(\frac{\cos\alpha}{1-\cos\alpha}\right), \tag{4.1}$$

$$\lim_{\alpha\to\pi/3} a = 0; \quad \lim_{\alpha\to\pi/2} a = \pi/2; \quad \lim_{\alpha\to2\pi/3} a = \pi - \cos^{-1} 1/3. \tag{4.2}$$

If $\alpha = 2\pi/3$, then a tetrahedron is a unit two-dimensional sphere. There are infinitely many simple closed geodesics on it. In the following, we suppose that α satisfies $\pi/3 < \alpha < 2\pi/3$.

A spherical space $\mathbb{S}^3$ of the curvature 1 is realized as a unite tree-dimensional sphere in four-dimensional Euclidean space. Hence the regular tetrahedron $A_1A_2A_3A_4$ is in an open hemisphere. Consider a Euclidean space tangent to this hemisphere at the center of circumscribed sphere of the tetrahedron. A central projection of the hemisphere to this tangent space maps the regular tetrahedron from $\mathbb{S}^3$ onto the regular tetrahedron in Euclidean tangent space. A simple closed geodesic γ on $A_1A_2A_3A_4$ is mapped into an abstract geodesic on a regular tetrahedron in $\mathbb{E}^3$. Proposition 3.4 states that there exists a simple closed geodesic on a regular tetrahedron in Euclidean space equivalent to this generalized geodesic. It follows that a *simple closed geodesic on a regular tetrahedron in* $\mathbb{S}^3$ *is also characterized uniquely by a pair of coprime integers* (p, q) *and has the same combinatorical structure as a closed geodesic on a regular tetrahedron in* $\mathbb{E}^3$.

Lemma 4.1 ([10])**.**

1) *On a regular tetrahedron with the planar angle* $\alpha \in (\pi/3, 2\pi/3)$ *in spherical space there exist three different simple closed geodesics of type* $(0, 1)$*. They coincide under isometries of the tetrahedron.*
2) *Geodesics of type* $(0, 1)$ *exhaust all simple closed geodesics on a regular tetrahedron with the planar angle* $\alpha \in [\pi/2, 2\pi/3)$ *in spherical space.*
3) *On a regular tetrahedron with the planar angle* $\alpha \in (\pi/3, \pi/2)$ *in spherical space there exist three different simple closed geodesics of type* $(1, 1)$*.*

Proof. 1) Consider a regular tetrahedron $A_1A_2A_3A_4$ in $\mathbb{S}^3$ with the planar angle $\alpha \in (\pi/3, 2\pi/3)$. Let X_1 and X_2 be the midpoints of A_1A_4 and A_3A_2, and let Y_1, Y_2 be the midpoints of A_4A_2 and A_1A_3. Join these points consecutively with the segments through the faces. Since the points X_1, Y_1, X_2, and Y_2 are midpoints, the triangles $X_1A_4Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$, and $Y_2A_1X_1$ are equal. It follows that the closed polyline $X_1Y_1X_2Y_2$ is a simple closed geodesic of type $(0, 1)$ on a regular tetrahedron in spherical space (see Fig. 4.1). Choosing the midpoints of other pairs of opposite edges, we can construct other two geodesics of type $(0, 1)$ on the tetrahedron.

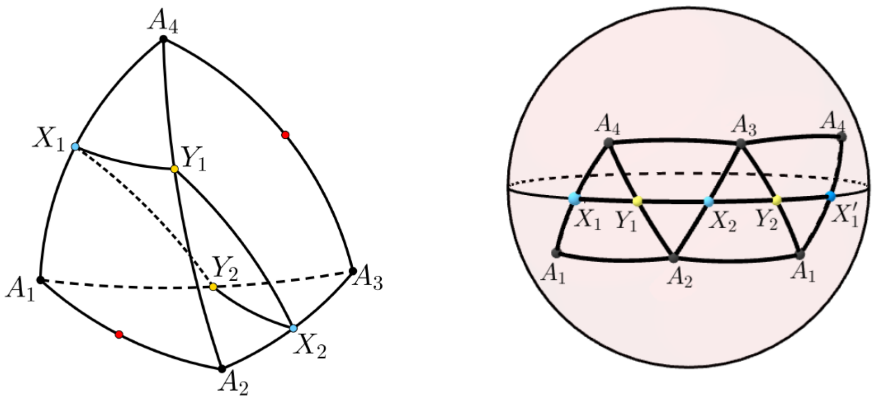

Fig. 4.1

2) Consider a regular tetrahedron with the planar angle $\alpha \geq \pi/2$. Since a geodesic is a line segment inside the development of the tetrahedron, it cannot intersect three edges of the tetrahedron, coming out from the same vertex, in succession.

If a simple closed geodesic on the tetrahedron is of type (p, q), where $p = q = 1$ or $1 < p < q$, then this geodesic intersects three edges, with the common vertex, in succession (see [41]). Only a simple closed geodesic of type $(0, 1)$ intersects two edges of the tetrahedron, which have a common vertex, and does not intersect the third edge. It follows that on a regular tetrahedron in spherical space with the planar angle $\alpha \in [\pi/2, 2\pi/3)$ there exist only three simple closed geodesics of type $(0, 1)$ and there are no other geodesics.

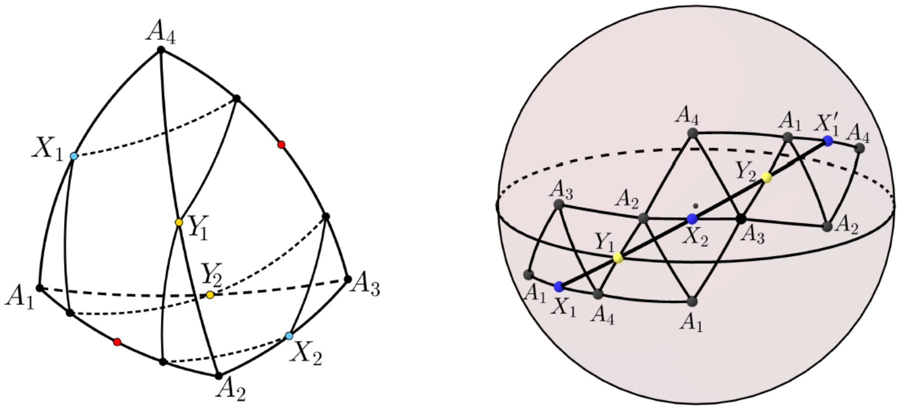

Fig. 4.2

3) Consider a regular tetrahedron $A_1A_2A_3A_4$ in $\mathbb{S}^3$ with the planar angle $\alpha \in (\pi/3, \pi/2)$. As above, the points X_1, X_2, Y_1, and Y_2 are the midpoints of A_1A_4, A_3A_2, A_4A_2, and A_1A_3, respectively.

Unfold two adjacent faces $A_1A_4A_3$ and $A_4A_3A_2$ into the plane and draw a geodesic line segment X_1Y_1. Since $\alpha < \pi/2$, the segment X_1Y_1 is contained inside the development and intersects the edge A_4A_2 at the right angle. Then unfold other two adjacent faces $A_4A_1A_2$ and $A_1A_2A_3$ and construct the segment Y_1X_2. In the same way, join the points X_2 and Y_2 within the faces $A_2A_3A_4$ and $A_3A_4A_1$, and join Y_2 and X_1 within $A_1A_2A_3$ and $A_4A_1A_2$ (see Fig. 4.2). Since the points X_1, Y_1, X_2, and Y_2 are the midpoints of their edges, the triangles $X_1A_4Y_1$, $Y_1A_2X_2$, $X_2A_3Y_2$, and $Y_2A_1X_1$ are equal. Hence, the segments X_1Y_1, Y_1X_2, X_2Y_2, and Y_2X_1 form a simple closed geodesic of type $(1, 1)$ on the tetrahedron.

Two other simple closed geodesics of type $(1, 1)$ on a tetrahedron can be constructed in the same way by connecting the midpoints of other pairs of opposite edges of the tetrahedron. □

In the following, we assume that α satisfies $\pi/3 < \alpha < \pi/2$.

4.2. The properties of simple closed geodesics on a regular tetrahedron in $\mathbb{S}^3$.

Lemma 4.2. *The length of a simple closed geodesic on a regular tetrahedron in spherical space is less than* 2π.

In [10], this lemma was proved by using Proposition 3.5 about the construction of a simple closed geodesic on a regular tetrahedron. However, Lemma 4.2 can be considered as a particular case of the result proved by A. Borisenko [8] about the generalization of V. Toponogov's theorem [48] to the case of two-dimensional Alexandrov space.

Lemma 4.3 ([10]). *On a regular tetrahedron in spherical space a simple closed geodesic intersects midpoints of two pairs of opposite edges.*

Proof. Let γ be a simple closed geodesic on a regular tetrahedron $A_1A_2A_3A_4$ in $\mathbb{S}^3$. As it was shown above, there exists a simple closed geodesic $\widetilde{\gamma}$ on a regular tetrahedron in Euclidean space such that $\widetilde{\gamma}$ is equivalent to γ. From Theorem 3.1, we assume that $\widetilde{\gamma}$ intersects the midpoints $\widetilde{X}_1$ and $\widetilde{X}_2$ of the edges A_1A_2 and A_3A_4 on the tetrahedron in $\mathbb{E}^3$. Denote by X_1 and X_2 the vertices of γ at the edges A_1A_2 and A_3A_4 on the tetrahedron in $\mathbb{S}^3$ such that X_1 and X_2 are equivalent to the points $\widetilde{X}_1$ and $\widetilde{X}_2$.

Consider the development of the tetrahedron along γ starting from the point X_1 on a two-dimensional unite sphere. The geodesic γ is unrolled into the line segment X_1X_1' of length less than 2π inside the development. Denote the parts of the development along X_1X_2 and X_2X_1' by T_1 and T_2.

Let M_1 and M_2 be the midpoints of the edges A_1A_2 and A_3A_4 on the tetrahedron in $\mathbb{S}^3$. The rotation by the angle π over the line M_1M_2 is an isometry of the tetrahedron. Then the development of the tetrahedron is central symmetric with the center M_2.

In addition, the symmetry over M_2 swaps the parts T_1 and T_2. The point X_1' at the edge A_1A_2 of T_2 is mapped into the point $\widehat{X}_1'$ at the edge A_2A_1 containing X_1 on T_1, and the lengths of A_2X_1 and $\widehat{X}_1'A_1$ are equal.

The image of the point X_1 on T_1 is a point $\widehat{X}_1$ at the edge A_1A_2 on T_2. Since M_2 is the midpoint of A_3A_4, the symmetry maps the point X_2 at A_3A_4 onto the point $\widehat{X}_2$ at the same edge A_3A_4 such that the lengths of A_4X_2 and $\widehat{X}_2A_3$ are equal. Thus, the segment X_1X_1' is mapped into the segment $\widehat{X}_1'\widehat{X}_1$ inside the development.

Suppose the segments $\widehat{X}_1'\widehat{X}_2$ and X_1X_2 intersect at the point Z_1 inside T_1. Then the segments $\widehat{X}_2\widehat{X}_1$ and X_2X_1' intersect at the point Z_2 inside T_2, and the point Z_2 is central symmetric to Z_1 with respect to M_2 (see Fig. 4.3). Inside the polygon on the sphere, we obtain two circular arcs X_1X_1' and $\widehat{X}_1'\widehat{X}_1$ intersecting in two points. Therefore Z_1 and Z_2 are antipodal points on the sphere and the length of the geodesic segment $Z_1X_2Z_2$ is π.

Now, consider the development of the tetrahedron along γ starting from the point X_2. This development also consists of spherical polygons T_2 and T_1, but in this case they are glued by the edge A_1A_2 and are central symmetric with respect to M_1.

Similarly to the above, apply the symmetry over M_1. The segments $X_2X_1X_2'$ and $\widehat{X}_2\widehat{X}_1\widehat{X}_2'$ are swapped inside the development. Since the symmetries over M_1 and over M_2 correspond to the same isometry of the tetrahedron, the arcs $X_2X_1X_2'$ and $\widehat{X}_2\widehat{X}_1\widehat{X}_2'$ also intersect at the points Z_1 and Z_2. It follows that the

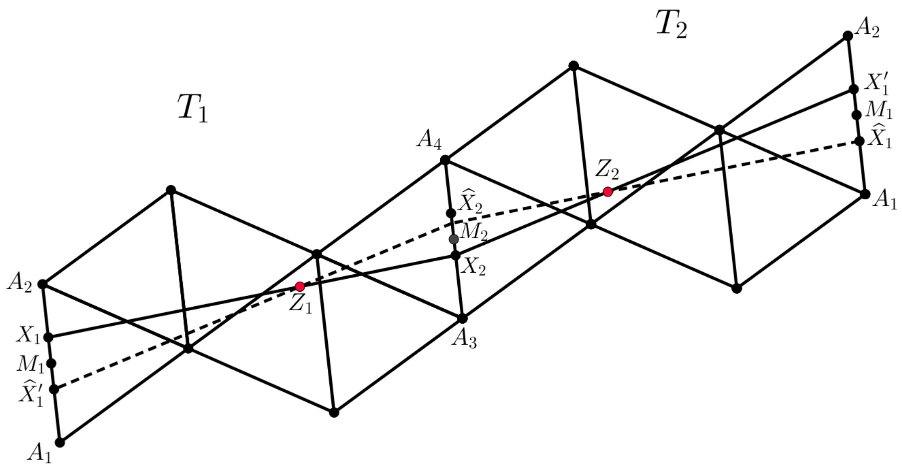

Fig. 4.3

length of the geodesic segment $Z_1X_1Z_2$ is also equal to π. Hence the length of the geodesic γ on a regular tetrahedron in spherical space is 2π, which contradicts Lemma 4.2. We get that the segments $\widehat{X}_1'\widehat{X}_2$ and X_1X_2 on T_1 either do not intersect or coincide.

If X_1X_2 and $\widehat{X}_1'\widehat{X}_2$ do not intersect, then they form a quadrilateral $X_1X_2\widehat{X}_2\widehat{X}_1'$ inside T_1. Since γ is a closed geodesic, $\angle A_1X_1X_2 + \angle A_2\widehat{X}_1'\widehat{X}_2 = \pi$. Furthermore, $\angle X_1X_2A_3 + \angle \widehat{X}_1'\widehat{X}_2A_4 = \pi$. We obtain the convex quadrilateral on a sphere with the sum of inner angles 2π. It follows that the integral of the Gaussian curvature over the interior of $X_1X_2\widehat{X}_2\widehat{X}_1'$ on a sphere is equal to zero. Hence the segments X_1X_2 and $\widehat{X}_1'\widehat{X}_2$ coincide under the symmetry of the development. Then the points X_1 and X_2 of the geodesic γ are the midpoints of the edges A_1A_2 and A_3A_4.

The statement that γ intersects the midpoints of the second pair of the opposite edges of the tetrahedron can be proved in a similar way. □

Corollary 4.4 ([10]). *If two simple closed geodesics on a regular tetrahedron in spherical space intersect the edges of the tetrahedron in the same order, then they coincide.*

4.3. The estimation on the angle α for which there is no simple closed geodesic of type (p, q).

Theorem 4.5 ([10]). *On a regular tetrahedron with the planar angle α in spherical space such that*

$$\alpha > 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}}, \tag{4.3}$$

where (p, q) is a pair of coprime integers, there is no simple closed geodesic of type (p, q).

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in $\mathbb{S}^3$ with the planar angle $\alpha \in (\pi/3, \pi/2)$, and let γ be a simple closed geodesic of type (p, q) on it.

Each face of the tetrahedron is a regular spherical triangle. Consider a two-dimensional unit sphere containing the face $A_1A_2A_3$. Construct the Euclidean plane Π passing through the points A_1, A_2, and A_3. The intersection of the sphere with the plane Π is a small circle. Draw the rays starting at the sphere center O to the points at the spherical triangle $A_1A_2A_3$. This defines the geodesic map between the sphere and the plane Π. The image of the spherical triangle $A_1A_2A_3$ is the triangle $\widetilde{\triangle} A_1A_2A_3$ at the Euclidean plane Π. The edges of $\widetilde{\triangle} A_1A_2A_3$ are the chords joining the vertices of the spherical triangle. From (4.1), it follows that the length $\widetilde{a}$ of an edge of $\widetilde{\triangle} A_1A_2A_3$ equals

$$\widetilde{a} = \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{\sin(\alpha/2)}. \tag{4.4}$$

The segments of the geodesic γ lying inside $A_1A_2A_3$ are mapped into the straight line segments inside $\widetilde{\triangle} A_1A_2A_3$ (see Fig. 4.4).

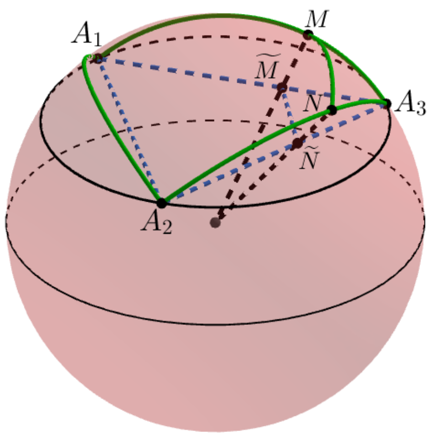

Fig. 4.4

In the same way, the other tetrahedron faces $A_2A_3A_4$, $A_2A_4A_1$, and $A_1A_4A_3$ are mapped into the plane triangles $\widetilde{\triangle} A_2A_3A_4$, $\widetilde{\triangle} A_2A_4A_1$, and $\widetilde{\triangle} A_1A_4A_3$, respectively. Since the spherical tetrahedron is regular, the constructed plane triangles are equal. We can glue them together identifying the edges with the same labels. Hence we obtain the regular tetrahedron in Euclidean space. Since the segments of γ are mapped into the straight line segments within the plane triangles, they form an abstract geodesic $\widetilde{\gamma}$ on the regular tetrahedron in $\mathbb{E}^3$, and $\widetilde{\gamma}$ is equivalent to γ.

Let us show that the length of γ is greater than the length of $\widetilde{\gamma}$. Consider an arc MN of the geodesic γ within the face $A_1A_2A_3$. The rays OM and ON intersect the plane Π at the points $\widetilde{M}$ and $\widetilde{N}$. The line segment $\widetilde{M}$ and $\widetilde{N}$ lying into $\widetilde{\triangle} A_1A_2A_3$ is the image of the arc MN under the geodesic map (see Fig. 4.4). Suppose that the length of the arc MN is equal to 2φ, then the length of the segment $\widetilde{M}\widetilde{N}$ equals $2\sin\varphi$. Thus, the length of γ on a regular tetrahedron in spherical space is greater than the length of its image $\widetilde{\gamma}$ on a regular tetrahedron in Euclidean space.

From Proposition 3.4, we know that on a regular tetrahedron in Euclidean space there exists a simple closed geodesic $\widehat{\gamma}$ equivalent to $\widetilde{\gamma}$. On the development of the tetrahedron, the geodesic $\widehat{\gamma}$ is a straight line segment, and the generalized geodesic $\widetilde{\gamma}$ is a polyline, and thus the length of $\widehat{\gamma}$ is less than the length of $\widetilde{\gamma}$.

This implies that on a regular tetrahedron $A_1A_2A_3A_4$ in $\mathbb{S}^3$ with the planar angle α the length $L_{p,q}$ of a simple closed geodesic γ of type (p,q) is greater than the length of a simple closed geodesic $\widehat{\gamma}$ of type (p,q) on a regular tetrahedron with the edge length $\widetilde{a}$ in $\mathbb{E}^3$. From equations (3.1) and (4.4), we get that

$$L_{p,q} > 2\sqrt{p^2+pq+q^2}\,\frac{\sqrt{4\sin^2(\alpha/2)-1}}{\sin(\alpha/2)}.$$

If α is such that the following inequality holds:

$$2\sqrt{p^2+pq+q^2}\,\frac{\sqrt{4\sin^2(\alpha/2)-1}}{\sin(\alpha/2)} > 2\pi, \tag{4.5}$$

then the necessary condition for the existence of a simple closed geodesic of type (p,q) on a regular tetrahedron with the face angle α in spherical space is failed. Therefore, if

$$\alpha > 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}},$$

then there are no simple closed geodesics of type (p,q) on the tetrahedron with the planar angle α in spherical space. □

Corollary 4.6 ([10])**.** *On a regular tetrahedron in spherical space there exist a finite number of simple closed geodesics.*

Proof. If the integers p and q go to infinity, then

$$\lim_{p,q\to\infty} 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}} = 2\arcsin\frac{1}{2} = \frac{\pi}{3}.$$

From inequality (4.3), we get that for large numbers p and q a simple closed geodesic of type (p,q) can exist on a regular tetrahedron with the planar angle α closed to $\pi/3$ in spherical space. □

The pairs $p=0, q=1$ and $p=1, q=1$ do not satisfy the condition (4.3). Geodesics of these types are described in Lemma 4.1.

4.4. The estimation on the angle α for which there is a simple closed geodesic of type (p, q). In the previous sections, we assumed that the Gaussian curvature of faces of a regular tetrahedron in spherical space was equal to 1. In that case, the length a of the edges of the regular tetrahedron was the function of α given by (4.1). In the current section, we will assume that the faces of the tetrahedron are spherical triangles with the angle α on a sphere of radius $R = 1/a$. Then the length of the tetrahedron edges equals 1, and the faces curvature is a^2.

Since $\alpha > \pi/3$, we can write $\alpha = \pi/3 + \varepsilon$, where $\varepsilon > 0$. Taking into account Lemma 4.1, we also expect $\varepsilon < \pi/6$.

Theorem 4.7 ([10]). *Let (p, q) be a pair of coprime integers, $0 \le p < q$, and let ε satisfy*

$$\varepsilon < \min\left\{ \frac{\sqrt{3}}{4c_0\sqrt{p^2+q^2+pq}\ \sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\left(c_l(i) + \sum_{j=0}^{i} c_\alpha(j)\right)}; \frac{1}{8\cos\frac{\pi}{12}(p+q)^2} \right\},$$

where

$$c_0 = \frac{3 - \frac{(p+q+2)}{\pi\cos\frac{\pi}{12}(p+q)^2} - 16\sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\tan^2\left(\frac{\pi i}{2(p+q)}\right)}{1 - \frac{(p+q+2)}{2\pi\cos\frac{\pi}{12}(p+q)^2} - 8\sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\tan^2\left(\frac{\pi i}{2(p+q)}\right)},$$

$$c_l(i) = \frac{\cos\frac{\pi}{12}(p+q)^2\left(4+\pi^2(2i+1)^2\right)}{(p+q-i-1)^2},$$

$$c_\alpha(j) = 4\left(8\pi(p+q)^2\cos\frac{\pi}{12}\tan^2\frac{\pi j}{2(p+q)} + 1\right).$$

Then on a regular tetrahedron in spherical space with the planar angle $\alpha = \pi/3 + \varepsilon$ there exists a unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p, q).

First, let us prove some auxiliary lemmas.

Lemma 4.8 ([10]). *The edge length of a regular tetrahedron in spherical space of curvature 1 satisfies the inequality*

$$a < \pi\sqrt{2\cos(\pi/12)}\,\sqrt{\varepsilon}, \tag{4.6}$$

where $\alpha = \pi/3 + \varepsilon$ is the planar angle of the face of the tetrahedron.

Proof. From (4.1), we have

$$\sin a = \frac{\sqrt{4\sin^2(\alpha/2) - 1}}{2\sin^2(\alpha/2)}.$$

Substituting $\alpha = \pi/3 + \varepsilon$, we get

$$\sin a = \frac{\sqrt{\sin(\varepsilon/2)\cos(\pi/6 - \varepsilon/2)}}{\sin^2(\pi/6 + \varepsilon/2)}.$$

Since $\varepsilon < \pi/6$, we have

$$\cos(\pi/6 - \varepsilon/2) < \cos\pi/12, \quad \sin(\pi/6 + \varepsilon/2) > \sin\pi/6, \quad \text{and} \quad \sin(\varepsilon/2) < \varepsilon/2.$$

Using these estimations, we obtain

$$\sin a < 2\sqrt{2\cos(\pi/12)}\,\sqrt{\varepsilon}.$$

The inequality $a < \pi/2$ implies that $\sin a > (2/\pi)a$. Then

$$a < \pi\sqrt{2\cos(\pi/12)}\,\sqrt{\varepsilon}. \qquad \square$$

Consider a parametrization of a two-dimensional sphere S^2 of radius R in $\mathbb{E}^3$:

$$\begin{cases} x = R\sin\varphi\cos\theta \\ y = R\sin\varphi\sin\theta \\ z = -R\cos\varphi \end{cases}, \tag{4.7}$$

where $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$. Let the point P have the coordinates $\varphi = r/R$, $\theta = 0$, where $r/R < \pi/2$, and let the point X_1 correspond to $\varphi = 0$. Apply a central projection of the hemisphere $\varphi \in [0, \pi/2]$, $\theta \in [0, 2\pi)$ onto the tangent plane at X_1 (see Fig. 4.5).

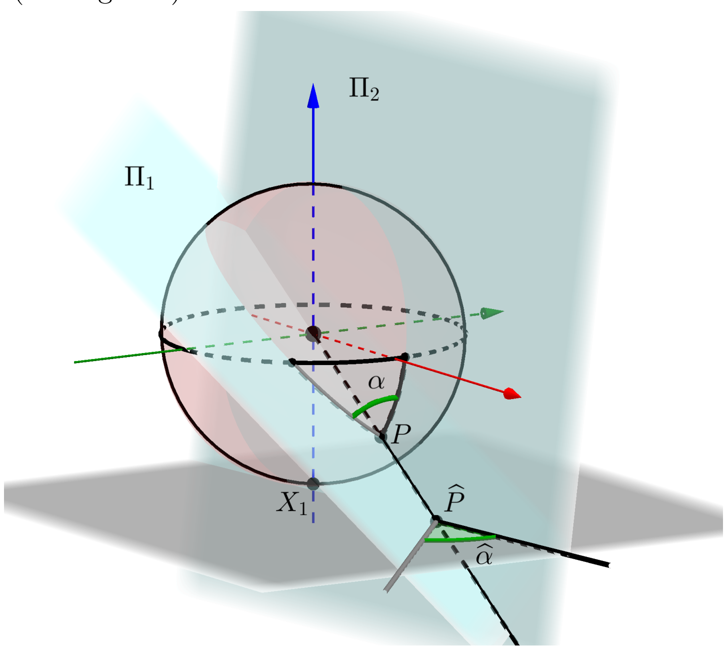

Fig. 4.5

Lemma 4.9 ([10]). *Under the central projection of the hemisphere of radius $R = 1/a$ onto the tangent plane at X_1, the angle $\alpha = \pi/3 + \varepsilon$ with the vertex $P\,(R\sin(r/R), 0, -R\cos(r/R))$ on the hemisphere is mapped to the angle $\widehat{\alpha}_r$ on the plane, which satisfies the inequality*

$$\left|\widehat{\alpha}_r - \pi/3\right| < \pi\tan^2(r/R) + \varepsilon. \tag{4.8}$$

Proof. Construct the planes Π_1 and Π_2 through the center of a hemisphere and the point $P\,(R\sin(r/R), 0, -R\cos(r/R))$:

$$\Pi_1 : a_1\cos(r/R)\,x + \sqrt{1-a_1^2}\;y + a_1\sin(r/R)\,z = 0,$$
$$\Pi_2 : a_2\cos(r/R)\,x + \sqrt{1-a_2^2}\;y + a_2\sin(r/R)\,z = 0,$$

where

$$|a_1|, |a_2| \le 1. \tag{4.9}$$

If the angle between these two planes, Π_1 and Π_2, equals α, then

$$\cos\alpha = a_1a_2 + \sqrt{(1-a_1^2)(1-a_2^2)}. \tag{4.10}$$

The tangent plane to S^2 at X_1 is given by $z = -R$. The planes Π_1 and Π_2 intersect the tangent plane along the lines that form the angle $\widehat{\alpha}_r$ (see Fig. 4.5), and

$$\cos\widehat{\alpha}_r = \frac{a_1a_2\cos^2(r/R) + \sqrt{(1-a_1^2)(1-a_2^2)}}{\sqrt{1-a_1^2\sin^2(r/R)}\sqrt{1-a_2^2\sin^2(r/R)}}. \tag{4.11}$$

From equations (4.10) and (4.11), we get

$$|\cos\widehat{\alpha}_r - \cos\alpha| < \frac{|a_1a_2\sin^2(r/R)|}{\sqrt{1-a_1^2\sin^2(r/R)}\sqrt{1-a_2^2\sin^2(r/R)}}. \tag{4.12}$$

Inequalities (4.9) and (4.12) imply that

$$|\cos\widehat{\alpha}_r - \cos\alpha| < \tan^2(r/R). \tag{4.13}$$

It is true that

$$|\cos\widehat{\alpha}_r - \cos\alpha| = \left|2\sin\frac{\widehat{\alpha}_r-\alpha}{2}\sin\frac{\widehat{\alpha}_r+\alpha}{2}\right|.$$

Then $\alpha > \pi/3$ and $\widehat{\alpha}_r < \pi$ together with the inequities

$$\left|\sin\frac{\widehat{\alpha}_r+\alpha}{2}\right| > \sin\frac{\pi}{6} \quad\text{and}\quad \left|\sin\frac{\widehat{\alpha}_r-\alpha}{2}\right| > \frac{2}{\pi}\left|\frac{\widehat{\alpha}_r-\alpha}{2}\right|$$

imply that

$$\frac{2}{\pi}\left|\frac{\widehat{\alpha}_r-\alpha}{2}\right| < |\cos\widehat{\alpha}_r - \cos\alpha|.$$

From (4.14), (4.13) and $\alpha = \pi/3 + \varepsilon$, we obtain

$$\left|\widehat{\alpha}_r - \pi/3\right| < \pi\tan^2(r/R) + \varepsilon. \qquad \square$$

On a sphere (4.7), let us consider the arc of length one starting at the point P with the coordinates $\varphi = r/R, \theta = 0$, where $r/R < \pi/2$. Apply the central projection of this arc to the plane $z = -R$, which is tangent to the sphere at the point $X_1(\varphi = 0)$ (see Fig. 4.6).

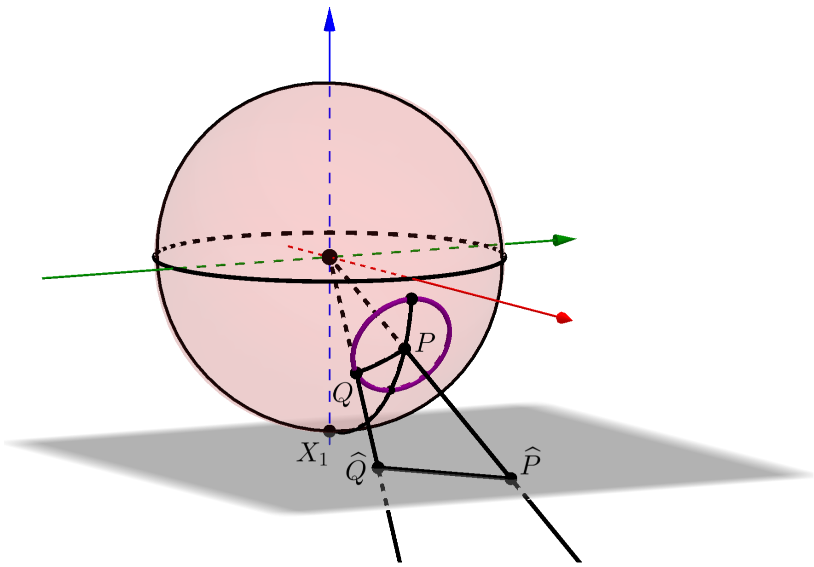

Fig. 4.6

Lemma 4.10 ([10])**.** *Under the central projection of the hemisphere of radius $R = 1/a$ onto the tangent plane at X_1, the arc of the length one starting from the point $P\,(R\sin(r/R), 0, -R\cos(r/R))$ is mapped to the segment of length $\widehat{l}_r$ satisfying the inequality*

$$\widehat{l}_r - 1 < \frac{\cos(\pi/12)\left(4 + \pi^2(2r+1)^2\right)}{\left(1 - (2\pi)a(r+1)\right)^2} \cdot \varepsilon. \tag{4.14}$$

Proof. The point $P\,(R\sin(r/R), 0, -R\cos(r/R))$ on the sphere S^2 is mapped to $\widehat{P}(R\tan(r/R), 0, -R)$ on the tangent plane $z = -R$.

Take the point $Q(Ra_1, Ra_2, Ra_3)$ on the sphere such that the spherical distance PQ equals 1. Then $\angle POQ = 1/R$, where O is the center of the sphere S^2 (see Fig. 4.6). We obtain the following conditions for the constants a_1, a_2, a_3:

$$a_1 \sin(r/R) - a_3 \cos(r/R) = \cos(1/R); \tag{4.15}$$

$$a_1^2 + a_2^2 + a_3^2 = 1. \tag{4.16}$$

The central projection into the plane $z = -R$ maps the point Q to the point $\widehat{Q}\left(-\frac{a_1}{a_3}R, -\frac{a_2}{a_3}R, -R\right)$. The length of $\widehat{P}\widehat{Q}$ equals

$$|\widehat{P}\widehat{Q}| = R\sqrt{\left(a_1/a_3 - \tan(r/R)\right)^2 + a_2^2/a_3^2}. \tag{4.17}$$

Using the Lagrange multiplier method to find the local extremum of the length $\widehat{P}\widehat{Q}$, we get that the minimum of $|\widehat{P}\widehat{Q}|$ is reached when Q has the coordinates

$$\left(R\sin\left((r-1)/R\right), 0, R\cos\left((r-1)/R\right)\right).$$

Then

$$|\widehat{P}\widehat{Q}|_{\min} = R\,|\tan(r/R) - \tan((r-1)/R)| = \frac{R\sin(1/R)}{\cos(r/R)\cos((r-1)/R)}.$$

It should be noticed that $|\widehat{P}\widehat{Q}|_{\min} > 1$.

The maximum of $|\widehat{P}\widehat{Q}|$ is reached at the point

$$Q\,(R\sin((r+1)/R)\,, 0, R\cos((r+1)/R))\,.$$

This maximum value equals

$$|\widehat{P}\widehat{Q}|_{\max} = R\,|\tan(r/R) - \tan((r+1)/R)| = \frac{R\sin(1/R)}{\cos(r/R)\cos((r+1)/R)}.$$

Since $R = 1/a$, the length $\widehat{l}_r$ of the projection of PQ satisfies

$$\widehat{l}_r < \frac{\sin a}{a\cos(ar)\cos\big(a(r+1)\big)}.$$

From $\sin a < a$, we obtain

$$\widehat{l}_r - 1 < \frac{2 - \cos a - \cos\big(a(2r+1)\big)}{2\cos(ar)\cos\big(a(r+1)\big)}. \tag{4.18}$$

Equation (4.6) implies that

$$1 - \cos a = \frac{\sin^2 a}{1+\cos a} \leq 8\cos(\pi/12)\;\varepsilon. \tag{4.19}$$

Analogously, from inequality (4.6), we have

$$1 - \cos(a(2r+1)) \leq 2\pi^2\cos(\pi/12)(2r+1)^2\;\varepsilon. \tag{4.20}$$

Estimate the denominator of (4.18) using the inequality $\cos x > 1-(2/\pi)x$, where $x < \pi/2$. Using (4.19) and (4.20), we get

$$\widehat{l}_r - 1 < \frac{4\cos(\pi/12) + \pi^2\cos(\pi/12)(2r+1)^2}{(1-(2/\pi)a\,(r+1))^2}\cdot\varepsilon. \qquad \square$$

Proof of Theorem 4.7. Fix a pair of coprime integers (p, q) such that $0 < p < q$. Consider a simple closed geodesic $\widetilde{\gamma}$ of type (p, q) on a regular tetrahedron $\widetilde{A}_1\widetilde{A}_2\widetilde{A}_3\widetilde{A}_4$ with the edge of length 1 in $\mathbb{E}^3$. Assume that $\widetilde{\gamma}$ passes through the midpoints $\widetilde{X}_1$, $\widetilde{X}_2$ and $\widetilde{Y}_1$, $\widetilde{Y}_2$ of the edges $\widetilde{A}_1\widetilde{A}_2$, $\widetilde{A}_3\widetilde{A}_4$ and $\widetilde{A}_1\widetilde{A}_3$, $\widetilde{A}_4\widetilde{A}_2$, respectively.

Consider the development $\widetilde{T}_{pq}$ of the tetrahedron along $\widetilde{\gamma}$ starting from the point $\widetilde{X}_1$. The geodesic unfolds to the segment $\widetilde{X}_1\widetilde{Y}_1\widetilde{X}_2\widetilde{Y}_2\widetilde{X}_1'$ inside the development $\widetilde{T}_{pq}$. From Corollary 3.2, we know that the parts of the development along the geodesic segments $\widetilde{X}_1\widetilde{Y}_1$, $\widetilde{Y}_1\widetilde{X}_2$, $\widetilde{X}_2\widetilde{Y}_2$, and $\widetilde{Y}_2\widetilde{X}_1'$ are equal, and any two adjacent polygons can be transformed into each other by a rotation through an angle π around the midpoint of their common edge.

Now, consider a two-dimensional sphere S^2 of radius $R = 1/a$, where a depends on α according to (4.1). On this sphere, we take several copies of the regular spherical triangles with the angle $\alpha \in (\pi/3, \pi/2)$ at vertices. Fold these triangles up in the same order as the faces of the Euclidean tetrahedron were unfolded along $\widetilde{\gamma}$ into the plane. In other words, we construct a polygon T_{pq} on a sphere S^2 formed by the same sequence of regular triangles as the polygon $\widetilde{T}_{pq}$ in $\mathbb{E}^3$. Denote the vertices of T_{pq} in accordance with the vertices of $\widetilde{T}_{pq}$. By the construction, the spherical polygon T_{pq} has the same properties of the central symmetry as the Euclidean $\widetilde{T}_{pq}$. Since the groups of isometries of regular tetrahedra in $\mathbb{S}^3$ and in $\mathbb{E}^3$ are equal, T_{pq} corresponds to the development of a regular tetrahedron with the planar angle α in spherical space.

Denote by X_1, X_1' and X_2, Y_1, Y_2 the midpoints of the edges A_1A_2, A_3A_4, A_1A_3, A_4A_2 on T_{pq}, respectively. These midpoints correspond to the points $\widetilde{X}_1$, $\widetilde{X}_1'$ and $\widetilde{X}_2$, $\widetilde{Y}_1$, $\widetilde{Y}_2$ on the Euclidean development $\widetilde{T}_{pq}$. Construct the great circle arcs X_1Y_1, Y_1X_2, X_2Y_2, and Y_2X_1'. The central symmetry of T_{pq} implies that these arcs form one great arc X_1X_1' on S^2. If α is such that X_1X_1' lies inside T_{pq}, then X_1X_1' corresponds to a simple closed geodesic of type (p, q) on a regular tetrahedron with the planar angle α in $\mathbb{S}^3$.

In what follows, we consider the part of the polygon T_{pq} only along X_1Y_1, but we also denote it as T_{pq} for the convenience. This part consists of $p + q$ regular spherical triangles with the edges of length 1. The polygon T_{pq} is contained inside the open hemisphere if

$$a(p + q) < \pi/2. \tag{4.21}$$

Since $\alpha = \pi/3 + \varepsilon$, the condition (4.6) implies that (4.21) holds if

$$\varepsilon < \frac{1}{8\cos(\pi/12)(p+q)^2}. \tag{4.22}$$

In this case, the length of the arc X_1Y_1 is less than $\pi/2a$, so X_1Y_1 satisfies the necessary condition from Lemma 4.2.

Apply the central projection of T_{pq} into the tangent plane $T_{X_1}S^2$ at the point X_1 to the sphere S^2. The image of the spherical polygon T_{pq} on $T_{X_1}S^2$ is a polygon $\widehat{T}_{pq}$.

Denote by $\widehat{A}_i$ the vertex of $\widehat{T}_{pq}$, which is an image of the vertex A_i on T_{pq}. The arc X_1Y_1 maps into the line segment $\widehat{X}_1\widehat{Y}_1$ on $T_{X_1}S^2$ that joins the midpoints of the edges $\widehat{A}_1\widehat{A}_2$ and $\widehat{A}_1\widehat{A}_3$. If, for some α, the segment $\widehat{X}_1\widehat{Y}_1$ lies inside the polygon $\widehat{T}_{pq}$, then the arc X_1Y_1 is also inside T_{pq} on the sphere.

The vector $\widehat{X}_1\widehat{Y}_1$ equals

$$\widehat{X}_1\widehat{Y}_1 = \widehat{a}_0 + \widehat{a}_1 + \cdots + \widehat{a}_s + \widehat{a}_{s+1}, \tag{4.23}$$

where $\widehat{a}_i$ are the sequential vectors of the $\widehat{T}_{pq}$ boundary, $\widehat{a}_0 = \widehat{X_1}\widehat{A_2}$, $\widehat{a}_{s+1} = \widehat{A_1}\widehat{Y_1}$, and $s = \left[\frac{p+q}{2}\right] + 1$ (if we take the boundary of $\widehat{T}_{pq}$ from the other side of $\widehat{X}_1\widehat{Y}_1$, then $s = \left[\frac{p+q}{2}\right]$), (see Fig. 4.7).

Furthermore, at the Euclidean plane $T_{X_1}S^2$ there exists a development $\widetilde{T}_{pq}$ of a regular Euclidean tetrahedron $\widetilde{A}_1\widetilde{A}_2\widetilde{A}_3\widetilde{A}_4$ with the edge of length 1 along a simple closed geodesic $\widetilde{\gamma}$. The development $\widetilde{T}_{pq}$ is equivalent to T_{pq}, and thus it is equivalent to $\widehat{T}_{pq}$. The segment $\widetilde{X}_1\widetilde{Y}_1$ lies inside $\widetilde{T}_{pq}$ and corresponds to the segment of $\widetilde{\gamma}$ (see Fig. 4.7).

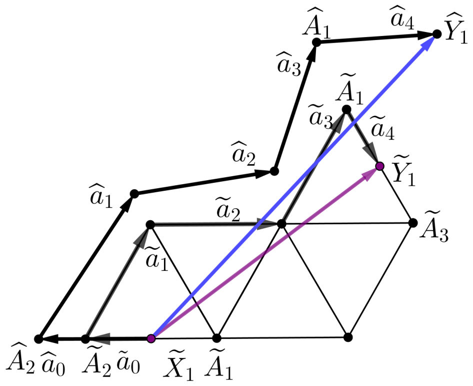

Fig. 4.7

Let the development $\widetilde{T}_{pq}$ be placed such that the point $\widetilde{X}_1$ coincides with $\widehat{X}_1$ of $\widehat{T}_{pq}$, and the vector $\widehat{X}_1\widehat{A}_2$ has the same direction as $\widetilde{X}_1\widetilde{A}_2$. Similarly to the above, we have

$$\widetilde{X}_1\widetilde{Y}_1 = \widetilde{a}_0 + \widetilde{a}_1 + \cdots + \widetilde{a}_s + \widetilde{a}_{s+1}, \tag{4.24}$$

where $\widetilde{a}_i$ are the sequential vectors of the $\widetilde{T}_{pq}$ boundary, $s = \left[\frac{p+q}{2}\right] + 1$ and $\widetilde{a}_0 = \widetilde{X}_1\widetilde{A}_2$, $\widetilde{a}_{s+1} = \widetilde{A}_1\widetilde{Y}_1$ (see Fig. 4.7).

Suppose the minimal distance from the vertices of $\widetilde{T}_{pq}$ to the segment $\widetilde{X}_1\widetilde{Y}_1$ is at the vertex $\widetilde{A}_k$ and equals $\widetilde{h}$ by formula (3.5). Let us estimate the distance $\widehat{h}$ between the segment $\widehat{X}_1\widehat{Y}_1$ and the corresponding vertex $\widehat{A}_k$ on $\widehat{T}_{pq}$. A geodesic on a regular tetrahedron in $\mathbb{E}^3$ intersects at most three edges starting from the same vertex of the tetrahedron. It follows that the interior angles of the polygon $\widetilde{T}_{pq}$ are not greater than $4\pi/3$. Hence the angles of the corresponding vertices on $\widehat{T}_{pq}$ are not greater than $4\widehat{\alpha}_i$. Applying (4.8) for $1 \le i \le s$, we get that the angle between $\widehat{a}_i$ and $\widetilde{a}_i$ satisfies the inequality

$$\angle(\widehat{a}_i, \widetilde{a}_i) < \sum_{j=0}^{i} 4\left(\pi \tan^2 \frac{j}{R} + \varepsilon\right). \tag{4.25}$$

Since $R = 1/a$, then, using (4.6), we obtain

$$\tan \frac{j}{R} < \tan\left(j\pi\sqrt{2\cos\frac{\pi}{12}}\,\sqrt{\varepsilon}\right). \tag{4.26}$$

Inequality (4.21) holds if the following condition fulfills:

$$\tan\left(j\pi\sqrt{2\cos\frac{\pi}{12}}\,\sqrt{\varepsilon}\right) < \tan\frac{\pi j}{2(p+q)}. \tag{4.27}$$

If $\tan x < \tan x_0$, then $\tan x < \frac{\tan x_0}{x_0} x$. From (4.27), it follows that

$$\tan\left(j\pi\sqrt{2\cos\frac{\pi}{12}}\,\sqrt{\varepsilon}\right) < 2(p+q)\tan\frac{\pi j}{2(p+q)}\sqrt{2\cos\frac{\pi}{12}}\,\sqrt{\varepsilon}. \tag{4.28}$$

Therefore, from (4.26) and (4.28), we get

$$\tan\frac{j}{R} < 2(p+q)\tan\frac{\pi j}{2(p+q)}\sqrt{2\cos\frac{\pi}{12}}\,\sqrt{\varepsilon}. \tag{4.29}$$

Using (4.25) and (4.29), we obtain the final estimation for the angle between the vectors $\widehat{a}_i$ and $\widetilde{a}_i$:

$$\angle(\widehat{a}_i,\widetilde{a}_i) < \sum_{j=0}^{i} 4\left(8\pi(p+q)^2\cos\frac{\pi}{12}\tan^2\frac{\pi j}{2(p+q)}+1\right)\varepsilon. \tag{4.30}$$

Now, estimate the length of the vector $\widehat{a}_i-\widetilde{a}_i$. The following inequality holds:

$$|\widehat{a}_i-\widetilde{a}_i| \le \left|\frac{\widehat{a}_i}{|\widehat{a}_i|}-\widetilde{a}_i\right| + \left|\widehat{a}_i-\frac{\widehat{a}_i}{|\widehat{a}_i|}\right|. \tag{4.31}$$

Since $\widetilde{a}_i$ is a unite vector,

$$\left|\frac{\widehat{a}_i}{|\widehat{a}_i|}-\widetilde{a}_i\right| \le \angle(\widehat{a}_i,\widetilde{a}_i) \quad\text{and}\quad \left|\widehat{a}_i-\frac{\widehat{a}_i}{|\widehat{a}_i|}\right| \le \widehat{l}_i-1. \tag{4.32}$$

From inequality (4.14), we get

$$\left|\widehat{a}_i-\frac{\widehat{a}_i}{|\widehat{a}_i|}\right| < \frac{\cos\frac{\pi}{12}\left(4+\pi^2(2i+1)^2\right)}{\left(1-\frac{2}{\pi}a(i+1)\right)^2}\cdot\varepsilon. \tag{4.33}$$

Estimate the denominator in (4.33) using (4.21). Thus,

$$\left|\widehat{a}_i-\frac{\widehat{a}_i}{|\widehat{a}_i|}\right| < \frac{\cos\frac{\pi}{12}(p+q)^2\left(4+\pi^2(2i+1)^2\right)}{(p+q-i-1)^2}\cdot\varepsilon. \tag{4.34}$$

From (4.31), (4.30) and (4.34), we obtain

$$|\widehat{a}_i-\widetilde{a}_i| \le \left(c_l(i)+\sum_{j=0}^{i}c_\alpha(j)\right)\varepsilon, \tag{4.35}$$

where

$$c_l(i) = \frac{\cos\frac{\pi}{12}(p+q)^2\left(4+\pi^2(2i+1)^2\right)}{(p+q-i-1)^2}, \tag{4.36}$$

$$c_\alpha(j) = 4\left(8\pi(p+q)^2\cos\frac{\pi}{12}\tan^2\frac{\pi j}{2(p+q)}+1\right). \tag{4.37}$$

We estimate the length of $\widehat{Y}_1\widetilde{Y}_1$ using (4.35),

$$|\widehat{Y}_1\widetilde{Y}_1| < \sum_{i=0}^{s+1} |\widehat{a}_i - \tilde{a}_i| < \sum_{i=0}^{s+1} \left(c_l(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon. \tag{4.38}$$

From (4.30), it follows that the angle $\angle\widehat{Y}_1\widehat{X}_1\widetilde{Y}_1$ satisfies

$$\angle\widehat{Y}_1\widehat{X}_1\widetilde{Y}_1 < \sum_{i=0}^{s+1} c_\alpha(i)\varepsilon. \tag{4.39}$$

The distance between the vertices $\widehat{A}_k$ and $\widetilde{A}_k$ equals

$$|\widehat{A}_k\widetilde{A}_k| < \sum_{i=0}^{k} \left(c_l(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon. \tag{4.40}$$

We drop a perpendicular $\widehat{A}_k\widehat{H}$ from the vertex $\widehat{A}_k$ into the segment $\widehat{X}_1\widehat{Y}_1$. The length of $\widehat{A}_k\widehat{H}$ equals $\widehat{h}$. Then we drop the perpendicular $\widetilde{A}_k\widetilde{H}$ into the segment $\widetilde{X}_1\widetilde{Y}_1$ and the length of $\widetilde{A}_k\widetilde{H}$ equals $\widetilde{h}$ (see Fig. 4.8).

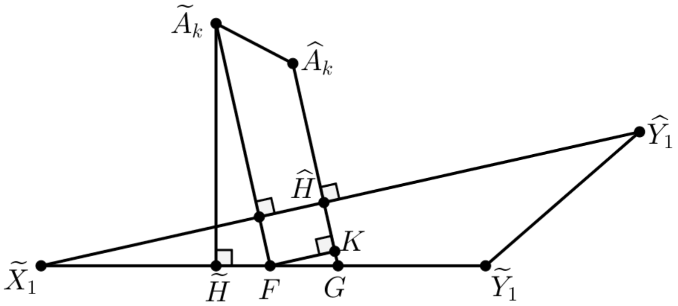

Fig. 4.8

Let the point F on $\widetilde{X}_1\widetilde{Y}_1$ be such that the segment $\widetilde{A}_kF$ is perpendicular to $\widehat{X}_1\widehat{Y}_1$. Then the length of $\widetilde{A}_kF$ is at least $\widetilde{h}$. Let G be the point of intersection of $\widetilde{X}_1\widetilde{Y}_1$ and the extension of $\widehat{A}_k\widehat{H}$. Let FK be perpendicular to $\widehat{H}G$ (see Fig. 4.8). Then the length of FK is not greater than the length of $\widehat{A}_k\widetilde{A}_k$, and $\angle KFG = \angle\widehat{Y}_1\widehat{X}_1\widetilde{Y}_1$. From the triangle GFK, we obtain

$$|FG| = \frac{|FK|}{\cos\angle\widehat{Y}_1\widehat{X}_1\widetilde{Y}_1}. \tag{4.41}$$

Applying the inequality $\cos x > 1 - \frac{2}{\pi}x$, for $x < \frac{\pi}{2}$, to (4.41), we obtain

$$|FG| < \frac{|\widehat{A}_k\widetilde{A}_k|}{1 - \frac{2}{\pi}\angle\widehat{Y}_1\widehat{X}_1\widetilde{Y}_1}. \tag{4.42}$$

Inequalities (4.39), (4.40) and (4.42) imply

$$|FG| < \frac{\sum_{i=0}^{k} \left(c_l(i) + \sum_{j=0}^{i} c_\alpha(j) \right) \varepsilon}{1 - \sum_{i=0}^{s} \left(64\pi(p+q)^2 \cos\frac{\pi}{12} \tan^2\frac{\pi i}{2(p+q)} + \frac{8}{\pi} \right) \varepsilon}. \tag{4.43}$$

Applying (4.22) to the denominator in (4.43), we obtain

$$|FG| < \frac{\sum_{i=0}^{k}\left(c_l(i)+\sum_{j=0}^{i}c_\alpha(j)\right)\varepsilon}{1-\frac{(p+q+2)}{2\pi\cos\frac{\pi}{12}(p+q)^2}-8\sum_{i=0}^{s+1}\tan^2\left(\frac{\pi i}{2(p+q)}\right)}. \tag{4.44}$$

Therefore, we have

$$\widetilde{h} \le \widetilde{A}_kF \le \widehat{h} + |\widehat{H}G| + |\widehat{A}_k\widetilde{A}_k| + |FG|. \tag{4.45}$$

Notice that $|\widehat{H}G| < |\widehat{Y}_1\widetilde{Y}_1|$. Lemma 3.3 implies that

$$\widetilde{h} > \frac{\sqrt{3}}{4\sqrt{p^2+q^2+pq}}.$$

From (4.45), it follows that

$$\widehat{h} > \frac{\sqrt{3}}{4\sqrt{p^2+q^2+pq}} - |\widehat{Y}_1\widetilde{Y}_1| - |\widehat{A}_k\widetilde{A}_k| - |FG|. \tag{4.46}$$

Applying estimations (4.38), (4.40), (4.44) and the identity $s = \left[\frac{p+q}{2}\right]+1$, we obtain

$$\widehat{h} > \frac{\sqrt{3}}{4\sqrt{p^2+q^2+pq}} - c_0\sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\left(c_l(i)+\sum_{j=0}^{i}c_\alpha(j)\right)\varepsilon, \tag{4.47}$$

where $c_l(i)$ is from (4.36), $c_\alpha(j)$ is from (4.37), and

$$c_0 = \frac{3-\frac{(p+q+2)}{\pi\cos\frac{\pi}{12}(p+q)^2}-16\sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\tan^2\left(\frac{\pi i}{2(p+q)}\right)}{1-\frac{(p+q+2)}{2\pi\cos\frac{\pi}{12}(p+q)^2}-8\sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\tan^2\left(\frac{\pi i}{2(p+q)}\right)},$$

Inequality (4.47) implies that if ε satisfies the condition

$$\varepsilon < \frac{\sqrt{3}}{4c_0\sqrt{p^2+q^2+pq}\ \sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\left(c_l(i)+\sum_{j=0}^{i}c_\alpha(j)\right)}, \tag{4.48}$$

then the distance from the vertices of the polygon $\widehat{T}_{pq}$ to $\widehat{X}_1\widehat{Y}_1$ is nonzero.

By using estimation (4.22), we get that if

$$\varepsilon < \min\left\{\frac{\sqrt{3}}{4c_0\sqrt{p^2+q^2+pq}\ \sum_{i=0}^{\left[\frac{p+q}{2}\right]+2}\left(c_l(i)+\sum_{j=0}^{i}c_\alpha(j)\right)};\frac{1}{8\cos\frac{\pi}{12}(p+q)^2}\right\}, \tag{4.49}$$

then the segment $\widehat{X}_1\widehat{Y}_1$ lies inside the polygon $\widehat{T}_{pq}$. This implies that the arc X_1Y_1 on a sphere lies inside the polygon T_{pq}. The arc X_1Y_1 corresponds to a simple closed geodesic γ of type (p,q) on a regular tetrahedron with the planar angle

$\alpha = \pi/3 + \varepsilon$ in spherical space. From Corollary 4.4, we get that this geodesic is unique up to the rigid motion of the tetrahedron.

Note that the geodesic γ is invariant under the rotation of the tetrahedron of the angle π over the line passing through the midpoints of the opposite edges of the tetrahedron. The rotation of the tetrahedron through the angle $2\pi/3$ or $4\pi/3$ over the altitude dropped from the vertex to the center of its opposite face changes γ into other simple closed geodesics of type (p, q).

The rotation over the lines connecting other vertices of the tetrahedron with the center of the opposite faces does not give us any new geodesics. So, if ε satisfies the condition (4.49), then on a regular tetrahedron with the planar angle $\alpha = \pi/3 + \varepsilon$ in spherical space there exist three different simple closed geodesics of type (p, q), disregarding isometries of the tetrahedron. □

4.5. The necessary and sufficient condition for the existence of a simple closed geodesic. Let $T(\alpha)$ be a regular tetrahedron with the planar angles α in spherical space $\mathbb{S}^3$ of curvature 1. Consider a development $R_{p,q}(\alpha)$ of $T(\alpha)$ in $\mathbb{S}^3$ along a simple closed geodesic $\gamma_{p,q}$ of type (p, q), for $\alpha \in (\pi/3, \pi/3 + \varepsilon)$, where ε is from Theorem 4.7. It follows from Lemma 3.2 that the development $R_{p,q}(\alpha)$ has four points of symmetry $X_1(\alpha)$, $X_2(\alpha)$, $Y_1(\alpha)$, $Y_2(\alpha)$, and $X_1'(\alpha)$ that correspond to the midpoints of two pairs of opposite edges of the tetrahedron. The geodesic $\gamma_{p,q}$ passes through these midpoints.

Now, for fixed (p, q), consider a one-parameter family of closed polygons $R_{p,q}(\alpha)$, where $\alpha \in (\pi/3, 2\pi/3)$. Then $R_{p,q}(\alpha)$ may have overlaps on the sphere. However, $R_{p,q}(\alpha)$ is considered as an abstract polygon homeomorphic to a disc, with intrinsic metric since each interior point of this polygon has a neighborhood isometric to the interior of a disc on the unit sphere $\mathbb{S}^2$. This polygon is locally isometrically immersed in the sphere $\mathbb{S}^2$ (see Fig. 4.9). The development $R_{p,q}(\alpha)$ also has a symmetry property for any $\alpha \in (\pi/3, 2\pi/3)$ with the corresponding points $X_1(\alpha)$, $X_2(\alpha)$, $Y_1(\alpha)$, $Y_2(\alpha)$, and $X_1'(\alpha)$ on them.

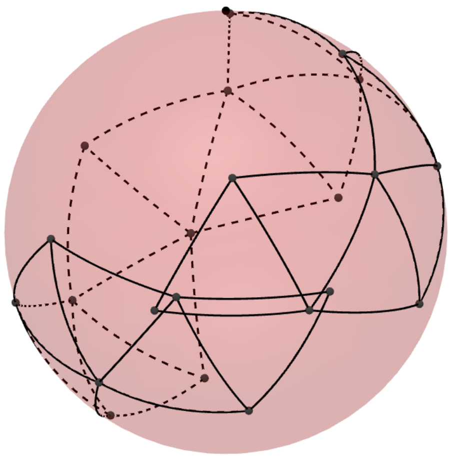

Fig. 4.9

Next, consider rectifiable curves $\sigma_{p,q}(\alpha)$ on $R_{p,q}(\alpha)$ that connect the points $X_1(\alpha)$, $X_1'(\alpha)$ and pass through $X_2(\alpha)$, $Y_1(\alpha)$, and $Y_2(\alpha)$. If $X_1(\alpha)X_1'(\alpha)$ lies inside the development $R_{p,q}(\alpha)$, then $\sigma_{p,q}(\alpha)$ corresponds to the simple closed geodesic on the regular tetrahedron $T(\alpha)$. From Theorem 4.7, it follows that this is true if α is close to $\pi/3$. Then, from Lemma 4.2, we get that the length of $\sigma_{p,q}(\alpha)$ is less than 2π. In [7], Borisenko proved that this condition is also sufficient for the existence of a simple closed geodesic on a regular tetrahedron in $\mathbb{S}^3$.

The infimum $L_{p,q}(\alpha)$ of the lengths of the curves $\sigma_{p,q}(\alpha)$ is referred to as *the length of the abstract shortest curve* in the development.

Theorem 4.11 ([7])**.** *On a regular tetrahedron in spherical space of curvature one there exists a simple closed geodesic of type (p, q) if and only if the length of the abstract shortest curve in the development is less than 2π.*

Proof. 1. *Necessity.* If there exists a simple closed geodesic of type (p, q) on a tetrahedron $T(\alpha)$, then by unfolding along this geodesic we obtain $R_{p,q}(\alpha)$. The geodesic unfolds into an arc of great circle, which lies inside $R_{p,q}(\alpha)$, connects the points $X_1(\alpha)$ and $X_1'(\alpha)$ and passes through the points of symmetry of $R_{p,q}(\alpha)$. Lemma 4.2 implies that $L_{p,q}(\alpha)$ equals the length of this geodesic, and $L_{p,q}(\alpha)$ is less than 2π (see Fig. 4.10).

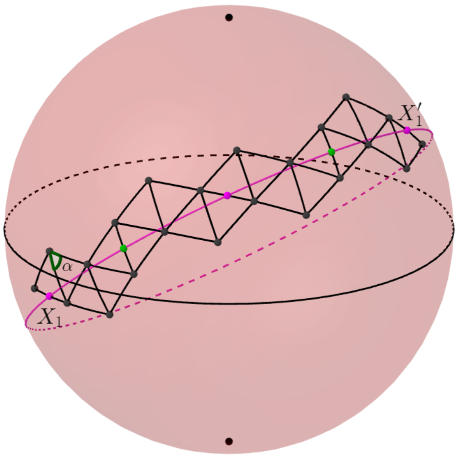

Fig. 4.10

2. *Sufficiency.* Let us prove the monotonicity of $L_{p,q}(\alpha)$. Let the infimum $L_{p,q}(\alpha)$ be attained on a curve $\sigma_{p,q}(\alpha)$ on $R_{p,q}(\alpha)$. Consider the geodesic mapping of the sphere $\mathbb{S}^3$ onto Euclidean tangent space $T_O\mathbb{S}^3$, where O is the center of the inscribed sphere in the tetrahedron $T(\alpha)$. Then $T(\alpha)$ is mapped onto the regular tetrahedron $\widehat{T}(\alpha)$ in $\mathbb{E}^3$, and the curve $\sigma_{p,q}(\alpha)$ is mapped onto $\widehat{\sigma}_{p,q}(\alpha)$.

Let $\widehat{T}(\alpha(\lambda)) = \lambda\widehat{T}(\alpha)$ be a tetrahedron homothetic to $T(\alpha)$ with center O and ratio $\lambda < 1$ such that $\alpha(\lambda) < \alpha$. This homothety takes $\widehat{\sigma}_{p,q}(\alpha)$ to a curve $\widehat{\sigma}_{p,q}(\alpha(\lambda))$.

Consider the inverse geodesic mapping of $T_O\mathbb{S}^3$ onto $\mathbb{S}^3$. It takes $\widehat{T}(\alpha(\lambda))$ to a regular tetrahedron $T(\alpha(\lambda))$, where $\alpha(\lambda) < \alpha$. The curve $\widehat{\sigma}_{p,q}(\alpha(\lambda))$ is mapped to $\sigma_{p,q}(\alpha(\lambda))$ that belongs to our class of curves. Let us show that the length of the curve $\sigma_{p,q}(\alpha(\lambda))$ is less than $L_{p,q}(\alpha)$ for $\lambda < 1$.

The curve $\widehat{\sigma}_{p,q}(\alpha)$ consists of a finite number of segments with endpoints on edges of the regular tetrahedron. Consider one of these segments, $\widehat{z}(\alpha)$, on the face $A_1A_2A_3$ of $\widehat{T}(\alpha)$. The family of segments $\lambda\widehat{z}(\alpha)$ on $\lambda\widehat{T}(\alpha)$ is homothetic to $\widehat{z}(\alpha)$ with respect to the center O. The great circle arcs $z(\lambda) = z(\alpha(\lambda))$ are the inverse geodesic images of $\lambda\widehat{z}(\alpha)$. We show that the length of $z(\lambda)$ is a monotonically increasing function of λ.

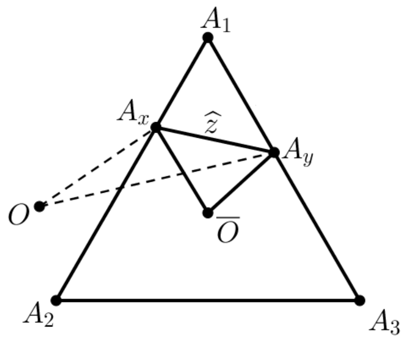

Fig. 4.11

Denote by A_x and A_y the endpoints of $\widehat{z}(\alpha)$ on A_1A_2 and A_1A_3. Then

$$|A_xA_y|^2 = |A_1A_x|^2 + |A_1A_y|^2 - |A_1A_x||A_1A_y|.$$

The radius of the inscribed sphere of the tetrahedron $\widehat{T}(\alpha)$ with edge length a is $r = a/(2\sqrt{6})$. The distance from the center of $\widehat{T}(\alpha)$ to the points A_x and A_y can be found from the triangles $\triangle A_1\bar{O}A_x$, where $\bar{O}$ is the center of the face $A_1A_2A_3$ (see Fig. 4.11):

$$|\bar{O}A_x|^2 = |A_1A_x|^2 + \frac{a^2}{3} - a|A_1A_x|.$$

From the triangle $\triangle O\bar{O}A_x$, we get

$$|\bar{O}A_x|^2 = \frac{3}{8}a^2 + |A_1A_x|^2 - a|A_1A_x|.$$

From the triangle $\triangle O\bar{O}A_y$, we have

$$|\bar{O}A_y|^2 = \frac{3}{8}a^2 + |A_1A_y|^2 - a|A_1A_y|.$$

From the triangles $\triangle OSA_x$ and $\triangle OSA_y$, where S is the center of the sphere $\mathbb{S}^3$, we obtain

$$|SA_x|^2 = 1 + |\bar{O}A_x|^2; \;\; |SA_y|^2 = 1 + |\bar{O}A_y|^2.$$

From $\triangle A_xSA_y$, it follows that

$$\cos z = \frac{\left(1+|\bar{O}A_x|^2\right)+\left(1+|\bar{O}A_y|^2\right)-|A_xA_y|^2}{2\sqrt{1+|\bar{O}A_x|^2}\sqrt{1+|\bar{O}A_y|^2}},$$

where z is the angle at the vertex S.

Similarly, for the homothetic tetrahedron $\lambda\widehat{T}(\alpha)$, we have

$$\cos z(\lambda) = \frac{\left(1+\lambda^2|\bar{O}\widehat{A}_x|^2\right)+\left(1+\lambda^2|\bar{O}A_y|^2\right)-\lambda^2|A_xA_y|^2}{2\sqrt{1+\lambda^2|\bar{O}A_x|^2}\sqrt{1+\lambda^2|\bar{O}A_y|^2}}.$$

The derivative of $z(\lambda)$ at $\lambda = 1$ is positive. This implies that the length of $\sigma_{p,q}(\alpha(\lambda))$ is less than the length of $\sigma_{p,q}(\alpha)$ for $\lambda < 1$. Hence, $L_{p,q}(\alpha(\lambda)) < L_{p,q}(\alpha)$ for $\lambda < 1$ and $\alpha(\lambda) < \alpha$.

For $\pi/3 < \alpha < \pi/3+\varepsilon$, where ε is from Theorem 4.7, there is a simple closed geodesic of type (p, q) on a regular tetrahedron in $\mathbb{S}^3$. This geodesic unfolds into a curve $\sigma_{p,q}(\alpha)$ of length $L_{p,q}(\alpha) < 2\pi$ inside the development $R_{p,q}(\alpha)$.

Now, increase the angle α starting from $\pi/3+\varepsilon$. As $\sigma_{p,q}(\alpha)$ lies inside the development $R_{p,q}(\alpha)$, it corresponds to a simple closed geodesic on a regular tetrahedron $T(\alpha)$. Let β be the first value of α for which $\sigma_{p,q}(\alpha)$ attains the boundary of $R_{p,q}(\alpha)$. This value exists by Theorem 4.5, which implies that there exists $\alpha_2 \in (\pi/3, \pi/2)$ such that there is no simple closed geodesic on $T(\alpha)$ for $\alpha > \alpha_2$.

The point of intersection of $\sigma_{p,q}(\beta)$ with the boundary of the development $R_{p,q}(\beta)$ is a vertex of the tetrahedron. Since $R_{p,q}(\beta)$ consists of congruent polygons, the segment $\sigma_{p,q}(\beta)$ 'touches' the boundary of $R_{p,q}(\beta)$ at four vertices. The property of symmetry of $R_{p,q}(\beta)$ implies that these 'touchings' alternate and there are two of them from each side of $\sigma_{p,q}(\beta)$ (see Fig. 4.12).

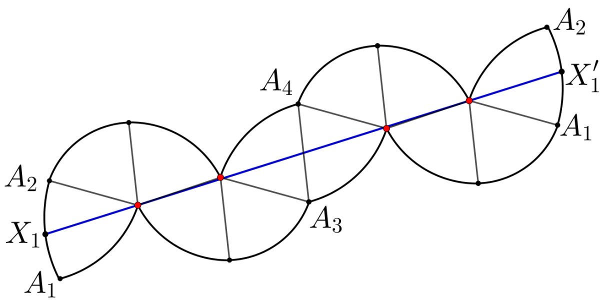

Fig. 4.12

The segment $\sigma_{p,q}(\alpha)$ cannot 'touch' the boundary of the development $R_{p,q}(\beta)$ at five points. Otherwise the curve $\sigma_{p,q}(\alpha)$ passes twice through some vertex of $T(\beta)$. For any line segment, the full angle on one side is π. The full angle at any vertex is less than 2π, and thus the segments l_1 and l_2 of the curve $\sigma_{p,q}(\alpha)$ intersect at a nonzero angle at that vertex. The geodesics $\sigma_{p,q}(\alpha)$ with $\alpha < \beta$

and α close to β also intersect themselves, which contradicts the fact that these geodesics are simple.

The case when two points of intersection (for example, the vertices A_2 and A_3) merge is also impossible. These two vertices are not connected by an edge, because, if we take $\alpha < \beta$ and $\lim \alpha = \beta$, then we can see that the length of the edge connecting these two vertices of intersection tends to zero. As $\alpha \to \beta$, the full angles at A_2 and A_3 tend to angles $\geq \pi$. Otherwise the geodesics $\sigma_{p,q}(\alpha)$ cross the boundary of the development for some $\alpha < \beta$. Without loss of generality, we can assume that $\beta \leq \beta_0 = 2 \arcsin \sqrt{7/18} < 2\pi$ since there are only three simple closed geodesics for $\beta \geq \pi/2$ (see Lemma 4.1). This bound follows from the case $p = 2, q = 1$ of inequality (4.3) from Theorem 4.5. For the full angles at the vertices A_2 and A_3 to tend to the limits $\geq \pi$, it is necessary that at least three triangles meet at A_2 and at A_3 and that for α close to β two edges meeting at A_2 belong to triangles in the development traversed by the line segment $\sigma_{p,q}(\alpha)$.

The same is observed for A_3. Then four different edges of triangles would meet at the merged vertex. Thus, four edges come out of a vertex of the tetrahedron, which is a contradiction.

As a result, for $\alpha = \beta$, the segment $\sigma_{p,q}(\alpha)$ 'touches' the boundary of $R_{p,q}(\beta)$ at four points, which correspond to the vertices of the tetrahedron. The curve $\sigma_{p,q}(\alpha)$ divides the tetrahedron into two regions homeomorphic to a circle. Each interior point has a neighborhood isometric to a disc on the sphere $\mathbb{S}^2$ of curvature 1, and the boundary is a digon. The edges of this digon have the same length, the full angles at both vertices are $3\beta - \pi$, and the geodesic curvature of the digon is zero. Therefore, the perimeter of the digon is 2π. Hence the length of $\sigma_{p,q}(\alpha)$ is 2π, which implies that $L_{p,q}(\alpha) = 2\pi$.

If a simple closed geodesic exists for a fixed α, then $L_{p,q}(\alpha)$ is equal to the length of this geodesic, and therefore it is $< 2\pi$ for $\alpha < \beta$. If $\alpha > \beta$, then, due to the monotonicity of $L_{p,q}(\alpha)$, the length of $L_{p,q}(\alpha)$ is greater than 2π, and there are no simple closed geodesics of type (p, q) on the tetrahedron $T(\alpha)$. □

Corollary 4.12 ([7]). *If the edge a of a regular tetrahedron in the spherical space satisfies the inequality*

$$a < 2 \arcsin \frac{\pi}{\sqrt{p^2 + pq + q^2} + \sqrt{(p^2 + pq + q^2) + 2\pi^2}}, \tag{4.50}$$

then this tetrahedron has a simple closed geodesic of type (p, q).

Proof. Let O be the centre of the inscribed and circumscribed spheres of a regular tetrahedron $T(\alpha)$ in spherical space $\mathbb{S}^3$.

Consider a geodesic mapping of the open hemisphere of $\mathbb{S}^3$ containing $T(\alpha)$ onto the tangent space $T_O\mathbb{S}^3$. The tetrahedron $T(\alpha)$ is mapped to a regular tetrahedron $\widehat{T}(\alpha)$ with center at O in Euclidean space $T_O\mathbb{S}^3$. The midpoints of the edges are mapped to the midpoints. Let $\widehat{a}$ be the edge length of $\widehat{T}(\alpha)$.

Let $\widehat{\gamma}_{p,q}(\alpha)$ be a simple closed geodesic of type (p, q) that passes through the midpoints of two pairs of opposite edges of $\widehat{T}(\alpha)$. Then the length of $\widehat{\gamma}_{p,q}(\alpha)$ is equal to

$$\widehat{L}_{p,q}(\alpha) = 2\widehat{a}\sqrt{p^2 + pq + q^2}. \tag{4.51}$$

Take α such that $\widehat{L}_{p,q}(\alpha) < 2\pi$. The inverse image $\gamma_{p,q}(\alpha)$ of the geodesic $\widehat{\gamma}_{p,q}(\alpha)$ on $T(\alpha)$ has the length less than $\widehat{L}_{p,q}(\alpha)$, and therefore less than 2π. The curve $\gamma_{p,q}(\alpha)$ belongs to the class of admissible curves $\sigma_{p,q}(\alpha)$ in the definition of $L_{p,q}(\alpha)$. Therefore, $L_{p,q}(\alpha) < 2\pi$, and Theorem 4.11 implies that there exists a simple closed geodesic of type (p, q) on $T(\alpha)$. It remains to use the inequality

$$2\widehat{a}\sqrt{p^2 + pq + q^2} < 2\pi$$

to obtain a bound on α, or, equivalently, on a. Formula (4.1) implies that

$$2\sin(a/2)\cos(a/2) = 1.$$

We apply a geodesic mapping of the sphere $\mathbb{S}^3$ from its centre S onto the tangent space $T_O\mathbb{S}^3$. Consider the triangle $\triangle SOB$, where B is the midpoint of A_1A_2. Let $\widehat{B}$ be the image of B under the geodesic mapping (Fig. 4.13). Then

$$|O\widehat{B}| = \tan|OB|.$$

The edge A_1A_2 of the spherical triangle maps to the edge $\widehat{A}_1\widehat{A}_2$ of the regular tetrahedron in Euclidean space, and $\widehat{A}_1\widehat{A}_2$ is perpendicular to $O\widehat{B}$. From the triangle $\triangle S\widehat{A}_1\widehat{B}$, we obtain

$$\frac{\widehat{a}}{2} = |\widehat{A}_1\widehat{B}| = |S\widehat{B}|\tan\frac{a}{2} = \frac{\tan(a/2)}{\cos|OB|}. \tag{4.52}$$

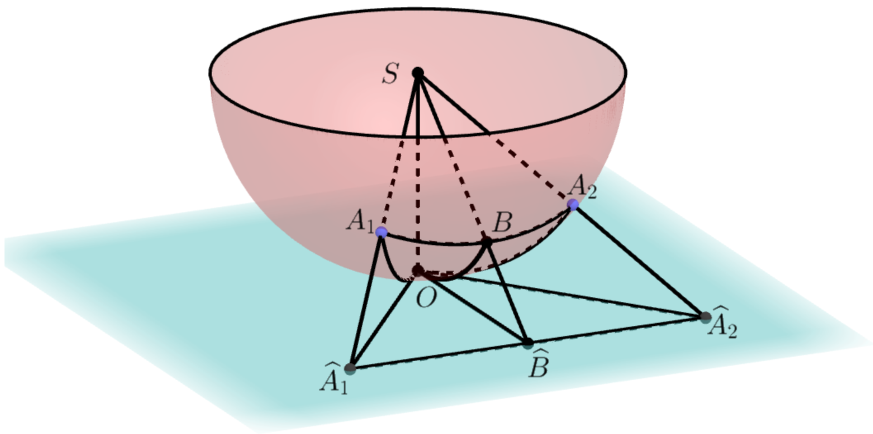

Fig. 4.13

From the triangle $\triangle PA_1A_2$ on a face of the tetrahedron in spherical space, where P is the centre of the inscribed and circumscribed circles of the face, we obtain

$$\cos a = \cos^2 R_{bas} - \frac{1}{2}\sin^2 R_{bas},$$

where $R_{bas} = |PA_1| = |PA_2|$. Hence,

$$\cos R_{bas} = \sqrt{\frac{1 + 2\cos a}{3}}. \tag{4.53}$$

From $\triangle A_4PA_1$ (Fig. 4.14), we obtain

$$\cos a = \cos(R + r)\cos R_{bas}, \tag{4.54}$$

where R is the radius of the circumscribed sphere of the tetrahedron $A_1A_2A_3A_4$, r is the radius of the inscribed ball, and $|A_4P| = R + r$. Then (4.54) implies that

$$\cos R > \frac{\cos a}{\cos R_{bas}}. \tag{4.55}$$

From $\triangle OA_1B$, we obtain

$$\cos R = \cos|OB|\cos(a/2). \tag{4.56}$$

Expressions (4.55) and (4.56) imply that

$$\frac{1}{\cos|OB|} = \frac{\cos(a/2)}{\cos R} < \frac{\cos(a/2)\cos R_{bas}}{\cos a}. \tag{4.57}$$

From (4.52), (4.53) and (4.57), we get

$$\widehat{a}/2 < \frac{\sin(a/2)}{\cos a}\sqrt{\frac{1 + 2\cos a}{3}} \le \frac{\sin(a/2)}{\cos a}. \tag{4.58}$$

Therefore, from (4.51) and (4.58), we obtain the following estimation for the length of a simple closed geodesic $\widehat{\gamma}_{p,q}(\alpha)$ of type (p, q) on $\widehat{T}(\alpha)$:

$$\widehat{L}_{p,q}(\alpha) \le 4\frac{\sin(a/2)}{\cos a}\sqrt{p^2 + pq + q^2}.$$

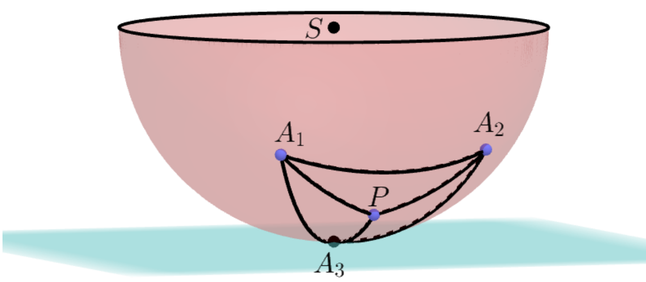

Fig. 4.14

Remind that from Theorem 4.11 it follows that if $\widehat{L}_{p,q}(\alpha) < 2\pi$, then there exists a simple closed geodesic of type (p, q) on $T(\alpha)$ in $\mathbb{S}^3$. Resolving the quadratic inequality

$$4\frac{\sin(a/2)}{\cos a}\sqrt{p^2 + pq + q^2} < 2\pi$$

with respect to $\sin(a/2)$, we obtain the required inequality. $\square$

5. Simple closed geodesics on regular tetrahedra in $\mathbb{H}^3$

5.1. Necessary conditions for a closed geodesic to be simple. We assume that the Gaussian curvature of *hyperbolic space* (*Lobachevsky space*) $\mathbb{H}^3$ is -1. A *regular tetrahedron* in $\mathbb{H}^3$ is a closed convex polyhedron whose all faces are regular geodesic triangles and all vertices are regular trihedral angles. The planar angle α of the face satisfies the inequality $0 < \alpha < \pi/3$ and the length a of edges is equal to

$$a = \operatorname{arcosh}\left(\frac{\cos\alpha}{1-\cos\alpha}\right). \tag{5.1}$$

Consider the Cayley–Klein model of hyperbolic space. In this model, the points are represented by the points in the interior of the unit ball. Geodesics in this model are the chords of the ball. Assume that the center of the circumscribed sphere of a regular tetrahedron coincides with the center of the model. Then the regular tetrahedron in hyperbolic space is represented by a regular tetrahedron in Euclidean space.

Lemma 5.1 ([9])**.** *If a geodesic on a regular tetrahedron in hyperbolic space intersects three edges meeting at a common vertex consecutively, and intersects one of these edges twice, then this geodesic has a point of self-intersection.*

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in $\mathbb{H}^3$. Suppose the geodesic γ intersects A_4A_1, A_4A_2, and A_4A_3 consecutively at the points X_1, X_2, and X_3, respectively, and then intersects the edge A_4A_1 again at the point Y_1.

Suppose also that the length of A_4X_1 is less than the length of A_4Y_1.

Unfold the faces $A_1A_2A_4$, $A_4A_2A_3$, and $A_4A_3A_1$ to the hyperbolic plane. Consider the Cayley–Klein model of the hyperbolic plane and place the vertex A_4 at the center of the model. Then the part $X_1X_2X_3Y_1$ of the geodesic is a straight line segment on the development. We obtain a triangle $X_1A_4Y_1$ on the development.

Let $\rho(X)$ be the distance function between the vertex A_4 and a point X on γ. It is known that if γ is a geodesic in a complete simply connected Riemannian manifold M of nonpositive curvature, then the function $\rho(X)$ of a distance from the fixed point A on M to the points X on γ is a convex function. The minimum of $\rho(X)$ is achieved at the point H_0 such that A_4H_0 is orthogonal to γ, and $\angle H_0A_4Y_1 > 3\alpha/2$.

Let Z_1 be the point on the segment H_0Y_1 such that $\angle H_0A_4Z_1 = 3\alpha/2$. On the opposite side of H_0, we choose the point Z_2 such that $\angle H_0A_4Z_2 = 3\alpha/2$. The point Z_2 also lies on the face at the vertex A_4 of the tetrahedron.

Since $\angle H_0A_4Z_1 = \angle H_0A_4Z_2 = 3\alpha/2$, it follows that the points Z_1 and Z_2 correspond to the same point Z on the generatrix A_4Z opposite to A_4H_0 on the tetrahedron. This point is the self-intersection point of the geodesic γ (Fig. 5.1). The lemma is proved. □

Lemma 5.2 ([9])**.** *Let d be the minimum distance from the vertices of a regular tetrahedron in hyperbolic space to a simple closed geodesic on the tetrahedron.*

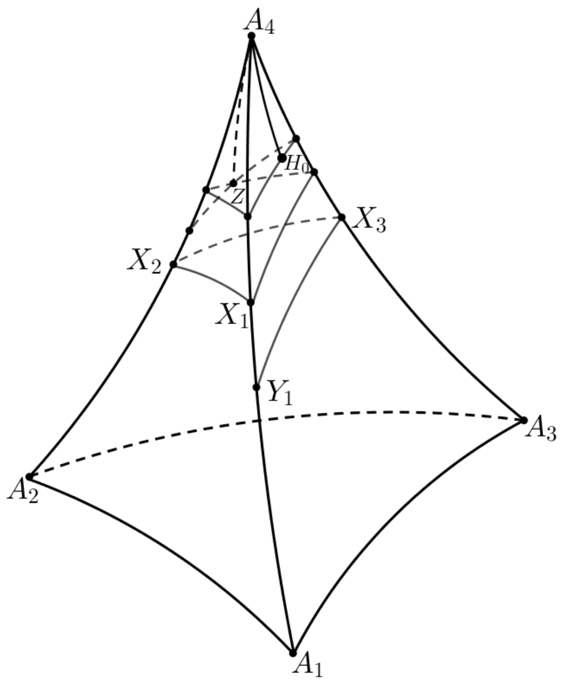

Fig. 5.1

Then

$$d > \frac{1}{2}\ln\left(\frac{\sqrt{2\pi^3} + (\pi - 3\alpha)^{\frac{3}{2}}}{\sqrt{2\pi^3} - (\pi - 3\alpha)^{\frac{3}{2}}}\right), \tag{5.2}$$

where α is the planar angle of a face of the tetrahedron.

Proof. Let γ be a simple closed geodesic on a regular tetrahedron $A_1A_2A_4A_3$ in hyperbolic space $\mathbb{H}^3$. Assume that the minimum distance d from the vertices of the tetrahedron to γ is achieved at the vertex A_4 on the face $A_2A_4A_3$. Draw a generatrix A_4H orthogonal to γ at the point H_0. Denote the angle $\angle A_2A_4H$ by β. Without loss of generality, we assume that $0 \le \beta \le \alpha/2$.

We draw a generatrix A_4K such that the planar angle between A_4K and A_4H is $3\alpha/2$. Then A_4K lies in the face $A_1A_4A_3$, and $\angle A_1A_4K = \alpha/2 - \beta$. Notice that if $\beta = \alpha/2$, then A_4K coincides with A_4A_1. If $\beta = 0$, then A_4K coincides with the altitude in the face of the tetrahedron and has the smallest length h (Fig. 5.2).

We cut the trihedral angle at A_4 along the generatrix A_4K and develop it to the hyperbolic plane in the Cayley-Klein model. We put the vertex A_4 at the centre of the boundary circle. The trihedral angle unfolds into a convex polygon $K_1A_4K_2A_3A_2A_1$. The angle $K_1A_4K_2$ equals 3α. The segment A_4H corresponds to the bisector of the angle $K_1A_4K_2$. The geodesic γ is a straight line ortogonal to A_4H at H_0.

On the lines A_4K_1 and A_4K_2, choose the points P_1 and P_2 such that

$$|A_4P_1| = |A_4P_2| = h.$$

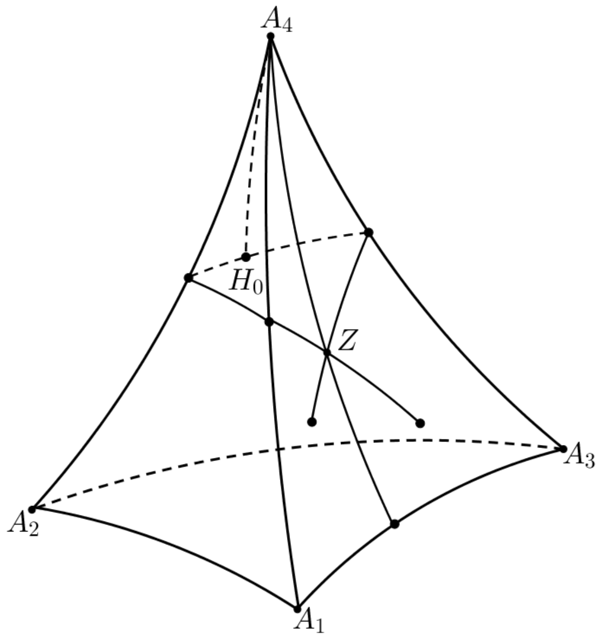

Fig. 5.2

The line segment P_1P_2 is ortogonal to A_4H at the point H_p, and

$$\tanh|A_4H_p| = \cos(3\alpha/2)\tanh h.$$

If $d \leq |A_4H_p|$, then γ lies above the segment P_1P_2, and therefore γ intersects the lines A_4K_1 and A_4K_2 at the points Z_1 and Z_2. When we fold the development back to the tetrahedron, the segments A_4K_1 and A_4K_2 are mapped to the segment A_4K on the tetrahedron, and Z_1 and Z_2 are mapped to the same point Z on A_4K. This point Z is the point of self- intersection of the geodesic γ.

Therefore, in order that γ have no points of self-intersection, it is necessary that $d > |A_4H_p|$. This implies

$$\tanh d > \cos(3\alpha/2)\tanh h. \tag{5.3}$$

The altitude h of the face of the tetrahedron satisfies

$$\tanh h = \tanh a \cos\alpha/2 = \cos\alpha/2\ \frac{\sqrt{2\cos\alpha - 1}}{\cos\alpha}. \tag{5.4}$$

Combining (5.4) and (5.3), we obtain

$$\tanh d > \cos\alpha/2\cos(3\alpha/2)\frac{\sqrt{2\cos\alpha - 1}}{\cos\alpha}, \tag{5.5}$$

Now we estimate the expression on the right-hand side of (5.5) from below. Consider the function $\sqrt{2\cos\alpha - 1}$:

$$2\cos\alpha - 1 = 4\sin(\pi/6 - \alpha/2)\sin(\pi/6 + \alpha/2).$$

Since the function $\sin(\pi/6 + \alpha/2)$ increases on the interval $(0, \pi/3)$, we get

$$\sin(\pi/6 + \alpha/2) > 1/2 \quad \text{when } \alpha \in (0, \pi/3).$$

The function $\sin(\pi/6 - \alpha/2)$ decreases on the interval $(0, \pi/3)$. It is known that $\sin y > (2/\pi)y$ when $0 < y < \pi/2$. These imply

$$\sin(\pi/6 - \alpha/2) > \frac{1}{\pi}(\pi/3 - \alpha).$$

We obtain

$$\sqrt{2\cos\alpha - 1} > \sqrt{\frac{2}{3\pi}(\pi - 3\alpha)}. \tag{5.6}$$

The function $\cos(3\alpha/2)$ is decreasing for $0 < \alpha < \pi/3$. It is true that $\cos y > 1 - (2/\pi)y$ when $0 < y < \pi/2$. Therefore,

$$\cos(3\alpha/2) > \frac{1}{\pi}(\pi - 3\alpha). \tag{5.7}$$

We have $\cos\alpha/2 > \sqrt{3}/2$ when $0 < \alpha < \pi/3$.

These inequalities, together with (5.6) and (5.7), give the following bound:

$$\tanh d > \frac{1}{\sqrt{2\pi^3}}(\pi - 3\alpha)^{3/2}. \tag{5.8}$$

Inequality (5.8) implies inequality (5.2) as required. □

5.2. Uniqueness of a simple closed geodesic on a regular tetrahedron in $\mathbb{H}^3$. For a regular tetrahedron in hyperbolic space the following analogue of Lemma 4.3 holds.

Lemma 5.3 ([9]). *A simple closed geodesic on a regular tetrahedron in hyperbolic space passes through the midpoints of two pairs of opposite edges on the tetrahedron.*

Proof. Let γ be a simple closed geodesic on a regular tetrahedron T in hyperbolic space $\mathbb{H}^3$. Consider the Cayley–Klein model of $\mathbb{H}^3$ and place the tetrahedron such that the center of the circumscribed sphere of the tetrahedron coincides with the center of the model. Then T is represented by a regular tetrahedron $\tilde{T}$ in Euclidean space $\mathbb{E}^3$.

A simple closed geodesic γ on T is represented by an abstract geodesic on $\tilde{T}$. From Proposition 3.4, we get that this generalized geodesic is equivalent to a simple closed geodesic $\tilde{\gamma}$ on $\tilde{T}$ in $\mathbb{E}^3$. From Theorem 3.1, we assume that $\tilde{\gamma}$ passes through the midpoints of two pairs of opposite edges on this tetrahedron.

Label the vertices of the tetrahedron T and the corresponding vertices of $\tilde{T}$ with A_1, A_2, A_3, and A_4. Suppose that $\tilde{\gamma}$ passes through the midpoints $\tilde{X}_1$ and $\tilde{X}_2$ of the edges A_1A_2 and A_3A_4. Consider the development of $\tilde{T}$ along $\tilde{\gamma}$ starting from $\tilde{X}_1$. From Corollary 3.2, it follows that this development is central symmetric with respect to the point $\tilde{X}_2$.

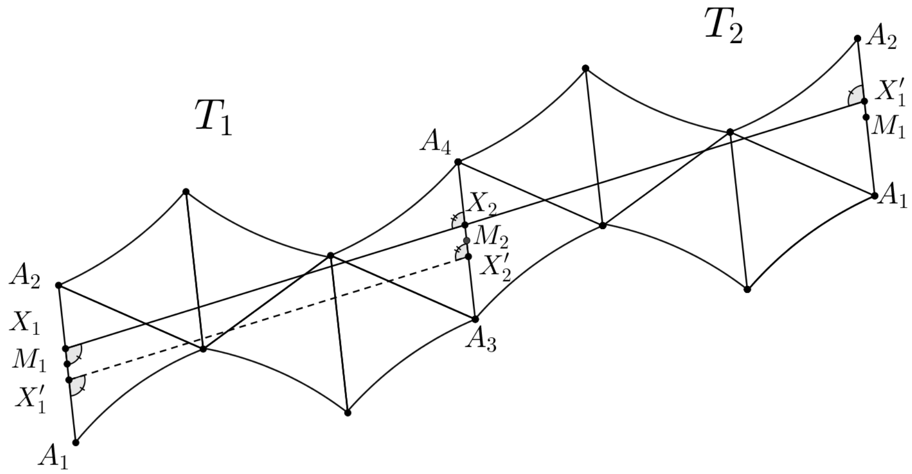

Fig. 5.3

Let X_1 and X_2 be the corresponding points on γ on the edges A_1A_2 and A_3A_4 of T. Consider the development of T onto hyperbolic plane along γ starting from the point X_1. Then γ is a line segment X_1X_1' on the development.

Denote the midpoints of the edges A_1A_2 and A_3A_4 by M_1 and M_2. Since the rotation of the tetrahedron trough π around M_1M_2 in hyperbolic space is the isometry of the tetrahedron, the development of T along $X_1X_2X_1'$ on hyperbolic plane is central symmetric with the center at M_2.

Denote by T_1 and T_2 the parts of the development along the segments X_1X_2 and X_2X_1'. The central symmetry of the development around the point M_2 swaps T_1 and T_2.

The edge A_1A_2 containing X_1' is mapped onto A_2A_1 with the point X_1. Then the point X_1' belongs to the edge A_1A_2 of T_1, and the lengths of A_2X_1 and $X_1'A_1$ are equal.

The edge A_3A_4 is mapped into itself with the opposite orientation. The point X_2 on A_3A_4 is mapped to the point X_2' on A_3A_4 such that the lengths of A_4X_2 and $X_2'A_3$ are equal. Moreover, $\angle X_1X_2A_4 = \angle X_1'X_2'A_4$. Since the geodesic is closed, we have $\angle A_1X_1X_2 = \angle A_1X_1'X_2'$ (Fig. 5.3).

We obtain the quadrilateral $X_1X_2X_2'X_1'$ inside T_1 the sum of whose interior angles is 2π. Then the integral of the Gaussian curvature over the interior of $X_1X_2X_2'X_1'$ in hyperbolic plane is zero. This implies that the rotation takes the part $X_2'X_1'$ of the geodesic to the part X_1X_2. Hence the points X_1 and X_2 are the midpoints of the corresponding edges (Fig. 5.3).

In the same way, it can be proved that γ passes through the midpoints of other two opposite edges on the regular tetrahedron in $\mathbb{H}^3$. □

Corollary 5.4 ([9]). *If two closed geodesics on a regular tetrahedron in hyperbolic space intersect the edges of the tetrahedron in the same order, then they coincide.*

5.3. The existence of a simple closed geodesic of type (p, q) on a regular tetrahedron.

Theorem 5.5 ([9]). *On a regular tetrahedron in hyperbolic space for each ordered pair of coprime integers (p, q) there exists a unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p, q). The geodesics of type (p, q) exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space.*

Proof. Let $\widetilde{\gamma}$ be a simple closed geodesic on a regular tetrahedron $A_1A_2A_3A_4$ in Euclidean space. Assume that $\widetilde{\gamma}$ passes through the midpoints $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{Y}_1$, and $\widetilde{Y}_2$ of the edges A_1A_2, A_3A_4, A_1A_3, and A_2A_4, respectively.

Consider the development $\widetilde{T}$ of the tetrahedron along $\widetilde{\gamma}$ from the point $\widetilde{X}_1$ to the point $\widetilde{X}_1'$. The polygon $\widetilde{T}$ consists of four equal polygons. Any two adjacent polygons can be transformed into each other by a rotation through an angle π around the midpoint of their common edge. The interior angles of $\widetilde{T}$ are $\pi/3$, $2\pi/3$, π, or $4\pi/3$. The angle of $4\pi/3$ is obtained if $\widetilde{\gamma}$ intersects three edges having a common vertex consecutively.

Now we take regular triangles on the hyperbolic plane with angle α at the vertices. We put these triangles in the same order in which the faces of the tetrahedron were unfolded in Euclidean space along $\widetilde{\gamma}$.

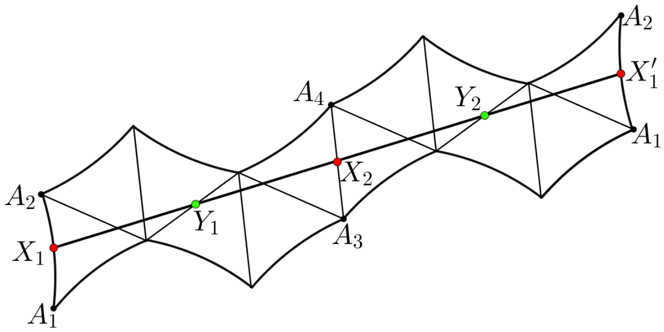

Fig. 5.4

In other words, we construct a polygon T on a hyperbolic plane that is formed by the same sequence of regular triangles as the polygon $\widetilde{T}$ on the Euclidean plane. Label the vertices of T according to the vertices of $\widetilde{T}$. Then the polygon T corresponds to a development of a regular tetrahedron with the planar angle α in hyperbolic space (see Fig. 5.4).

Moreover, T has the same property of central symmetry with respect to the midpoint of the same edge as the polygon $\widetilde{T}$. Denote by X_1, X_2, Y_1, Y_2, and X_1' the midpoints of the edges A_1A_2, A_3A_4, A_1A_3, and A_2A_4 of T, respectively. We draw the geodesic line segment X_1X_1'.

By the construction, the interior angles at the vertices of T are equal to α, 2α, 3α, or 4α, according to the development on the Euclidean plane.

First, assume that $\alpha \in (0, \pi/4]$. Then the polygon T is convex and the segment X_1X_1' lies inside T. Furthermore, X_1X_1' passes through the points X_2, Y_1, Y_2 that are the centers of symmetry of T. Therefore X_1X_1' is a simple closed geodesic γ on the regular tetrahedron with the planar angle $\alpha \in (0, \pi/4]$ in hyperbolic space.

Now we increase the angle α starting from $\alpha = \pi/4$. Then the polygon T is not convex because it contains the interior angles $4\alpha > \pi$.

Let α_0 be the supremum of α for which the segment X_1X_2 lies inside T. Suppose $\alpha_0 < \pi/3$. For all $\alpha < \alpha_0$, the segment X_1X_1' lies entirely inside T and it is a simple closed geodesic γ on the regular tetrahedron in $\mathbb{H}^3$. The distance d from the vertices of the tetrahedron to γ satisfies (5.2). Therefore, there exists $\alpha_1 = \alpha_0 + \varepsilon$ such that the segment X_1X_2 lies entirely inside T. This contradicts the maximality of α_0. Thus $\alpha_0 = \pi/3$.

It follows that for any $\alpha \in (0, \pi/3)$ there is a simple closed geodesic of type (p, q) on a regular tetrahedron with the planar angle α in hyperbolic space.

The uniqueness of a simple closed geodesic of type (p, q) on a regular tetrahedron in $\mathbb{H}^3$ follows from Corollary 5.4. This geodesic has p points on each of two opposite edges of the tetrahedron, q points on each of other two opposite edges, and $(p + q)$ points on each edge of the third pair of opposite edges. For any coprime integers (p, q), $0 \le p < q$, there exist three simple closed geodesics of type (p, q) on a regular tetrahedron in $\mathbb{H}^3$. They coincide if the tetrahedron is rotated by the angle $2\pi/3$ or $4\pi/3$ around the altitude constructed from a vertex to the opposite face.

Since any simple closed geodesic on a regular tetrahedron in $\mathbb{H}^3$ is equivalent to a simple closed geodesic on a regular tetrahedron in $\mathbb{E}^3$, there is not another simple closed geodesic on a regular tetrahedron in $\mathbb{H}^3$. □

5.4. The existence of a simple closed geodesic of type (p, q) on a generic tetrahedron. In Euclidean space $\mathbb{E}^3$, there is no simple closed geodesic on a generic tetrahedron. Protasov [41] gave an upper bound for the number of simple closed geodesics depending on the largest deviation from π of the sum of planar angles at the vertices of the tetrahedron. The situation in hyperbolic space is quite different provided that the planar angles of the tetrahedron are sufficiently small. Borisenko proved the following result.

Theorem 5.6 ([7])**.** *If the planar angles of a tetrahedron in hyperbolic space are at most $\pi/4$, then for any pair of coprime natural numbers (p, q) there exist a simple closed geodesics of type (p, q).*

Proof. Let $\widetilde{\gamma}$ be a simple closed geodesic on a regular tetrahedron $A_1A_2A_3A_4$ in Euclidean space. Consider the development $\widetilde{T}$ of the tetrahedron along $\widetilde{\gamma}$ from the point $\widetilde{X}_1$ on A_1A_2 to the point $\widetilde{X}_1'$.

Consider a generic tetrahderon in hyperbolic space. For more convenience, we can also label the vertices of the tetrahedron with A_1, A_2, A_3, and A_4. Develop this tetrahedron onto the hyperbolic plane in the same order as the development $\widetilde{T}$ is unfolded, starting from the edge A_1A_2.

As it was shown in the proof of Theorem 5.5, at most four faces can meet at one vertex of the development. Hence, if $\alpha \le \pi/4$, then the development is a convex polygon.

However, there are at most two faces meeting at each of the vertices A_1, A_2, A_1', and A_2', where A_1A_2 is a starting edge and $A_1'A_2'$ is a finishing edge. Therefore the angles at these vertices are at most $\pi/2$.

Consider the quadrilateral $A_1A_2A_2'A_1'$. Take the points $X(s)$ on A_1A_2 and $X'(s)$ on $A_1'A_2'$ such that $X(0) = A_1$, $X'(0) = A_1'$, and the lengths of $A_1X(s)$ and $A_1'X'(s)$ are both equal to s (Fig. 5.5).

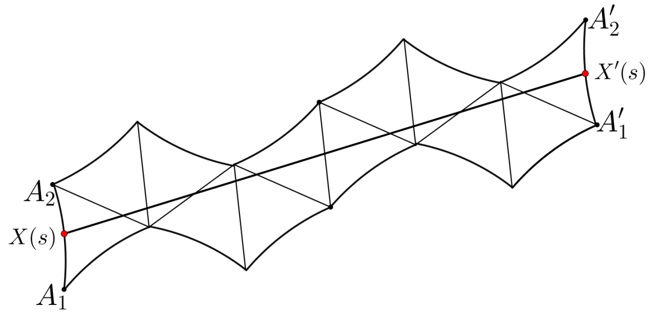

Fig. 5.5

For $s = 0$, the sum of the angles $\angle A_1$ and $\angle A_1'$ measured from inside the polygon is less than π. For $s = |A_1A_2|$, the sum of $\angle A_2$ and $\angle A_2'$ measured from outside the polygon is greater than π. Therefore, there is s_0 such that the sum of $\angle X(s_0)$ and $\angle X'(s_0)$ equals π. The line segment $X(s_0)X'(s_0)$ on the development corresponds to the simple closed geodesic of type (p, q) on the tetrahedron in $\mathbb{H}^3$.

Since for any ordered pair of coprime integers (p, q) there exist three simple closed geodesics of type (p, q) on a regular tetrahedron in $\mathbb{E}^3$, disregarding isometries of the tetrahedron, in a similar way, we can construct three simple closed geodesics of type (p, q) on a tetrahedron in $\mathbb{H}^3$ with the planar angle at most $\pi/4$. □

5.5. The number of simple closed geodesics. Let $N(L, \alpha)$ be the number of simple closed geodesics of length not greater than L on a regular tetrahedron with the planar angle α in hyperbolic space. In [9], it was shown that

$$N(L, \alpha) = c(\alpha)L^2 + O(L \ln L),$$

where $O(L \ln L) \le CL \ln L$ when $L \to +\infty$, and

$$c(\alpha) = \frac{27}{16\left(\ln \frac{1-\frac{\sqrt{3}}{2}\left(1-\frac{3\alpha}{\pi}\right)^3\left(1-\frac{\alpha^2}{4}\right)}{1-\frac{\sqrt{3}}{2}\left(1-\frac{3\alpha}{\pi}\right)^3\left(1+\frac{\alpha^2}{4}\right)} + \ln \frac{1+\frac{\sqrt{3}}{4}\left(1-\frac{3\alpha}{\pi}\right)}{1-\frac{\sqrt{3}}{4}\left(1-\frac{3\alpha}{\pi}\right)}\right)^2},$$

$$\lim_{\alpha \to \frac{\pi}{3}} c(\alpha) = +\infty; \quad \lim_{\alpha \to 0} c(\alpha) = \frac{27}{16\left(\ln \frac{1+\frac{\sqrt{3}}{4}}{1-\frac{\sqrt{3}}{4}}\right)^2}.$$

This result was proved using Proposition 3.5 about the structure of a simple closed geodesic on a regular tetrahedron.

In the current paper, we improve the constant $c(\alpha)$ by using the estimations obtained in [9].

Lemma 5.7. *If the length of a simple closed geodesic of type (p, q) on a regular tetrahedron in hyperbolic space is not greater than L, then*

$$L \geq 2(p+q) \ln \left(2\sqrt{3} \left(1 - \frac{3\alpha}{\pi} \right) + 1 \right),$$

where α is the plane angle of a face of the tetrahedron.

Proof. Let γ be a simple closed geodesic of type (p, q), $0 \leq q < p$, on a regular tetrahedron $A_1A_2A_3A_4$ in hyperbolic space.

Assume that γ has q points on the edges A_1A_2 and A_3A_4, p points on A_1A_4 and A_2A_3 and $p+q$ points on A_2A_4 and A_1A_3. Denote by $B_1, \ldots, B_{p+q}$ points of γ on A_1A_3 and by $B'_1, \ldots, B'_{p+q}$ points of γ on A_2A_4.

Consider the development of the faces $A_3A_1A_4$ and $A_1A_4A_2$ onto the plane. The geodesic segment starting at the point B_i, where $i = 1, \ldots, p$, goes through the edge A_1A_4 to the point B'_{q+i}. Analogously, on the development of the faces $A_1A_2A_3$ and $A_2A_3A_4$ there are p segments of γ connecting B'_i and B_{q+i}, $i = 1, \ldots, p$, and passing through the edge A_2A_3.

On the faces $A_4A_1A_2$ and $A_1A_2A_3$, the geodesic segments $B_iB'_{q-(i-1)}$, $i = 1, \ldots, q$, pass through the edge A_1A_2. Analogously, on the development of the faces $A_2A_4A_3$ and $A_4A_3A_1$ there are q geodesic segments $B_{p+i}B'_{(p+q)-(i-1)}$, $i = 1, \ldots, q$ (see Fig. 5.6).

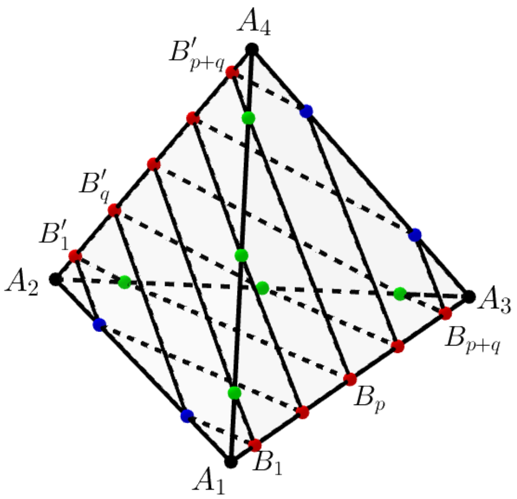

Fig. 5.6

Therefore, the geodesic γ consists of $2(p+q)$ segments that connect opposite edges of the tetrahedron. Let us evaluate from below the length of these segments. Consider the quadrilateral obtained by unfolding the faces $A_2A_1A_4$ and $A_1A_4A_3$. The minimum distance between the points on the edges A_2A_4 and A_1A_3 is achieved at H_1H_2 perpendicular to these edges. Since the planar angle of the tetrahedron $\alpha < \pi/3$, H_1H_2 lies inside the quadrilateral $A_3A_1A_4A_2$ and passes through the midpoint M of the edge A_1A_4 (see Fig. 5.7).

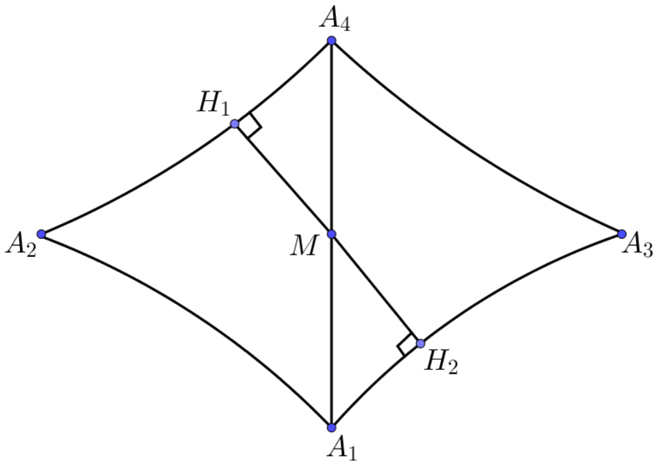

Fig. 5.7

From the triangle A_4MH_1, we have

$$\sinh|MH_1| = \sinh(a/2)\sin\alpha$$

Using (5.1), we get

$$\sinh|MH_1| = \cos(\alpha/2)\sqrt{2\cos\alpha - 1}.$$

Using

$$2\cos\alpha - 1 = \frac{\cos(3\alpha/2)}{\cos(\alpha/2)},$$

we obtain

$$\sinh|MH_1| = \sqrt{\cos(\alpha/2)\cos(3\alpha/2)}.$$

Inequality (5.7) together with $\cos\alpha/2 > \sqrt{3}/2$ implies

$$\sinh|MH_1| \geq \sqrt{\frac{\sqrt{3}}{2}\left(1 - \frac{3\alpha}{\pi}\right)}. \tag{5.9}$$

Consider the function arsinh(x):

$$2\text{arsinh(x)} = 2\ln\left(x + \sqrt{x^2+1}\right) = \ln(2x^2 + 1 + 2x\sqrt{x^2+1}) > \ln(4x^2+1).$$

This inequality implies

$$|H_1H_2| \geq \ln\left(2\sqrt{3}\,(1 - 3\alpha/\pi) + 1\right).$$

We obtain that the length L of a simple closed geodesic γ of type (p, q) satisfies

$$L \geq 2(p+q)\ln\left(2\sqrt{3}\,(1 - 3\alpha/\pi) + 1\right). \qquad \square$$

Euler's function $\phi(n)$ is equal to the number of positive integers not greater than n and prime to $n \in \mathbb{N}$. From [24, Theorem 330], we know that

$$\sum_{n=1}^{x} \phi(n) = \frac{3}{\pi^2}x^2 + O(x \ln x), \tag{5.10}$$

where $O(x \ln x) < Cx \ln x$ when $x \to +\infty$.

Denote by $\psi(x)$ the number of pairs of coprime integers (p, q) such that $p < q$ and $p + q \leq x$, $x \in \mathbb{R}$. Suppose $\hat{\psi}(y)$ is equal to the number of pairs of coprime integers (p, q) such that $p < q$ and $p + q = y$, $y \in \mathbb{N}$. From the definitions, we get

$$\psi(x) = \sum_{y=1}^{x} \hat{\psi}(y). \tag{5.11}$$

If $(p, q) = 1$ and $p + q = y$, then $(p, y) = 1$ and $(q, y) = 1$. Consider Euler's function $\phi(y)$. We obtain that the set of integers not greater than and prime to y are separated into the pairs of coprime integers (p, q) such that $p < q$ and $p + q = y$. It follows that $\phi(y)$ is even and $\hat{\psi}(y) = \phi(y)/2$. From (5.11), we have

$$\psi(x) = \frac{1}{2}\sum_{y=1}^{x} \phi(y).$$

Then (5.10) implies

$$\psi(x) = \frac{3}{2\pi^2}x^2 + O(x \ln x) \quad \text{as } x \to +\infty \tag{5.12}$$

The following result can be proved by using this asymptotic.

Theorem 5.8. *Let $N(L, \alpha)$ be the number of simple closed geodesics of length not greater than L on a regular tetrahedron with plane angles of the faces equal to α in hyperbolic space. Then*

$$N(L, \alpha) = c(\alpha)L^2 + O(L \ln L) \quad \textit{as } L \to +\infty, \tag{5.13}$$

where

$$c(\alpha) = \frac{9}{8\pi^2 \left(\ln\left(2\sqrt{3}\,(1 - 3\alpha/\pi) + 1\right)\right)^2},$$

$$\lim_{\alpha \to \frac{\pi}{3}} c(\alpha) = +\infty; \quad \lim_{\alpha \to 0} c(\alpha) = \frac{9}{8\pi^2 \ln\left(2\sqrt{3} + 1\right)}.$$

Proof. To each ordered pair of coprime integers (p, q), $p < q$, there correspond three different geodesics on the regular tetrahedron. We have

$$N(L, \alpha) = 3\psi\left(\frac{L}{2\ln\left(2\sqrt{3}\,(1 - 3\alpha/\pi) + 1\right)}\right).$$

Using (5.12), we get

$$N(L, \alpha) = \frac{9}{8\pi^2 \left(\ln\left(2\sqrt{3}\,(1 - 3\alpha/\pi) + 1\right)\right)^2}L^2 + O(L \ln L) \quad \text{as } L \to +\infty. \qquad \square$$

In [43], I. Rivin showed that for any hyperbolic structure on a sphere with n boundary components, the number of simple closed geodesics of length bounded by L on it grows like L^{2n-6} as $L \to \infty$.

From Lemma 5.2, we know that there is no simple closed geodesic on a regular tetrahedron on a distance $< d_0(\alpha)$, where $d_0(\alpha)$ is from (5.2). The estimation (5.2) holds also for a generic tetrahedron in hyperbolic space.

We can consider the tetrahedron as a non-compact surface with regular Riemannian metric of constant negative curvature with 4 boundary components. From Lemma 5.1, it follows that there are no simple closed geodesics that are boundary parallel. From (5.13), we get that the number $N(L, \alpha)$ is asymptotic to L^2 as $L \to +\infty$.

If the planar angle α of the tetrahedron goes to zero, then the vertices of the tetrahedron tend to infinity. The limiting tetrahedron is homeomorphic to a sphere with four cusps with a complete regular Riemannian metric of constant negative curvature. The genus of this surface is zero. In work of I. Rivin [43] it was shown that the number of simple closed geodesics on this surface has order of growth L^2. Thus the the number of simple closed geodesics of length at most L on a regular hyperbolic surface with four cusps and on a regular tetrahedron in hyperbolic space has order of grows L^2.

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Прості замкнені геодезичні на правильних тетраедрах у просторах постійної кривини

Darya Sukhorebska

У даному огляді представлені результати о поведінці простих замкнених геодезичних на правильних тетраедрах у тривимірних просторах постійної кривини.

Ключові слова: замкнені геодезичні, правильний тетраедр, простір Лобачевського, сферичний простір