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The Plasticity of Fittable Cones for a Given Quadruple of Points on the Surface of a Unit 2-sphere

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We construct a family of fittable cones for a given quadruple of points on a unit 2-sphere $S^2(1)$, which form a weighted Fermat–Torricelli (tree) network on $S^2(1)$, such that one of the four given points is the weighted Fermat–Torricelli point that has got a positive subconscious quantity (remaining weight). We describe five types of weighted Fermat–Torricelli trees located on these fittable cones, which depend on the sign of the subconscious quantity that corresponds to the same weighted Fermat–Torricelli point derived on $S^2(1)$ (plasticity of fittable cones).

Key words: weighted Fermat–Torricelli tree, sphere, circular cone, geodesic triangle

Mathematical Subject Classification 2010: 51K05, 52A15, 53A05, 51E10

1. Introduction

We start with the weighted Fermat–Torricelli problem for a geodesic quadrilateral $A_1A_2A_3A_4$ on a C^2 complete convex surface M with curvature bounded from above by K .

Problem 1.1 (Weighted Fermat–Torricelli problem for $A_1A_2A_3A_4$ on M). Let $A_1A_2A_3A_4 \subset M$ be a quadrilateral whose perimeter is less than $\frac{\pi}{2\sqrt{K}}$. Suppose that a positive number (weight) w_i, corresponds to the vertex A_i. Find a weighted Fermat–Torricelli point A_0 such that

$$f(A_0) = \sum_{i=1}^{4} w_i (a_{0i})_g \to \min, \tag{1.1}$$

where $(a_{0i})_g$ is the length of the shortest geodesic arc A_0A_i.

The solution of the weighted Fermat–Torricelli problem is the unique weighted Fermat–Torrcelli point A_0. The following lemmas give two characterizations of the weighted Fermat–Torricelli point A_0 with respect to the geometric structure of $A_1A_2A_3A_4$ and the four weights, which correspond to its vertices ([9]).

Let D be a domain of M bounded by $A_1A_2A_3A_4$, and let $\vec{U}_{A_iA_j}$ be the unit tangent vector of the geodesic arc A_iA_j at A_i :

$$\vec{U}_{A_iA_j} = \frac{\exp_{A_i}^{-1}(A_j)}{(a_{ij})_g} = \frac{\tilde{X}_{A_iA_j}}{\|\tilde{X}_{A_iA_j}\|}, \tag{1.2}$$

where $\vec{U}_{A_iA_j}$ belongs to the tangent plane $T_{A_i}(M)$ at A_i.

Lemma 1.2 (Floating Case [9, Proposition 4,p. 91]). *Conditions (i), (ii), and (iii) are equivalent on M:*

(i) *All the following inequalities are satisfied simultaneously:*

$$\left\| w_2\vec{U}_{A_1A_2} + w_3\vec{U}_{A_1A_3} + w_4\vec{U}_{A_1A_4} \right\| > w_1, \tag{1.3}$$

$$\left\| w_1\vec{U}_{A_2A_1} + w_3\vec{U}_{A_2A_3} + w_4\vec{U}_{A_2A_4} \right\| > w_2, \tag{1.4}$$

$$\left\| w_1\vec{U}_{A_3A_1} + w_2\vec{U}_{A_3A_2} + w_4\vec{U}_{A_3A_4} \right\| > w_3, \tag{1.5}$$

$$\left\| w_1\vec{U}_{A_4A_1} + w_2\vec{U}_{A_4A_2} + w_3\vec{U}_{A_4A_3} \right\| > w_4. \tag{1.6}$$

(ii) *The point A_0 is an interior point of $A_1A_2A_3A_4$ and does not belong to the geodesic arcs A_1A_2, A_2A_3, A_3A_4 and A_4A_1.*

(iii) $\sum_{i=1}^{4} w_i\vec{U}_{A_0A_i} = \vec{0}$.

Lemma 1.3 (Absorbed Case [9, Proposition 5,p. 91]). *Conditions (i) and (ii) are equivalent on M.*

(i) *One of the following inequalities is satisfied:*

$$\left\| w_2\vec{U}_{A_1A_2} + w_3\vec{U}_{A_1A_3} + w_4\vec{U}_{A_1A_4} \right\| \le w_1, \tag{1.7}$$

$$\left\| w_1\vec{U}_{A_2A_1} + w_3\vec{U}_{A_2A_3} + w_4\vec{U}_{A_2A_4} \right\| \le w_2, \tag{1.8}$$

$$\left\| w_1\vec{U}_{A_3A_1} + w_2\vec{U}_{A_3A_2} + w_4\vec{U}_{A_3A_4} \right\| \le w_3, \tag{1.9}$$

$$\left\| w_1\vec{U}_{A_4A_1} + w_2\vec{U}_{A_4A_2} + w_3\vec{U}_{A_4A_3} \right\| \le w_4. \tag{1.10}$$

(ii) *The point A_0 is attained at A_1 or A_2 or A_3 or A_4, respectively.*

The solution of the weighted Fermat–Torricelli problem for $A_1A_2A_3A_4$ with respect to Lemma 1.1 yields a weighted floating Fermat–Torricelli tree, which consists of the geodesic arcs $\{A_1A_0, A_2A_0, A_3A_0, A_4A_0\}$.

The solution of the weighted Fermat–Torricelli problem for $A_1A_2A_3A_4$ with respect to Lemma 1.2 yields a weighted absorbed Fermat–Torricelli tree, which consists of the geodesic arcs $\{A_iA_0, A_jA_0, A_kA_0\}$, for $A_l \equiv A_0$, for $i, j, k, l = 1, 2, 3, 4$, $i \neq j \neq k \neq l$, such that

$$\left\| w_i\vec{U}_{A_lA_i} + w_j\vec{U}_{A_lA_j} + w_k\vec{U}_{A_lA_k} \right\| \le w_l.$$

By setting $w_4 = 0$ in Lemma 1.2, the solution of the weighted Fermat–Torricelli problem for $A_1A_2A_3A_4$ yields a weighted floating Fermat–Torricelli tree, which consists of the geodesic arcs $\{A_1A_0, A_2A_0, A_3A_0\}$.

By setting $w_k = 0$ in Lemma 1.3, the solution of the weighted Fermat–Torricelli problem for $A_1A_2A_3A_4$ yields a weighted absorbed Fermat–Torricelli tree, which consists of the geodesic arcs $\{A_iA_0, A_jA_0\}$ $A_l \equiv A_0$, for $i, j, l = 1, 2, 3, 4$, $i \neq j \neq l$, such that

$$\left\| w_i \vec{U}_{A_l A_i} + w_j \vec{U}_{A_l A_j} \right\| \leq w_l.$$

In 2014, we introduced a problem of the plasticity of fittable surfaces on a given quadruple of points in $\mathbb{R}^3$ in [7].

The problem of plasticity of fittable surfaces in $\mathbb{R}^3$ states the following.

Suppose that F is the corresponding weighted Fermat–Torricelli point of a geodesic triangle $\triangle A_1A_2A_3$ on a C^2 complete surface M with weights w_1, w_2 and w_3. Find a fittable Alexandrov surface M' of a bounded curvature which passes A_1, A_2 A_3 and F such that F is the corresponding weighted Fermat–Torricelli point of $\triangle A_1A_2A_3$ on M' with weights w'_1, w'_2 and w'_3.

The problem is solved for the case of a fittable sphere and a circular cone on a circular cylinder in [7].

Euclidean surfaces having conical singularities are given in [2]. Troyanov established metrics on a sphere with two conical singularities in [3], and Umehara and Yamada extended these studies for metrics on a sphere with three conical singularities in [5].

In [1], Ivanov and Tuzhilin investigated the behavior of shortest networks under deformation of their boundary sets. They proved that the analyticity of the boundary set guarantees preserving of the network type for minimal spanning trees.

In this paper, we describe five types of shortest networks (weighted Fermat–Torricelli trees) for a fixed boundary quad of points on a unit 2-sphere $S^2(1)$, which are located on fittable circular cones (deformation of the metric) that pass through the same quad of points (Section 3, Theorems 3.4–3.8). The characterization of these types of networks depends on the sign of the subconscious quantity that remains at the same weighted Fermat–Torricelli (node) on these fittable cones (Plasticity of fittable cones).

2. Extrinsic geodesic flow along some weighted Fermat–Torricelli trees that have got a subconscious on $S^2(1)$

Let $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$ be the vertices of an equilateral geodesic triangle $\triangle A_1A_2A_3$, and let A_0 be an interior point of $\triangle A_1A_2A_3$ on the unit sphere $S^2(1)$.

We denote by $(a_{ij})_{S^2(1)}$ the length of the geodesic arc A_iA_j, which is part of a great circle of unit radius, by $\vec{U}_{A_iA_j}$ the unit tangent vector of A_iA_j at A_i, and by α_{ikj} the angle between A_iA_k and A_kA_j for $i, j, k = 0, 1, 2, 3$, $i \neq j \neq k$.

The weighted Fermat–Torricelli problem for $\triangle A_1A_2A_3$ on $S^2(1)$ states the following.

Problem 2.1. Find a point A_0 (weighted Fermat–Torricelli point) such that

$$\sum_{i=1}^{3} w_i (a_{0i})_{S^2(1)} \to \min. \tag{2.1}$$

If $\left\|w_i\vec{U}_{A_jA_i}+w_k\vec{U}_{A_jA_k}\right\| > w_j$, for $i,j,k=1,2,3$, $i\neq j\neq k$, then the weighted Fermat–Torricelli point A_0 is an interior point of $\triangle A_1A_2A_3$ and the solution of the weighted Fermat–Torricelli problem consists of three geodesic arcs $\{A_1A_0, A_2A_0, A_3A_0\}$, which intersect at A_0 (weighted floating Fermat–Torricelli tree, [10, Proposition 2]).

If $\left\|w_i\vec{U}_{A_jA_i}+w_k\vec{U}_{A_jA_k}\right\| < w_j$, for $i,j,k=1,2,3$, $i\neq j\neq k$, then the weighted Fermat–Torricelli point A_0 is the vertex A_j of $\triangle A_1A_2A_3$ and the solution of the weighted Fermat–Torricelli problem consists of two geodesic arcs $\{A_iA_j, A_kA_j\}$, which intersect at A_j, (weighted absorbing Fermat–Torricelli tree, [10, Proposition 2]).

The inverse weighted Fermat–Torricelli problem for $\triangle A_1A_2A_3$ on $S^2(1)$ states the following.

Problem 2.2. Given a point A_0, which belongs to the interior of $\triangle A_1A_2A_3$ on $S^2(1)$. The question is whether there exists a unique set of positive weights $\{w_1, w_2, w_3\}$ such that

$$w_1 + w_2 + w_3 = 1,$$

for which A_0 minimizes

$$f(A_0) = \sum_{i=1}^{3} w_i (a_{0i})_{S^2(1)}.$$

Lemma 2.3 ([7, Lemma 3, p. 488]). *The solution of the inverse weighted Fermat–Torricelli problem for $\triangle A_1A_2A_3$ on $S^2(1)$ is given by*

$$w_i = \frac{1}{1+\frac{\sin\alpha_{i0j}}{\sin\alpha_{j0k}}+\frac{\sin\alpha_{i0k}}{\sin\alpha_{j0k}}} \tag{2.2}$$

for $i,j,k=1,2,3$, $i\neq j\neq k$.

The idea of assigning a residual weight (subconscious) at a weighted Fermat–Torricelli point (generalized Fermat–Torricelli point) is given in [11]. It is assumed that a weighted Fermat–Torricelli tree is a two-way communication network and the weights w_1, w_2, w_3 are three small masses that may move through the branches of the weighted Fermat–Torricelli tree. By assuming mass flow continuity of this network, we obtain the generalized inverse weighted Fermat–Torricelli problem (INVSFT problem).

The INVSFT problem is the inverse weighted Fermat–Torricelli problem such that the weighted Fermat–Torricelli point acquires a subconscious quantity w_0.

We denote by w_i a mass flow, which is transferred from A_i to A_0 for $i = 1, 2$, by w_0 a residual weight, which remains at A_0, by w_3 a mass flow, which is transferred from A_0 to A_3, by $\tilde{w_i}$ a mass flow, which is transferred from A_0 to A_i, $i = 1, 2$, by $\tilde{w_0}$ a residual weight, which remains at A_0, and by $\tilde{w_3}$ a mass flow, which is transferred from A_3 to A_0.

The following equations are derived by this mass flow along the geodesic arcs A_1A_0, A_2A_0, A_3A_0 :

$$w_1 + w_2 = w_3 + w_0, \tag{2.3}$$

$$\tilde{w}_1 + \tilde{w}_2 + \tilde{w}_0 = \tilde{w}_3. \tag{2.4}$$

By taking into account (2.3) and (2.4) and by setting $\bar{w}_i = w_i - \tilde{w}_i$, for $i = 0, 1, 2, 3$, we get

$$\bar{w}_1 + \bar{w}_2 = \bar{w}_3 + \bar{w}_0 \tag{2.5}$$

such that

$$\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = c > 0. \tag{2.6}$$

Problem 2.4. Given a point A_0, which belongs to the interior of $\triangle A_1A_2A_3$ on $S^2(1)$. The question is whether there exists a unique set of positive weights $\bar{w}_i$ such that

$$\bar{w}_1 + \bar{w}_2 + \bar{w}_3 = c, \tag{2.7}$$

for which A_0 minimizes

$$f(A_0) = w_1(a_{01})_{S^2(1)} + w_2(a_{02})_{S^2(1)} + w_3(a_{03})_{S^2(1)},$$

$$f(A_0) = \tilde{w}_1(a_{01})_{S^2(1)} + \tilde{w}_2(a_{02})_{S^2(1)} + \tilde{w}_3(a_{03})_{S^2(1)},$$

$$f(A_0) = \bar{w}_1(a_{01})_{S^2(1)} + \bar{w}_2(a_{02})_{S^2(1)} + \bar{w}_3(a_{03})_{S^2(1)}.$$

$$w_i + \tilde{w}_i = \bar{w}_i \tag{2.8}$$

under the condition for the weights

$$\bar{w}_i + \bar{w}_j = \bar{w}_0 + \bar{w}_k \tag{2.9}$$

for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

Theorem 2.5. *If the g.FT point A_0 is an interior point of the triangle $\triangle A_1A_2A_3$ with the vertices lying on three geodesic arcs that meet at A_0 and from the two given values of α_{103}, α_{102}, then the positive real weights*

$$\bar{w}_1 = -\left(\frac{\sin(\alpha_{103} + \alpha_{102})}{\sin\alpha_{102}}\right)\frac{c - \bar{w}_0}{2}, \tag{2.10}$$

$$\bar{w}_2 = \left(\frac{\sin\alpha_{103}}{\sin\alpha_{102}}\right)\frac{c - \bar{w}_0}{2}, \tag{2.11}$$

$$\bar{w}_3 = \frac{c - \bar{w}_0}{2} \tag{2.12}$$

give a negative answer with respect to the inverse s.FT problem on $S^2(1)$.

We note that Theorem 2.5 is proved in [11] for the case of $\mathbb{R}^2$. Without loss of generality, we may use the normalized unit constant $c/\sum_{i=1}^{3}\bar{w}_i$ instead of c.

By expressing $A_0 \in S^2(1)$ in terms of the spherical coordinates (ω, φ), $A_0 = (\cos\omega\cos\varphi, \cos\omega\sin\varphi, \sin\omega)$, we found the exact position of the weighted Fermat–Torricelli tree on $S^2(1)$ in ([10]). By replacing the weight w_i with $\bar{w}_i$, the exact position of A_0 depends on the extrinsic geodesic flow produced by $\bar{w}_1$, $\bar{w}_2$, $\bar{w}_3$, and $\bar{w}_0$.

The exact location of A_0 is given by

$$\varphi = \arccos\left(\sqrt{\frac{w_1^2 + w_3^2 - w_2^2}{2w_3^2}}\right) \tag{2.13}$$

and

$$\omega = \arccos\left(\sqrt{\frac{w_1^2 + w_2^2 - w_3^2}{2w_1 w_2\sqrt{(1-(\frac{w_1^2-w_2^2-w_3^2}{2w_2w_3})^2)(1-(\frac{w_2^2-w_1^2-w_3^2}{2w_1w_3})^2)}}}\right). \tag{2.14}$$

Lemma 2.6 ([10, Corollary 1]). *If $w_1 = w_2 = w_3$, then $\varphi = \frac{\pi}{4}$, $\omega = \arccos\sqrt{\frac{2}{3}}$ and $A_0 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.*

3. The plasticity of fittable circular cones for a given quadruple of points on $S^2(1)$

It is well known that for every three non-collinear (not located on a great circle) points $\in \{A_1, A_2, A_3, A_0,\}$ there is one circle on the unit 2-sphere $S^2(1)$ passing through them. Hence, four "small" circles pass through A_1, A_2, A_3, A_0 located on $S^2(1)$. The constructions on the sphere using a spherical compasses are given in [4].

Theorem 3.1. *Four circular cones pass through A_1, A_2, A_3, A_0 such that each triad of these points determines their base circles.*

Proof. We use a spherical compass to construct each center of the four circles, which pass through A_1, A_2, A_3, A_0. The intersection of two geodesics orthogonal to two sides in their midpoints is the center of the circle on $S^2(1)$, (the same as in Euclidean geometry). A circular cone passes through this circle and contains its interior. There are four circles on $S^2(1)$ passing through them. Hence, there are four circular cones in R^3 that pass across these four circles. □

We consider the parametric and Cartesian forms of a right circular cone $S'(r_1, H)$ with a base circle $C(M, r_1)$ with center M, base radius r_1, vertex A, height H,

$$\vec{r}(u,v) = \left(r_1\left(1 - \frac{u}{H}\right)\cos v + x_0, r_1\left(1 - \frac{u}{H}\right)\sin v + y_0, u\right),$$

$$(x-x_0)^2+(y-y_0)^2=\frac{r_1^2}{H^2}(z-H)^2$$

for $0<u, 0<v<2\pi, r_1>0, M=(x_0,y_0), A=(x_0,x_0,H)$.

Theorem 3.2. *If $H>1$, then there exists a circular cone, which passes through A_1, A_2, A_3, A_0 such that A_1, A_2 lie on their base circles.*

Proof. We have

$$(1-x_0)^2+y_0^2=r_1^2, \tag{3.1}$$

$$x_0^2+(1-y_0)^2=r_1^2, \tag{3.2}$$

$$x_0^2+y_0^2=\left(\frac{r_1}{H}\right)^2(H-1)^2, \tag{3.3}$$

$$(x_F-x_0)^2+(y_F-y_0)^2=\left(\frac{r_1}{H}\right)^2(z_F-H)^2, \tag{3.4}$$

where

$$x_F^2+y_F^2+z_F^2=1. \tag{3.5}$$

By subtracting (3.1) from (3.2), we get $x_0=y_0$. By substituting $x_0=y_0$ in (3.2), (3.3) and (3.4), we get:

$$2x_0^2-2x_0+1=r_1^2, \tag{3.6}$$

$$2x_0^2=(\frac{r_1}{H})^2(H-1)^2, \tag{3.7}$$

$$(x_F-x_0)^2+(y_F-x_0)^2=(\frac{r_1}{H})^2(z_F-H)^2. \tag{3.8}$$

We distinguish two cases.

A. If $x_0>0$, then (3.7) yields

$$\sqrt{2}x_0=\frac{r_1}{H}(H-1). \tag{3.9}$$

By substituting (3.9) in (3.6), we get

$$\frac{r_1}{H}=\frac{\sqrt{2}}{2}\frac{H-1-\sqrt{H^2+2H-1}}{1-2H}>0. \tag{3.10}$$

By substituting (3.10), (3.9) in (3.8), we obtain

$$H=1+\frac{x_F^2+y_F^2+(\frac{\sqrt{2}}{2}\frac{H-1-\sqrt{H^2+2H-1}}{1-2H})^2(z_F-1)(2H-1-z_F)}{2(x_F+y_F)(\frac{H-1-\sqrt{H^2+2H-1}}{1-2H})}, \tag{3.11}$$

which yields $H-f(H)=0$, where f is a rational (continuous) function with respect to H.

Taking into account that $1-f(1)<0$, $10-f(10)>0$, $(1-f(1))(10-f(10))<0$, by Bolzano's theorem, there exists $H_0\in(1,10): H_0=f(H_0)$.

B. If $x_0 < 0$, then (3.7) yields

$$\sqrt{2}x_0 = \frac{r_1}{H}(1-H). \tag{3.12}$$

By substituting (3.12) in (3.6), we get

$$\frac{r_1}{H} = \frac{\sqrt{2}}{2}\frac{1-H-\sqrt{H^2+2H-1}}{1-2H} > 0. \tag{3.13}$$

By substituting (3.13), (3.12) in (3.8), we obtain

$$H = 1 + \frac{x_F^2 + y_F^2 + (\frac{\sqrt{2}}{2}\frac{1-H-\sqrt{H^2+2H-1}}{1-2H})^2(z_F-1)(2H-1-z_F)}{2(x_F+y_F)(\frac{1-H-\sqrt{H^2+2H-1}}{1-2H})}. \tag{3.14}$$

which yields $H - g(H) = 0$, where g is a rational (continuous) function with respect to H.

In a similar way, by using Bolzano's theorem, we can find a suitable $H_0' \in [1,10] : H_0' = g(H_0')$.

The theorem is proved. □

Theorem 3.3. *An infinite number of circular cones pass through A_1, A_2, A_3, A_0 such that each triad of these points does not determine their base circles.*

Proof. Let Π_1 be a given plane $a_1x+b_1y+c_1z+d_1=0$, which passes through the points A_1, A_2 and a circle $C(M_1(x_0,y_0,z_0),r_1)$ with center $M_1(x_0,y_0,z_0) \in \Pi_1$. The points A_1, A_3, A_0 define the plane $\Pi_2 : a_2x+b_2y+c_2z+d_2=0$. Thus, there is an ellipse, which passes through A_1, A_3, A_F, having as a projection the circle $C(M_1(x_0,y_0,z_0),r_1)$. This construction leads to a fittable circular cone for A_1, A_2, A_3, A_0 on $S^2(1)$. Thus, taking an infinite number of planes, which pass through A_1, A_2, one yields an infinite number of fittable cones to the quadruple of points $\{A_1, A_2, A_3, A_0\}$. □

There are five different types of weighted Fermat–Torricelli trees located on a circular cone, which fits the weighted (floating) Fermat–Torricelli tree $\{(A_1A_0)_{S^2(1)}, (A_2A_0)_{S^2(1)}, (A_3A_0)_{S^2(1)},\}$ at a given quad of points of $\{A_1, A_2, A_3, A_0\}$ on $S^2(1)$, or the weighted (floating) Fermat–Torricelli tree $\{(A_1'A_0)_{S^2(1)}, (A_2'A_0)_{S^2(1)}, (A_3'A_0)_{S^2(1)},\}$ at a given quad of points of $\{A_1', A_2', A_3', A_0\}$ on $S^2(1)$ such that each point lies on a geodesic arc A_0A_i for $i = 1,2,3$. The type of weighted Fermat–Torricelli trees located on these fittable cones depends on the sign of the subconscious quantity that corresponds to the weighted Fermat–Torricelli point A_0.

We proceed by giving notations for the weights that correspond to the points A_1, A_2, A_2, A_4 and a description of mass flow, by assuming mass flow continuity along the geodesics on a fittable cone C. We denote by $(w_i)_C$ a mass flow, which is transferred from A_i to A_0, by (w_0) a residual weight, which remains at A_0, by $(w_j)_C$ a mass flow, which is transferred from A_0 to A_k, by $(\tilde{w}_i)_C$ a mass flow, which is transferred from A_0 to A_i, $i = 1,2$, by $(\tilde{w}_0)_C$ a residual weight, which remains at A_0, and by $(\tilde{w}_k)_C$ a mass flow, which is transferred from A_k to A_0, for $i, j, k = 1, 2, 3$, $i \neq j \neq k$.

By setting $(\bar{w}_i)_C = (w_i)_C - (\tilde{w}_i)_C$ for $i = 0, 1, 2, 3$, we obtain (3.25) and (3.26).

Theorem 3.4 (Type A. Positive subconscious of a weighted minimum geodesic network on C)**.** *If the g.FT point A_0 is an interior point of $(\triangle A_1A_2A_3)_C$ with the vertices lying on three geodesic arcs that meet at A_0, then the dynamic plasticity equations of $\{A_1, A_2, A_3, A_0\}$ are given by*

$$(\bar{w}_1)_C = -\left(\frac{\sin((\alpha_{103})_C + (\alpha_{102})_C)}{\sin(\alpha_{102})_C}\right)\frac{c - (\bar{w}_0)_C}{2}, \tag{3.15}$$

$$(\bar{w}_2)_C = \left(\frac{(\sin\alpha_{103})_C}{\sin(\alpha_{102})_C}\right)\frac{c - (\bar{w}_0)_C}{2}, \tag{3.16}$$

$$(\bar{w}_3)_C = \frac{c - (\bar{w}_0)_C}{2}, \tag{3.17}$$

where

$$(\bar{w}_i)_C + (\bar{w}_j)_C = (\bar{w}_0)_C + (\bar{w}_k)_C \tag{3.18}$$

for $i, j, k = 1, 2, 3$, $i \neq j \neq k$, and

$$(\bar{w}_1)_C + (\bar{w}_2)_C + (\bar{w}_3)_C = c. \tag{3.19}$$

Proof. The solution of the INVSFT problem for $\triangle A_1A_2A_3$ on a circular cone is given by (3.22), (3.23) and (3.24), by substituting the weight $(\bar{w}_i)_C \to w_i$ and taking into account the weighted condition of mass flow continuity (3.25) and (3.26). □

Theorem 3.5 (Type B. Positive subconscious of a weighted minimum geodesic network on C)**.** *If the g.FT point A_0 is an exterior point of $(\triangle A_1A_2A_3)_C$ with the vertices lying on three geodesic arcs that meet at A_0, then the dynamic plasticity of $\{A_1, A_2, A_3, A_0\}$ is given by*

$$\begin{aligned}((\bar{w}_0)_C)^2 \geq \sum_{i=1}^{3}((\bar{w}_i)_C)^2 &+ 2(\bar{w}_1)_C(\bar{w}_2)_C\cos(\alpha_{102})_C \\ &+ 2(\bar{w}_1)_C(\bar{w}_3)_C\cos(\alpha_{103})_C + 2(\bar{w}_2)_C(\bar{w}_3)_C\cos(\alpha_{203})_C,\end{aligned} \tag{3.20}$$

and

$$\sum_{i=1}^{3}(\bar{w}_i)_C = (\bar{w}_0)_C + (\bar{w}_{0'})_C. \tag{3.21}$$

Proof. Assume that we select four weights $(\bar{w}_i)_C(0) \equiv (\bar{w}_i)_C(0)$ such that (1.10) holds, by substituting the weight $(\bar{w}_0)_C \to (\bar{w}_4)_C$. Thus, we get

$$\left\|\bar{w}_1\vec{U}_{A_0A_1} + \bar{w}_2\vec{U}_{A_0A_2} + \bar{w}_3\vec{U}_{A_0A_3}\right\| \leq \bar{w}_0,$$

which yields (3.20). Hence, the weighted Fermat–Torricelli point $A_{0'} \equiv A_0$. If we select as a new weight $(\bar{w}_0)_C + (\bar{w}_{0'})_C$ the weight, which corresponds to the point A_0, then the weighted absorbed inequality holds (1.10), by substituting the weight $(\bar{w}_0)_C + (\bar{w}_{0'})_C \to ((\bar{w}_4)_C$. Therefore, $A_{0'} \equiv A_0$. The positive subconscious quantity is defined by the weight $((\bar{w}_0)_C + (\bar{w}_{0'})_C$ that may remain at A_0, taking into account a two-way communication network described by the weighted condition (3.21). □

We denote by P the center of the base circle of C, with radius r_1, by A the vertex of C, by A_{ip} the intersection of the line defined by the line segment AA_i with the base circle $c(P, r_1)$, by φ_i the angle $\angle A_1 P A_{ip}$, and we set $(x_i)_g \equiv A_i P$ for $i = 0, 1, 2, 3$. We assume that $\varphi_0 > \pi$. Let $A_{0'}$ be a point on C, which is derived by rotating A_0 with respect to AP by $\varphi_0 + \pi$.

Theorem 3.6 (Type C. Negative subconscious of a weighted minimum geodesic network on C)**.** *If the g.FT point $A_{0'}$ is an interior point of $(\triangle A_1A_2A_3)_C$ with the vertices lying on three geodesic arcs that meet at A_0, then the dynamic plasticity equations of $\{A_1, A_2, A_3, A_{0'}\}$ are given by*

$$(\bar{w}_1)_C = -\left(\frac{\sin((\alpha_{10'3})_C + (\alpha_{10'2})_C)}{\sin(\alpha_{10'2})_C}\right)\frac{c - (\bar{w}_{0'})_C}{2}, \tag{3.22}$$

$$(\bar{w}_2)_C = \left(\frac{(\sin\alpha_{10'3})_C}{\sin(\alpha_{10'2})_C}\right)\frac{c - (\bar{w}_{0'})_C}{2}, \tag{3.23}$$

$$(\bar{w}_3)_C = \frac{c - (\bar{w}_{0'})_C}{2}, \tag{3.24}$$

where

$$(\bar{w}_i)_C + (\bar{w}_j)_C = (\bar{w}_{0'})_C + (\bar{w}_k)_C \tag{3.25}$$

for $i, j, k = 1, 2, 3$, $i \neq j \neq k$, and

$$(\bar{w}_1)_C + (\bar{w}_2)_C + (\bar{w}_3)_C = c. \tag{3.26}$$

Proof. It is a direct consequence of Theorem 3.4, taking into account that a negative subconscious $(\bar{w}_0)_C$ quantity remains at the point A_0 and changes to a positive subconscious quantity $(\bar{w}_{0'})_C$ that remains at the point $A_{0'}$. □

Theorem 3.7 (Type D. Negative subconscious of a weighted minimum geodesic network on C)**.** *If $A_{0'}$ is an exterior point of $(\triangle A_1A_2A_3)_C$ with the vertices lying on three geodesic arcs that meet at A_0, then the dynamic plasticity of $\{A_1, A_2, A_3, A_{0'}\}$ is given by*

$$\begin{aligned}((\bar{w}_{0'})_C)^2 \geq \sum_{i=1}^{3}((\bar{w}_i)_C)^2 &+ 2(\bar{w}_1)_C(\bar{w}_2)_C\cos(\alpha_{102})_C \\ &+ 2(\bar{w}_1)_C(\bar{w}_3)_C\cos(\alpha_{103})_C + 2(\bar{w}_2)_C(\bar{w}_3)_C\cos(\alpha_{203})_C\end{aligned} \tag{3.27}$$

and

$$\sum_{i=1}^{3}((\bar{w}_i)_C = (\bar{w}_0)_C + (\bar{w}_{0''})_C.$$

Proof. It is a direct consequence of Theorem 3.5, taking into account that a negative subconscious $(\bar{w}_0)_C$ quantity remains at the point A_0 and changes to a positive subconscious quantity $(\bar{w}_{0'})_C + (\bar{w}_{0''})_C$ that remains at the point $A_{0'}$. □

Theorem 3.8 (Type E. Vanishing subconscious of a weighted minimum geodesic network on C)**.** *If A_0, A_i, A_j determine a base circle of a fittable cone C, and A_k belongs to C, then $(\bar{w}_k)_C \equiv 0$ and A_0 is the weighted Fermat–Torricelli point of the geodesic arc A_iA_j with vanishing subconscious $(\bar{w}_0)_C \equiv (\bar{w}_0)_{S^2(1)}$, such that $(\bar{w}_i)_C \equiv (\bar{w}_i)_{S^2(1)}$, $(\bar{w}_j)_C \equiv (\bar{w}_j)_{S^2(1)}$, $(\bar{w}_i)_C, (\bar{w}_j)_C < \frac{1}{2}$, and*

$$(\bar{w}_1)_C + (\bar{w}_2)_C + (\bar{w}_3)_C = 1.$$

Proof. We show that the weighted Fermat–Torricelli point A_0 cannot lie on the side (geodesic arc) A_iA_j of a geodesic triangle $\triangle A_i(A_0)A_jA_k$ on C for $\bar{w}_i, \bar{w}_j, \bar{w}_k > 0$. If A_0 is an interior point of $\triangle A_iA_jA_k$, then the weighted condition (III) for $w_4 = 0$ of the floating case (Lemma 1.2) yields

$$\cos\angle A_iA_0A_j = \frac{(\bar{w}_k)_C^2 - (\bar{w}_i)_C^2 - (\bar{w}_j)_C^2}{2(\bar{w}_i)_C(\bar{w}_j)_C^2}.$$

If A_0 lies on the geodesic arc A_iA_j, then, by substituting $\angle A_iA_0A_j = \pi$ in the weighted cosine condition, we get

$$(\bar{w}_i)_C = (\bar{w}_k)_C + (\bar{w}_j)_C$$

for $(\bar{w}_i)_C > (\bar{w}_j)_C$, or

$$(\bar{w}_j)_C = (\bar{w}_k)_C + (\bar{w}_i)_C$$

for $(\bar{w}_j)_C > (\bar{w}_i)_C$. Thus, $A_0 \equiv A_i$ or $A_0 \equiv A_j$. Therefore, if A_0 is a weighted Fermat–Torricelli point, which lies on A_iA_j, we get $(\bar{w}_k)_C = 0$.

Therefore, by substituting $(\bar{w}_k)_C \equiv 0$, we may locate A_0 at the geodesic arc $A_i\ A_j$. By setting $(\bar{w}_0)_C \equiv (\bar{w}_0)_{S^2(1)}$, $(\bar{w}_i)_C \equiv (\bar{w}_i)_{S^2(1)} < \frac{1}{2}$, $(\bar{w}_j)_C \equiv (\bar{w}_j)_{S^2(1)} < \frac{1}{2}$, A_0 is the weighted Fermat–Torricelli point of A_iA_j with the corresponding weight $(\bar{w}_0)_{S^2(1)}$ such that

$$(\bar{w}_1)_{S^2(1)} + (\bar{w}_2)_{S^2(1)} + (\bar{w}_3)_{S^2(1)} = 1. \qquad \square$$

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Пластичність пристосованих конусів для заданої четвірки точок на поверхні одиничної 2-сфери

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Ми будуємо сім'ю пристосованих конусів для заданої четвірки точок на одиничній 2-сфері $S^2(1)$, що утворюють зважену мережу (дерево) Ферма–Торрічеллі на $S^2(1)$, таких, що одна з чотирьох заданих точок є зваженою точкою Ферма–Торрічеллі з додатною підсвідомою величиною (залишковою вагою). Ми описуємо п'ять типів зважених дерев Ферма–Торрічеллі, розташованих на пристосованих конусах, у залежності від знаку підсвідомої величини, що відповідає тій же самій зваженій точці Ферма–Торрічеллі, обчисленій на $S^2(1)$ (пластичність пристосованих конусів).

Ключові слова: зважене дерево Ферма–Торрічеллі, сфера, круговий конус, геодезичний трикутник