Journal of Mathematical Physics, Analysis, Geometry
2023, Vol. 19, No. 1, pp. 3–58
doi: https://doi.org/10.15407/mag19.01.003

The Generalized Marchenko Method in the Inverse Scattering Problem for a First-Order Linear System with Energy-Dependent Potentials

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Dedicated to Professor Vladimir A. Marchenko for his 100th birthday

The Marchenko method is developed in the inverse scattering problem for a linear system of first-order differential equations containing potentials proportional to the spectral parameter. The corresponding Marchenko system of integral equations is derived in such a way that the method can be applied to some other linear systems for which a Marchenko method is not yet available. It is shown how the potentials and the Jost solutions to the linear system are constructed from the solution to the Marchenko system. The bound-state information for the linear system with any number of bound states and any multiplicities is described in terms of a pair of constant matrix triplets. When the potentials in the linear system are reflectionless, some explicit solution formulas are presented in closed form for the potentials and for the Jost solutions to the linear system. The theory is illustrated with some explicit examples.

Key words: Marchenko method, generalized Marchenko integral equation, inverse scattering, first-order linear system, energy-dependent potentials, Jost solutions

Mathematical Subject Classification 2020: 34A55, 34L25, 34L40, 47A40

1. Introduction

Our main goal in this paper is to develop the Marchenko method for the linear system of ordinary differential equations

$$\frac{d}{dx}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-i\zeta^2 & \zeta\, q(x)\\ \zeta\, r(x) & i\zeta^2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \quad -\infty<x<+\infty, \tag{1.1}$$

where x is the spacial coordinate, ζ is the spectral parameter, the scalar functions q and r are some complex-valued potentials, and the column vector $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is the wavefunction depending on x and ζ. We assume that the potentials q and r belong to the Schwartz class, i.e. the class of functions of x on the real axis $\mathbb{R}$ for which the derivatives of all orders exist and all those derivatives decay faster than any negative power of x as $x \to \pm\infty$. Even though our results hold for potentials satisfying weaker restrictions, in order to provide insight into the development of the Marchenko method, for simplicity and clarity we assume that the potentials belong to the Schwartz class.

The linear system (1.1), when the potentials q and r depend also on the additional parameter t, is associated with the first-order system of nonlinear partial differential equations given by

$$\begin{cases} iq_t + q_{xx} - i(qrq)_x = 0, \\ ir_t - r_{xx} - i(rqr)_x = 0, \end{cases} \qquad x \in \mathbb{R}, \ t > 0, \tag{1.2}$$

where the subscripts denote the appropriate partial derivatives. The nonlinear system (1.2) is known [1, 3, 25, 33] as the derivative NLS (nonlinear Schrödinger) system or as the Kaup–Newell system. The derivative NLS equations have important physical applications in plasma physics, propagation of hydromagnetic waves traveling in a magnetic field, and transmission of ultra short nonlinear pulses in optical fibers [1, 25]. Hence, the study of (1.1) is physically relevant, and the development of the Marchenko method for (1.1) is significant. We remark that our concentration in this paper is not on integrable nonlinear systems such as (1.2) but rather on the linear system (1.1). We refer the reader to [10] for the use of the Marchenko method to solve the initial value problem for (1.2) via the inverse scattering transform method [24] and also for some explicit solution formulas for (1.2).

We present our Marchenko method for (1.1) in such a way that the method can be applied on other linear systems and also on their discrete versions. We have already developed [9] the Marchenko method for the discrete analog of the linear system (1.1), and hence our emphasis in this paper is the development of the Marchenko method for the linear system (1.1) of ordinary differential equations.

A linear system of differential equations such as (1.1), which contains the spectral parameter ζ and some potentials that are functions of the spacial variable x with sufficiently fast decay at infinity, yields a scattering scenario. We associate the potentials in the linear system to an appropriate scattering data set, which consists of some scattering coefficients that are functions of the spectral parameter ζ and the bound-state information related to the values of the spectral parameter at which the linear system has square-integrable solutions. The direct scattering problem for (1.1) consists of the determination of the scattering data set when the potentials q and r are known. On the other hand, the inverse scattering problem for (1.1) consists of the determination of the potentials when the scattering data set is known.

One of the most effective methods in the solution to an inverse scattering problem is the Marchenko method, originally developed by Vladimir Marchenko

[4, 28] for the half-line Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\,\psi = k^2\,\psi, \quad 0 < x < +\infty.$$

The Marchenko method was later extended by Faddeev [23] to solve the inverse scattering problem for the full-line Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\,\psi = k^2\,\psi, \quad -\infty < x < +\infty. \tag{1.3}$$

In the Marchenko method, the potential is recovered from the solution to a linear integral equation, usually called the Marchenko equation, where the kernel and the nonhomogeneous term are constructed from the scattering data set with the help of a Fourier transformation. The Marchenko equation for (1.3) has the form

$$K(x,y) + \Omega(x+y) + \int_x^\infty dz\, K(x,z)\,\Omega(z+y) = 0, \quad x < y, \tag{1.4}$$

if the scattering data set is related to the measurements at $x = +\infty$, and the corresponding Marchenko equation has the form

$$\tilde K(x,y) + \tilde\Omega(x+y) + \int_{-\infty}^x dz\, \tilde K(x,z)\,\tilde\Omega(z+y) = 0, \quad y < x, \tag{1.5}$$

if the scattering data set is related to the measurements at $x = -\infty$. The integral kernels and the nonhomogeneous terms in (1.4) and (1.5) are constructed from the corresponding scattering data sets, and the potential V is obtained from the solution $K(x,y)$ to (1.4) as

$$V(x) = -2\,\frac{dK(x,x)}{dx}, \tag{1.6}$$

where $K(x,x)$ denotes the limiting value $K(x,x^+)$, or it is constructed from the solution $\tilde K(x,y)$ to (1.5) as

$$V(x) = 2\,\frac{d\tilde K(x,x)}{dx},$$

where $\tilde K(x,x)$ denotes the limiting value $\tilde K(x,x^-)$.

The Marchenko method is applicable to various other differential equations as well as systems of differential equations. For example, when applied to the AKNS system [1, 2]

$$\frac{d}{dx}\begin{bmatrix}\xi\\ \eta\end{bmatrix} = \begin{bmatrix}-i\lambda & u(x)\\ v(x) & i\lambda\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}, \quad -\infty < x < +\infty, \tag{1.7}$$

the corresponding Marchenko integral equation still has the form given in (1.4), except that $K(x,y)$ and $\Omega(x+y)$ are now 2×2 matrices. The nonhomogeneous term and the kernel are constructed from the scattering data set in a similar manner as done for (1.3), and the two potentials u and v in (1.7) are recovered from the solution to the relevant Marchenko equation by using a slight variation of (1.6), i.e.

$$u(x) = -2 \begin{bmatrix} 1 & 0 \end{bmatrix} K(x,x) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v(x) = -2 \begin{bmatrix} 0 & 1 \end{bmatrix} K(x,x) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $K(x,x)$ is the 2×2 matrix $K(x,x^+)$ with $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$ denoting the corresponding row vectors and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ denoting the corresponding column vectors.

The Marchenko method is also applicable to various inverse scattering problems for linear difference equations such as the discrete Schrödinger equation on the half-line lattice given by

$$-\psi_{n+1} + 2\psi_n - \psi_{n-1} + V_n \psi_n = \lambda \psi_n, \quad n \geq 1, \tag{1.8}$$

where λ is the spectral parameter and the quantities ψ_n and V_n denote the values of the wavefunction and the potential, respectively, at the lattice location n. Assuming that V_n is real valued and $\sum_{n=1}^{\infty} nV_n$ is finite, we can supplement (1.8) with the Dirichlet boundary condition $\psi_0 = 0$ and obtain the discrete Schrödinger operator on a half-line lattice. In this case, the corresponding Marchenko equation has the discrete form given by

$$K_{nm} + \Omega_{n+m} + \sum_{j=n+1}^{\infty} K_{nj}\, \Omega_{j+m} = 0, \quad 0 \leq n < m. \tag{1.9}$$

The nonhomogeneous term and the kernel are still constructed from the corresponding scattering data set, and the potential value V_n is recovered [13] from the double-indexed solution K_{nm} to (1.9) via

$$V_n = K_{(n-1)n} - K_{n(n+1)}, \quad n \geq 1,$$

with the understanding that $K_{01} = 0$.

There are still many other inverse scattering problems described by various differential or difference equations, or system of differential or difference equations, for which a Marchenko method is not yet available, and (1.1) is one of them. In this paper, we develop the Marchenko method for (1.1) and present the corresponding matrix-valued Marchenko integral equation in (4.40). We note that (4.40) resembles (1.4), but the integral kernel in (4.40) slightly differs from that in (1.4). In (4.58) and (4.59), we present the recovery of the potentials q and r from the solution to (4.40).

The main result presented in this paper, i.e. the derivation of the Marchenko system for (1.1) and the recovery of the potentials q and r from the solution to that Marchenko system, is significant because not only it extends the powerful Marchenko method to (1.1) but it also provides a procedure that can be applied to various other inverse problems.

In our extension of the Marchenko method to solve the inverse scattering problem for (1.1), we use the following guidelines in order to refer to the extension still as the Marchenko method. First, the derived Marchenko system should resemble (1.4), where the nonhomogeneous term and the kernel should both be obtained from the scattering data for (1.1) with the help of a Fourier transform, but by allowing some minor modifications. Next, the potentials in (1.1) should be readily obtained from the solution to the derived Marchenko system, but by allowing some appropriate modifications. Some similar guidelines can also be used to establish a Marchenko method for other differential and difference equations, or systems of differential and difference equations.

Let us remark that, in the literature related to the inverse scattering transform method [1,3,19,24,26,32] used to solve integrable nonlinear evolution equations, some authors refer to the Marchenko equation as the Gel'fand–Levitan–Marchenko equation, but this is a misnomer [20, 22, 30]. The Gel'fand–Levitan integral equation [12,15,20,23,27,29,31] is different from the Marchenko integral equation. The standard Gel'fand–Levitan equation has the form

$$A(x,y)+G(x,y)+\int_0^x dz\,A(x,z)\,G(z,y)=0,\quad 0<y<x, \tag{1.10}$$

where $G(x,y)$ appearing in the kernel and the nonhomogeneous term. We note that that the integral limits in the Marchenko equation (1.4) are x and $+\infty$, whereas the integral limits in the Gel'fand–Levitan equation (1.10) are 0 and x. The main difference between the two methods is that the kernel and the nonhomogeneous term in the Gel'fand–Levitan equation are constructed from the corresponding spectral measure, whereas in the Marchenko integral equation the kernel and the nonhomogeneous term are constructed from the corresponding scattering data.

Our paper is organized as follows. In Section 2 we provide the preliminaries by introducing the Jost solutions and the scattering coefficients for the linear system (1.1), and we present their relevant properties needed in the development of our Marchenko method. In Section 3 we introduce the relevant information on the bound states for (1.1), and we show that the bound-state information can be presented in a simple and elegant way for any number of bound states and for any multiplicities, and this is done by using a pair of constant matrix triplets. In Section 4 we present the matrix-valued Marchenko system for (1.1), where the input to the Marchenko system consists of a pair of reflection coefficients and the bound-state information. We also show that the Marchenko system can be written in an equivalent but uncoupled format, and we describe how the potentials and the Jost solutions are obtained from the solution to the Marchenko system. In Section 5, when the reflection coefficients are zero, with the general bound-state information expressed in terms of a pair of matrix triplets, we obtain the closed-form solution to the Marchenko system. This allows us to present some explicit solution formulas for the potentials and the Jost solutions for (1.1) expressed in closed form in terms of our matrix triplets. In Section 5, we also prove a relevant restriction on the bound states for (1.1) when the potentials q and r are reflectionless; namely, we prove that the bound-state poles of the corresponding transmission coefficients must be equally distributed in the four quadrants of the complex ζ-plane. The same restriction also holds for the AKNS system (1.7), i.e. in the reflectionless case the bound-state poles of the corresponding transmission coefficients must be equally distributed in the upper and lower halves of the complex λ-plane. Finally, in Section 6, we illustrate the theory developed in the earlier sections, and in particular we provide some examples of potentials and Jost solutions for (1.1) in terms of elementary functions when the sizes of our matrix triplets are small. We remark that some unorthodox choices for the norming constants in the bound-state data set result in some explicit examples of potentials that are periodic or that have nonzero asymptotics at infinity.

2. Preliminaries

In this section, in order to prepare for the derivation of the Marchenko system for (1.1), we introduce the Jost solutions and the scattering coefficients for (1.1) and present their relevant properties. We use the notation of [8] and rely some of the results presented there.

We let $\psi(\zeta,x)$, $\bar\psi(\zeta,x)$, $\phi(\zeta,x)$, $\bar\phi(\zeta,x)$ denote the four Jost solutions to (1.1) satisfying the respective spacial asymptotics

$$\psi(\zeta,x)=\begin{bmatrix} o(1)\\ e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to+\infty, \tag{2.1}$$

$$\bar\psi(\zeta,x)=\begin{bmatrix} e^{-i\zeta^2x}\,[1+o(1)]\\ o(1)\end{bmatrix},\qquad x\to+\infty, \tag{2.2}$$

$$\phi(\zeta,x)=\begin{bmatrix} e^{-i\zeta^2x}\,[1+o(1)]\\ o(1)\end{bmatrix},\qquad x\to-\infty, \tag{2.3}$$

$$\bar\phi(\zeta,x)=\begin{bmatrix} o(1)\\ e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to-\infty. \tag{2.4}$$

We remark that the overbar does not denote complex conjugation.

There are six scattering coefficients associated with (1.1), i.e. the transmission coefficients $T(\zeta)$ and $\bar T(\zeta)$, the right reflection coefficients $R(\zeta)$ and $\bar R(\zeta)$, and the left reflection coefficients $L(\zeta)$ and $\bar L(\zeta)$. Because the trace of the coefficient matrix in (1.1) is zero, the transmission coefficients from the left and from the right are equal to each other, and hence we do not need to make a distinction between the left and right transmission coefficients. The six scattering coefficients can be defined in terms of the spacial asymptotics of the Jost solutions given by

$$\psi(\zeta,x)=\begin{bmatrix} \dfrac{L(\zeta)}{T(\zeta)}\,e^{-i\zeta^2x}\,[1+o(1)]\\[2mm] \dfrac{1}{T(\zeta)}\,e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{2.5}$$

$$\bar{\psi}(\zeta,x)=\begin{bmatrix}\dfrac{1}{\bar{T}(\zeta)}\,e^{-i\zeta^2x}\,[1+o(1)]\\ \dfrac{\bar{L}(\zeta)}{\bar{T}(\zeta)}\,e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to-\infty, \tag{2.6}$$

$$\phi(\zeta,x)=\begin{bmatrix}\dfrac{1}{T(\zeta)}\,e^{-i\zeta^2x}\,[1+o(1)]\\ \dfrac{R(\zeta)}{T(\zeta)}\,e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to+\infty, \tag{2.7}$$

$$\bar{\phi}(\zeta,x)=\begin{bmatrix}\dfrac{\bar{R}(\zeta)}{\bar{T}(\zeta)}\,e^{-i\zeta^2x}\,[1+o(1)]\\ \dfrac{1}{\bar{T}(\zeta)}\,e^{i\zeta^2x}\,[1+o(1)]\end{bmatrix},\qquad x\to+\infty. \tag{2.8}$$

In order to present the relevant properties of the Jost solutions, we use the subscripts 1 and 2 to denote their first and second components, respectively, i.e. we let

$$\begin{bmatrix}\psi_1(\zeta,x)\\ \psi_2(\zeta,x)\end{bmatrix}:=\psi(\zeta,x),\qquad \begin{bmatrix}\bar{\psi}_1(\zeta,x)\\ \bar{\psi}_2(\zeta,x)\end{bmatrix}:=\bar{\psi}(\zeta,x), \tag{2.9}$$

$$\begin{bmatrix}\phi_1(\zeta,x)\\ \phi_2(\zeta,x)\end{bmatrix}:=\phi(\zeta,x),\qquad \begin{bmatrix}\bar{\phi}_1(\zeta,x)\\ \bar{\phi}_2(\zeta,x)\end{bmatrix}:=\bar{\phi}(\zeta,x). \tag{2.10}$$

We recall that the Wronskian of any two column-vector solutions to (1.1) is defined as the determinant of the 2×2 matrix formed from those columns. For example, the Wronskian of $\psi(\zeta,x)$ and $\phi(\zeta,x)$ is given by

$$[\psi;\phi]:=\begin{vmatrix}\psi_1 & \phi_1\\ \psi_2 & \phi_2\end{vmatrix}. \tag{2.11}$$

Due to the fact that the coefficient matrix in (1.1) has the zero trace, the value of the Wronskian of any two solutions to (1.1) is independent of x, and hence the six scattering coefficients appearing in (2.5)–(2.8) can equivalently be expressed [8] in terms of Wronskians of the Jost solutions as

$$T(\zeta)=\frac{1}{[\phi(\zeta,x);\psi(\zeta,x)]},\qquad \bar{T}(\zeta)=\frac{1}{[\bar{\psi}(\zeta,x);\bar{\phi}(\zeta,x)]}, \tag{2.12}$$

$$R(\zeta)=\frac{[\phi(\zeta,x);\bar{\psi}(\zeta,x)]}{[\psi(\zeta,x);\phi(\zeta,x)]},\qquad \bar{R}(\zeta)=\frac{[\bar{\phi}(\zeta,x);\psi(\zeta,x)]}{[\bar{\psi}(\zeta,x);\bar{\phi}(\zeta,x)]}, \tag{2.13}$$

$$L(\zeta)=\frac{[\psi(\zeta,x);\bar{\phi}(\zeta,x)]}{[\phi(\zeta,x);\psi(\zeta,x)]},\qquad \bar{L}(\zeta)=\frac{[\phi(\zeta,x);\bar{\psi}(\zeta,x)]}{[\bar{\psi}(\zeta,x);\bar{\phi}(\zeta,x)]}. \tag{2.14}$$

We relate the spectral parameter ζ appearing in (1.1) to the parameter λ in (1.7) as

$$\lambda=\zeta^2,\qquad \zeta=\sqrt{\lambda}, \tag{2.15}$$

with the square root denoting the principal branch of the complex-valued square-root function. We use $\mathbb{C}^+$ and $\mathbb{C}^-$ for the upper-half and lower-half, respectively, of the complex plane $\mathbb{C}$, and we let $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \mathbb{R}$ and $\overline{\mathbb{C}^-} := \mathbb{C}^- \cup \mathbb{R}$.

It is possible to connect (1.1) to the AKNS system (1.7) by using (2.15) and by choosing the potentials u and v in terms of the potentials q and r as

$$u(x) = q(x)\, E(x)^{-2}, \tag{2.16}$$

$$v(x) = \left[-\frac{i}{2}\, r'(x) + \frac{1}{4}\, q(x)\, r(x)^2\right] E(x)^2, \tag{2.17}$$

where the prime denotes the derivative and the quantity $E(x)$ is defined as

$$E(x) := \exp\left(\frac{i}{2}\int_{-\infty}^{x} dz\, q(z)\, r(z)\right). \tag{2.18}$$

Since the potentials q and r are complex valued, we remark that in general $E(x)$ does not have the unit modulus. From (2.18) it follows that

$$E(-\infty) = 1, \quad E(+\infty) = e^{i\mu/2},$$

where we have defined the complex constant μ as

$$\mu := \int_{-\infty}^{\infty} dz\, q(z)\, r(z). \tag{2.19}$$

Besides (1.7), it is possible to relate (1.1) to another AKNS system given by

$$\frac{d}{dx}\begin{bmatrix}\gamma\\ \epsilon\end{bmatrix} = \begin{bmatrix}-i\lambda & p(x)\\ s(x) & i\lambda\end{bmatrix}\begin{bmatrix}\gamma\\ \epsilon\end{bmatrix}, \qquad x \in \mathbb{R}, \tag{2.20}$$

by choosing the potentials p and s in terms of q and r as

$$p(x) = \left[\frac{i}{2}\, q'(x) + \frac{1}{4}\, q(x)^2\, r(x)\right] E(x)^{-2}, \tag{2.21}$$

$$s(x) = r(x)\, E(x)^2. \tag{2.22}$$

Let us remark that it is possible to analyze the direct and inverse scattering problems for (1.1) without relating (1.1) to the AKNS systems (1.7) or (2.20). As done for (1.3) [20,23,27,29], this can be accomplished for (1.1) by first determining the integral relations satisfied by the four Jost solutions to (1.1), where those integral relations are obtained by combining (1.1) and the asymptotic conditions (2.1)–(2.4). Using those integral relations, one can express the scattering coefficients for (1.1) in terms of certain integrals involving the potentials q and r. The relevant properties of the scattering coefficients can be determined from those integral relations. In a similar manner, the small and large ζ-asymptotics of the scattering coefficients, the bound states, and the inverse scattering problem for (1.1) can all be analyzed without relating (1.1) to (1.7) or (2.20). On the other hand, the analysis of the direct and inverse scattering problems for (1.1), by relating (1.1) to (1.7) or (2.20), brings some physical insight and intuition because the analysis of those two problems for an AKNS system is better understood. Note that (1.1) differs from the AKNS systems (1.7) or (2.20) because the off-diagonal entries of the coefficient matrix in (1.1) contain the potentials as multiplied by the spectral parameter ζ. This complicates the analysis of the direct and inverse scattering problems for (1.1). On the other hand, the three linear systems (1.1), (1.7), and (2.20) can all be viewed as some perturbations of the first-order unperturbed system

$$\frac{d}{dx}\begin{bmatrix}\mathring{\alpha}\\ \mathring{\beta}\end{bmatrix}=\begin{bmatrix}-i\lambda & 0\\ 0 & i\lambda\end{bmatrix}\begin{bmatrix}\mathring{\alpha}\\ \mathring{\beta}\end{bmatrix},\qquad x\in\mathbb{R},$$

and this helps us to understand the connections among (1.1), (1.7), and (2.20).

In the next theorem we provide the relations among the Jost solutions to (1.1), (1.7), and (2.20), respectively, when (2.15)–(2.17), (2.21), (2.22) hold. We omit the proof and refer the reader to Theorems 3.1 and 3.2 of [8].

Theorem 2.1. *Suppose that the potentials q and r in (1.1) belong to the Schwartz class. Let E denote the quantity $E(x)$ defined in (2.18), and μ be the complex constant defined in (2.19). Further, assume that the spectral parameter ζ is related to the parameter λ as in (2.15). We have the following:*

(a) *The linear system (1.1) can be transformed into the AKNS system (1.7), where the potential pair (u,v) is related to the potential pair (q,r) as in (2.16) and (2.17). It follows that the potentials u and v also belong to the Schwartz class. The four Jost solutions to (1.1) appearing in (2.1)–(2.4), respectively, and the four Jost solutions $\psi^{(u,v)}$, $\bar\psi^{(u,v)}$, $\phi^{(u,v)}$, $\bar\phi^{(u,v)}$ to (1.7), satisfying the corresponding asymptotics in (2.1)–(2.4), respectively, are related to each other as*

$$\psi(\zeta,x)=e^{i\mu/2}\begin{bmatrix}\sqrt{\lambda}\,E & 0\\ \dfrac{i}{2}\,r(x)\,E & E^{-1}\end{bmatrix}\psi^{(u,v)}(\lambda,x),\tag{2.23}$$

$$\bar\psi(\zeta,x)=e^{-i\mu/2}\begin{bmatrix}E & 0\\ \dfrac{i}{2\sqrt{\lambda}}\,r(x)\,E & \dfrac{1}{\sqrt{\lambda}}\,E^{-1}\end{bmatrix}\bar\psi^{(u,v)}(\lambda,x),\tag{2.24}$$

$$\phi(\zeta,x)=\begin{bmatrix}E & 0\\ \dfrac{i}{2\sqrt{\lambda}}\,r(x)\,E & \dfrac{1}{\sqrt{\lambda}}\,E^{-1}\end{bmatrix}\phi^{(u,v)}(\lambda,x),\tag{2.25}$$

$$\bar\phi(\zeta,x)=\begin{bmatrix}\sqrt{\lambda}\,E & 0\\ \dfrac{i}{2}\,r(x)\,E & E^{-1}\end{bmatrix}\bar\phi^{(u,v)}(\lambda,x).\tag{2.26}$$

(b) *The system (1.1) can be transformed into the system (2.20), where the potential pair (p,s) is related to (q,r) as in (2.21) and (2.22). It follows that* *the potentials p and s belong to the Schwartz class. The four Jost solutions to (1.1) and the four Jost solutions $\psi^{(p,s)}$, $\bar\psi^{(p,s)}$, $\phi^{(p,s)}$, $\bar\phi^{(p,s)}$ to (2.20), satisfying the corresponding asymptotics in (2.1)–(2.4), respectively, are related to each other as*

$$\psi(\zeta,x)=e^{i\mu/2}\begin{bmatrix}\dfrac{1}{\sqrt{\lambda}}\,E & -\dfrac{i}{2\sqrt{\lambda}}\,q(x)\,E^{-1}\\ 0 & E^{-1}\end{bmatrix}\psi^{(p,s)}(\lambda,x), \tag{2.27}$$

$$\bar\psi(\zeta,x)=e^{-i\mu/2}\begin{bmatrix}E & -\dfrac{i}{2}\,q(x)\,E^{-1}\\ 0 & \sqrt{\lambda}\,E^{-1}\end{bmatrix}\bar\psi^{(p,s)}(\lambda,x), \tag{2.28}$$

$$\phi(\zeta,x)=\begin{bmatrix}E & -\dfrac{i}{2}\,q(x)\,E^{-1}\\ 0 & \sqrt{\lambda}\,E^{-1}\end{bmatrix}\phi^{(p,s)}(\lambda,x), \tag{2.29}$$

$$\bar\phi(\zeta,x)=\begin{bmatrix}\dfrac{1}{\sqrt{\lambda}}\,E & -\dfrac{i}{2\sqrt{\lambda}}\,q(x)\,E^{-1}\\ 0 & E^{-1}\end{bmatrix}\bar\phi^{(p,s)}(\lambda,x). \tag{2.30}$$

Next, we present the relevant analyticity and symmetry properties of the Jost solutions to (1.1), which are needed to establish the Marchenko method for (1.1).

Theorem 2.2. *Let the potentials q and r in (1.1) belong to the Schwartz class. Assume that the spectral parameter ζ is related to the parameter λ as in (2.15). Then, we have the following:*

(a) *For each fixed $x\in\mathbb{R}$, the Jost solutions $\psi(\zeta,x)$ and $\phi(\zeta,x)$ to (1.1) are analytic in the first and third quadrants in the complex ζ-plane and are continuous in the closures of those regions. Similarly, the Jost solutions $\bar\psi(\zeta,x)$ and $\bar\phi(\zeta,x)$ are analytic in the second and fourth quadrants in the complex ζ-plane and are continuous in the closures of those regions.*

(b) *The components of the Jost solutions defined in (2.9) and (2.10) have the following properties. The components $\psi_1(\zeta,x)$, $\bar\psi_2(\zeta,x)$, $\phi_2(\zeta,x)$, and $\bar\phi_1(\zeta,x)$ are odd in ζ; whereas the components $\psi_2(\zeta,x)$, $\bar\psi_1(\zeta,x)$, $\phi_1(\zeta,x)$, and $\bar\phi_2(\zeta,x)$ are even in ζ. Furthermore, for each fixed $x\in\mathbb{R}$, the four scalar functions $\psi_1(\zeta,x)/\zeta$, $\psi_2(\zeta,x)$, $\phi_1(\zeta,x)$, and $\phi_2(\zeta,x)/\zeta$ are even in ζ; are analytic in $\lambda\in\mathbb{C}^+$, and continuous in $\lambda\in\overline{\mathbb{C}^+}$. Similarly, for each fixed $x\in\mathbb{R}$, the four scalar functions $\bar\psi_1(\zeta,x)$, $\bar\psi_2(\zeta,x)/\zeta$, $\bar\phi_1(\zeta,x)/\zeta$, and $\bar\phi_2(\zeta,x)$ are even in ζ; are analytic in $\lambda\in\mathbb{C}^-$, and continuous in $\lambda\in\overline{\mathbb{C}^-}$.*

Proof. The proof of (a) can be obtained by converting (1.1) and each of the asymptotics in (2.1)–(2.4) into an integral equation, then by solving the resulting four integral equations via iteration, and by expressing the Jost solutions as uniformly convergent infinite series of terms that are analytic in the appropriate domains in the complex ζ-plane and are continuous in the closures of those domains. Alternatively, the proof of (a) can be obtained with the help of

Theorem 2.1 and by using the corresponding analyticity and continuity properties [2,21] in λ of the Jost solutions to the AKNS systems (1.7) and (2.20). The proof of (b) is obtained by using the results in (a) and either the relations given in (2.23)–(2.26) or in (2.27)–(2.30). $\square$

In the following theorem, we present the asymptotics of the Jost solutions to (1.1) as $\zeta \to 0$. Those asymptotics are crucial for the establishment of the Marchenko method for (1.1).

Theorem 2.3. *Let the potentials q and r in (1.1) belong to the Schwartz class. Then, for each fixed $x \in \mathbb{R}$, as $\zeta \to 0$ in their domains of continuity, the Jost solutions to (1.1) appearing in (2.1)–(2.4) satisfy*

$$\psi(\zeta, x) = \begin{bmatrix} -\zeta \displaystyle\int_x^\infty dz\, q(z) + O\left(\zeta^3\right) \\ 1 + O\left(\zeta^2\right) \end{bmatrix}, \tag{2.31}$$

$$\bar{\psi}(\zeta, x) = \begin{bmatrix} 1 + O\left(\zeta^2\right) \\ \zeta \displaystyle\int_x^\infty dz\, r(z) + O\left(\zeta^3\right) \end{bmatrix}, \tag{2.32}$$

$$\phi(\zeta, x) = \begin{bmatrix} 1 + O\left(\zeta^2\right) \\ \zeta \displaystyle\int_{-\infty}^x dz\, r(z) + O\left(\zeta^3\right) \end{bmatrix}, \tag{2.33}$$

$$\bar{\phi}(\zeta, x) = \begin{bmatrix} \zeta \displaystyle\int_{-\infty}^x dz\, q(z) + O\left(\zeta^3\right) \\ 1 + O\left(\zeta^2\right) \end{bmatrix}. \tag{2.34}$$

Proof. The domains of continuity for the Jost solutions are specified in Theorem 2.2. The proof of (2.31) and (2.34) can be obtained by using (2.23) and (2.26), respectively, and the known small λ-asymptotics [8,21] of the Jost solutions $\psi^{(u,v)}(\lambda, x)$ and $\bar{\phi}^{(u,v)}(\lambda, x)$ to (1.7), and by taking into account the relationship between ζ and λ specified in (2.15). Similarly, the proof of (2.32) and (2.33) can be obtained by using (2.28) and (2.29) and the known small λ-asymptotics [8,21] of the Jost solutions $\bar{\psi}^{(p,s)}(\lambda, x)$ and $\phi^{(p,s)}(\lambda, x)$. $\square$

In relation to Theorem 2.3, let us remark that the small λ-asymptotics of the Jost solutions to (1.7) and (2.20) expressed in terms of the quantities relevant to (1.1) can be found in Proposition 6.1 of [8].

In order to prepare for the derivation of the Marchenko system for (1.1), we also need the large ζ-asymptotics of the Jost solutions to (1.1). For convenience, in the following theorem those asymptotics are expressed in terms of λ, which is related to ζ as in (2.15).

Theorem 2.4. *Let the potentials q and r in (1.1) belong to the Schwartz class, and let the parameter λ be related to the spectral parameter ζ as in (2.15).*

Then, for each fixed $x \in \mathbb{R}$, as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$ the Jost solutions $\psi(\zeta, x)$ and $\phi(\zeta, x)$ to (1.1) appearing in (2.1) and (2.3), respectively, satisfy

$$\psi(\zeta,x)=\begin{bmatrix} \sqrt{\lambda}\, e^{i\mu/2+i\lambda x} E(x) \left[\dfrac{q(x)\,E(x)^{-2}}{2i\lambda}+O\left(\dfrac{1}{\lambda^2}\right)\right] \\ \dfrac{e^{i\mu/2+i\lambda x}}{E(x)}\left[1+\dfrac{q(x)\,r(x)}{4\lambda}-\dfrac{1}{2i\lambda}\displaystyle\int_x^\infty dz\,\sigma(z)+O\left(\dfrac{1}{\lambda^2}\right)\right] \end{bmatrix}, \tag{2.35}$$

$$\phi(\zeta,x)=\begin{bmatrix} e^{-i\lambda x}E(x)\left[1-\dfrac{1}{2i\lambda}\displaystyle\int_{-\infty}^x dz\,\sigma(z)+O\left(\dfrac{1}{\lambda^2}\right)\right] \\ \sqrt{\lambda}\, e^{-i\lambda x}\left[\dfrac{i\,r(x)\,E(x)}{2\lambda}+O\left(\dfrac{1}{\lambda^2}\right)\right] \end{bmatrix},$$

where $E(x)$ and μ are the quantities appearing in (2.18) and (2.19), respectively, and the complex-valued scalar quantity $\sigma(x)$ is defined as

$$\sigma(x) := -\frac{i}{2}\, q(x)\, r'(x) + \frac{1}{4}\, q(x)^2\, r(x)^2. \tag{2.36}$$

Similarly, for each fixed $x \in \mathbb{R}$, as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$ the Jost solutions $\bar\psi(\zeta, x)$ and $\bar\phi(\zeta, x)$ to (1.1) appearing in (2.2) and (2.4), respectively, satisfy

$$\bar\psi(\zeta,x)=\begin{bmatrix} e^{-i\mu/2-i\lambda x}E(x)\left[1+\dfrac{1}{2i\lambda}\displaystyle\int_x^\infty dz\,\sigma(z)+O\left(\dfrac{1}{\lambda^2}\right)\right] \\ \sqrt{\lambda}\, e^{-i\mu/2-i\lambda x}\left[\dfrac{i\,r(x)\,E(x)}{2\lambda}+O\left(\dfrac{1}{\lambda^2}\right)\right] \end{bmatrix}, \tag{2.37}$$

$$\bar\phi(\zeta,x)=\begin{bmatrix} \sqrt{\lambda}\, e^{i\lambda x}\left[\dfrac{q(x)\,E(x)^{-1}}{2i\lambda}+O\left(\dfrac{1}{\lambda^2}\right)\right] \\ \dfrac{e^{i\lambda x}}{E(x)}\left[1+\dfrac{q(x)\,r(x)}{4\lambda}+\dfrac{1}{2i\lambda}\displaystyle\int_{-\infty}^x dz\,\sigma(z)+O\left(\dfrac{1}{\lambda^2}\right)\right] \end{bmatrix}.$$

Proof. The proof is obtained by using iteration on the integral representations of the Jost solutions aforementioned in the proof of Theorem 2.1 and by taking into consideration the fact that ζ is related to λ as in (2.15). Alternatively, the proof can be given by using (2.23)–(2.26) and the known large λ-asymptotics [2,8,21] of the Jost solutions to (1.7), and by taking into account the fact that the quantity $\sigma(x)$ defined in (2.36) corresponds to the product $u(x)\,v(x)$ when $u(x)$ and $v(x)$ are chosen as in (2.16) and (2.17), respectively. Equivalently, the proof can be obtained by using (2.27)–(2.30) and the known large λ-asymptotics [2,8,21] of the Jost solutions to (2.20), and by taking into consideration the fact that the quantity $\sigma(x)$ defined in (2.36) corresponds to the product $p(x)\,s(x)$ when $p(x)$ and $s(x)$ are chosen as in (2.21) and (2.22), respectively. □

In the next theorem, in preparation for the establishment of the Marchenko method for (1.1), we present the relevant properties of the scattering coefficients for (1.1).

Theorem 2.5. *Assume that the potentials q and r in (1.1) belong to the Schwartz class. Let λ be related to the spectral parameter ζ as in (2.15), and let μ be the complex constant defined in (2.19). Then, the scattering coefficients $T(\zeta)$, $\bar T(\zeta)$, $R(\zeta)$, $\bar R(\zeta)$, $L(\zeta)$, and $\bar L(\zeta)$ appearing in (2.5)–(2.8) have the following properties:*

(a) *The transmission coefficient $T(\zeta)$ is continuous in $\zeta\in\mathbb{R}$ and has a meromorphic extension from $\zeta\in\mathbb{R}$ to the first and third quadrants in the complex ζ-plane. Furthermore, $T(\zeta)$ is an even function of ζ, and thus it is a function of λ in $\overline{\mathbb{C}^+}$. The quantity $1/T(\zeta)$ is analytic in $\lambda\in\mathbb{C}^+$ and continuous in $\lambda\in\overline{\mathbb{C}^+}$. Moreover, $T(\zeta)$ is meromorphic in $\lambda\in\mathbb{C}^+$ with a finite number of poles there, where those poles are not necessarily simple but have finite multiplicities. We have the large ζ-asymptotics of $T(\zeta)$ expressed in λ as*

$$T(\zeta)=e^{-i\mu/2}\left[1+O\left(\frac{1}{\lambda}\right)\right],\quad \lambda\to\infty \text{ in } \overline{\mathbb{C}^+}. \tag{2.38}$$

(b) *The transmission coefficient $\bar T(\zeta)$ is continuous in $\zeta\in\mathbb{R}$ and has a meromorphic extension from $\zeta\in\mathbb{R}$ to the second and fourth quadrants in the complex ζ-plane. Furthermore, $\bar T(\zeta)$ is an even function of ζ, and thus it is a function of λ in $\overline{\mathbb{C}^-}$. The quantity $1/\bar T(\zeta)$ is analytic in $\lambda\in\mathbb{C}^-$, and it is continuous in $\lambda\in\overline{\mathbb{C}^-}$. Moreover, $\bar T(\zeta)$ is meromorphic in $\lambda\in\mathbb{C}^-$ with a finite number of poles, where the poles are not necessarily simple but have finite multiplicities. We have the large ζ-asymptotics of $\bar T(\zeta)$ expressed in λ as*

$$\bar T(\zeta)=e^{i\mu/2}\left[1+O\left(\frac{1}{\lambda}\right)\right],\quad \lambda\to\infty \text{ in } \overline{\mathbb{C}^-}. \tag{2.39}$$

(c) *Each of the four reflection coefficients $R(\zeta)$, $\bar R(\zeta)$, $L(\zeta)$, and $\bar L(\zeta)$ is continuous in $\zeta\in\mathbb{R}$, is an odd function of ζ, and behave as $O(1/\zeta^{5/2})$ as $\zeta\to\pm\infty$. Moreover, the four quantities $R(\zeta)/\zeta$, $\bar R(\zeta)/\zeta$, $L(\zeta)/\zeta$, $\bar L(\zeta)/\zeta$ are even in ζ; are continuous functions of $\lambda\in\mathbb{R}$; and expressed in λ they have the behavior $O(1/\lambda^3)$ as $\lambda\to\pm\infty$.*

(d) *The small ζ-asymptotics of the scattering coefficients $T(\zeta)$, $\bar T(\zeta)$, $R(\zeta)$, $\bar R(\zeta)$, $L(\zeta)$, and $\bar L(\zeta)$ are expressed in λ as*

$$T(\zeta)=1+O(\lambda),\quad \lambda\to 0 \text{ in } \overline{\mathbb{C}^+}, \tag{2.40}$$

$$\bar T(\zeta)=1+O(\lambda),\quad \lambda\to 0 \text{ in } \overline{\mathbb{C}^-}, \tag{2.41}$$

$$R(\zeta)=-\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, r(z)+O(\lambda)\right],\quad \lambda\to 0 \text{ in } \mathbb{R}, \tag{2.42}$$

$$\bar R(\zeta)=\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, q(z)+O(\lambda)\right],\quad \lambda\to 0 \text{ in } \mathbb{R}, \tag{2.43}$$

$$L(\zeta)=-\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, q(z)+O(\lambda)\right],\quad \lambda\to 0 \text{ in } \mathbb{R},$$

$$\bar L(\zeta)=\sqrt{\lambda}\left[\int_{-\infty}^{\infty}dz\, r(z)+O(\lambda)\right],\quad \lambda\to 0 \text{ in } \mathbb{R}.$$

(e) *The scattering coefficients satisfy*

$$T(\zeta)\,\bar T(\zeta)+R(\zeta)\,\bar R(\zeta)=1,\quad T(\zeta)\,\bar T(\zeta)+L(\zeta)\,\bar L(\zeta)=1,\quad \lambda\in\mathbb{R}. \tag{2.44}$$

(f) *The left reflection coefficients are determined in terms of the right reflection coefficients and the transmission coefficients, and we have*

$$L(\zeta)=-\frac{\bar R(\zeta)\,T(\zeta)}{\bar T(\zeta)},\quad \bar L(\zeta)=-\frac{R(\zeta)\,\bar T(\zeta)}{T(\zeta)},\quad \lambda\in\mathbb{R}. \tag{2.45}$$

Conversely, as seen from (2.45), the right reflection coefficients are determined in terms of the left reflection coefficients and the transmission coefficients. Consequently, if the right reflection coefficients $R(\zeta)$ and $\bar R(\zeta)$ vanish at some ζ-value, then the left reflection coefficients $L(\zeta)$ and $\bar L(\zeta)$ also vanish there.

Proof. Since the scattering coefficients can be expressed in terms of the Wronskians of the Jost solutions as in (2.12)–(2.14), their stated properties can be established by using the properties of the Jost solutions provided in Theorem 2.1. Alternatively, the proof can be obtained by using the relationships between the six scattering coefficients for (1.1) and the corresponding scattering coefficients for the two associated AKNS systems given in (1.7) and (2.20), respectively, when the potential pairs (u,v) and (p,s) are chosen as in (2.16), (2.17), (2.21), and (2.22). In fact, we have [8, 21]

$$T(\zeta)=e^{-i\mu/2}\,T^{(u,v)}(\lambda)=e^{-i\mu/2}\,T^{(p,s)}(\lambda), \tag{2.46}$$

$$\bar T(\zeta)=e^{i\mu/2}\,\bar T^{(u,v)}(\lambda)=e^{i\mu/2}\,\bar T^{(p,s)}(\lambda), \tag{2.47}$$

$$R(\zeta)=\frac{e^{-i\mu}}{\sqrt{\lambda}}\,R^{(u,v)}(\lambda)=e^{-i\mu}\sqrt{\lambda}\,R^{(p,s)}(\lambda), \tag{2.48}$$

$$\bar R(\zeta)=e^{i\mu}\sqrt{\lambda}\;\bar R^{(u,v)}(\lambda)=\frac{e^{i\mu}}{\sqrt{\lambda}}\,\bar R^{(p,s)}(\lambda), \tag{2.49}$$

$$L(\zeta)=\sqrt{\lambda}\,L^{(u,v)}(\lambda)=\frac{1}{\sqrt{\lambda}}\,L^{(p,s)}(\lambda), \tag{2.50}$$

$$\bar L(\zeta)=\frac{1}{\sqrt{\lambda}}\,\bar L^{(u,v)}(\lambda)=\sqrt{\lambda}\,\bar L^{(p,s)}(\lambda), \tag{2.51}$$

where the superscripts (u,v) and (p,s) are used to refer to the scattering coefficients for (1.7) and (2.20), respectively. Using (2.46)–(2.51) and the already known [2, 8, 21] properties of the scattering coefficients of the associated AKNS systems, the proof is established. □

Let us now consider the question whether the scattering coefficients for (1.1) can be determined from the knowledge of the scattering coefficients for (1.7) or (2.20), and vice versa. The presence of the factor $e^{i\mu/2}$ in (2.46)–(2.49) gives the impression that this is possible only if we know the value of $e^{i\mu/2}$ independently. The next theorem shows that the value of $e^{i\mu/2}$ is indeed determined by any one of the transmission coefficients for either (1.7) or (2.20), and hence the scattering coefficients for (1.7) and (2.20) can be explicitly expressed in terms of the scattering coefficients for (1.1). Similarly, the value of $e^{i\mu/2}$ is indeed determined by one of the transmission coefficients for (1.1), and hence the scattering coefficients for (1.7) and (2.20) can be determined from the knowledge of the scattering coefficients for (1.1).

Theorem 2.6. *Assume that the potentials q and r in (1.1) belong to the Schwartz class. Furthermore, suppose that the potential pairs (u,v) and (p,s) appearing in (1.7) and (2.20), respectively, are related to the potential pair (q,r) as in (2.16), (2.17), (2.21), and (2.22). Let λ be related to the spectral parameter ζ as in (2.15), and let μ be the complex constant defined in (2.19). Then, we have the following:*

(a) *The scalar constant $e^{i\mu/2}$ is uniquely determined by one of the transmission coefficients for either of (1.7) or (2.20). In fact, we have*

$$e^{i\mu/2} = T^{(u,v)}(0) = T^{(p,s)}(0), \tag{2.52}$$

$$e^{-i\mu/2} = \bar{T}^{(u,v)}(0) = \bar{T}^{(p,s)}(0), \tag{2.53}$$

where we recall that the superscripts (u,v) and (p,s) are used to refer to the scattering coefficients for (1.7) and (2.20), respectively.

(b) *The scattering coefficients for (1.1) are uniquely determined by the scattering coefficients for either of the linear systems (1.7) or (2.20). In fact, we have*

$$T(\zeta) = \frac{T^{(u,v)}(\lambda)}{T^{(u,v)}(0)} = \frac{T^{(p,s)}(\lambda)}{T^{(p,s)}(0)}, \tag{2.54}$$

$$\bar{T}(\zeta) = \frac{\bar{T}^{(u,v)}(\lambda)}{\bar{T}^{(u,v)}(0)} = \frac{\bar{T}^{(p,s)}(\lambda)}{\bar{T}^{(p,s)}(0)}, \tag{2.55}$$

$$R(\zeta) = \frac{R^{(u,v)}(\lambda)}{\sqrt{\lambda}\, T^{(u,v)}(0)^2} = \frac{\sqrt{\lambda}\, R^{(p,s)}(\lambda)}{\left[T^{(p,s)}(0)\right]^2}, \tag{2.56}$$

$$\bar{R}(\zeta) = \frac{\sqrt{\lambda}\, \bar{R}^{(u,v)}(\lambda)}{\bar{T}^{(u,v)}(0)^2} = \frac{\bar{R}^{(p,s)}(\lambda)}{\sqrt{\lambda}\left[\bar{T}^{(p,s)}(0)\right]^2}, \tag{2.57}$$

$$L(\zeta) = \sqrt{\lambda}\, L^{(u,v)}(\lambda) = \frac{1}{\sqrt{\lambda}}\, L^{(p,s)}(\lambda), \tag{2.58}$$

$$\bar{L}(\zeta) = \frac{1}{\sqrt{\lambda}}\, \bar{L}^{(u,v)}(\lambda) = \sqrt{\lambda}\, \bar{L}^{(p,s)}(\lambda), \tag{2.59}$$

where we remark that (2.58) and (2.59) are the same as (2.50) and (2.51), respectively, because the constant $e^{i\mu/2}$ does not appear in (2.50) and (2.51) and hence the left reflection coefficients for (1.1) are determined by the left reflection coefficients for either of (1.7) or (2.20) without using the value of $e^{i\mu/2}$.

(c) *The scalar constant $e^{i\mu/2}$ is uniquely determined by one of the transmission coefficients for (1.1). Hence, the scattering coefficients for (1.7) and (2.20)* *can be determined from the knowledge of the scattering coefficients for* (1.1) *by using* (2.46)–(2.51).

Proof. From (2.40) we see that $T(0) = 1$, and hence by evaluating (2.46) at $\lambda = 0$ we obtain (2.52). Similarly, from (2.41) we get $\bar{T}(0) = 1$, and hence by evaluating (2.47) at $\lambda = 0$ we have (2.53). Thus, the proof of (a) is complete. By using the value of $e^{i\mu/2}$ from (2.52) or (2.53) in (2.46)–(2.51), we obtain (2.54)–(2.59), respectively. Thus, the proof of (b) is also complete. Finally, from (2.38) or (2.39) we see that the value of $e^{i\mu/2}$ is uniquely determined by one of the transmission coefficients for (1.1), and hence (2.46)–(2.51) can be used to express the scattering coefficients for (1.7) and (2.20) from the knowledge of the scattering coefficients for (1.1), which completes the proof of (c). □

3. The bound states

The bound states for (1.1) correspond to square-integrable column vector solutions to (1.1). The existence and nature of the bound states are completely determined by the potentials q and r appearing in the coefficient matrix in (1.1). When the potentials q and r belong to the Schwartz class, the following are known [8] about the bound states for (1.1):

(a) The bound states cannot occur at any real ζ value in (1.1). In particular, there is no bound state at $\zeta = 0$. The bound states can only occur at a complex value of ζ at which the transmission coefficient $T(\zeta)$ has a pole in the first or third quadrant in the complex ζ-plane or at which the transmission coefficient $\bar{T}(\zeta)$ has a pole in the second or the fourth quadrant. In fact, as indicated in Theorem 2.5 the parameter ζ appears as ζ^2 in the transmission coefficients $T(\zeta)$ and $\bar{T}(\zeta)$, and hence the ζ-values corresponding to the bound states must be symmetrically located with respect to the origin in the complex ζ-plane.

(b) When the potential pairs (u, v) and (p, s) appearing in (1.7) and (2.20), respectively, are related to the potential pair (q, r) as in (2.16), (2.17), (2.21), (2.22), respectively, as seen from (2.46) and (2.47), the poles of the corresponding transmission coefficients for the linear systems (1.1), (1.7), and (2.20) coincide. Hence, the λ-values at which the bound states occurring for (1.1), (1.7), and (2.20) must coincide. We recall that λ and ζ are related to each other as in (2.15).

(c) The number of poles of $T(\zeta)$ in the upper-half complex λ-plane is finite and we use λ_j for $1 \le j \le N$ to denote those distinct poles and we use N to denote their number without taking into account their multiplicities. Similarly, the number of poles of $\bar{T}(\zeta)$ in the lower-half complex λ-plane is finite and we use $\bar{\lambda}_j$ for $1 \le j \le \bar{N}$ to denote those distinct poles and we use $\bar{N}$ to denote their number without taking into account their multiplicities. The multiplicity of each of those poles is finite, and we use m_j to denote the multiplicity of the pole at $\lambda = \lambda_j$ and use $\bar{m}_j$ to denote the corresponding multiplicity of the pole at $\lambda = \bar{\lambda}_j$. We remark that the bound-state poles are not necessarily simple. In the literature [25, 33], it is often unnecessarily assumed that the bound states are simple because of the difficulty to deal with bound states of multiplicities. However, we have an elegant method of handling any number of bound states with any multiplicities, and hence there is no reason to artificially assume the simplicity of bound states.

(d) As indicated in the previous steps, the bound-state information for (1.1) contains the sets $\{\lambda_j, m_j\}_{j=1}^N$ and $\{\bar\lambda_j, \bar m_j\}_{j=1}^{\bar N}$. Furthermore, for each bound state and multiplicity we specify a norming constant. As the bound-state norming constants, we use the double-indexed quantities c_{jk} for $1 \le j \le N$ and $0 \le k \le (m_j - 1)$ and the double-indexed quantities $\bar c_{jk}$ for $1 \le j \le \bar N$ and $0 \le k \le (\bar m_j - 1)$. The construction of the bound-state norming constants c_{jk} from the transmission coefficient $T(\zeta)$ and the Jost solutions $\phi(\zeta, x)$ and $\psi(\zeta, x)$ and the construction of the bound-state norming constants $\bar c_{jk}$ from the transmission coefficient $\bar T(\zeta)$ and the Jost solutions $\bar\phi(\zeta, x)$ and $\bar\psi(\zeta, x)$ are analogous to the constructions presented for the discrete version of (1.1), and we refer the reader to [9] for the details. Such a construction involves the determination of the double-indexed "residues" t_{jk} with $1 \le j \le N$ and $1 \le k \le m_j$ and the determination of the double-indexed "residues" $\bar t_{jk}$ with $1 \le j \le \bar N$ and $1 \le k \le \bar m_j$, respectively, by using the expansions of the transmission coefficients at the bound-state poles, which are given by

$$T(\zeta) = \frac{t_{jm_j}}{(\lambda - \lambda_j)^{m_j}} + \frac{t_{j(m_j-1)}}{(\lambda - \lambda_j)^{m_j-1}} + \cdots + \frac{t_{j1}}{(\lambda - \lambda_j)} + O(1), \quad \lambda \to \lambda_j, \tag{3.1}$$

$$\bar T(\zeta) = \frac{\bar t_{j\bar m_j}}{(\lambda - \bar\lambda_j)^{\bar m_j}} + \frac{\bar t_{j(\bar m_j-1)}}{(\lambda - \bar\lambda_j)^{\bar m_j-1}} + \cdots + \frac{\bar t_{j1}}{(\lambda - \bar\lambda_j)} + O(1), \quad \lambda \to \bar\lambda_j. \tag{3.2}$$

Next, we construct the double-indexed dependency constants γ_{jk}, where we have $1 \le j \le N$ and $0 \le k \le (m_j - 1)$. The dependency constants γ_{jk} appear in the coefficients when we express at $\lambda = \lambda_j$ the value of each $d^k\phi(\zeta, x)/d\lambda^k$ for $0 \le k \le (m_j - 1)$ in terms of the set of values $\{d^k\psi(\zeta, x)/d\lambda^k\}_{k=0}^{m_j-1}$. We get

$$\frac{d^k\phi(\zeta_j, x)}{d\lambda^k} = \sum_{l=0}^{k} \binom{k}{l} \gamma_{j(k-l)} \frac{d^l\psi(\zeta_j, x)}{d\lambda^l}, \quad 0 \le k \le m_j - 1, \tag{3.3}$$

where $\binom{k}{l}$ denotes the binomial coefficient. Note that (3.3) is obtained as follows. From the first equality of (2.12), we have

$$\frac{1}{T(\zeta)} = [\phi(\zeta, x); \psi(\zeta, x)], \tag{3.4}$$

where we recall that the Wronskian is defined as in (2.11). Using (3.1) and the fact that ζ appears as ζ^2 in $T(\zeta)$, from (3.4) it follows that the λ-derivatives of order k for $0 \le k \le (m_j - 1)$ vanish when $\lambda = \lambda_j$ or equivalently when

$\zeta = \zeta_j$. We then recursively obtain (3.3). For the details of the procedure, we refer the reader to [9]. Similarly, the double-indexed dependency constants $\bar{\gamma}_{jk}$ with $1 \le j \le \bar{N}$ and $0 \le k \le (\bar{m}_j - 1)$ appear in the coefficients when we express at $\lambda = \bar{\lambda}_j$ the value of each $d^k\bar{\phi}(\zeta,x)/d\lambda^k$ for $0 \le k \le (\bar{m}_j - 1)$ in terms of the set of values $\{d^k\bar{\psi}(\zeta,x)/d\lambda^k\}_{k=0}^{\bar{m}_j-1}$. We have

$$\frac{d^k\bar{\phi}(\bar{\zeta}_j,x)}{d\lambda^k} = \sum_{l=0}^{k} \binom{k}{l} \bar{\gamma}_{j(k-l)} \frac{d^l\bar{\psi}(\bar{\zeta}_j,x)}{d\lambda^l}, \quad 0 \le k \le \bar{m}_j - 1. \tag{3.5}$$

We remark that (3.5) is derived with the help of the Wronskian relation

$$\frac{1}{\bar{T}(\zeta)} = [\bar{\psi}(\zeta,x); \bar{\phi}(\zeta,x)], \tag{3.6}$$

which follows from the second equality of (2.12). Using (3.2) and the fact that ζ appears as ζ^2 in $\bar{T}(\zeta)$, from (3.6) it follows that the λ-derivatives of order k for $0 \le k \le (\bar{m}_j - 1)$ vanish when $\lambda = \bar{\lambda}_j$ or equivalently when $\zeta = \bar{\zeta}_j$. We then recursively obtain (3.5). The double-indexed norming constants c_{jk} are formed in an explicit manner by using the set of residues $\{t_{jk}\}_{k=1}^{m_j}$ and the set of dependency constants $\{\gamma_{jk}\}_{k=0}^{m_j-1}$, and this procedure is explained in the proof of Theorem 4.2 in the next section and it is similar to the procedure described in Theorem 15 of [9]. In a similar manner, the double-indexed norming constants $\bar{c}_{jk}$ are formed with the help of the set of residues $\{\bar{t}_{jk}\}_{k=1}^{\bar{m}_j}$ and the set of dependency constants $\{\bar{\gamma}_{jk}\}_{k=0}^{\bar{m}_j-1}$. Thus, we obtain the bound-state information for (1.1) consisting of the two sets

$$\left\{\lambda_j, m_j, \{c_{jk}\}_{k=0}^{m_j-1}\right\}_{j=1}^{N}, \quad \left\{\bar{\lambda}_j, \bar{m}_j, \{\bar{c}_{jk}\}_{k=0}^{\bar{m}_j-1}\right\}_{j=1}^{\bar{N}}. \tag{3.7}$$

In the first two examples in Section 6 we illustrate the formulas connecting the norming constants to the residues and the dependency constants.

(e) Let us remark that it is extremely cumbersome to use the bound-state information in the format specified in (3.7) unless that information is organized in an efficient format. In fact, this is the primary reason why it is artificially assumed in the literature that the bound states are simple. The bound-state information contained in (3.7) can be organized in an efficient and elegant manner by introducing a pair of matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ in such a way that the specification of the matrix triplet pair is equivalent to the specification of the bound-state information in (3.7). Furthermore, in the Marchenko method, the bound-state information is easily and in an elegant manner incorporated in the nonhomogeneous term and in the integral kernel in the corresponding Marchenko system if that incorporation is done through the use of matrix triplets. The use of the matrix triplets enables us to deal with any number of bound states with any number of multiplicities in a simple and elegant manner, as if we only have one bound state of multiplicity one. Let us remark that the use of the matrix triplets in the Marchenko method is not confined to any particular linear system, but it can be used on any linear system for which a Marchenko method is available. In fact, this is one of the main reasons why we are interested in establishing the Marchenko method for the linear system given in (1.1).

(f) Without loss of any generality, the matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ can be chosen as the minimal special triplets described later in this section. We refer the reader to [6,16] for the description of the minimality. Essentially, the minimality amounts to choosing each of the square matrices A and $\bar{A}$ with the smallest sizes by removing any columns of zeros or any rows of zeros. By the special triplets, we mean choosing the matrices A and $\bar{A}$ in their Jordan canonical forms and choosing the column vectors B and $\bar{B}$ in the special forms consisting of zeros and ones, as described in (3.9), (3.11), (3.14), and (3.17). The choice of the special forms for the matrix triplets is unique up to the permutations of the corresponding Jordan blocks. We refer the reader to Theorem 3.1 of [6] for the details and for the proof why there is no loss of generality in using the matrix triplets in their minimal special forms.

Next, we show how to convert the bound-state information given in (3.7) into the matrix triplet pair (A,B,C) and $(\bar{A},\bar{B},\bar{C})$. Since there is no loss of generality in choosing the matrix triplets in their special forms, we only deal with those special forms. For simplicity and clarity, we outline the main steps of the procedure by omitting the details. We refer the reader to [9] where the details of the procedure are presented for the discrete version of (1.1). The steps presented in [9] are general enough to apply to (1.1) and to other linear systems. Let us also remark that for linear systems for which the potentials appear in diagonal blocks in the corresponding coefficient matrix, only one matrix triplet (A,B,C) is needed. On the other hand, for linear systems for which the potentials appear in off-diagonal blocks in the corresponding coefficient matrix, a pair of matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ is used. The potentials q and r appear in the off-diagonal entries in the coefficient matrix in (1.1), and hence we convert the bound-state information into the format consisting of the triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$. For the use of matrix triplets for some other linear systems, we refer the reader to [5–7,14,17,18].

The conversion of the bound-state information from (3.7) to the matrix triplet pair (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ involves the following steps:

(a) For each bound state at $\lambda=\lambda_j$ with $1\le j\le N$, we form the matrix subtriplet (A_j,B_j,C_j) as

$$A_j:=\begin{bmatrix}\lambda_j & 1 & 0 & \cdots & 0 & 0\\ 0 & \lambda_j & 1 & \cdots & 0 & 0\\ 0 & 0 & \lambda_j & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_j & 1\\ 0 & 0 & 0 & \dots & 0 & \lambda_j\end{bmatrix}, \tag{3.8}$$

$$B_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_j := \begin{bmatrix} c_{j(m_j-1)} & c_{j(m_j-2)} & \cdots & c_{j1} & c_{j0} \end{bmatrix}, \tag{3.9}$$

where A_j is the $m_j \times m_j$ square matrix in the Jordan canonical form with λ_j appearing in the diagonal entries, B_j is the column vector with m_j components that are all zero except for the last entry which is 1, and C_j is the row vector with m_j components containing all the norming constants in the order indicated in (3.9). Note that if the bound state at $\lambda = \lambda_j$ is simple, then we have

$$A_j = \begin{bmatrix} \lambda_j \end{bmatrix}, \quad B_j = \begin{bmatrix} 1 \end{bmatrix}, \quad C_j = \begin{bmatrix} c_{j0} \end{bmatrix}.$$

Similarly, for each bound state at $\lambda = \bar{\lambda}_j$ with $1 \le j \le \bar{N}$ we form the matrix subtriplet $(\bar{A}_j, \bar{B}_j, \bar{C}_j)$ as

$$\bar{A}_j := \begin{bmatrix} \bar{\lambda}_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{\lambda}_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & \bar{\lambda}_j & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{\lambda}_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \bar{\lambda}_j \end{bmatrix}, \tag{3.10}$$

$$\bar{B}_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \bar{C}_j := \begin{bmatrix} \bar{c}_{j(\bar{m}_j-1)} & \bar{c}_{j(\bar{m}_j-2)} & \cdots & \bar{c}_{j1} & \bar{c}_{j0} \end{bmatrix}, \tag{3.11}$$

where $\bar{A}_j$ is the $\bar{m}_j \times \bar{m}_j$ square matrix in the Jordan canonical form with $\bar{\lambda}_j$ appearing in the diagonal entries, $\bar{B}_j$ is the column vector with $\bar{m}_j$ components that are all zero except for the last entry which is 1, and $\bar{C}_j$ is the row vector with $\bar{m}_j$ components containing all the norming constants in the order indicated in (3.11).

(b) Using A_j with $1 \le j \le N$, we form the $\mathcal{N} \times \mathcal{N}$ block-diagonal matrix A as

$$A := \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & A_N \end{bmatrix}, \tag{3.12}$$

where the zeros are the zero matrices of appropriate sizes. Here, the quantity $\mathcal{N}$ is defined as

$$\mathcal{N} := \sum_{j=1}^{N} m_j, \tag{3.13}$$

and it represents the number of bound-state poles in the upper-half complex λ-plane by including the multiplicities. We also form the column vector B with $\mathcal{N}$ components and the row vector C with $\mathcal{N}$ components as

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_N \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 & \cdots & C_N \end{bmatrix}. \tag{3.14}$$

Similarly, we define $\bar{\mathcal{N}}$ as

$$\bar{\mathcal{N}} := \sum_{j=1}^{\bar{N}} \bar{m}_j, \tag{3.15}$$

which represents the number of bound-state poles in the lower-half complex λ-plane by including the multiplicities. We then use $\bar{A}_j$ with $1 \le j \le \bar{N}$ in order to form the $\bar{\mathcal{N}} \times \bar{\mathcal{N}}$ block-diagonal matrix $\bar{A}$ as

$$\bar{A} := \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 & 0 \\ 0 & \bar{A}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{A}_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \bar{A}_N \end{bmatrix}, \tag{3.16}$$

where the zeros denote the zero matrices of appropriate sizes. We also form the column vector $\bar{B}$ with $\bar{\mathcal{N}}$ components and the row vector $\bar{C}$ with $\bar{\mathcal{N}}$ components as

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \vdots \\ \bar{B}_N \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \cdots & \bar{C}_N \end{bmatrix}. \tag{3.17}$$

4. The Marchenko method

In this section we develop the Marchenko method for (1.1) by deriving the corresponding Marchenko system of linear integral equations and also by showing how the Jost solutions and the potentials are recovered from the solution to that Marchenko system. We present the derivation of the Marchenko system in such a way that the procedure can be applied to other linear systems and to their discrete analogs. For the simplicity of the presentation, we first provide the derivation in the absence of bound states, and then we indicate the main modification needed to include the bound-state information in the Marchenko system.

In the following we outline the basic steps in the development of our Marchenko method for (1.1) by showing the similarities and differences with the development of the standard Marchenko method:

(a) We start with the Riemann–Hilbert problem for (1.1) by expressing the two Jost solutions $\phi(\zeta, x)$ and $\bar{\phi}(\zeta, x)$ as a linear combination of the Jost solutions $\psi(\zeta, x)$ and $\bar{\psi}(\zeta, x)$. This eventually yields the Marchenko system for (1.1) with $x < y < +\infty$ as an analog of (1.4). Note that this is also the step used in the derivation of the standard Marchenko method. In order to derive the Marchenko system for (1.1) with $-\infty < y < x$ as an analog of (1.5), we need to express the Jost solutions $\psi(\zeta, x)$ and $\bar{\psi}(\zeta, x)$ as a linear combination of the Jost solutions $\phi(\zeta, x)$ and $\bar{\phi}(\zeta, x)$. However, in this paper we only present the derivation of the former Marchenko system and hence only deal with the Riemann–Hilbert problem for the former case. We remark that the coefficients in the Riemann–Hilbert problem associated with the Marchenko system with $x < y < +\infty$ are directly related to the scattering coefficients $T(\zeta)$, $\bar{T}(\zeta)$, $R(\zeta)$, and $\bar{R}(\zeta)$, and the coefficients in the Riemann–Hilbert problem associated with the Marchenko system with $-\infty < y < x$ are directly related to the scattering coefficients $T(\zeta)$, $\bar{T}(\zeta)$, $L(\zeta)$, and $\bar{L}(\zeta)$.

(b) Next, we combine the two column-vector equations arising in the formulation of the Riemann–Hilbert problem into a 2×2 matrix-valued system. This step is also used in the development of the standard Marchenko method.

(c) We slightly modify our 2×2 matrix-valued system obtained in the previous step. This modification is not needed in the development of the standard Marchenko method. The modification involving the diagonal entries is carried out in order to take into account the large ζ-asymptotics of the Jost solutions. The modification involving the off-diagonal entries is carried out in order to formulate the 2×2 matrix-valued Riemann–Hilbert problem in the spectral parameter λ rather than in ζ, where λ and ζ are related to each other as in (2.15).

(d) With the modifications described in the previous step, we are able to take the Fourier transform from the λ-space to the y-space. This yields the 2×2 coupled Marchenko system. This step is also used in the development of the standard Marchenko method.

(e) We uncouple the 2×2 matrix-valued Marchenko system and obtain the associated uncoupled scalar Marchenko integral equations. This is also the step used in the development of the standard Marchenko method.

(f) With the help of the inverse Fourier transform, we show how the Jost solutions to (1.1) are constructed from the solution to the Marchenko system. This is also the step used in the development of the standard Marchenko method.

(g) Finally, we describe how the potentials q and r appearing in (1.1) are recovered from the solution to our Marchenko system. This step is slightly more involved than the step used in the development of the standard Marchenko method. However, the formulas for the potentials are explicit in terms of the solution to our Marchenko system.

In the next theorem we introduce the 2×2 matrix-valued Marchenko integral system for (1.1) in the absence of bound states.

Theorem 4.1. *Assume that the potentials q and r in (1.1) belong to the Schwartz class, and assume that there are no bound states. Then, the corresponding Marchenko system for (1.1) is given by*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar K_1(x,y) & K_1(x,y) \\ \bar K_2(x,y) & K_2(x,y) \end{bmatrix} + \begin{bmatrix} 0 & \hat{\bar R}(x+y) \\ \hat R(x+y) & 0 \end{bmatrix}$$
$$+ \int_x^\infty dz \begin{bmatrix} -iK_1(x,z)\,\hat R'(z+y) & \bar K_1(x,z)\,\hat{\bar R}(z+y) \\ K_2(x,z)\,\hat R(z+y) & i\bar K_2(x,z)\,\hat{\bar R}'(z+y) \end{bmatrix}, \quad x<y, \tag{4.1}$$

where $\hat R(y)$ and $\hat{\bar R}(y)$ are related to the reflection coefficients $R(\zeta)$ and $\bar R(\zeta)$ for (1.1) via the Fourier transforms given by

$$\hat R(y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, \frac{R(\zeta)}{\zeta}\, e^{i\lambda y}, \quad \hat{\bar R}(y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\, \frac{\bar R(\zeta)}{\zeta}\, e^{-i\lambda y}, \tag{4.2}$$

with $\hat R'(y)$ and $\hat{\bar R}'(y)$ denoting the derivatives of $\hat R(y)$ and $\hat{\bar R}(y)$, respectively, and λ being related to ζ as in (2.15). We also have

$$K_1(x,y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \left[\frac{e^{i\mu/2}\ \psi_1(\zeta,x)}{\zeta\, E(x)}\right] e^{-i\lambda y}, \tag{4.3}$$

$$K_2(x,y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \left[e^{-i\mu/2}\, E(x)\,\psi_2(\zeta,x) - e^{i\lambda x}\right] e^{-i\lambda y}, \tag{4.4}$$

$$\bar K_1(x,y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \left[\frac{e^{i\mu/2}\,\bar\psi_1(\zeta,x)}{E(x)} - e^{-i\lambda x}\right] e^{i\lambda y}, \tag{4.5}$$

$$\bar K_2(x,y) := \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda \left[\frac{e^{-i\mu/2}\, E(x)\,\bar\psi_2(\zeta,x)}{\zeta}\right] e^{i\lambda y}, \tag{4.6}$$

with $E(x)$ and μ being the quantities defined in (2.18) and (2.19), respectively, and $\psi_1(\zeta,x)$, $\psi_2(\zeta,x)$, $\bar\psi_1(\zeta,x)$, and $\bar\psi_2(\zeta,x)$ are the components of the Jost solutions given in (2.9).

Proof. For notational simplicity, we suppress the arguments and write ψ for $\psi(\zeta,x)$, $\bar\psi$ for $\bar\psi(\zeta,x)$, ϕ for $\phi(\zeta,x)$, $\bar\phi$ for $\bar\phi(\zeta,x)$, T for $T(\zeta)$, $\bar T$ for $\bar T(\zeta)$, R for $R(\zeta)$, $\bar R$ for $\bar R(\zeta)$, and E for $E(x)$. From the asymptotics in (2.1) and (2.2) we see that the columns of the Jost solutions ψ and $\bar\psi$ to (1.1) are linearly independent, and hence those four columns form a fundamental set of column-vector solutions to (1.1). Thus, each of the other two Jost solutions ϕ and $\bar\phi$ can be expressed as linear combinations of ψ and $\bar\psi$. With the help of (2.1), (2.2), (2.7), and (2.8), for $\zeta\in\mathbb{R}$ we obtain

$$\begin{cases} \phi = \dfrac{1}{T}\,\bar\psi + \dfrac{R}{T}\,\psi, \\[2mm] \bar\phi = \dfrac{\bar R}{\bar T}\,\bar\psi + \dfrac{1}{\bar T}\,\psi, \end{cases} \tag{4.7}$$

or equivalently

$$\begin{cases} T\,\phi = \bar\psi + R\,\psi, \\ \bar T\,\bar\phi = \bar R\,\bar\psi + \psi, \end{cases} \tag{4.8}$$

which forms our Riemann–Hilbert problem. The solution to the Riemann–Hilbert problem consists of the construction of the Jost solutions from the knowledge of T, $\bar T$, R, and $\bar R$. Let us now derive our Marchenko system starting from (4.8). We first combine the two column-vector equations in (4.8) and obtain the 2×2 matrix-valued system

$$\begin{bmatrix} T\,\phi & \bar T\,\bar\phi \end{bmatrix} = \begin{bmatrix} \bar\psi & \psi \end{bmatrix} + \begin{bmatrix} R\,\psi & \bar R\,\bar\psi \end{bmatrix}. \tag{4.9}$$

Using (2.9) and (2.10), we write (4.9) as

$$\begin{bmatrix} T\,\phi_1 & \bar T\,\bar\phi_1 \\ T\,\phi_2 & \bar T\,\bar\phi_2 \end{bmatrix} = \begin{bmatrix} \bar\psi_1 & \psi_1 \\ \bar\psi_2 & \psi_2 \end{bmatrix} + \begin{bmatrix} R\,\psi_1 & \bar R\,\bar\psi_1 \\ R\,\psi_2 & \bar R\,\bar\psi_2 \end{bmatrix}. \tag{4.10}$$

We first premultiply (4.10) by the diagonal matrix $\operatorname{diag}\{e^{i\mu/2}E^{-1}, e^{-i\mu/2}E\}$ and then divide by ζ the off-diagonal entries in the resulting matrix-valued system. From the resulting 2×2 matrix-valued equation, we subtract the diagonal matrix $\operatorname{diag}\{e^{-i\lambda x}, e^{i\lambda x}\}$ from both sides, and we obtain

$$\begin{aligned} &\begin{bmatrix} e^{i\mu/2}\,E^{-1}\,T\,\phi_1 - e^{-i\lambda x} & \dfrac{1}{\zeta} e^{i\mu/2}\,E^{-1}\bar T\,\bar\phi_1 \\ \dfrac{1}{\zeta} e^{-i\mu/2}\,E\,T\,\phi_2 & e^{-i\mu/2}\,E\,\bar T\,\bar\phi_2 - e^{i\lambda x} \end{bmatrix} \\ &\qquad = \begin{bmatrix} e^{i\mu/2}\,E^{-1}\,\bar\psi_1 - e^{-i\lambda x} & \dfrac{1}{\zeta} e^{i\mu/2}\,E^{-1}\psi_1 \\ \dfrac{1}{\zeta} e^{-i\mu/2}\,E\,\bar\psi_2 & e^{-i\mu/2}\,E\,\psi_2 - e^{i\lambda x} \end{bmatrix} \\ &\qquad\quad + \begin{bmatrix} e^{i\mu/2}\,E^{-1}\,R\,\psi_1 & \dfrac{1}{\zeta} e^{i\mu/2}\,E^{-1}\,\bar R\,\bar\psi_1 \\ \dfrac{1}{\zeta} e^{-i\mu/2}\,E\,R\,\psi_2 & e^{-i\mu/2}\,E\,\bar R\,\bar\psi_2 \end{bmatrix}. \end{aligned} \tag{4.11}$$

We now take the Fourier transform of (4.11) with $\int_{-\infty}^{\infty} d\lambda\, e^{i\lambda y}/2\pi$ in the first columns and with $\int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda y}/2\pi$ in the second columns. This yields the 2×2 matrix-valued equation

$$\text{LHS} = K(x,y) + \text{RHS}, \tag{4.12}$$

where we have defined

$$K(x,y) := \begin{bmatrix} \bar K_1(x,y) & K_1(x,y) \\ \bar K_2(x,y) & K_2(x,y) \end{bmatrix}, \tag{4.13}$$

with the entries $K_1(x,y)$, $K_2(x,y)$, $\bar K_1(x,y)$, and $\bar K_2(x,y)$ are as in (4.3)–(4.6), respectively, and

$$\text{LHS} := \begin{bmatrix} \text{LHS}_{11} & \text{LHS}_{12} \\ \text{LHS}_{21} & \text{LHS}_{22} \end{bmatrix}, \tag{4.14}$$

$$\text{RHS} := \begin{bmatrix} \text{RHS}_{11} & \text{RHS}_{12} \\ \text{RHS}_{21} & \text{RHS}_{22} \end{bmatrix}. \tag{4.15}$$

We remark that the matrix entries in (4.14) and (4.15) are defined as

$$\text{LHS}_{11} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left[e^{i\mu/2} E^{-1} T \phi_1 - e^{-i\lambda x} \right] e^{i\lambda y}, \tag{4.16}$$

$$\text{LHS}_{12} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{i\mu/2}\, E^{-1} \bar T\, \frac{\bar\phi_1}{\zeta}\, e^{-i\lambda y}, \tag{4.17}$$

$$\text{LHS}_{21} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{-i\mu/2}\, E\, T\, \frac{\phi_2}{\zeta}\, e^{i\lambda y}, \tag{4.18}$$

$$\text{LHS}_{22} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left[e^{-i\mu/2}\, E\, \bar T\, \bar\phi_2 - e^{i\lambda x} \right] e^{-i\lambda y}, \tag{4.19}$$

$$\text{RHS}_{11} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{i\mu/2}\, E^{-1}\, R\, \psi_1\, e^{i\lambda y}, \tag{4.20}$$

$$\text{RHS}_{12} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{i\mu/2}\, E^{-1}\, \frac{\bar R}{\zeta}\, \bar\psi_1\, e^{-i\lambda y}, \tag{4.21}$$

$$\text{RHS}_{21} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{-i\mu/2}\, E\, \frac{R}{\zeta}\, \psi_2\, e^{i\lambda y}, \tag{4.22}$$

$$\text{RHS}_{22} := \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi}\, e^{-i\mu/2}\, E\, \bar R\, \bar\psi_2\, e^{-i\lambda y}. \tag{4.23}$$

Using the continuity properties of the Jost solutions stated in Theorem 2.2, the continuity and asymptotic properties of the scattering coefficients presented in Theorem 2.5, and the small and large ζ-asymptotics of the Jost solutions stated in Theorems 2.3 and 2.4, respectively, we see that each integrand in (4.3)–(4.6) and (4.16)–(4.23) is continuous in $\lambda \in \mathbb{R}$ and $O(1/\lambda)$ as $\lambda \to \pm\infty$. Thus, the L^2-Fourier transforms in (4.3)–(4.6) and (4.16)–(4.23) are all well defined. Furthermore, in the absence of bound states, for $y > x$ the integrands in (4.3) and (4.4) are analytic in $\lambda \in \mathbb{C}^+$ and uniformly $o(1)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Similarly, in the absence of bound states, for $y > x$ the integrands in (4.5) and (4.6) are analytic in $\lambda \in \mathbb{C}^-$ and uniformly $o(1)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. Thus, from Jordan's lemma it follows that the four entries of the 2×2 matrix $K(x,y)$ defined in (4.13) are each equal to zero when $x > y$. Hence, using the inverse Fourier transform, from (4.3)–(4.6) we get

$$e^{i\mu/2}\, \frac{\psi_1(\zeta,x)}{\zeta\, E(x)} = \int_x^{\infty} dz\, K_1(x,z)\, e^{i\lambda z}, \tag{4.24}$$

$$e^{-i\mu/2}\, E(x)\, \psi_2(\zeta,x) = e^{i\lambda x} + \int_x^{\infty} dz\, K_2(x,z)\, e^{i\lambda z}, \tag{4.25}$$

$$e^{i\mu/2}\,\frac{\bar\psi_1(\zeta,x)}{E(x)}=e^{-i\lambda x}+\int_x^\infty dz\,\bar K_1(x,z)\,e^{-i\lambda z},\tag{4.26}$$

$$e^{-i\mu/2}\,E(x)\,\frac{\bar\psi_2(\zeta,x)}{\zeta}=\int_x^\infty dz\,\bar K_2(x,z)\,e^{-i\lambda z}.\tag{4.27}$$

Let us now show that each of the four entries of RHS defined in (4.15) is a convolution. By using the inverse Fourier transform, from (4.2) we have

$$\frac{R(\zeta)}{\zeta}=\int_{-\infty}^\infty ds\,\hat R(s)\,e^{-i\lambda s},\qquad \frac{\bar R(\zeta)}{\zeta}=\int_{-\infty}^\infty ds\,\hat{\bar R}(s)\,e^{i\lambda s}.\tag{4.28}$$

Also, by taking the derivatives, from (4.2) we obtain

$$\hat R'(y)=\frac{i}{2\pi}\int_{-\infty}^\infty d\lambda\,\frac{R(\zeta)}{\zeta}\lambda\,e^{i\lambda y},\quad \hat{\bar R}'(y)=-\frac{i}{2\pi}\int_{-\infty}^\infty d\lambda\,\frac{\bar R(\zeta)}{\zeta}\lambda\,e^{-i\lambda y}.\tag{4.29}$$

Using the inverse Fourier transform, from (4.29) we get

$$\frac{R(\zeta)}{\zeta}\,\lambda=-i\int_{-\infty}^\infty ds\,\hat R'(s)\,e^{-i\lambda s},\qquad \frac{\bar R(\zeta)}{\zeta}\,\lambda=i\int_{-\infty}^\infty ds\,\hat{\bar R}'(s)\,e^{i\lambda s}.\tag{4.30}$$

Note that (4.20) is equivalent to

$$\text{RHS}_{11}=\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\,e^{i\lambda y}\left(e^{i\mu/2}\,E^{-1}\frac{\psi_1}{\zeta}\right)\left(\frac{R}{\zeta}\,\lambda\right).\tag{4.31}$$

Using (4.24) and the first equality of (4.30) on the right-hand side of (4.31), we get the convolution

$$\text{RHS}_{11}=-i\int_x^\infty dz\,K_1(x,z)\,\hat R'(z+y).\tag{4.32}$$

Proceeding in a similar manner, we write (4.23) as

$$\text{RHS}_{22}=\int_{-\infty}^\infty \frac{d\lambda}{2\pi}\,e^{-i\lambda y}\left(e^{-i\mu/2}\,E\,\frac{\bar\psi_2}{\zeta}\right)\left(\frac{\bar R}{\zeta}\,\lambda\right).\tag{4.33}$$

Using (4.27) and the second equality of (4.30) on the right-hand side of (4.33), we obtain the convolution

$$\text{RHS}_{22}=\int_x^\infty dz\,\bar K_2(x,z)\,\hat{\bar R}'(z+y).\tag{4.34}$$

In a similar manner, by using (4.25), (4.26), and (4.28), we write (4.21) and (4.22), respectively, as

$$\text{RHS}_{12}=\hat{\bar R}(x+y)+\int_x^\infty dz\,\bar K_1(x,z)\,\hat{\bar R}(z+y),\tag{4.35}$$

$$\text{RHS}_{21}=\hat R(x+y)+\int_x^\infty dz\,K_2(x,z)\,\hat R(z+y).\tag{4.36}$$

Hence, using (4.32), (4.34), (4.35), and (4.36) in (4.12), we see that RHS is equal to the sum of the second and third terms on the right-hand side of (4.1). Thus, in order to complete the derivation of (4.1), it is sufficient to show that LHS is the 2×2 zero matrix when $x < y$ in the absence of bound states. This is proved as follows. When $x < y$, with the help of Theorems 2.2–2.5, we observe that the integrands in (4.16) and (4.18) are analytic in $\lambda \in \mathbb{C}^+$, are continuous in $\lambda \in \overline{\mathbb{C}^+}$, and behave uniformly as $O(1/\lambda)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. Hence, when $x < y$, using Jordan's lemma and the residue theorem we conclude that LHS_{11} and LHS_{21} are both zero. Similarly, when $x < y$, with the help of Theorems 2.2–2.5, we observe that the integrands in (4.17) and (4.19) are analytic in $\lambda \in \mathbb{C}^-$, continuous in $\lambda \in \overline{\mathbb{C}^-}$, and uniformly $O(1/\lambda)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. Hence, when $x < y$, using Jordan's lemma and the residue theorem we conclude that LHS_{12} and LHS_{22} are both zero. Thus, the proof is complete. □

The Marchenko integral system we have established in (4.1) is valid provided (1.1) has no bound states. When the bound states are present, the only modification needed in the proof of Theorem 4.1 is that the quantity LHS appearing in (4.12) and (4.14) is no longer equal to the zero matrix due to the fact that we must take into account the bound-state poles of the transmission coefficients in evaluating the integrals (4.16)–(4.19). It turns out that, using the matrix triplet pair (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ appearing in (3.12), (3.14), (3.16), (3.17), we can express the effect of the bound states in the Marchenko system in a simple and elegant manner. This amounts to replacing $\hat{R}(y)$ and $\hat{\bar{R}}(y)$ appearing in the Marchenko system (4.1) with $\Omega(y)$ and $\bar{\Omega}(y)$, respectively, where we have defined

$$\Omega(y) := \hat{R}(y) + C\, e^{iAy} B, \quad \bar{\Omega}(y) := \hat{\bar{R}}(y) + \bar{C}\, e^{-i\bar{A}y} \bar{B}. \tag{4.37}$$

By taking the derivatives, from (4.37) we get

$$\Omega'(y) = \hat{R}'(y) + i\, C A\, e^{iAy} B, \quad \bar{\Omega}'(y) = \hat{\bar{R}}'(y) - i\, \bar{C} \bar{A}\, e^{-i\bar{A}y} \bar{B}, \tag{4.38}$$

and hence in (4.1) we also replace $\hat{R}'(y)$ and $\hat{\bar{R}}'(y)$ with $\Omega'(y)$ and $\bar{\Omega}'(y)$, respectively.

In fact, in the Marchenko equations for any linear system, the two substitutions

$$\hat{R}(y) \mapsto \hat{R}(y) + C\, e^{iAy} B, \quad \hat{\bar{R}}(y) \mapsto \hat{\bar{R}}(y) + \bar{C}\, e^{-i\bar{A}y} \bar{B}, \tag{4.39}$$

are all that is needed in order to take into consideration the effect of any number of bound states with any multiplicities. Certainly, for linear systems where the potentials appear in the diagonal blocks in the coefficient matrix rather than in the off-diagonal blocks, we only use one matrix triplet (A, B, C), and in that case (4.39) still holds with the understanding that the second matrix triplet $(\bar{A}, \bar{B}, \bar{C})$ is absent. We remark that (4.39) is elegant for several reasons. When there is only one simple bound state, the matrix A has size 1×1. Hence, as far as algebraic operations are concerned, the eigenvalue of the matrix A can be viewed as the matrix itself. In that sense, there is an apparent correspondence between the factor $e^{i\lambda y}$ in (4.2) and e^{iAy} in (4.39) induced by $\lambda \leftrightarrow A$. This also indicates the usefulness of matrix exponentials in dealing with bound states. The same is also true for the correspondence between the factor $e^{-i\lambda y}$ in (4.2) and $e^{-i\bar{A}y}$ in (4.39) induced by $\lambda \leftrightarrow \bar{A}$. The information containing any number of bound states with any multiplicities and with the corresponding bound-state norming constants is all imbedded in (4.39) through the structure of the two matrix triplets there.

In the next theorem we present the Marchenko integral system for (1.1) in the presence of bound states.

Theorem 4.2. *Let the potentials q and r in (1.1) belong to the Schwartz class. In the presence of bound states, the corresponding Marchenko system for (1.1) is obtained from (4.1) by using the substitutions (4.39), where (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ are the pair of matrix triplets appearing in (3.12), (3.14), (3.16), (3.17). Hence, the Marchenko system for (1.1) is given by*

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{K}_1(x,y) & K_1(x,y) \\ \bar{K}_2(x,y) & K_2(x,y) \end{bmatrix} + \begin{bmatrix} 0 & \bar{\Omega}(x+y) \\ \Omega(x+y) & 0 \end{bmatrix}$$
$$+ \int_x^\infty dz \begin{bmatrix} -iK_1(x,z)\,\Omega'(z+y) & \bar{K}_1(x,z)\,\bar{\Omega}(z+y) \\ K_2(x,z)\,\Omega(z+y) & i\bar{K}_2(x,z)\,\bar{\Omega}'(z+y) \end{bmatrix}, \quad x<y, \tag{4.40}$$

where $\Omega(y)$ and $\bar{\Omega}(y)$ are the quantities defined in (4.37); $\Omega'(y)$ and $\bar{\Omega}'(y)$ are the derivatives appearing in (4.38); and $K_1(x,y)$, $K_2(x,y)$, $\bar{K}_1(x,y)$, and $\bar{K}_2(x,y)$ are the quantities defined in (4.3)–(4.6), respectively.

Proof. As indicated in the proof of Theorem 4.1, the quantity LHS in (4.14) is no longer equal to the 2×2 zero matrix when the bound states are present. For $x<y$, the integrands in (4.16) and (4.18) are continuous in $\lambda \in \mathbb{R}$, are $O(1/\lambda)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$, and are meromorphic in $\lambda \in \mathbb{C}^+$ with the poles at $\lambda = \lambda_j$ with multiplicity m_j for $1 \le j \le N$, where those poles are the bound-state poles of $T(\zeta)$. Hence, when $x<y$ those integrals can be evaluated by using the residue theorem. The resulting expressions contain the residues t_{jk} appearing in (3.1) and $d^k\phi(\zeta_j,x)/d\lambda^k$ for $1 \le j \le N$ and $0 \le k \le (m_j-1)$. Using (3.3) in the resulting expressions, we express those integrals in terms of the residues t_{jk} and the dependency constants γ_{jk} appearing in (3.3). In an analogous manner, for $x<y$ the integrands in (4.17) and (4.19) are continuous in $\lambda \in \mathbb{R}$, are $O(1/\lambda)$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$, and are meromorphic in $\lambda \in \mathbb{C}^-$ with the poles at $\lambda = \bar{\lambda}_j$ with multiplicity $\bar{m}_j$ for $1 \le j \le \bar{N}$, where those poles are the bound-state poles of $\bar{T}(\zeta)$. Thus, when $x<y$ those integrals can be evaluated by using the residue theorem. The resulting expressions contain the residues $\bar{t}_{jk}$ appearing in (3.2) and $d^k\bar{\phi}(\bar{\zeta}_j,x)/d\lambda^k$ for $1 \le j \le \bar{N}$ and $0 \le k \le (\bar{m}_j-1)$. Using (3.5) in the resulting expressions, we express those integrals in terms of the residues $\bar{t}_{jk}$ and the dependency constants $\bar{\gamma}_{jk}$ appearing in (3.5). We omit the details because the procedure is similar to that given in the proof of Theorem 15 of [9]. The only effect of the contribution from LHS to (4.12) amounts to the substitutions specified in (4.39). Hence, with the help of (4.1), (4.37), and (4.38) we obtain (4.40), where the norming constants c_{jk} are explicitly expressed in terms of t_{jk},

γ_{jk}, and ζ_j, and the norming constants $\bar{c}_{jk}$ are explicitly expressed in terms of $\bar{t}_{jk}$, $\bar{\gamma}_{jk}$, and $\bar{\zeta}_j$. □

Let us remark that the 2×2 matrix-valued coupled Marchenko system presented in (4.40) can readily be uncoupled, and it is equivalent to the respective uncoupled scalar Marchenko integral equations for $K_1(x, y)$ and $\bar{K}_2(x, y)$ given by

$$\begin{cases} K_1(x,y) + \bar{\Omega}(x+y) + i \displaystyle\int_x^\infty dz \int_x^\infty ds\, K_1(x,z)\, \Omega'(z+s)\, \bar{\Omega}(s+y) = 0, \\ \bar{K}_2(x,y) + \Omega(x+y) - i \displaystyle\int_x^\infty dz \int_x^\infty ds\, \bar{K}_2(x,z)\, \bar{\Omega}'(z+s)\, \Omega(s+y) = 0, \end{cases} \tag{4.41}$$

where $x < y$, with the auxiliary equations given by

$$\begin{cases} \bar{K}_1(x,y) = i \displaystyle\int_x^\infty dz\, K_1(x,z)\, \Omega'(z+y), \quad x < y, \\ K_2(x,y) = -i \displaystyle\int_x^\infty dz\, \bar{K}_2(x,z)\, \bar{\Omega}'(z+y), \quad x < y. \end{cases} \tag{4.42}$$

Having established the Marchenko system for (1.1), our goal now is to recover the potentials q and r in (1.1) from the solution $K(x, y)$ to the Marchenko system (4.40) or from the equivalent system of uncoupled equations given in (4.41) and (4.42). In preparation for this, in the next theorem we relate the entries of $K(x, x)$ to some key quantities for (1.1).

Proposition 4.3. *Assume that the potentials q and r appearing in (1.1) belong to the Schwartz class. Let $K(x, y)$ be the solution to the Marchenko system (4.40), with the components $K_1(x, y)$, $K_2(x, y)$, $\bar{K}_1(x, y)$, $\bar{K}_2(x, y)$ as in (4.13). In the limit $y \to x^+$ we have*

$$K_1(x,x) = -\frac{e^{i\mu}}{2} \frac{q(x)}{E(x)^2}, \tag{4.43}$$

$$K_2(x,x) = -\frac{iq(x)\, r(x)}{4} + \frac{1}{2} \int_x^\infty dy\, \sigma(y), \tag{4.44}$$

$$\bar{K}_1(x,x) = \frac{1}{2} \int_x^\infty dy\, \sigma(y), \tag{4.45}$$

$$\bar{K}_2(x,x) = -\frac{e^{-i\mu}}{2}\, r(x)\, E(x)^2, \tag{4.46}$$

where $E(x)$, μ, and $\sigma(x)$ are the quantities defined in (2.18), (2.19), and (2.36), respectively.

Proof. Let us recall that ζ and λ are related to each other as in (2.15). We obtain the proof by establishing the large λ-asymptotics of the Jost solutions $\psi(\zeta, x)$ and $\bar{\psi}(\zeta, x)$ expressed in terms of the Fourier transforms given in (4.24)–(4.27) and by comparing the results with the corresponding asymptotic expressions given in Theorem 2.4. For example, in order to establish (4.43), we write (4.24) as

$$\frac{e^{i\mu/2}\,\psi_1(\zeta,x)}{\zeta\,E(x)}=\int_x^\infty dy\left[K_1(x,y)\,\frac{d}{dy}\,\frac{e^{i\lambda y}}{i\lambda}\right], \tag{4.47}$$

and using integration by parts, from (4.47) we obtain

$$\frac{e^{i\mu/2}\,\psi_1(\zeta,x)}{\zeta\,E(x)}=K_1(x,y)\,\frac{e^{i\lambda y}}{i\lambda}\Bigg|_{y=x}^{y=\infty}-\int_x^\infty dy\,\frac{e^{i\lambda y}}{i\lambda}\,\frac{\partial\,K_1(x,y)}{\partial y}. \tag{4.48}$$

Since the potentials in (1.1) belong to the Schwartz class, the corresponding Jost solutions and their Fourier transforms are sufficiently smooth. Letting $\lambda\to\pm\infty$ in (4.48) and using the Riemann–Lebesgue lemma, from (4.48) we get

$$\frac{e^{i\mu/2}\,\psi_1(\zeta,x)}{\zeta\,E(x)}=-\frac{K_1(x,x)\,e^{i\lambda x}}{i\lambda}+o\left(\frac{1}{\lambda}\right). \tag{4.49}$$

The large ζ-asymptotics of $\psi_1(\zeta,x)$ is given in the first component of (2.35), and we use it on the left-hand side of (4.49) and obtain

$$e^{i\mu+i\lambda x}\left[\frac{q(x)}{2i\lambda\,E(x)^2}+O\left(\frac{1}{\lambda^2}\right)\right]=-\frac{K_1(x,x)\,e^{i\lambda x}}{i\lambda}+o\left(\frac{1}{\lambda}\right),\quad \lambda\to\pm\infty. \tag{4.50}$$

By comparing the first-order terms on both sides of (4.50), we get (4.43). We then establish (4.44)–(4.46) by proceeding in a similar manner, i.e. by using integration by parts in (4.25)–(4.27), obtain the large λ-asymptotics in the resulting expressions with the help of the Riemann–Lebesgue lemma, then by using the large ζ-asymptotics from (2.35) and (2.37) in the resulting equalities, and finally by comparing the first-order terms in the corresponding asymptotic expressions. □

In the next theorem we show how to recover the relevant quantities for (1.1), including the potentials and the Jost solutions, from the solution to the corresponding Marchenko system (4.40).

Theorem 4.4. *Let the potentials q and r in (1.1) belong to the Schwartz class. The relevant quantities are recovered from the solution to the Marchenko system (4.40) or equivalently from the uncoupled counterpart given in (4.41) and (4.42) as follows:*

(a) *The scalar quantity $E(x)$ given in (2.18) is recovered from the solution to the Marchenko system as*

$$E(x)=\exp\left(2\int_{-\infty}^x dz\,Q(z)\right), \tag{4.51}$$

where $Q(x)$ is the auxiliary scalar quantity constructed from $\bar K_1(x,y)$ and $K_2(x,y)$ as

$$Q(x):=\bar K_1(x,x)-K_2(x,x). \tag{4.52}$$

Alternatively, one can recover $E(x)$ from the solution to the Marchenko system as

$$E(x) = \exp\left(2i \int_{-\infty}^{x} dz\, P(z)\right), \tag{4.53}$$

where $P(x)$ is the auxiliary scalar quantity constructed from $K_1(x,y)$ and $\bar{K}_2(x,y)$ as

$$P(x) := K_1(x,x)\, \bar{K}_2(x,x). \tag{4.54}$$

We remark that the quantities $Q(x)$ and $P(x)$ are related to each other as

$$Q(x) = i\, P(x). \tag{4.55}$$

(b) *The complex-valued scalar constant μ given in (2.19) is obtained from the solution to the Marchenko system as*

$$\mu = -4i \int_{-\infty}^{\infty} dz\, Q(z), \tag{4.56}$$

or alternatively as

$$\mu = 4 \int_{-\infty}^{\infty} dz\, P(z). \tag{4.57}$$

(c) *The potentials q and r are recovered from the solution to the Marchenko system as*

$$q(x) = -2K_1(x,x)\, e^{-4\mathcal{Q}(x)}, \tag{4.58}$$

$$r(x) = -2\bar{K}_2(x,x)\, e^{4\mathcal{Q}(x)}, \tag{4.59}$$

where the scalar-valued function $\mathcal{Q}(x)$ is related to the quantity $Q(x)$ appearing in (4.52) as

$$\mathcal{Q}(x) := \int_{x}^{\infty} dz\, Q(z). \tag{4.60}$$

Alternatively, the potentials q and r are recovered from the solution to the Marchenko system as

$$q(x) = -2K_1(x,x)\, e^{-4i\,\mathcal{P}(x)}, \tag{4.61}$$

$$r(x) = -2\bar{K}_2(x,x)\, e^{4i\,\mathcal{P}(x)}, \tag{4.62}$$

where the scalar-valued function $\mathcal{P}(x)$ is related to the quantity $P(x)$ appearing in (4.54) as

$$\mathcal{P}(x) := \int_{x}^{\infty} dz\, P(z). \tag{4.63}$$

(d) *The Jost solutions $\psi(\zeta,x)$ and $\bar{\psi}(\zeta,x)$ to (1.1) are recovered from the solution to the Marchenko system as*

$$\psi_1(\zeta,x) = \zeta \left(\int_{x}^{\infty} dy\, K_1(x,y)\, e^{i\zeta^2 y}\right) e^{-2\mathcal{Q}(x)}, \tag{4.64}$$

$$\psi_2(\zeta,x)=\left(e^{i\zeta^2 x}+\int_x^\infty dy\,K_2(x,y)\,e^{i\zeta^2 y}\right)e^{2\mathcal{Q}(x)},\tag{4.65}$$

$$\bar\psi_1(\zeta,x)=\left(e^{-i\zeta^2 x}+\int_x^\infty dy\,\bar K_1(x,y)\,e^{-i\zeta^2 y}\right)e^{-2\mathcal{Q}(x)},\tag{4.66}$$

$$\bar\psi_2(\zeta,x)=\zeta\left(\int_x^\infty dy\,\bar K_2(x,y)\,e^{-i\zeta^2 y}\right)e^{2\mathcal{Q}(x)},\tag{4.67}$$

where $\psi_1(\zeta,x)$, $\psi_2(\zeta,x)$, $\bar\psi_1(\zeta,x)$, and $\bar\psi_2(\zeta,x)$ are the components of the Jost solutions defined in (2.9). Alternatively, we recover the Jost solutions $\psi(\zeta,x)$ and $\bar\psi(\zeta,x)$ from the solution to the Marchenko system as

$$\psi_1(\zeta,x)=\zeta\left(\int_x^\infty dy\,K_1(x,y)\,e^{i\zeta^2 y}\right)e^{-2i\,\mathcal{P}(x)},\tag{4.68}$$

$$\psi_2(\zeta,x)=\left(e^{i\zeta^2 x}+\int_x^\infty dy\,K_2(x,y)\,e^{i\zeta^2 y}\right)e^{2i\,\mathcal{P}(x)},\tag{4.69}$$

$$\bar\psi_1(\zeta,x)=\left(e^{-i\zeta^2 x}+\int_x^\infty dy\,\bar K_1(x,y)\,e^{-i\zeta^2 y}\right)e^{-2i\,\mathcal{P}(x)},\tag{4.70}$$

$$\bar\psi_2(\zeta,x)=\zeta\left(\int_x^\infty dy\,\bar K_2(x,y)\,e^{-i\zeta^2 y}\right)e^{2i\,\mathcal{P}(x)}.\tag{4.71}$$

(e) *The Jost solutions $\phi(\zeta,x)$ and $\bar\phi(\zeta,x)$ to (1.1) are recovered from the solutions to the Marchenko system with the help of (4.7) and either of (4.64)–(4.67) or (4.68)–(4.71).*

Proof. From (4.44) and (4.45), we observe that the auxiliary scalar quantity $Q(x)$ defined in (4.52) is related to the potentials q and r as

$$Q(x)=\frac{i\,q(x)\,r(x)}{4}.\tag{4.72}$$

As a result, from (2.18) and (4.72) we see that $E(x)$ is recovered as in (4.51). Using (4.43) and (4.46) we get

$$q(x)\,r(x)=4\,K_1(x,x)\,\bar K_2(x,x).\tag{4.73}$$

From (4.52), (4.54), (4.72), and (4.73), it follows that (4.55) holds. Hence, the expression (4.51) implies (4.53). Thus, the proof of (a) is complete. From (2.19) and (4.72) we observe that μ is recovered as in (4.56). Alternatively, using (4.55) in (4.56) we get (4.57), and therefore the proof of (b) is also completed. Let us now prove (c). Having obtained $E(x)$ and μ, we see that we can recover the potential q with the help of (4.43). Thus, using (4.51) and (4.56) in (4.43) we get q as in (4.58). Similarly, having $E(x)$ and μ already recovered, we see that we can obtain the potential r from (4.46). Therefore, using (4.51) and (4.56) in (4.46) we get r as in (4.59). The alternate expressions (4.61) and (4.62) are obtained by using (4.55) in (4.58) and (4.59), respectively. Next, we move to the proof of (d). Having $E(x)$ and μ at hand, we use (2.15), (4.51), and (4.56) in (4.24)–(4.27), respectively, and get (4.64)–(4.67). Alternatively, using (4.57) in

(4.64)–(4.67), we obtain the alternate expressions (4.68)–(4.71). Hence, the proof of (d) is complete. The proof of (e) is obtained as follows. Having constructed the Jost solutions $\psi(\zeta, x)$ and $\bar{\psi}(\zeta, x)$ from the solution to the Marchenko system as in (4.64)–(4.67) or (4.68)–(4.71), we first use (2.5) and (2.6) to obtain $T(\zeta)$ and $\bar{T}(\zeta)$. Since $R(\zeta)$ and $\bar{R}(\zeta)$ are already included in the scattering data set, we then use (4.7) to obtain the Jost solutions $\phi(\zeta, x)$ and $\bar{\phi}(\zeta, x)$. Thus, the proof of (e) is complete. □

In Theorem 4.4, the listing of the alternate recovery formulas may at first sight look redundant because $P(x)$ and $Q(x)$ differ only by the scalar factor i, as seen from (4.55). However, the alternate recovery formulas involve more than a substitution suggested by (4.55), and that is why we provide the alternate formulas in Theorem 4.4. There may be some advantages or disadvantages of using the alternate formulas. One slight disadvantage might occur in the evaluation of the integral of $P(x)$ compared to the integral of $Q(x)$. As seen from (4.52), the quantity $Q(x)$ consists of the difference of two functions and hence the integral of $Q(x)$ can be obtained by evaluating the integrals of those two functions separately. On the other hand, as seen from (4.54) the quantity $P(x)$ consists of a product of two functions, and hence the computation of the integral of $P(x)$ may be more challenging. There are some advantages of using the alternate recovery formulas of Theorem 4.4. For example, the use of the recovery of the potentials q and r via the alternate formulas (4.61) and (4.62) involves only the solution to the uncoupled Marchenko system (4.41) without needing to solve the auxiliary system (4.42). On the other hand, the recovery of q and r via (4.58) and (4.59) requires also the solution to the auxiliary system (4.42). In fact, in the special case when q and r are related to each other as $r(x) = q(x)^*$ or $r(x) = -q(x)^*$, where we use an asterisk to denote complex conjugation, the use of the alternate recovery formulas becomes convenient. We refer the reader to Section 8 of [10] for the details related to the reductions $r(x) = \pm q(x)^*$. In each of those two reduced cases, the uncoupled Marchenko system of two equations in two unknowns is reduced to a single Marchenko integral equation with one unknown, namely,

$$K_1(x, y) \pm \Omega(x + y) \pm i \int_x^\infty dz\, K_1(x, z)\, \Omega'(z + s)\, \Omega(s + y)^* = 0, \quad y > x, \quad (4.74)$$

where $\Omega(y)$ is the quantity defined in the first equality of (4.37). Then, the potential q is recovered from the solution $K_1(x, y)$ to the Marchenko equation (4.74) as

$$q(x) = -2K_1(x, x) \exp\left(\mp 4i \int_x^\infty dz\, |K_1(z, z)|^2 \right). \quad (4.75)$$

As in any inverse problem, the inverse problem for (1.1) has four aspects: the existence, uniqueness, reconstruction, and characterization. The existence deals with the question whether there exists at least one pair of potentials q and r in some class corresponding to a given set of scattering data in a particular class. Once the existence problem is solved, the uniqueness deals with the question whether there is only one pair of potentials for that corresponding scattering data set or there are more such pairs. The reconstruction is concerned with the recovery of the potentials from the scattering data set. Finally, the characterization deals with the specification of the class of potentials and the class of scattering data sets so that there is a one-to-one correspondence between the elements of the class of potentials and the class of scattering data sets. It is clear that in this paper we only deal with the reconstruction aspect of the inverse problem for (1.1). The remaining three aspects are challenging and need to be investigated. Since the linear differential operator related to (1.1) is not selfadjoint, the analysis of the inverse problem for (1.1) is naturally complicated. We anticipate that the development of the Marchenko method in this paper will provide a motivation for the scientific community to analyze the other three aspects of the corresponding inverse problem.

5. Solution formulas with reflectionless scattering data

In this section we consider the linear system (1.1) with the potentials q and r when the corresponding reflection coefficients are zero. We refer to such potentials as reflectionless. From (2.45) we know that if the right reflection coefficients $R(\zeta)$ and $\bar{R}(\zeta)$ are both zero then the left reflection coefficients $L(\zeta)$ and $\bar{L}(\zeta)$ are also zero. Hence, we can describe the reflectionless case for (1.1) as

$$R(\zeta) \equiv 0, \quad \bar{R}(\zeta) \equiv 0. \tag{5.1}$$

In the reflectionless case, the quantities $\Omega(y)$ and $\bar{\Omega}(y)$ appearing in the kernel and the nonhomogeneous term of the Marchenko system (4.40) are determined by the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ alone. In fact, by using (5.1) in (4.37) and (4.38), we get

$$\Omega(y) = C\, e^{iAy}\, B, \qquad \bar{\Omega}(y) = \bar{C}\, e^{-i\bar{A}y}\, \bar{B}, \tag{5.2}$$

$$\Omega'(y) = i\, CA\, e^{iAy}\, B, \quad \bar{\Omega}'(y) = -i\bar{C}\bar{A}\, e^{-i\bar{A}y}\, \bar{B}. \tag{5.3}$$

With the input from (5.2) and (5.3), the Marchenko system (4.40) or the equivalent uncoupled Marchenko system (4.41) supplemented with (4.42) becomes explicitly solvable by using the methods of linear algebra. This is because the corresponding integral kernel is separable in either case. Consequently, we obtain the closed-form formulas for the potentials and Jost solutions for (1.1) corresponding to all reflectionless scattering data, where the formulas are explicitly expressed in terms of the two matrix triplets.

In [10] we have studied the nonlinear system (1.2) associated with the linear system (1.1) when the potentials q and r contain the parameter t in addition to the independent variable x. Consequently, most of the proofs of the results presented in this section can also be obtained by using the proofs in Section 4 of [10] by letting $t = 0$ in those proofs.

In the next theorem, we show that, in the reflectionless case (5.1), the integrals of the potentials q and r over $x \in \mathbb{R}$ each vanish.

Theorem 5.1. *Assume that the potentials q and r in (1.1) belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta)$ and $\bar R(\zeta)$ appearing in (2.7) and (2.8), respectively, are zero. Then, we have*

$$\int_{-\infty}^{\infty} dz\, q(z) = 0, \qquad \int_{-\infty}^{\infty} dz\, r(z) = 0. \tag{5.4}$$

Proof. From the leading asymptotics as $\zeta \to 0$ in (2.42) and (2.43), we see that (5.4) holds. □

In the next theorem, in the reflectionless case (5.1), it is shown that the transmission coefficients for (1.1) are uniquely determined by the matrices A and $\bar A$ appearing in the matrix triplets (A, B, C) and $(\bar A, \bar B, \bar C)$.

Theorem 5.2. *Assume that the potentials q and r in (1.1) belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta)$ and $\bar R(\zeta)$ appearing in (2.7) and (2.8), respectively, are zero. Let $T(\zeta)$ and $\bar T(\zeta)$ be the corresponding transmission coefficients that appear in (2.5) and (2.6), respectively, and let the parameters λ and ζ be related as in (2.15). Suppose that the corresponding bound-state information is described by the two sets in (3.7), with N distinct poles of $T(\zeta)$ occurring at $\lambda = \lambda_j$ in $\mathbb{C}^+$ with multiplicity m_j and with $\bar N$ distinct poles of $\bar T(\zeta)$ occurring at $\lambda = \bar\lambda_j$ in $\mathbb{C}^-$ with multiplicity $\bar m_j$. Equivalently, let the corresponding bound-state information be described by the pair of matrix triplets (A, B, C) and $(\bar A, \bar B, \bar C)$ appearing in (3.12), (3.14), (3.16), (3.17) with eigenvalues of A located in $\mathbb{C}^+$ and eigenvalues of $\bar A$ located in $\mathbb{C}^-$. We have the following:*

(a) *The total number of poles of $T(\zeta)$ including multiplicities in the upper-half complex λ-plane is equal to the total number of poles of $\bar T(\zeta)$ including multiplicities in the lower-half complex λ-plane. That is, we have*

$$\mathcal{N} = \bar{\mathcal{N}}, \tag{5.5}$$

where $\mathcal{N}$ and $\bar{\mathcal{N}}$ are the numbers defined in (3.13) and (3.15), respectively. Consequently, the matrices A and $\tilde A$ have the same size $\mathcal{N} \times \mathcal{N}$.

(b) *The corresponding complex constant $e^{i\mu/2}$ appearing in (2.38) and (2.39) is uniquely determined by the eigenvalues of the matrices A and $\bar A$ and their corresponding multiplicities. We have*

$$e^{i\mu/2} = \frac{\displaystyle\prod_{k=1}^{\bar N} (\bar\lambda_k)^{\bar m_k}}{\displaystyle\prod_{j=1}^{N} (\lambda_j)^{m_j}}, \tag{5.6}$$

with the restriction $\mathcal{N} = \bar{\mathcal{N}}$. The expression in (5.6) is equivalent to the determinant expression

$$e^{i\mu/2} = \frac{\det[\bar A]}{\det[A]}. \tag{5.7}$$

(c) *The transmission coefficients $T(\zeta)$ and $\bar T(\zeta)$ are determined by the eigenvalues of the matrices A and $\bar A$ and their corresponding multiplicities, and for $\lambda\in\mathbb{C}$ we have*

$$T(\zeta)=\left(\frac{\displaystyle\prod_{k=1}^{\bar N}\left((\lambda/\bar\lambda_k)-1\right)^{\bar m_k}}{\displaystyle\prod_{j=1}^{N}\left((\lambda/\lambda_j)-1\right)^{m_j}}\right),\quad \bar T(\zeta)=\left(\frac{\displaystyle\prod_{j=1}^{N}\left((\lambda/\lambda_j)-1\right)^{m_j}}{\displaystyle\prod_{k=1}^{\bar N}\left((\lambda/\bar\lambda_k)-1\right)^{\bar m_k}}\right),\tag{5.8}$$

with the restriction $\mathcal{N}=\bar{\mathcal{N}}$. The two expressions in (5.8) are equivalent to the pair of respective determinant equations given by

$$T(\zeta)=\frac{\det[(\lambda\bar A^{-1}-I)]}{\det[(\lambda A^{-1}-I)]},\quad \bar T(\zeta)=\frac{\det[(\lambda A^{-1}-I)]}{\det[(\lambda\bar A^{-1}-I)]},\quad \lambda\in\mathbb{C},\tag{5.9}$$

which implies that

$$\bar T(\zeta)=1/T(\zeta),\quad \lambda\in\mathbb{C}.\tag{5.10}$$

Thus, the zeros of $T(\zeta)$ corresponds to the poles of $\bar T(\zeta)$ and vice versa.

Proof. In the reflectionless case (5.1), from (2.44) we see that

$$T(\zeta)\,\bar T(\zeta)=1,\quad \lambda\in\mathbb{R}.\tag{5.11}$$

Let us write (5.11) as

$$e^{i\mu/2}\,T(\zeta)\left(\frac{\displaystyle\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}{\displaystyle\prod_{k=1}^{\bar N}(\lambda-\bar\lambda_k)^{\bar m_k}}\right)=\frac{1}{e^{-i\mu/2}\,\bar T(\zeta)}\left(\frac{\displaystyle\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}{\displaystyle\prod_{k=1}^{\bar N}(\lambda-\bar\lambda_k)^{\bar m_k}}\right),\quad \lambda\in\mathbb{R}.\tag{5.12}$$

Without loss of any generality, we can assume that $\mathcal{N}\ge\bar{\mathcal{N}}$. From Theorem 2.5(a) it follows that the left-hand side of (5.12) is analytic in $\lambda\in\mathbb{C}^+$, is continuous in $\lambda\in\overline{\mathbb{C}^+}$, and behaves as $\lambda^{\mathcal{N}-\bar{\mathcal{N}}}[1+O(1/\lambda)]$ as $\lambda\to\infty$ in $\overline{\mathbb{C}^+}$. From Theorem 2.5(b), it follows that the right-hand side of (5.12) is analytic in $\lambda\in\mathbb{C}^-$, is continuous in $\lambda\in\overline{\mathbb{C}^-}$, and and behaves as $\lambda^{\mathcal{N}-\bar{\mathcal{N}}}[1+O(1/\lambda)]$ as $\lambda\to\infty$ in $\overline{\mathbb{C}^-}$. Thus, with the help of Liouville's theorem we conclude that both sides of (5.12) must be equal to a monic polynomial in λ of degree $\mathcal{N}-\bar{\mathcal{N}}$. Next, by taking the reciprocals of both sides of (5.12) we get

$$\frac{1}{e^{i\mu/2}\,T(\zeta)}\left(\frac{\displaystyle\prod_{k=1}^{\bar N}(\lambda-\bar\lambda_k)^{\bar m_k}}{\displaystyle\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}\right)=e^{-i\mu/2}\,\bar T(\zeta)\left(\frac{\displaystyle\prod_{k=1}^{\bar N}(\lambda-\bar\lambda_k)^{\bar m_k}}{\displaystyle\prod_{j=1}^{N}(\lambda-\lambda_j)^{m_j}}\right),\quad \lambda\in\mathbb{R}.\tag{5.13}$$

Again, from Theorem 2.5(a) it follows that the left-hand side of (5.13) is analytic in $\lambda \in \mathbb{C}^+$, is continuous in $\lambda \in \overline{\mathbb{C}^+}$, and and behaves as $\lambda^{\bar{\mathcal{N}}-\mathcal{N}}[1+O(1/\lambda)]$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^+}$. From Theorem 2.5(b), it follows that the right-hand side of (5.13) is analytic in $\lambda \in \mathbb{C}^-$, continuous in $\lambda \in \overline{\mathbb{C}^-}$, and behaves as $\lambda^{\bar{\mathcal{N}}-\mathcal{N}}[1+O(1/\lambda)]$ as $\lambda \to \infty$ in $\overline{\mathbb{C}^-}$. Thus, with the help of Liouville's theorem we conclude that both sides of (5.13) must be identically equal to zero unless $\bar{\mathcal{N}} = \mathcal{N}$. From (2.40) and (2.41) we know that neither side of (5.13) can identically vanish. Thus, (5.5) must hold. Hence, the proof of (a) is complete. Since each side of (5.12) is a monic polynomial in λ of degree $\mathcal{N} - \bar{\mathcal{N}}$, we see that (5.5) implies that each side of (5.12) is identically equal to 1. Therefore, for $\lambda \in \mathbb{C}$ we have

$$T(\zeta) = e^{-i\mu/2} \left(\frac{\prod_{k=1}^{\bar{N}} (\lambda - \bar{\lambda}_k)^{\bar{m}_k}}{\prod_{j=1}^{N} (\lambda - \lambda_j)^{m_j}} \right), \quad \bar{T}(\zeta) = e^{i\mu/2} \left(\frac{\prod_{j=1}^{N} (\lambda - \lambda_j)^{m_j}}{\prod_{k=1}^{\bar{N}} (\lambda - \bar{\lambda}_k)^{\bar{m}_k}} \right). \tag{5.14}$$

Evaluating (5.14) at $\lambda = 0$ and using (2.40) and (2.41), we obtain (5.6) with the understanding that (5.5) is valid. Thus, the proof of (b) is complete. Using (5.6) in (5.14) we get (5.8). From (3.8), (3.10), (3.12), and (3.16) we see that A and $\bar{A}$ are upper-triangular matrices. Hence, (5.8) and (5.9) are equivalent. □

As we see from Theorem 5.2(b), in the reflectionless case (5.1) the matrices C and $\bar{C}$ in the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ appearing in (3.12), (3.14), (3.16), (3.17) play no role in the determination of the quantity $e^{i\mu/2}$, where μ is the complex constant determined by the potentials q and r via (2.19). In the next theorem, we show that C and $\bar{C}$ have some limited effect on the value of μ itself.

Theorem 5.3. *Assume that the potentials q and r in (1.1) belong to the Schwartz class and that the corresponding reflection coefficients $R(\zeta)$ and $\bar{R}(\zeta)$ appearing in (2.7) and (2.8), respectively, are zero. Let μ denote the corresponding complex constant defined in (2.19). The bound-state norming constants c_{jk} and $\bar{c}_{jk}$ appearing in the bound-state data specified in (3.7), or equivalently the two matrices C and $\bar{C}$ appearing in the corresponding matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, can affect the value of μ in a limited way. In fact, any change in C and $\bar{C}$ can only change the value of the integer n in the expression*

$$\mu = -2i \log \left[\det[\bar{A}\, A^{-1}]\right] + 4\pi n, \quad n \in \mathbb{Z}, \tag{5.15}$$

where log *denotes the principal branch of the complex logarithm function.*

Proof. By taking the logarithm of both sides of (5.7), we obtain (5.15). Since the complex logarithm function is infinitely many valued, we have the term $4\pi n$ in (5.15). □

Let us comment on Theorem 5.2(b) and Theorem 5.3. In the reflectionless case (5.1), even though $e^{i\mu/2}$ is uniquely determined by the matrices A and $\bar{A}$, the complex constant μ itself is not uniquely determined by A and $\bar{A}$ alone. This is expected because, if we change C and $\bar{C}$ while keeping A and $\bar{A}$ unchanged, the potentials q and r change and hence, as implied by (2.19) the value of μ may change. What is remarkable about Theorem 5.3 is that the change in μ can only occur in integer multiples of 4π. In Section 6 we illustrate Theorem 5.3 with some explicit examples. We pose it as an open problem whether a physical explanation can be found why the change in μ can only occur in integer multiples of 4π and whether such integers are restricted to a smaller set.

In the next theorem, we present the solution to the Marchenko system (4.40) in the reflectionless case (5.1), where the solution is explicitly expressed in terms of the two matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$.

Theorem 5.4. *When the reflectionless scattering data set given in* (5.2) *and* (5.3) *is used as input, the Marchenko system* (4.40) *corresponding to* (1.1) *has the solution expressed in closed form in terms of the triplets* (A, B, C) *and* $(\bar{A}, \bar{B}, \bar{C})$*, and we have*

$$K_1(x,y) = -\bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x)^{-1}\, e^{-i\bar{A}y}\, \bar{B}, \tag{5.16}$$

$$K_2(x,y) = C\, e^{iAx}\, \Gamma(x)^{-1}\, e^{iAx}\, M\, \bar{A}\, e^{-i\bar{A}(x+y)}\, \bar{B}, \tag{5.17}$$

$$\bar{K}_1(x,y) = \bar{C}\, e^{-i\bar{A}x}\, \bar{\Gamma}(x)^{-1}\, e^{-i\bar{A}x}\, \bar{M}\, A\, e^{iA(x+y)}\, B, \tag{5.18}$$

$$\bar{K}_2(x,y) = -C\, e^{iAx}\, \Gamma(x)^{-1}\, e^{iAy}\, B, \tag{5.19}$$

where $\Gamma(x)$, $\bar{\Gamma}(x)$, M, *and* $\bar{M}$ *are the matrices defined in terms of the two matrix triplets as*

$$\Gamma(x) := I - e^{iAx}\, M\, \bar{A}\, e^{-2i\bar{A}x}\, \bar{M}\, e^{iAx}, \tag{5.20}$$

$$\bar{\Gamma}(x) := I - e^{-i\bar{A}x}\, \bar{M}\, A\, e^{2iAx}\, M\, e^{-i\bar{A}x}, \tag{5.21}$$

$$M := \int_0^\infty dz\, e^{iAz}\, B\, \bar{C}\, e^{-i\bar{A}z}, \quad \bar{M} := \int_0^\infty dz\, e^{-i\bar{A}z}\, \bar{B}\, C\, e^{iAz}, \tag{5.22}$$

with I *denoting the identity matrix whose size is not necessarily the same in different appearances. The constant matrices* M *and* $\bar{M}$ *can alternatively be obtained by solving the two respective matrix-valued linear systems*

$$AM - M\bar{A} = iB\bar{C}, \quad \bar{M}A - \bar{A}\bar{M} = i\bar{B}C. \tag{5.23}$$

Proof. The Marchenko system (4.40) is equivalent to the uncoupled system (4.41) and the related auxiliary system (4.42). We use (5.2) and (5.3) as input to (4.41) and (4.42), and from the first line of (4.41) we get

$$K_1(x,y) + \bar{C}\, e^{-i\bar{A}x - i\bar{A}y}\, \bar{B} + i\int_x^\infty dz \int_x^\infty ds\, K_1(x,z)\, i\, C\, A\, e^{iAz+iAs}\, B\, \bar{C}\, e^{-i\bar{A}s - i\bar{A}y}\, \bar{B} = 0,$$

whose solution has the form

$$K_1(x,y) = H_1(x)\, e^{-i\bar{A}y}\, \bar{B}, \tag{5.24}$$

with $H_1(x)$ satisfying

$$H_1(x)\left[I-\int_x^\infty dz\int_x^\infty ds\, e^{-i\bar{A}z}\,\bar{B}\,C\,e^{iAz}\,A\,e^{iAs}\,B\bar{C}\,e^{-i\bar{A}s}\right]=-\bar{C}\,e^{-i\bar{A}x}. \quad (5.25)$$

The matrix in the brackets in (5.25) is equal to $\bar{\Gamma}(x)$ defined in (5.21), and this can be seen by observing that

$$\int_x^\infty dz\, e^{-i\bar{A}z}\,\bar{B}\,C\,e^{iAz}=e^{-i\bar{A}x}\,\bar{M}\,e^{iAx}, \quad (5.26)$$

$$\int_x^\infty ds\, e^{iAs}\,B\,\bar{C}\,e^{-i\bar{A}s}=e^{iAx}\,M\,e^{-i\bar{A}x}, \quad (5.27)$$

where M and $\bar{M}$ are the constant matrices defined in (5.22). Since the eigenvalues of A are located in $\mathbb{C}^+$ and the eigenvalues of $\bar{A}$ are located in $\mathbb{C}^-$, the integrals in (5.26) and (5.27) are well defined. From (5.26) and (5.27), by directly evaluating the right-hand sides in (5.23), we can confirm that the two matrix-valued linear equations in (5.23) are satisfied. From (5.24) we obtain

$$H_1(x)=-\bar{C}\,e^{-i\bar{A}x}\,\bar{\Gamma}(x)^{-1}, \quad (5.28)$$

and using (5.28) in (5.24) we get (5.16). We obtain (5.19) in a similar manner, by using (5.2) and (5.3) as input in the second line of (4.41). Finally, using (5.16) and (5.19) as input to (4.42), with the help of (5.3) we get (5.17) and (5.18). □

In the next theorem, in the reflectionless case (5.1), we present the explicit expressions for the key quantity $E(x)$ and the potentials q and r associated with (1.1). As indicated in Theorem 5.2(a), unless the two matrix triplets have the same size, the corresponding potentials cannot both belong to the Schwartz class.

Theorem 5.5. *When the reflectionless scattering data set given in (5.2) and (5.3) containing the two matrix triplets (A,B,C) and $(\bar{A},\bar{B},\bar{C})$ is used as input in the Marchenko system (4.40) associated with (1.1), we have the following:*

(a) *The corresponding key quantity $E(x)$ defined in (2.18) is expressed explicitly in terms of the two matrix triplets, and we have*

$$E(x)=\exp\left(2\int_{-\infty}^x dz\, Q(z)\right), \quad (5.29)$$

where $Q(x)$ is the scalar quantity defined in (4.52) with $\bar{K}_1(x,x)$ and $K_2(x,x)$ explicitly expressed in terms of the matrix triplets as

$$\bar{K}_1(x,x)=\bar{C}e^{-i\bar{A}x}\,\bar{\Gamma}(x)^{-1}e^{-i\bar{A}x}\bar{M}A\,e^{2iAx}B, \quad (5.30)$$

$$K_2(x,x)=Ce^{iAx}\,\Gamma(x)^{-1}e^{iAx}M\bar{A}\,e^{-2i\bar{A}x}\bar{B}, \quad (5.31)$$

with M and $\bar{M}$ being the constant matrices in (5.22), and $\Gamma(x)$ and $\bar{\Gamma}(x)$ being the matrices defined in (5.20) and (5.21), respectively. Alternatively, we have

$$E(x)=\exp\left(2i\int_{-\infty}^x dz\, P(z)\right), \quad (5.32)$$

where $P(x)$ is the scalar quantity defined in (4.54) with $K_1(x,x)$ and $\bar K_2(x,x)$ explicitly expressed in terms of the matrix triplets as

$$K_1(x,x) = -\bar C\, e^{-i\bar A x}\,\bar\Gamma(x)^{-1}\, e^{-i\bar A x}\,\bar B, \tag{5.33}$$

$$\bar K_2(x,x) = -C\, e^{iAx}\,\Gamma(x)^{-1}\, e^{iAx}\, B. \tag{5.34}$$

(b) *The corresponding potentials q and r in (1.1) are expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar A,\bar B,\bar C)$, and we have*

$$q(x) = \left(2\bar C\, e^{-i\bar A x}\,\bar\Gamma(x)^{-1}\, e^{-i\bar A x}\,\bar B\right) e^{-4\mathcal{Q}(x)}, \tag{5.35}$$

$$r(x) = \left(2C\, e^{iAx}\,\Gamma(x)^{-1}\, e^{iAx}\, B\right) e^{4\mathcal{Q}(x)}, \tag{5.36}$$

where we recall that $\mathcal{Q}(x)$ is related to $Q(x)$ as in (4.60). Alternatively, we have

$$q(x) = \left(2\bar C\, e^{-i\bar A x}\,\bar\Gamma(x)^{-1}\, e^{-i\bar A x}\,\bar B\right) e^{-4i\,\mathcal{P}(x)}, \tag{5.37}$$

$$r(x) = \left(2C\, e^{iAx}\,\Gamma(x)^{-1}\, e^{iAx}\, B\right) e^{4i\,\mathcal{P}(x)}, \tag{5.38}$$

where we recall that $\mathcal{P}(x)$ is related to $P(x)$ as in (4.63).

Proof. We obtain (5.30) and (5.31) from (5.18) and (5.17), respectively, by using $y = x$ there. Then, (5.29) directly follows from (4.51) and (4.52). In a similar way, (5.33) and (5.34) are obtained from (5.16) and (5.19), respectively, by letting $y = x$ there. Then, the alternate expression (5.32) is obtained with the help of (4.53) and (4.54). Hence, the proof of (a) is complete. We get (5.35) by using (4.58) with the help of (4.52), (5.30), (5.31), and (5.33). In a similar manner, we obtain (5.36) by using (4.59) with the help of (4.52), (5.30), (5.31), and (5.34). The alternate expressions (5.37) and (5.38) are obtained from (4.61) and (4.62), respectively, with the help of (5.33) and (5.34). □

In the next theorem, in the reflectionless case (5.1) we present the explicit formulas for the Jost solutions to (1.1) expressed explicitly in terms of the matrix triplet pair (A,B,C) and $(\bar A,\bar B,\bar C)$.

Theorem 5.6. *Assume that the quantities $\Omega(y)$ and $\bar\Omega(y)$ in (5.2) expressed in terms of the two matrix triplets (A,B,C) and $(\bar A,\bar B,\bar C)$ are used as input to the Marchenko system (4.40) associated with (1.1). Let the parameter λ be related to the spectral parameter ζ as in (2.15). Then, the corresponding four Jost solutions to (1.1) with the respective asymptotics in (2.1)–(2.4) can be expressed explicitly in terms of the matrix triplets (A,B,C) and $(\bar A,\bar B,\bar C)$, and we have the following:*

(a) *The Jost solutions $\psi(\zeta,x)$ and $\bar\psi(\zeta,x)$ are expressed in terms of the two matrix triplets as*

$$\psi_1(\zeta,x) = i\sqrt{\lambda}e^{i\lambda x}\left[\bar C\, e^{-i\bar A x}\bar\Gamma^{-1}\left(\bar A-\lambda I\right)^{-1} e^{-i\bar A x}\bar B\right] e^{-2\mathcal{Q}(x)}, \tag{5.39}$$

$$\psi_2(\zeta,x) = e^{i\lambda x}\left[1 - iC\, e^{iAx}\Gamma^{-1}\, e^{iAx} M\bar A\left(\bar A-\lambda I\right)^{-1} e^{-2i\bar A x}\bar B\right] e^{2\mathcal{Q}(x)}, \tag{5.40}$$

$$\bar\psi_1(\zeta,x)=e^{-i\lambda x}\left[1+i\bar C\,e^{-i\bar A x}\bar\Gamma^{-1}\,e^{-i\bar A x}\,\bar M A\,(A-\lambda I)^{-1}\,e^{2iAx}B\right]e^{-2\mathcal{Q}(x)},\tag{5.41}$$

$$\bar\psi_2(\zeta,x)=-i\sqrt{\lambda}\,e^{-i\lambda x}\left[C\,e^{iAx}\Gamma^{-1}\,(A-\lambda I)^{-1}\,e^{iAx}B\right]e^{2\mathcal{Q}(x)},\tag{5.42}$$

where M and $\bar M$ are the constant matrices defined in (5.22); Γ and $\bar\Gamma$ are the matrices appearing in (5.20) and (5.21), respectively; and $\mathcal{Q}(x)$ is related to $Q(x)$ as in (4.60) with $Q(x)$ being the quantity in (4.52) with its right-hand side expressed by using (5.30) and (5.31).

(b) *Alternatively, the Jost solutions $\psi(\zeta,x)$ and $\bar\psi(\zeta,x)$ are expressed in terms of the matrix triplet pair as*

$$\psi_1(\zeta,x)=i\sqrt{\lambda}\,e^{i\lambda x}\left[\bar C\,e^{-i\bar A x}\bar\Gamma^{-1}\left(\bar A-\lambda I\right)^{-1}e^{-i\bar A x}\bar B\right]e^{-2i\,\mathcal{P}(x)},\tag{5.43}$$

$$\psi_2(\zeta,x)=e^{i\lambda x}\left[1-iC\,e^{iAx}\Gamma^{-1}\,e^{iAx}M\bar A\left(\bar A-\lambda I\right)^{-1}e^{-2i\bar A x}\bar B\right]e^{2i\,\mathcal{P}(x)},\tag{5.44}$$

$$\bar\psi_1(\zeta,x)=e^{-i\lambda x}\left[1+i\bar C\,e^{-i\bar A x}\bar\Gamma^{-1}\,e^{-i\bar A x}\,\bar M A\,(A-\lambda I)^{-1}\,e^{2iAx}B\right]e^{-2i\,\mathcal{P}(x)},\tag{5.45}$$

$$\bar\psi_2(\zeta,x)=-i\sqrt{\lambda}\,e^{-i\lambda x}\left[C\,e^{iAx}\Gamma^{-1}\,(A-\lambda I)^{-1}\,e^{iAx}B\right]e^{2i\,\mathcal{P}(x)},\tag{5.46}$$

where $\mathcal{P}(x)$ is related to $P(x)$ as in (4.63) with $P(x)$ being the scalar quantity in (4.54) with its right-hand side expressed by using (5.33) and (5.34).

(c) *For the Jost solutions $\phi(\zeta,x)$ and $\bar\phi(\zeta)$, we have*

$$\phi(\zeta,x)=\bar T(\zeta)\,\bar\psi(\zeta,x),\quad \bar\phi(\zeta,x)=T(\zeta)\,\psi(\zeta,x),\tag{5.47}$$

where the transmission coefficients $T(\zeta)$ and $\bar T(\zeta)$ are expressed in terms of the matrices A and $\bar A$ as in (5.9), and where the Jost solutions $\psi(\zeta,x)$ and $\bar\psi(\zeta,x)$ are expressed in terms of the matrix triplet pair as in (5.39)–(5.42) or alternatively as in (5.43)–(5.46).

Proof. We obtain (5.39)–(5.42) by using (4.64)–(4.67) with the help of (5.16)–(5.19), (5.29)–(5.31), (5.33), and (5.34). Similarly, we get the alternate expressions (5.43)–(5.46) by using (4.68)–(4.71) with the help of (5.16)–(5.19) and (5.30)–(5.34). We obtain (5.47) by first using (5.1) in (4.7) and then by using (5.10) in the resulting equalities. □

As indicated at the end of Section 4, in this paper we only deal with the reconstruction aspect of the inverse problem for (1.1). Hence, the results presented in this section should be interpreted in the sense of the reconstruction. The potentials and the corresponding Jost solutions are reconstructed explicitly in Theorems 5.5 and 5.6, respectively, from their reflectionless scattering data expressed in terms of a pair of matrix triplets. When the potentials q and r belong to the Schwartz class, there are additional restrictions on the two matrix triplets used in Theorem 5.5. As seen from (5.35) and (5.36), those restrictions amount to the following: The determinants of the matrices $\Gamma(x)$ and $\bar{\Gamma}(x)$ defined in (5.20) and (5.21) should not vanish for any $x \in \mathbb{R}$, and the exponential terms in (5.35) and (5.36) should not cause an exponential increase and in fact should not yield a nonzero asymptotic value as $x \to \pm\infty$. In Section 6 we present some explicit examples of potentials violating such restrictions as well as some explicit examples satisfying those restrictions.

Let us remark that the result presented in Theorem 5.2(a) for (1.1) holds also for the AKNS system given in (1.7). Next, we present that result as a corollary because its proof follows by essentially repeating the proof given for Theorem 5.2(a). We recall that we use the superscript (u, v) to refer to the quantities relevant to (1.7).

Corollary 5.7. *Let the potentials u and v in the AKNS system (1.7) belong to the Schwartz class. Let us also assume that the corresponding reflection coefficients $R^{(u,v)}(\lambda)$ and $\bar{R}^{(u,v)}(\lambda)$ are zero for all $\lambda \in \mathbb{R}$. Then, the number of bound-state poles, including the multiplicities, of the transmission coefficient $T^{(u,v)}(\lambda)$ in $\mathbb{C}^+$ must be equal to the number of bound-state poles, including the multiplicities, of the transmission coefficient $\bar{T}^{(u,v)}(\lambda)$ in $\mathbb{C}^-$.*

We remark that, as indicated in Theorem 5.2(a) and Corollary 5.7, the equivalence of the respective number of bound states is when the multiplicities are included in the counting. This issue is illustrated in Section 6.

6. Explicit examples

In this section we elaborate on the results presented in the previous sections and provide some illustrative examples.

As indicated in Section 3, for the linear system (1.1) one can construct the norming constants c_{jk} appearing in (3.7) explicitly in terms of the residues t_{jk} for $1 \le k \le m_j$ and the dependency constants γ_{jk} for $0 \le k \le m_j - 1$. Similarly, one can construct the norming constants $\bar{c}_{jk}$ appearing in (3.7) explicitly in terms of the residues $\bar{t}_{jk}$ for $1 \le k \le \bar{m}_j$ and construct the dependency constants $\bar{\gamma}_{jk}$ for $0 \le k \le \bar{m}_j - 1$. In the first two examples, we illustrate that construction and observe that, especially in the case of bound states with multiplicities, it is cumbersome to deal with the individual norming constants, and it is better to use the bound-state information not in the form given in (3.7) but rather in the form of the matrix triplet pair (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$.

The first example considers the norming constants for simple bound states.

Example 6.1. Consider the linear system (1.1) with the potentials q and r in the Schwartz class. We elaborate on step (d) appearing in the beginning of Section 3. If the bound state at $\lambda = \lambda_j$ is simple, then we have $m_j = 1$ and hence there is only one norming constant c_{j0}. By proceeding as in [9] we obtain

$$c_{j0} = -\frac{i\, t_{j1}\, \gamma_{j0}}{\zeta_j}, \tag{6.1}$$

where ζ_j is the complex number in the first quadrant in $\mathbb{C}$ so that $\lambda_j = \zeta_j^2$, the complex constant t_{j1} corresponds to the residue in (3.1) in the expansion of the transmission coefficient $T(\zeta)$, i.e.

$$T(\zeta) = \frac{t_{j1}}{\lambda - \lambda_j} + O(1), \quad \lambda \to \lambda_j,$$

and γ_{j0} is the dependency constant appearing in (3.3), i.e.

$$\phi(\zeta_j, x) = \gamma_{j0}\, \psi(\zeta_j, x),$$

with $\psi(\zeta, x)$ and $\phi(\zeta, x)$ being the Jost solutions to (1.1) with the asymptotics (2.1) and (2.3), respectively. If the bound state at $\lambda = \bar{\lambda}_j$ is simple, we have $\bar{m}_j = 1$ and hence there is only one norming constant $\bar{c}_{j0}$, which is expressed as

$$\bar{c}_{j0} = \frac{i\, \bar{t}_{j1}\, \bar{\gamma}_{j0}}{\bar{\zeta}_j}, \tag{6.2}$$

where $\bar{\zeta}_j$ is the complex number in the fourth quadrant in $\mathbb{C}$ for which we have $\bar{\lambda}_j = \bar{\zeta}_j^2$, the complex constant $\bar{t}_{j1}$ corresponds to the residue in (3.2) in the expansion of the transmission coefficient $\bar{T}(\zeta)$, i.e.

$$\bar{T}(\zeta) = \frac{\bar{t}_{j1}}{\lambda - \bar{\lambda}_j} + O(1), \quad \lambda \to \bar{\lambda}_j,$$

and $\bar{\gamma}_{j0}$ is the dependency constant appearing in (3.5), i.e.

$$\bar{\phi}(\bar{\zeta}_j, x) = \bar{\gamma}_{j0}\, \bar{\psi}(\bar{\zeta}_j, x),$$

with $\bar{\psi}(\zeta, x)$ and $\bar{\phi}(\zeta, x)$ being the Jost solutions to (1.1) with the asymptotics (2.2) and (2.4), respectively. As seen from (6.1) and (6.2), one can get (6.2) from (6.1) by using the substitutions

$$\lambda_j \mapsto \bar{\lambda}_j, \quad t_{jk} \mapsto -\bar{t}_{jk}, \quad \gamma_{jk} \mapsto \bar{\gamma}_{jk}, \quad c_{jk} \mapsto \bar{c}_{jk}. \tag{6.3}$$

The next example considers the norming constants for bound states with multiplicities.

Example 6.2. As in Example 6.1, we again consider the linear system (1.1) with the potentials q and r in the Schwartz class, and we elaborate on step (d) appearing in the beginning of Section 3. If the bound state at $\lambda = \lambda_j$ is double, we have $m_j = 2$ and there are only two norming constant c_{j0} and c_{j1}, which are expressed in terms of the residues t_{j1} and t_{j2} and the dependency constants γ_{j0} and γ_{j1} as

$$\begin{cases} c_{j1} = -\dfrac{i\, t_{j2}\, \gamma_{j0}}{\zeta_j}, \\[2ex] c_{j0} = -\dfrac{i\, t_{j1}\, \gamma_{j0}}{\zeta_j} - \dfrac{i\, t_{j2}}{\zeta_j}\left(\gamma_{j1} - \dfrac{\gamma_{j0}}{2\lambda_j}\right), \end{cases} \tag{6.4}$$

where we recall that ζ_j is the complex constant in the first quadrant in $\mathbb{C}$ for which we have $\lambda_j = \zeta_j^2$. If the bound state at $\lambda = \bar{\lambda}_j$ is double, then we have $\bar{m}_j = 2$ and there are only two norming constant $\bar{c}_{j0}$ and $\bar{c}_{j1}$, which are obtained from (6.4) by using the substitutions given in (6.3). For a triple bound state at $\lambda = \lambda_j$, we have $m_j = 3$ and the three norming constants are expressed in terms of the residues t_{j1}, t_{j2}, t_{j3} and the dependency constants γ_{j0}, γ_{j1}, γ_{j2} as

$$\begin{cases} c_{j2} = -\dfrac{i\, t_{j3}\, \gamma_{j0}}{\zeta_j}, \\[2mm] c_{j1} = -\dfrac{i\, t_{j2}\, \gamma_{j0}}{\zeta_j} - \dfrac{i\, t_{j3}}{\zeta_j}\left(\gamma_{j1} - \dfrac{\gamma_{j0}}{2\lambda_j}\right), \\[2mm] c_{j0} = -\dfrac{i\, t_{j1}\, \gamma_{j0}}{\zeta_j} - \dfrac{i\, t_{j2}}{\zeta_j}\left(\gamma_{j1} - \dfrac{\gamma_{j0}}{2\lambda_j}\right) - \dfrac{i\, t_{j3}}{2\zeta_j}\left(\gamma_{j2} - \dfrac{\gamma_{j1}}{\lambda_j} + \dfrac{3\gamma_{j0}}{4\lambda_j^2}\right). \end{cases} \tag{6.5}$$

For a bound state at $\lambda = \bar{\lambda}_j$ of multiplicity three, we can obtain the norming constants $\bar{c}_{j0}$, $\bar{c}_{j1}$, $\bar{c}_{j2}$ from (6.5) by using the substitutions in (6.3). For bound states with higher multiplicities, the norming constants can be explicitly constructed by using the corresponding residues and the dependency constants. However, as already mentioned, the use of the matrix triplet pair (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ is the simplest and most elegant way to represent the bound-state information without having to deal with any cumbersome formulas involving the individual norming constants.

The formulas presented in Theorems 5.5 and 5.6 express, in the reflectionless case (5.1), the relevant quantities for (1.1) in a compact form with the help of matrix exponentials. We have prepared a Mathematica notebook, available from the first author's webpage [11], using the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as input and evaluating all the relevant quantities by expressing the matrix exponentials in terms of elementary functions. In particular, our Mathematica notebook provides in terms of elementary functions the solution to the Marchenko system as indicated in Theorem 5.4, the potentials q and r given in Theorem 5.5, the Jost solutions as described in Theorem 5.6, the transmission coefficients $T(\zeta)$ and $\bar{T}(\zeta)$ as described in (5.9), and the corresponding auxiliary quantities $E(x)$ and μ as given in (5.29) and (5.15), respectively. It also verifies that (1.1) is satisfied when those expressions for the potentials and the Jost solutions are used in (1.1). As the matrix sizes in the triplets get large, contrary to the compact expressions involving the matrix exponentials, the equivalent expressions presented in terms of elementary functions become lengthy.

In the next example, in the reflectionless case (5.1), we express the corresponding potentials and transmission coefficients for (1.1) explicitly in terms of two 1×1 matrix triplets. We recall that the parameter λ is related to the spectral parameter ζ as in (2.15).

Example 6.3. In the reflectionless case (5.1), let us choose our matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ as

$$A = [\alpha], \quad B = [1], \quad C = [\gamma], \quad \bar{A} = [-\beta], \quad \bar{B} = [1], \quad \bar{C} = [\delta], \tag{6.6}$$

where α, β, γ, and δ are some complex constants. Using the procedure described in Theorem 5.2, Theorem 5.4, and Theorem 5.5, we obtain the key quantity $E(x)$ defined in (2.18), the transmission coefficients $T(\zeta)$ and $\bar{T}(\zeta)$, and the potentials q and r explicitly in terms of α, β, γ, and δ as

$$E(x) = -\frac{\beta\left[(\alpha+\beta)^2 + \alpha\gamma\delta e^{2ix(\alpha+\beta)}\right]}{\alpha\left[(\alpha+\beta)^2 - \beta\gamma\delta e^{2ix(\alpha+\beta)}\right]}, \tag{6.7}$$

$$T(\zeta) = -\frac{\alpha(\lambda+\beta)}{\beta(\lambda-\alpha)}, \quad \bar{T}(\zeta) = -\frac{\beta(\lambda-\alpha)}{\alpha(\lambda+\beta)}, \tag{6.8}$$

$$q(x) = \frac{2\delta(\alpha+\beta)^2 e^{2ix\beta}\left[(\alpha+\beta)^2 + \alpha\gamma\delta e^{2ix(\alpha+\beta)}\right]}{\left[(\alpha+\beta)^2 - \beta\gamma\delta e^{2ix(\alpha+\beta)}\right]^2}, \tag{6.9}$$

$$r(x) = \frac{2\gamma(\alpha+\beta)^2 e^{2ix\alpha}\left[(\alpha+\beta)^2 - \beta\gamma\delta e^{2ix(\alpha+\beta)}\right]}{\left[(\alpha+\beta)^2 + \alpha\gamma\delta e^{2ix(\alpha+\beta)}\right]^2}. \tag{6.10}$$

We obtain the four Jost solutions explicitly expressed in terms of α, β, γ, and δ, but we do not display those expressions in our paper. We also verify that (1.1) is satisfied by those Jost solutions with the potentials q and r given in (6.9) and (6.10), respectively. If we choose the four parameters appearing in (6.6) as

$$(\alpha, \beta, \gamma, \delta) = (3i, 2i, 3, 1), \tag{6.11}$$

from (6.7)–(6.10) we get

$$T(\zeta) = -\frac{3(\lambda+2i)}{2(\lambda-3i)}, \quad \bar{T}(\zeta) = -\frac{2(\lambda-3i)}{3(\lambda+2i)}, \quad \mu = 2\pi + 2i\,\ln(3/2), \tag{6.12}$$

$$q(x) = \frac{50e^{6x}\left(25e^{10x} - 9i\right)}{\left(25e^{10x} + 6i\right)^2}, \quad r(x) = \frac{150e^{4x}\left(25e^{10x} + 6i\right)}{\left(25e^{10x} - 9i\right)^2}, \tag{6.13}$$

where μ is the complex constant defined in (2.19). The absolute values $|q(x)|$ and $|r(x)|$ corresponding to the potentials in (6.13) are plotted in Figure 6.1. From

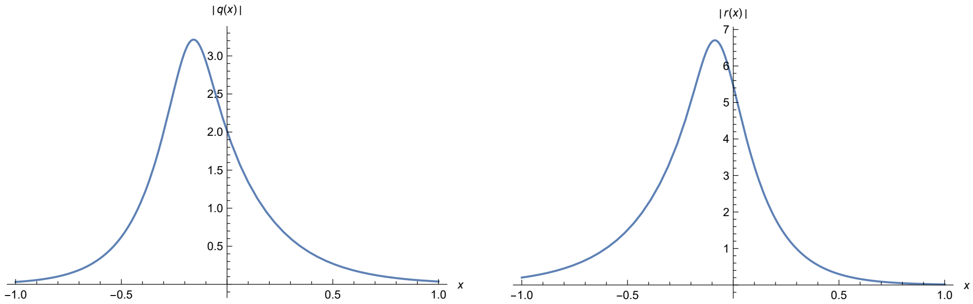

Fig. 6.1: The absolute potentials corresponding to (6.13) in Example 6.3.

(6.13) we confirm that both q and r belong to the Schwartz class. We also verify that the potentials presented in (6.13) satisfy (5.4). Without changing α and β in (6.11), if we instead use the input with

$$(\alpha, \beta, \gamma, \delta) = (3i, 2i, 1, -3),$$

we then get

$$q(x)=-\frac{150e^{6x}\left(25e^{10x}+9i\right)}{\left(25e^{10x}-6i\right)^2},\quad r(x)=\frac{50e^{4x}\left(25e^{10x}-6i\right)}{\left(25e^{10x}+9i\right)^2},\tag{6.14}$$

$$\mu=-2\pi+2i\,\ln(3/2),\tag{6.15}$$

with the same $T(\zeta)$ and $\bar{T}(\zeta)$ as in (6.12). The potentials q and r in (6.14) also

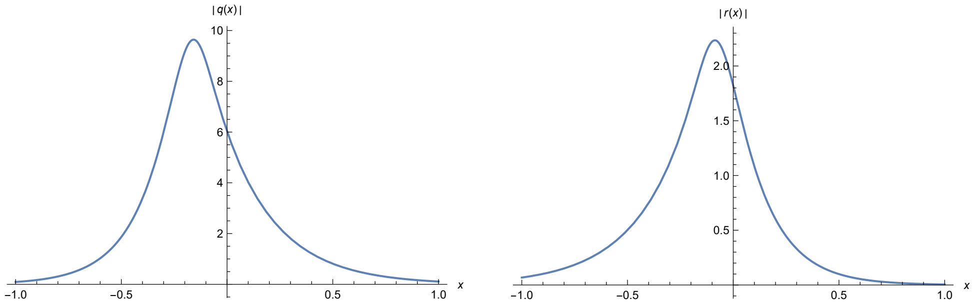

Fig. 6.2: The absolute potentials corresponding to (6.14) in Example 6.3.

belong to the Schwartz class, they satisfy (5.4). We remark that the two μ values appearing in (6.12) and (6.15) differ by 4π, which is compatible with the result of Theorem 5.3. The plots of absolute potentials corresponding to q and r in (6.14) are presented in Figure 6.2. We remark that, except for some scaling, the plots in Figures 6.1 and 6.2 look similar. Let us choose the four parameters appearing in (6.6) as

$$(\alpha,\beta,\gamma,\delta)=(-1,2,3,i),\tag{6.16}$$

which violates the necessary condition that α and β must have positive imaginary parts in order for the potentials q and r to belong to the Schwartz class. With the input in (6.16), the potentials and the transmission coefficients are given by the explicit expressions

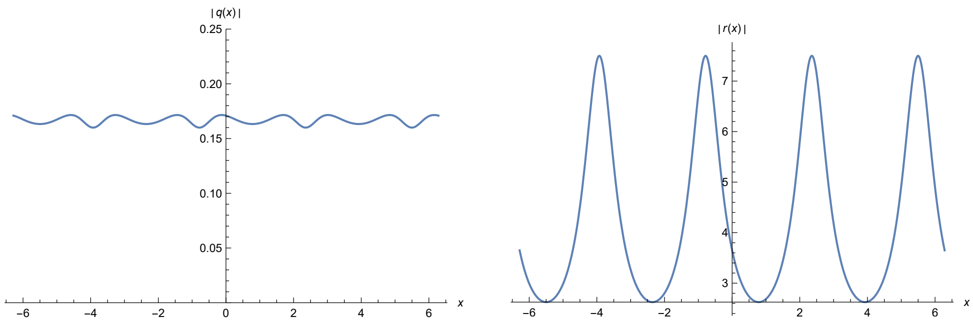

Fig. 6.3: The absolute potentials corresponding to (6.17) in Example 6.3.

$$q(x)=-\frac{2e^{4ix}\left(i+3e^{2ix}\right)}{\left(i+6e^{2ix}\right)^2},\quad r(x)=\frac{36i-6e^{-2ix}}{\left(i+3e^{2ix}\right)^2},\tag{6.17}$$

$$T(\zeta)=\frac{\lambda+2}{2(\lambda+1)},\quad \bar{T}(\zeta)=\frac{2(\lambda+1)}{\lambda+2}.\tag{6.18}$$

The transmission coefficients in (6.18) are unorthodox in the sense that they have zeros and poles for real values of λ. The potentials q and r presented in (6.17) are both periodic with the same period of π. We note that the absolute potentials corresponding to (6.17) are given by

$$|q(x)| = \frac{2\sqrt{10+6\sin(2x)}}{37+12\sin(2x)}, \quad |r(x)| = \frac{3\sqrt{37+12\sin(2x)}}{5+3\sin(2x)},$$

and they are plotted in Figure 6.3. Next, we use another unorthodox set of values for the four parameters, and we choose

$$(\alpha,\beta,\gamma,\delta) = (-1, 2i, 3, i), \tag{6.19}$$

and we get the corresponding potentials and transmission coefficients as

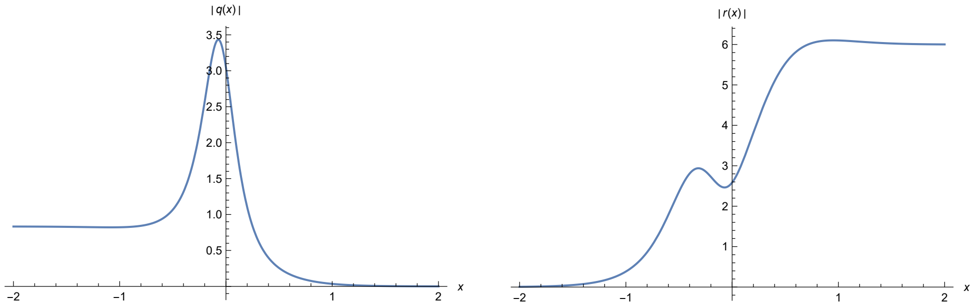

Fig. 6.4: The absolute potentials in (6.20) and (6.21) of Example 6.3.

$$q(x) = -\frac{(8-6i)e^{2ix}\left(3i+(3+4i)e^{2ix+4x}\right)}{[-6+(3+4i)e^{2ix+4x}]^2}, \tag{6.20}$$

$$r(x) = \frac{(18+24i)e^{4x}\left(-6+(3+4i)e^{2ix+4x}\right)}{[3i+(3+4i)e^{2ix+4x}]^2}, \tag{6.21}$$

$$T(\zeta) = \frac{-i(\lambda+2i)}{2(\lambda+1)}, \quad \bar{T}(\zeta) = \frac{2i(\lambda+1)}{\lambda+2i}. \tag{6.22}$$

The input in (6.19) is still unorthodox because α does not have a positive imaginary part. As a result, as seen from (6.22) the transmission coefficient $T(\zeta)$ has a pole at a real value of λ and the transmission coefficient $\bar{T}(\zeta)$ has a zero at a real value of λ. Consequently, we do not expect that the potentials q and r given in (6.20) and (6.21), respectively, belong to the Schwartz class. In fact, from (6.20) we obtain $|q(x)| \to 5/6$ as $x \to -\infty$ and from (6.21) we see that $|r(x)| \to 6$ as $x \to +\infty$. We present the plots of the absolute potentials corresponding to (6.20) and (6.21) in Figure 6.4. Finally in this example, we choose the four parameters as

$$(\alpha,\beta,\gamma,\delta) = (3i, 2i, 3, i). \tag{6.23}$$

Corresponding to the input in (6.23) we have the potentials and transmission coefficients given by

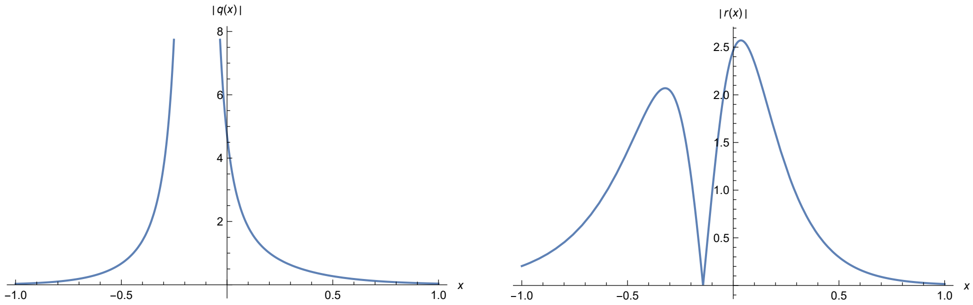

Fig. 6.5: The absolute potentials related to (6.24) in Example 6.3.

$$q(x)=\frac{50ie^{6x}\left(9+25e^{10x}\right)}{(-6+25e^{10x})^2},\quad r(x)=\frac{150e^{4x}\left(-6+25e^{10x}\right)}{(9+25e^{10x})^2},\tag{6.24}$$

$$T(\zeta)=\frac{-3(\lambda+2i)}{2(\lambda-3i)},\quad \bar T(\zeta)=\frac{-2(\lambda-3i)}{3(\lambda+2i)}.$$

Even though both α and β in (6.23) have positive imaginary parts, the choice of δ in (6.23) causes the potential q to have a singularity at $x=-(1/10)\ln(25/6)$. The plots of the absolute potentials corresponding to (6.24) is presented in Figure 6.4, where $|q(x)|$ becomes infinite at $x=-(1/10)\ln(25/6)$.

In the next example we illustrate the reflectionless case in the reduced case, where the potential q can alternatively be obtained as in (4.75) from the solution to the single Marchenko integral equation (4.74).

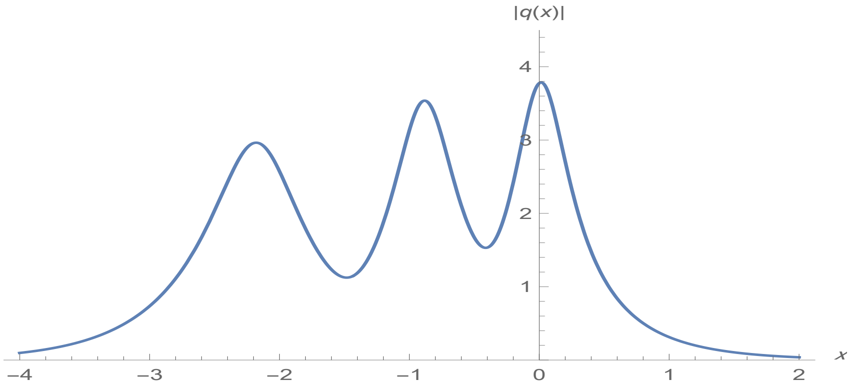

Fig. 6.6: The absolute potential related to (6.25) and (6.26) in Example 6.4.

Example 6.4. Consider the reflectionless scattering data described by the matrix triplets (A,B,C) and $(\bar A,\bar B,\bar C)$ given by

$$A=\begin{bmatrix} i & 0 & 0\\ 0 & 2i & 0\\ 0 & 0 & 3i\end{bmatrix},\quad B=\begin{bmatrix}1\\1\\1\end{bmatrix},\quad C=\begin{bmatrix}1 & 1 & 1\end{bmatrix},\tag{6.25}$$

$$\bar A=A^*,\quad \bar B=B,\quad \bar C=C^*.\tag{6.26}$$

We remark that both C and $\bar C$ are real in the special case of (6.25) and (6.26). From the reflectionless data in (6.25) and (6.26), we observe that the potentials q

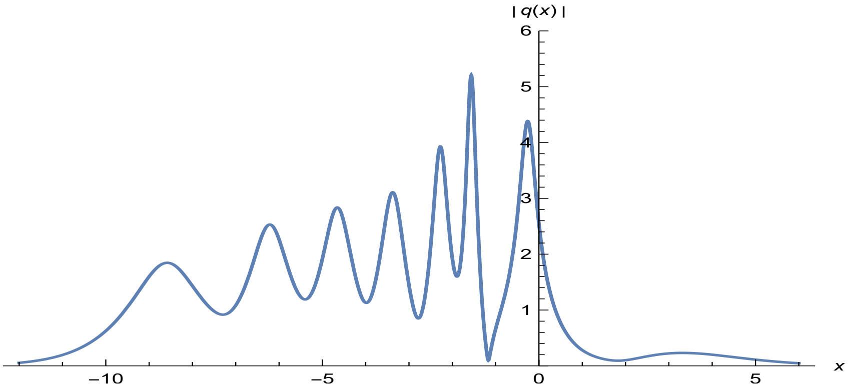

Fig. 6.7: The absolute potential related to (6.30)–(6.32) in Example 6.4.

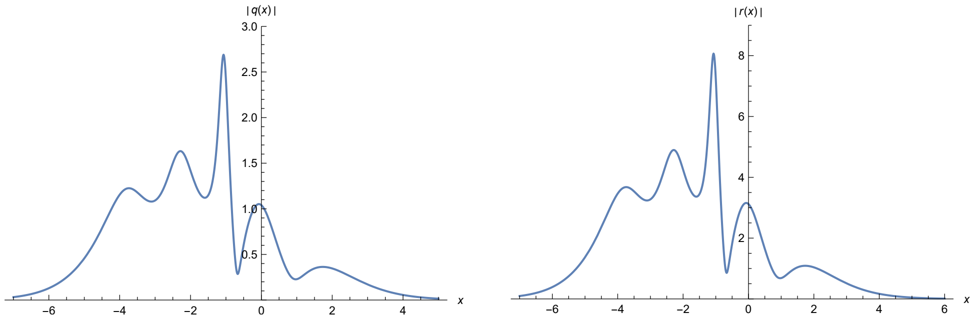

Fig. 6.8: The absolute potentials related to (6.35) with $\mu = -8\pi$ in Example 6.5.

and r in (1.1) are related to each other as $r(x) = q(x)^*$, where we recall that we use an asterisk to denote complex conjugation. Hence, it is enough to consider the potential q only. Using the matrix triplet (A, B, C) given in (6.25) as input to the first equality in (5.2), we solve the corresponding Marchenko equation (4.74) with the upper sign there and recover the potential q with the help of (4.75). The potential q in this case is given by

$$q(x) = \frac{48(\omega_3 + \omega_4)}{\omega_5 + \omega_6} \exp\left(2x + 4i \tan^{-1}(\omega_1/\omega_2)\right), \tag{6.27}$$

where we have defined

$$\begin{aligned}
\omega_1 &:= 24e^{4x}(-216 - 3600e^{2x} - 18675e^{4x} - 18000e^{6x} - 5000e^{8x} + 12960000e^{20x}),\\
\omega_2 &:= -1 + 1000e^{12x}\left[25920 + 62208e^{2x} + 116640e^{4x} + 103680e^{6x} + 77760e^{8x}\right],\\
\omega_3 &:= 6 + 75e^{2x} + 50e^{4x} + 43200ie^{6x} + 334800ie^{8x} + 648000ie^{10x},\\
\omega_4 &:= 10000e^{12x}\left(99i + 36ie^{2x} - 1296e^{4x} - 1296e^{6x} - 1296e^{8x}\right),\\
\omega_5 &:= -i + 5184e^{4x} + 86400e^{6x} + 448200e^{8x} + 432000e^{10x} + 10000(12+2592i)e^{12x},\\
\omega_6 &:= 1000e^{14x}\left(62208i + 116640ie^{2x} + 103680ie^{4x} + 77760ie^{6x} - 311040e^{10x}\right).
\end{aligned}$$

Corresponding to the input (6.25) and (6.26) and the potential q in (6.27), the constant μ appearing in (2.19) and the transmission coefficients $T(\zeta)$ and $\bar{T}(\zeta)$

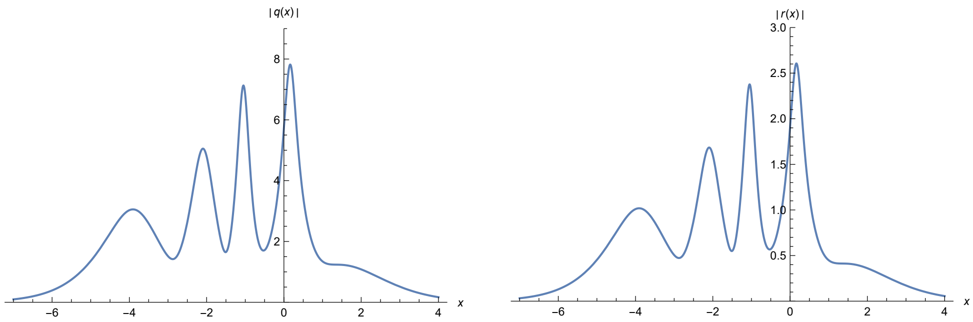

Fig. 6.9: The absolute potentials related to (6.36) with $\mu = 8\pi$ in Example 6.5.

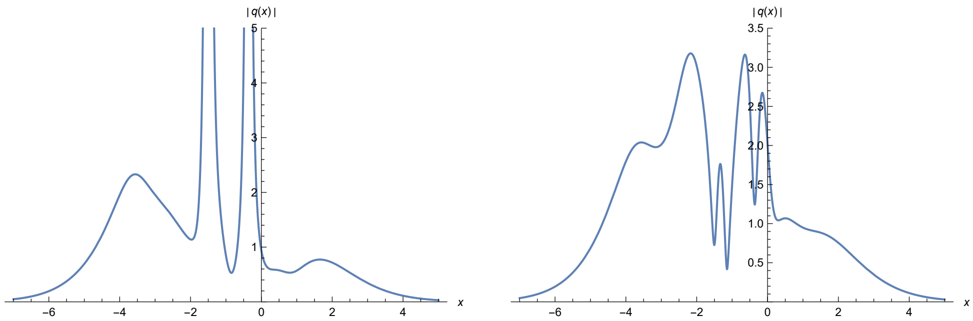

Fig. 6.10: The absolute potentials related to (6.37) with $\mu = -4\pi$ in Example 6.5.

are given by

$$\mu = 6\pi, \quad T(\zeta) = -\frac{(\lambda+i)(\lambda+2i)(\lambda+3i)}{(\lambda-i)(\lambda-2i)(\lambda-3i)}, \tag{6.28}$$

$$\bar{T}(\zeta) = -\frac{(\lambda-i)(\lambda-2i)(\lambda-3i)}{(\lambda+i)(\lambda+2i)(\lambda+3i)}. \tag{6.29}$$

The first plot in Figure 6.6 represents the absolute potential $|q|$ corresponding to the reflectionless input data (6.25). Instead of the reflectionless input (6.25), let us use another reflectionless input given by

$$A = \begin{bmatrix} i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{6.30}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \tag{6.31}$$

$$\bar{A} = A^*, \quad \bar{B} = B, \quad \bar{C} = C^*. \tag{6.32}$$

We note that both C and $\bar{C}$ appearing in (6.32) are real. Corresponding to the input data in (6.30)–(6.32), we have the analog of the expressions in (6.28) and (6.29) given by

$$\mu = 14\pi, \quad T(\zeta) = -\left(\frac{\lambda+i}{\lambda-i}\right)^7, \quad \bar{T}(\zeta) = -\left(\frac{\lambda-i}{\lambda+i}\right)^7. \tag{6.33}$$

It takes over 200 lines to display the explicit expression for the corresponding q in terms of elementary functions without the use of matrix exponentials, and hence we do not display it in our paper. We present the graph of the corresponding absolute potential $|q|$ in the second plot in Figure 6.6. With our Mathematica notebook we can explicitly display $q(x)$, $|q(x)|$, and all the corresponding Jost solutions. We can verify that the Jost solutions satisfy (1.1) with $r(x) = q(x)^*$. We remark that the number of poles of the transmission coefficients $T(\zeta)$ given in (6.28) and (6.33), respectively, agree with the number of peaks in the corresponding graphs in Figure 6.6 and Figure 6.7, respectively.

In the next example, we illustrate Theorem 5.3 and show that, in the reflectionless case, by only changing the matrices C and $\bar{C}$ in the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, we can change the constant μ only as in (5.15).

Example 6.5. In the reflectionless case (5.1), let us fix the matrices A, B, $\bar{A}$, and $\bar{B}$ in the matrix triplets (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$, and let us only change C and $\bar{C}$ to observe how the value of the constant μ defined in (2.19) changes. Let us use

$$A = \begin{bmatrix} i & 1 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{A} = A^*, \quad \bar{B} = B. \tag{6.34}$$

Corresponding to the reflectionless input data partially specified in (6.34), we have the transmission coefficients

$$T(\zeta) = \left(\frac{\lambda + i}{\lambda - i}\right)^4, \quad \bar{T}(\zeta) = \left(\frac{\lambda - i}{\lambda + i}\right)^4,$$

which are obtained by using (5.9). By using different C and $\bar{C}$ we would like to show that the constant μ defined in (2.19) takes the values 0, $\pm 4\pi$, and $\pm 8\pi$. By using the values of C and $\bar{C}$ given by

$$C = \begin{bmatrix} -3 & -3 & -3 & -3 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \tag{6.35}$$

we get $\mu = -8\pi$, and in Figure 6.8 we present the plots of the absolute potentials $|q|$ and $|r|$ corresponding to (6.35). Next, we use the values of C and $\bar{C}$ given by

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 3 & 3 & 3 & 3 \end{bmatrix}, \tag{6.36}$$

and obtain $\mu = 8\pi$. In Figure 6.9 we present the plots of the absolute potentials $|q|$ and $|r|$ corresponding to (6.36). We then use the values of C and $\bar{C}$ given by

$$C = \begin{bmatrix} 2 & 2 & 2 & -1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} -2 & -2 & -2 & -1 \end{bmatrix}, \tag{6.37}$$

which yields $\mu = -4\pi$. In Figure 6.10 we present the plots of the absolute potentials $|q|$ and $|r|$ corresponding to (6.37). We remark that, in Figure 6.10, the two peaks in $|q(x)|$ occur at the finite values $(-1.500\overline{7}, 46.733\overline{7}))$ and $(-0.361\overline{7}, 24.537\overline{1}))$, respectively. By using the values of C and $\bar{C}$ given by

$$C = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 2 & 2 & 1 \end{bmatrix}, \tag{6.38}$$

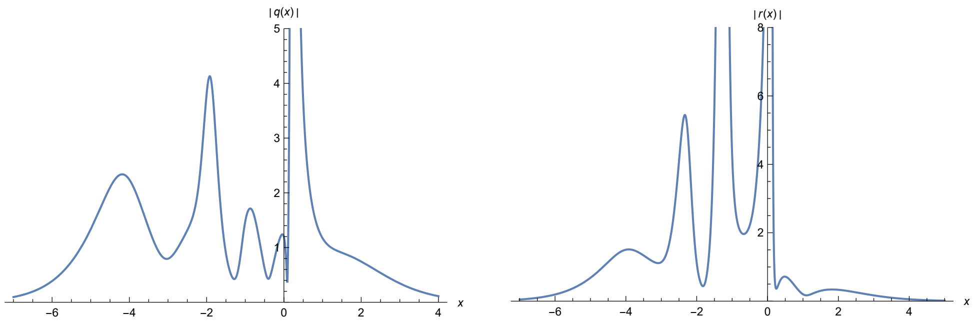

Fig. 6.11: The absolute potentials related to (6.38) with $\mu = 4\pi$ in Example 6.5.

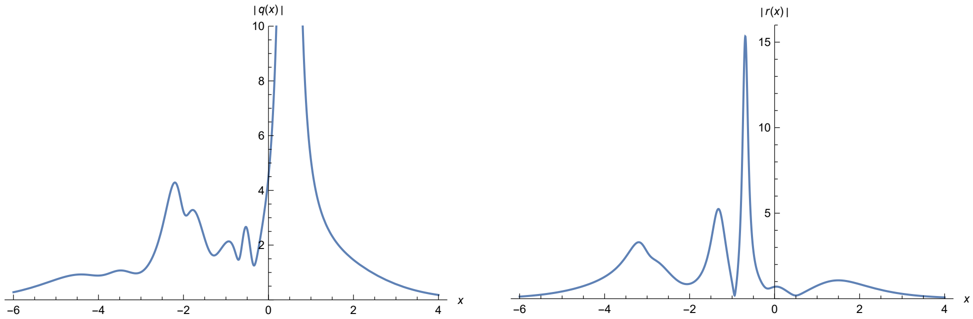

Fig. 6.12: The absolute potentials related to (6.39) with $\mu = 0$ in Example 6.5.

we obtain $\mu = 4\pi$. In Figure 6.11 we present the plots of the absolute potentials $|q|$ and $|r|$ corresponding to (6.38). We remark that, in Figure 6.11, the peak in $|q(x)|$ occurs at the finite value $(0.232\overline{0}, 31.38\overline{9})$ and the two peaks in $|r(x)|$ occur at the finite values $(-1.263\overline{6}, 36.706\overline{0})$ and $(0.0875\overline{0}, 533.04\overline{2})$, respectively. Finally, we use the values of C and $\bar{C}$ given by

$$C = \begin{bmatrix} 1 & 1 & -3 & -3 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 3 & 3 & 1 & 1 \end{bmatrix}, \tag{6.39}$$

we obtain $\mu = 0$. In Figure 6.12 we present the plots of the absolute potentials $|q|$ and $|r|$ corresponding to (6.39). We remark that, in Figure 6.12, the peak in $|q(x)|$ occurs at the finite value $(0.493\overline{7}, 87.688\overline{5})$.

In the final example, we illustrate Theorem 5.2(a) by using a pair of matrix triplets with different sizes as input to the Marchenko system, demonstrating that the corresponding potentials cannot both belong to the Schwartz class.

Example 6.6. Using the matrix triplet (A, B, C) and $(\bar{A}, \bar{B}, \bar{C})$ given by

$$A = \begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \bar{C} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

as input in (5.35) and (5.36), we obtain the corresponding potentials q and r as

$$q(x) = \frac{32\,\omega_9\,\omega_{10}}{\omega_{11} + \omega_{12} + \omega_{13}} \exp\left(2x - 2i\,\tan^{-1}\omega_7 - 2i\,\tan^{-1}\omega_8\right), \tag{6.40}$$

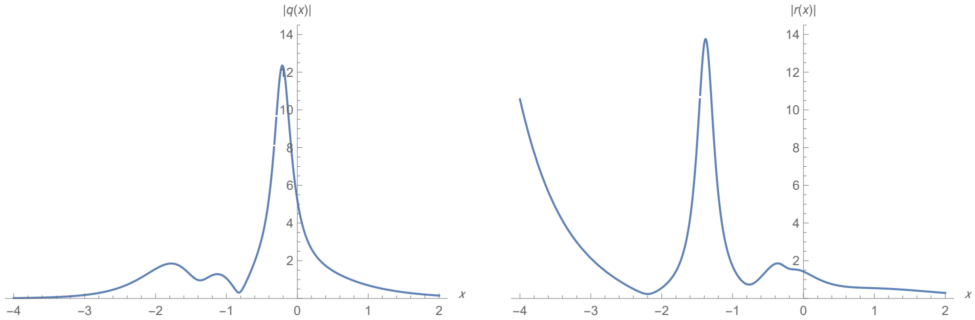

Fig. 6.13: The absolute potentials $|q|$ and $|r|$ in Example 6.6.

$$r(x) = \frac{\omega_{14}\,\omega_{15}}{4(\omega_{16}+\omega_{17}+\omega_{18})}\,\exp\left(-2x + 2i\,\tan^{-1}\omega_7 + 2i\,\tan^{-1}\omega_8\right), \tag{6.41}$$

where we have defined

$$\begin{aligned}
\omega_7 &:= \frac{2+4x+32e^{4x}(7+8x+8x^2)}{512e^{8x}+(1+2x)^2+64e^{4x}(-2+x+4x^2+8x^3)},\\
\omega_8 &:= \frac{-4(1+x)+32e^{4x}(3+8x^2)}{3+8x+4x^2+512e^{8x}-32e^{4x}(7+10x+16x^2+16x^3)},\\
\omega_9 &:= 1+2i-4(1-i)x-4x^2+32e^{4x}(i+2x),\\
\omega_{10} &:= 2-i+4(1-i)x-4ix^2-512ie^{8x}\\
&\qquad +32e^{4x}[7+4i+(8-2i)x+8(1-i)x^2-16ix^3],\\
\omega_{11} &:= (5+8x+4x^2)^2+262144e^{16x}-32768e^{12x}(7+10x+16x^2+16x^3),\\
\omega_{12} &:= -64e^{4x}(33+98x+188x^2+248x^3+192x^4+64x^5),\\
\omega_{13} &:= 1024e^{8x}(61+148x+376x^2+544x^3+640x^4+512x^5+256x^6),\\
\omega_{14} &:= (2+i+2x)^2+512e^{8x}\\
&\qquad -32(1-i)e^{4x}[2+5i+(4+12i)x^2+(1+i)(5x+8x^3)],\\
\omega_{15} &:= -1+4096e^{8x}(1+2ix-2x^2)\\
&\qquad +64e^{4x}[-6+5i+(8i-22)x+(8i-24)x^2-16x^3],\\
\omega_{16} &:= (1+2x)^2(5+4x+4x^2)+262144e^{16x}+65536e^{12x}(-2+x+4x^2+8x^3),\\
\omega_{17} &:= 128e^{4x}(5+15x+24x^2+44x^3+48x^4+32x^5),\\
\omega_{18} &:= 2048e^{8x}(33+50x+60x^2+16x^3+96x^4+128x^5+128x^6).
\end{aligned}$$

In Figure 6.13 we present the plots of the absolute values of the potentials q and r given in (6.40) and (6.41), respectively. From (6.40) we observe that q belongs to the Schwartz class. On the other hand, from the graph in Figure 6.13 it is clear that r cannot belong to the Schwartz class because $|r(x)|$ becomes unbounded as $x \to -\infty$. In this example, as $x \to -\infty$ we have

$$\omega_{14} = 4x^2[1+o(1)], \quad \omega_{15} = -1+o(1), \quad \omega_{16}+\omega_{17}+\omega_{18} = 16x^4[1+o(1)], \tag{6.42}$$

and hence the term responsible for the blow up of $|r(x)|$ as $x \to -\infty$ is the term e^{-2x} appearing in (6.42).

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Received November 29, 2022, revised February 9, 2023.

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Узагальнений метод Марченко в оберненій задачі розсіювання для лінійної системи першого порядку з енергетично залежними потенціалами

Tuncay Aktosun, Ramazan Ercan, and Mehmet Unlu

Метод Марченка розповсюджено на обернену задачу розсіювання для системи лінійних диференціальних рівнянь першого порядку, які містять потенціали пропорційні спектральному параметру. Відповідну систему інтегральних рівнянь Марченка одержано таким чином, що цей метод може бути застосованим до певних систем, для яких раніше застосування методу Марченка було неможливим. Показано як потенціали і розв'язки Йоста лінійної системи будуються з розв'язків системи Марченка. Інформація про зв'язані стани для лінійної системи з будь-якою кількістю зв'язаних станів і будь-якими кратностями описана в термінах пари трійок сталих матриць. У випадку, коли потенціали в лінійній системі є безвідбивними, знайдено деякі формули явних розв'язків в замкненій формі для потенціалів і для розв'язків Йоста лінійної системи. Теорія ілюстрована деякими явними прикладами.

Ключові слова: метод Марченка, узагальнене інтегральне рівняння Марченка, зворотнє розсіювання, лінійна система першого порядку, енергетично залежний потенціал, розв'язки Йоста