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On Conformal Metrics of Constant Positive Curvature in the Plane

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Dedicated to Professor Vladimir A. Marchenko on the occasion of his 100th birthday

We prove three theorems about solutions of $\Delta u + e^{2u} = 0$ in the plane. The first two describe explicitly all concave and quasiconcave solutions. The third theorem says that the diameter of the plane with respect to the metric with line element $e^u|dz|$ is at least $4\pi/3$, except for two explicitly described families of solutions u.

Key words: Liouville equation, positive curvature, meromorphic function, spherical derivative

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1. Introduction

The general solution of the differential equation

$$\Delta u + e^{2u} = 0 \tag{1.1}$$

in a simply connected region in the plane was written by Liouville as

$$u = \log \frac{2|f'|}{1+|f|^2}, \tag{1.2}$$

where f is a meromorphic local homeomorphism, that is a meromorphic function with only simple poles which satisfies $f'(z) \neq 0$. The geometric interpretation is that the metric σ with the line element

$$\frac{2|f'(z)|}{1+|f(z)|^2}|dz|$$

is the pull-back of the standard metric on the unit sphere via f. Here f is called the developing map of the metric, and the relation (1.2) will be assumed throughout the paper.

Expression (1.2) is due to J. Liouville [15, 16], though the equivalent result that every two metrics of the same constant curvature are locally isometric is contained in the earlier paper of F. Minding [17]. Formula (1.2) for the general solution of (1.1) is widely used in modern literature, see, for example [4, 14].

In this paper, we discuss equation (1.1) in the plane.

In [5, Theorem 1.6], solutions of (1.1) which are bounded from above are completely described: they are exactly those for which f in (1.2) is either linear-fractional or of the form

$$f(z) = L(e^{az+b}), \tag{1.3}$$

where L is a linear-fractional transformation and $a, b \in \mathbf{C}$, $a \neq 0$. It was noticed in [5] that all solutions of (1.1) with developing map of the form (1.3) are one-dimensional: after a complex affine change of the variable, they depend on one real variable only.

In this paper we prove two conjectures stated in [5]. Our first theorem proves Conjecture 2 of that paper.

Theorem 1.1. *If u is concave, then f is of the form* (1.3).

This was proved in [5] under the additional condition that u is bounded from above. In Section 2, we give a direct proof of Theorem 1.1, but an alternative approach via differential equations delivers a stronger conclusion when f is transcendental. A function $h\colon \mathbf{C} \to \mathbf{R}$ is called *quasiconcave* if, for any $a_1, a_2 \in \mathbf{C}$, we have $h(z) \geq \min\{h(a_1), h(a_2)\}$ on the line segment from a_1 to a_2: this is equivalent to the condition that, for every $c \in \mathbf{R}$, the set $\{z\colon h(z) \geq c\}$ is convex. If f is linear-fractional then via a rotation of the Riemann sphere it may be assumed that $f(\infty) = \infty$, so that each set $\{z\colon u(z) \geq c\}$ is a disk and u is quasiconcave.

Theorem 1.2. *If u is quasiconcave, then f is linear-fractional or of the form* (1.3). *Indeed, for transcendental f not of the form* (1.3), *and for any $M > 0$, there exist $a_1, a_2 \in \mathbf{C}$ such that*

$$u\left(\frac{a_1 + a_2}{2}\right) < \min\{u(a_1), u(a_2)\} - M. \tag{1.4}$$

If f is linear-fractional or of the form $f = \phi(e^{az+b})$, where ϕ is a rotation of the sphere, then the diameter of the plane with respect to the metric σ is π. It was conjectured in [7, Question 8.1], [5, Conjecture 1] that the diameter is strictly greater than π otherwise. We shall prove the following stronger result.

Theorem 1.3. *The diameter of the plane with respect to the metric σ is at least $4\pi/3$ unless f is linear-fractional or of the form* (1.3).

It is proved in [5, Corollary 3.4] that if $f(z) = e^z + t$, then the diameter is equal to $\pi + 2 \arctan |t|$. So it can be any number in $[\pi, 2\pi)$.

We do not know whether the estimate $4\pi/3$ in Theorem 1.3 is the best possible. At the end of the paper we give an example of a metric σ such that the plane with this metric has infinite diameter. This example answers another question asked in [7, Question 8.1].

2. Proof of Theorem 1.1

Since the case when u is bounded from above has been treated in [5], we may assume that u is unbounded from above. We distinguish the cases whether f in (1.2) has finite or infinite order, see [6, Section 2.1] for the definition of the order of a meromorphic function. The asymptotic formula (1.6) in [5, Theorem 1.9] shows that u cannot be concave when f is of finite order, unless f is of the form (1.3); for further details in this case, see Section 3.

Thus we limit ourselves to the case that f is of infinite order. Let us call a point $a \in \overline{\mathbf{C}}$ exceptional if $f(z) \to a$ as $z \to \infty$ uniformly with respect to $\arg z$ in a sector of opening $\pi/3$. Since there can be at most 6 exceptional points, we can apply a rotation of the sphere to f to ensure that ∞ is not exceptional.

Then f' is a meromorphic function of infinite order without zeros, so $f' = 1/w$, where w is entire of infinite order, that is

$$\limsup_{z\to\infty} \frac{\log\log|w(z)|}{\log|z|} = \infty. \tag{2.1}$$

Consider the sets

$$E = \{z \colon u(z) \geq 0\} \quad \text{and} \quad D = \mathbf{C}\backslash E.$$

Since u is concave, E is convex. Since u is unbounded from above, E is unbounded. Let us assume without loss of generality that $0 \in E$ and

$$u(0) > 0.$$

This can be achieved by translation of the independent variable. Since E is unbounded, closed and convex, and contains 0, there is at least one ray

$$\ell_\theta = \{z = te^{i\theta} \colon t \geq 0\}$$

contained in E. Let I be the set of arguments $\theta \in \mathbf{R}/(2\pi\mathbf{Z})$ of the rays ℓ_θ which are contained in E. Unless E is a parallel strip, in which case I consists of two points, I is a closed interval of length at most π.

Let us call $\theta_0 \in \mathbf{R}/(2\pi\mathbf{Z})$ a *direction of fast decrease* if there is a sequence (z_n) tending to ∞ such that

$$\lim_{n\to\infty} \frac{\log^+(-u(z_n))}{\log|z_n|} = \infty \quad \text{and} \quad \arg z_n \to \theta_0. \tag{2.2}$$

Since

$$u(z) \leq \log 2|f'(z)| = -\log|w(z)| + \log 2, \tag{2.3}$$

(2.1) implies that there exists a direction of fast decrease.

We claim that there exists a direction of fast decrease which does not belong to I. Clearly an interior point of I cannot be a direction of fast decrease, so the claim will follow if we show that there are more than two directions of fast decrease. Let (z_n) be a sequence which satisfies (2.2), and let $\theta_0 \in (\theta_1, \theta_2)$ where

$|\theta_1 - \theta_2| < \pi$. Draw a segment $[a_n, b_n]$ through z_n such that $\arg a_n = \theta_1$ and $\arg b_n = \theta_2$. By concavity, the minimum of u on $[a_n, b_n]$ is attained either at a_n or at b_n, so one of θ_1 and θ_2 is also a direction of fast decrease. Since θ_1 and θ_2 can be chosen in many ways there are many directions of fast decrease.

We conclude that there is a half-plane $H \subset D$ which contains a ray in a direction of fast decrease.

By rotating the independent variable, we assume that H is a left half-plane, say $H = \{z\colon \operatorname{Re} z < -c\}$ where $c > 0$.

Considering the restriction of u to the intervals $[-c + iy, 0]$, we see that the derivatives $(d/dt)u((-c + iy)t)$ are negative somewhere in these intervals. The concavity of u implies that for every $\epsilon > 0$ we have

$$u(re^{i\theta}) \leq -kr, \quad |\theta - \pi| \leq \frac{\pi}{2} - \epsilon, \quad r > r_0, \tag{2.4}$$

where k and r_0 are some positive constants that depend on ϵ. We denote the angular sector in (2.4) by A, and fix ϵ so that A contains a ray of fast decrease.

Consider now the set $G = \{z\colon |w(z)| > 2\}$. By (2.3) we have $G \subset D$. On the other hand, we claim that

$$\log |w(z)| \geq -u(z) - C, \quad z \in A, \tag{2.5}$$

if r_0 is large enough. Here C is a positive constant. To prove (2.5) we first notice that for every sequence (z_n) in A, tending to infinity, the sequence $(f(z_n))$ on the Riemann sphere is a Cauchy sequence (with respect to the spherical metric). This follows from the estimate of the spherical distance

$$\operatorname{dist}(f(z_n), f(z_m)) \leq \int_{z_m}^{z_n} e^{u(z)} |dz|,$$

and the estimate (2.4). Therefore there exists $a \in \overline{\mathbf{C}}$ such that $f(z) \to a$ as $z \to \infty$, $z \in A$. This limit a is an exceptional point as defined in the beginning of the proof, and thus $a \neq \infty$. Using (2.4) we see that

$$\operatorname{dist}(a, f(z)) \leq \int_{-\infty}^{z} e^{u(z)} |dz| \leq \frac{1}{k} e^{-kr_0}.$$

Choosing r_0 large enough, we achieve that there exists a constant C_1 such that $|f(z)| \leq C_1$ for $z \in A$. Thus

$$u(z) \geq \log(2|f'(z)|) - \log\left(1 + C_1^2\right) = -\log |w(z)| + \log 2 - \log\left(1 + C_1^2\right)$$

for $z \in A$ and we obtain (2.5).

Since $u(z) \to -\infty$ in A we conclude from (2.5) that $\log |w|$ is bounded from below in A, say $\log |w(z)| \geq C_2$ for $z \in A$. Thus $v = \log |w| - C_2$ is a positive harmonic function in A. It follows (see [18, p. 87]) that in any proper subsector of A, the function v cannot grow faster than a power. In particular, for a sequence (z_n) as in (2.2), where θ_0 is a direction of fast decrease with $\ell_{\theta_0} \subset A$, we have $v(z_n) \leq |z_n|^{C_3}$ for some $C_3 > 0$. Together with (2.5) this yields that

$$-u(z_n) \leq \log |w(z_n)| + C = v(z_n) + C_2 + C \leq |z_n|^{C_3} + C_2 + C,$$

contradicting (2.2).

3. Proof of Theorem 1.2

Assume that f is transcendental but not of the form (1.3). Then the Schwarzian $2A$ of f is a non-constant entire function [10, 12] and

$$f = \frac{f_1}{f_2}, \tag{3.1}$$

where f_1, f_2 are linearly independent solutions of

$$w'' + Aw = 0. \tag{3.2}$$

Suppose first that A is a polynomial of degree $d > 0$. Then by the classical theory of asymptotic integration [5, 10] there are $d + 2$ equally spaced Stokes rays which divide the plane into open sectors, on each of which $f(z)$ tends to some asymptotic value, these values being different on adjacent sectors. Thus by a rotation of the independent variable it may be assumed that $f(z)$ tends to a finite asymptotic value on a sector of opening $2\pi/(d+2)$, symmetric about the positive real axis. Hence the sectorial asymptotics for (3.2) give a constant $c > 0$ with the following property. Let M, δ be positive constants with δ small: then

$$u(re^{i\theta}) = -cr^{(d+2)/2} \cos\left(\frac{(d+2)\theta}{2}\right) + o\left(r^{(d+2)/2}\right)$$

as $r \to \infty$, uniformly for real θ with $|\theta| \leq \pi/(d+2) - \delta$ (see [10] and [5, Theorem 1.9, formula (1.6)]). Since δ is small it is then clear that (1.4) holds with $a_j = r\exp((-1)^j i(\pi/(d+2) - \delta))$ and r sufficiently large.

Assume henceforth that A is transcendental. The proof will use estimates, analogous to the sectorial asymptotics in the polynomial case, which were proved in [11–13] for solutions of (3.2) on a neighborhood of a maximum modulus point of A. Let $N(r)$ be the central index of A, and take $\phi(r) = N(r)^{1/3}$ in the notation of [12, Sections 2, 3]. Let $r > 0$ be large and lie outside the exceptional set E_0 arising from applying the Wiman-Valiron theory [8] to A, and take z_r with $|z_r| = r$ and $|A(z_r)| = M(r, A)$. Then

$$\phi(r) = N(r)^{1/3} = o(\log|A(z_r)|) \tag{3.3}$$

and

$$A(z) \sim \left(\frac{z}{z_r}\right)^N A(z_r), \quad \frac{A'(z)}{A(z)} \sim \frac{N}{z}, \quad \frac{A''(z)}{A(z)} \sim \frac{N^2}{z^2}, \quad N = N(r), \tag{3.4}$$

for $z \in D(z_r, 8)$, where $D(z_r, L)$ denotes the logarithmic rectangle

$$D(z_r, L) = \{z_r e^{\tau} \colon |\mathrm{Re}\,\tau| \leq LN(r)^{-2/3}, \quad |\mathrm{Im}\,\tau| \leq LN(r)^{-2/3}\}. \tag{3.5}$$

Let $w_r = z_r \exp(-4N(r)^{-2/3})$; then [12, formula (10)] gives, on $D(z_r, 4)$,

$$Z = \frac{2w_r A(w_r)^{1/2}}{N+2} + \int_{w_r}^{z} A(t)^{1/2}\, dt$$

$$\begin{aligned} &\sim \frac{2zA(z)^{1/2}}{N+2} \sim Z(z_r)\left(\frac{z}{z_r}\right)^{(N+2)/2}, \\ \log\frac{Z(z)}{Z(z_r)} &= \frac{N+2}{2}\log\frac{z}{z_r} + o(1), \\ \frac{d\log Z}{\log z} &= \frac{zA(z)^{1/2}}{Z} \sim \frac{N+2}{2}. \end{aligned} \tag{3.6}$$

As in the previous case, fix $M > 0$ and let δ be small and positive. The following is a slightly stronger assertion than [12, Lemma 3.1].

Lemma 3.1. *Let Q be a large positive integer, and let $r \notin E_0$ be large. Then $\log Z$ is a univalent function of $\log z$ on $D(z_r, 7/2)$ and there exist at least Q pairwise disjoint simple islands H_q in $D(z_r, 3)$ each mapped univalently by Z onto the closed logarithmic rectangle*

$$J_1 = \{Z \colon R \le |Z| \le S,\ |\arg Z| \le \pi - \delta\},$$

in which

$$R = |Z(z_r)| \exp\left(-N(r)^{1/3}\right), \quad S = |Z(z_r)| \exp\left(N(r)^{1/3}\right),$$

while R and S/R are both large.

Proof. The first two assertions follow from (3.6), exactly as in [12]. To see that R is large, use (3.6) to write

$$\log R \ge \frac{1}{2}\log M(r, A) + \log r - \log(N+2) - N(r)^{1/3} - O(1),$$

the right-hand side being large and positive by (3.3). $\square$

Next, for z in some H_q, apply to (3.2) the Liouville transformation [10]

$$W(Z) = A(z)^{1/4} w(z), \tag{3.7}$$

so that W satisfies

$$\frac{d^2 W}{dZ^2} + (1 - F_0(Z))W = 0, \tag{3.8}$$

in which, by (3.4),

$$F_0(Z) = \frac{A''(z)}{4A(z)^2} - \frac{5A'(z)^2}{16A(z)^3}, \quad |F_0(Z)| \le \frac{3}{|Z|^2} \quad \text{on } J_1.$$

Then [13, Lemma 2.1] gives solutions $U_1(Z), U_2(Z)$ of (3.8) which satisfy

$$U_1(Z) \sim e^{-iZ}, \quad U_2(Z) \sim e^{iZ}, \quad W(U_1, U_2) \sim 2i, \tag{3.9}$$

uniformly for Z in the set

$$J_2 = J_1 \setminus \{Z \colon \operatorname{Re} Z < 0, \quad |\operatorname{Im} Z| < R\}. \tag{3.10}$$

The restriction to J_2 is a consequence of the method of proof, which requires removal of the "shadow" of the disk $B(0, R)$ [10]. Hence (3.7) and (3.9) deliver solutions u_1, u_2 of (3.2) satisfying

$$u_1(z) \sim A(z)^{-1/4} e^{-iZ}, \quad u_2(z) \sim A(z)^{-1/4} e^{iZ}, \quad W(u_1, u_2) \sim 2i, \tag{3.11}$$

on the preimage H'_q of J_2 in H_q.

3.1. Two line segments lying in the same H'_q. The next step is to choose two line segments, both lying in the same H'_q, one of which will be used to show that u is not concave. By (3.6) there exist $p \in \mathbf{R}$ and z'_r satisfying

$$-\frac{4\pi}{N+2} < p < \frac{4\pi}{N+2}, \quad z'_r = z_r e^{ip}, \quad |Z(z'_r)| \sim |Z(z_r)|, \quad \arg Z(z'_r) = 0. \tag{3.12}$$

Now set

$$\begin{aligned} \zeta_1^+ &= z'_r e^{i4\delta/(N+2)}, & \zeta_2^+ &= z'_r e^{i2(\pi-2\delta)/(N+2)}, \\ \zeta_1^- &= z'_r e^{-i4\delta/(N+2)}, & \zeta_2^- &= z'_r e^{-i2(\pi-2\delta)/(N+2)}. \end{aligned} \tag{3.13}$$

Let S^+ be the line segment from ζ_1^+ to ζ_2^+, let S^- be that from ζ_1^- to ζ_2^-, and let Σ be the arc of the circle $|z| = r$ from ζ_1^+ to ζ_1^- via z'_r: these lie in $D(z_r, 3)$, by (3.5), (3.12) and the fact that N is large. Indeed, elementary trigonometry gives

$$r \geq |z| \geq r \cos\left(\frac{\pi - 4\delta}{N+2}\right) \geq r\left(1 - O\left(\frac{1}{N^2}\right)\right) \quad \text{for } z \in S^+ \cup S^-,$$

from which it follows, in view of (3.4) and (3.6), that

$$|A(z)| \sim |A(z_r)| = M(r, A) \quad \text{and} \quad |Z(z)| \sim T = |Z(z_r)| \tag{3.14}$$

for $z \in S^+ \cup S^- \cup \Sigma$. On the other hand (3.6), (3.12) and (3.13) also yield

$$-\pi + 2\delta + o(1) \leq \arg Z(z) \leq \pi - 2\delta + o(1) \quad \text{for } z \in S^+ \cup S^- \cup \Sigma.$$

Combining this estimate with (3.10), (3.14) and the fact that $T/R = \sqrt{S/R}$ is large then shows that S^+, S^- indeed lie in the same H'_q.

Next, let ζ_3^+ be the midpoint of S^+, and ζ_3^- that of S^-. Then (3.6), (3.12) and (3.13) deliver

$$\begin{aligned} \arg(Z(\zeta_1^\pm)) &= \pm 2\delta + o(1), \\ \arg(Z(\zeta_2^\pm)) &= \pm(\pi - 2\delta) + o(1), \\ \arg(Z(\zeta_3^\pm)) &= \pm\frac{\pi}{2} + o(1). \end{aligned} \tag{3.15}$$

3.2. Estimates for u. On that H'_q which contains $S^+ \cup S^-$, write

$$f_1 = C_1 u_1 + C_2 u_2, \quad f_2 = D_1 u_1 + D_2 u_2, \quad C_j, D_j \in \mathbf{C}. \tag{3.16}$$

Since the spherical derivatives of f and $1/f$ agree, it may be assumed that, among C_1, C_2, D_1, D_2, either D_1 or D_2 has maximal modulus. Hence (3.1) and (3.16) yield constants α, β, γ with

$$f = \frac{f_1}{f_2} = \frac{\alpha v + \beta}{\gamma v + 1}, \quad \max\{|\alpha|, |\beta|, |\gamma|\} \leq 1, \quad v = \left(\frac{u_1}{u_2}\right)^{\pm 1}. \tag{3.17}$$

Suppose first that D_1 has maximal modulus. Then $v = u_2/u_1$ and, on H'_q, (3.11) delivers

$$\log|v| = -2\operatorname{Im} Z + o(1), \tag{3.18}$$

as well as

$$\begin{aligned} f' &= \frac{(\alpha-\beta\gamma)v'}{(\gamma v+1)^2} = \frac{(\alpha-\beta\gamma)}{(\gamma v+1)^2}\,\frac{W(u_1,u_2)}{u_1^2} \\ &\sim \left(\frac{\alpha-\beta\gamma}{(\gamma v+1)^2}\right)\frac{2i}{u_1^2} \sim \left(\frac{\alpha-\beta\gamma}{(\gamma v+1)^2}\right) 2iA^{1/2}v. \end{aligned} \tag{3.19}$$

Thus (3.14), (3.15), and (3.18) imply that

$$\log|v(\zeta_1^+)| \sim \log|v(\zeta_2^+)| \sim -2T\sin(2\delta), \quad \log|v(\zeta_3^+)| \sim -2T. \tag{3.20}$$

It then follows from (3.17) that, at the three points ζ_1^+, ζ_2^+, ζ_3^+, both v and $f-\beta$ are small, so that,

$$u(\zeta_j^+) = \log\frac{|\alpha-\beta\gamma|}{1+|\beta|^2} + \log 4 + \frac{1}{2}\log M(r,A) + \log|v(\zeta_j^+)| + o(1).$$

Since δ is small, while T is large, (3.20) now implies that (1.4) holds with $a_j = \zeta_j^+$. This completes the proof that u is not quasiconcave, provided that D_1 has the largest modulus among C_1, C_2, D_1, D_2. On the other hand, if $|D_2|$ is maximal then the same argument goes through using $v = u_1/u_2$ and the points ζ_1^-, ζ_2^-, ζ_3^-. □

4. Diameter of the plane with the metric σ. Proof of Theorem 1.3

Since the case of linear-fractional f has been dealt with in [5], and a rational function of degree greater than 1 cannot be locally univalent in $\mathbf{C}$, we assume f is transcendental.

We begin by recalling some notation and facts from the theory of singularities of inverses of meromorphic functions; see, for example, [1–3].

Let a be a point in $\overline{\mathbf{C}}$. We denote by $\mathfrak{A}(a)$ the set of all open simply connected neighborhoods of a. A (transcendental) *singularity* of f^{-1} over a is a function $\Omega \mapsto V(\Omega)$ which assigns to every $\Omega \in \mathfrak{A}(a)$ a connected component $V(\Omega)$ of the preimage $f^{-1}(\Omega)$ such that

$$\Omega_1 \subset \Omega_2 \Rightarrow V(\Omega_1) \subset V(\Omega_2),$$

and

$$\bigcap_{\Omega\in\mathfrak{A}} V(\Omega) = \varnothing.$$

A singularity over a exists if and only if a is an *asymptotic value*; that is, there exists a curve $\gamma\colon [0,1) \to \mathbf{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to a$ as $t \to 1$.

If a region $D \subset \overline{\mathbf{C}}$ contains no asymptotic values, then the restriction of f on any component of $f^{-1}(D)$ is a covering.

Singularities can be considered as elements of a completion of $\mathbf{C}$ with respect to a certain metric adapted to f, see [2]. The open sets $V(\Omega)$ are called *neighborhoods* of a singularity. A singularity is *isolated* if it has a neighborhood $V(\Omega)$ which is not a neighborhood of any other singularity. This condition implies that the restriction

$$f\colon V(\Omega)\backslash f^{-1}(a) \to \Omega\backslash\{a\} \tag{4.1}$$

is a covering. So it must be a universal covering, since we assumed that f has an essential singularity at ∞. Singularities for which (4.1) is a universal covering are called the *logarithmic singularities* of f^{-1} in the classical literature.

In the case when all singularities are isolated, the metric completion of $(\mathbf{C},\sigma)$ consists of $\mathbf{C}$ and one point for each singularity. In the general case of non-isolated singularities, the singularities can also be interpreted as elements of a metric completion, but with respect to a different metric; see [2]. We will denote the metric completion of $(\mathbf{C},\sigma)$ by $\widetilde{\mathbf{C}}$. It will be used only in the case when all singularities are isolated.

We will need several lemmas.

Lemma 4.1. *Let f be a non-constant meromorphic function and γ a simple curve which consists of two asymptotic curves with common starting point and distinct asymptotic values. Let D be one of the two components of $\mathbf{C}\backslash\gamma$. Then the restriction $f|_D$ has a dense image in $\overline{\mathbf{C}}$.*

Proof. A theorem of Lindelöf (see [6, p. 179] or [18, p. 81]) yields that $f|_D$ cannot be bounded. Applying this to $1/(f-a)$ shows that $f|_D$ cannot omit a neighborhood of a point $a \in \mathbf{C}$. □

Remark 4.2. Using a deeper result of Heins [9, Theorem 4] one can actually show that $f|_D$ omits at most two values in $\overline{\mathbf{C}}$.

Lemma 4.3. *Let $K \subset \overline{\mathbf{C}}$ be a geodesic arc of length $t \in (0,\pi)$: then*

$$\sup_{z\in\overline{\mathbf{C}}} \operatorname{dist}(z,K) = \pi - \frac{t}{2}.$$

Proof. Without loss of generality, we may assume that

$$K = \{e^{i\phi}\colon\ -t/2 \le \phi \le t/2\}.$$

We claim that the maximal distance from z to K is attained for $z=-1$. The conclusion follows from this, since the points in K that are closest to $z=-1$ are the points $e^{\pm it/2}$, implying that $\operatorname{dist}(-1,K) = \pi - t/2$.

Since the spherical distance is a strictly increasing function of the chordal distance, the above claim follows if we show that $z=-1$ has the maximal chordal distance from K. Let χ denote the chordal metric and $\operatorname{dist}_\chi(z,K)$ the distance from z to K with respect to this metric.

By symmetry, it is sufficient to consider the case that $|z| \leq 1$ and $\operatorname{Im} z \geq 1$ so that z has the form $z = re^{i\theta}$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. If $t/2 < \theta \leq \pi$, then, using that $0 < t < \pi$ and hence $\cos(\pi - t/2) < 0$, we find that

$$\begin{aligned}
\operatorname{dist}_\chi(z, K)^2 \leq \chi(z, e^{it/2})^2 &= \frac{4|z - e^{it/2}|^2}{(1 + |z|^2)(1 + |e^{it/2}|^2)} \\
&= \frac{2(1 + r^2 - 2r\cos(\theta - t/2))}{1 + r^2} = 2 - \frac{4r}{1 + r^2}\cos(\theta - t/2) \\
&\leq 2 - \frac{4r}{1 + r^2}\cos(\pi - t/2) \leq 2 - 2\cos(\pi - t/2) \\
&= \chi(-1, e^{it/2})^2 = \operatorname{dist}_\chi(-1, K)^2.
\end{aligned}$$

On the other hand, if $0 \leq \theta \leq t/2$, then

$$\operatorname{dist}_\chi(z, K)^2 \leq \chi(z, e^{i\theta}) = 2 - \frac{4r}{1 + r^2} \leq 2 < \operatorname{dist}_\chi(-1, K)^2.$$

It follows that the maximum of $\operatorname{dist}_\chi(z, K)$ and hence of $\operatorname{dist}(z, K)$ is attained for $z = -1$, as claimed above. □

Our proof of Theorem 1.3 splits into two parts:

Proposition 4.4. *If f has a non-isolated singularity, then the diameter of $(\mathbf{C}, \sigma)$ is at least 2π.*

Proof. By rotating the sphere we may assume that there is a non-isolated singularity over 0. Let D_ϵ be the open spherical disk of radius $\epsilon > 0$ about zero. Then $V(D_\epsilon)$ must be a neighborhood of some other singularity. We claim that this other singularity can be chosen with asymptotic value $b \neq 0$. Indeed, if all singularities of which $V(D_\epsilon)$ is a neighborhood were to lie over 0, then $f \colon V(D_\epsilon) \to D \backslash \{0\}$ would be a covering, contradicting our assumption that the singularity over 0 is not isolated.

Thus there exists a curve $\gamma \subseteq V(D_\epsilon)$, both ends of which tend to ∞ in $\mathbf{C}$, and which is asymptotic for two values, that is, $f(z) \to 0$ and $f(z) \to b$ as $z \to \infty$ along the two ends of γ. By removing any loops we may assume that γ is simple. So it divides the plane into two parts, which we denote by D_1 and D_2. By construction, $f(\gamma) \subset D_\epsilon$.

By Lemma 4.1, $f(D_1)$ and $f(D_2)$ are dense open subsets of $\overline{\mathbf{C}}$. Thus there exists a point w in the spherical disk of radius ϵ centered at ∞ which is contained in both $f(D_1)$ and $f(D_2)$, say $w = f(z_j)$ where $z_j \in D_j$ for $j = 1, 2$. Then $\operatorname{dist}(z_j, f(\gamma)) \geq \pi - 2\epsilon$ for $j = 1, 2$. Hence the distance between z_1 and z_2 is at least $2\pi - 4\epsilon$. Since $\epsilon > 0$ can be taken arbitrarily small, the conclusion follows. □

Proposition 4.5. *If all singularities of f are isolated, and there are at least 3 of them, then the diameter of $(\mathbf{C}, \sigma)$ is at least $4\pi/3$.*

Proof. Our strategy is to find a geodesic $\gamma \subset \widetilde{\mathbf{C}}$ which connects two singularities, and such that the length of γ is at most $2\pi/3$. Then application of Lemma 4.1 and Lemma 4.3 will give the required diameter estimate, similar to the argument in the proof of Proposition 4.4.

To do this, it is sufficient to find an open metric disk in $\widetilde{\mathbf{C}}$ which contains no singularities, and has at least three singularities on the boundary. Indeed, since the length of the boundary of any spherical disk is at most 2π, there will be two singularities with the distance between them at most $2\pi/3$, and the shortest curve between them (which exists since $\widetilde{\mathbf{C}}$ is complete) will contain a geodesic arc of length at most $2\pi/3$ connecting two singularities.

It remains to show that such a disk can be found unless f is an exponential. We start with some point $w^* \in \overline{\mathbf{C}}$ which has an f-preimage $z^* \in \mathbf{C}$. Let ϕ be the germ of f^{-1} such that $\phi(w^*) = z^*$.

Let D be the open spherical disk of the largest radius centered at w^* to which ϕ has an analytic continuation. Then ∂D contains a singularity w_1 of ϕ. If this singularity is unique, then using the assumption that singularities are isolated, we can cover ∂D by finitely many disks, of which only one contains w_1. Then ϕ has an analytic continuation to a larger disk D' such that $D \subset D'$ and $w_1 \in \partial D'$. Among such disks D' there is one with maximal spherical radius, and its boundary must contain at least two singularities, w_1 and w_2. We denote this maximal disk D' by D_0, and its center by w_0. Let ϕ_0 be the germ at w_0 obtained by the analytic continuation of ϕ that we just described.

If D_0 has at least three singularities on the boundary, we are finished.

Otherwise, consider the curve β passing through w_0 such that the points on this curve are at equal distance from the two singularities. This curve is a great circle. We assume that the curve is parameterized by the arc length and that $\beta(0) = w_0$.

Since the disk D_0 contains only two singularities w_1 and w_2 on the boundary, we can cover the boundary by finitely many disks of which only two contain singularities. Then there is a one-parametric family of disks D_t centered at $\beta(t)$ whose radii are equal to the distances from $\beta(t)$ to w_1 and w_2. Since we assumed that the only singularities on ∂D_0 are w_1 and w_2, the germ ϕ_0 admits an immediate analytic continuation from D_0 to D_t with small t.

Now consider the supremum and infimum of the values of t for which this analytic continuation is possible. If either of them is finite, we obtain a disk with three singularities on the boundary.

If an analytic continuation is possible to all disks D_t for $t \in \mathbf{R}$, we will show that f is in fact a universal covering of $\overline{\mathbf{C}} \backslash \{w_1, w_2\}$, that is, f is an exponential function.

To prove this last statement we consider a curve $\delta \colon \mathbf{R} \to \overline{\mathbf{C}} \setminus \{w_1, w_2\}$ with $\delta(0) = w_0$. We "project" this curve δ onto the curve β as follows: For every $t \in \mathbf{R}$ there exists a unique circle which contains w_1, w_2 and $\delta(t)$ and intersects the circle β orthogonally. (This is easy to see by applying a linear-fractional transformation which sends the circle β to the equator of the sphere, and sends the points w_1 and w_2 to the poles.) The intersection of this circle with the circle

β is the projection of $\delta(t)$. We thus find a continuous function $g\colon \mathbf{C} \to \mathbf{C}$ such that $\delta(t)$ projects to $\beta(g(t))$. It is clear that the disk $D_{g(t)}$ contains $\delta(t)$. Since ϕ_0 can be continued analytically along β, this shows that an analytic continuation of ϕ_0 along δ is also possible. This completes the proof of Proposition 4.5 and Theorem 1.3. $\square$

Example 4.6. We construct a locally univalent meromorphic function for which the diameter of the plane with respect to pull-back metric is infinite.

We define f using a line complex [6, Chap. VII]. Let a, b, c, d be four distinct points in the Riemann sphere $\overline{\mathbf{C}}$. We call them the base points. Consider the cell decomposition Y of $\overline{\mathbf{C}}$ shown in Fig. 4.1 (right). It consists of two vertices $\times$

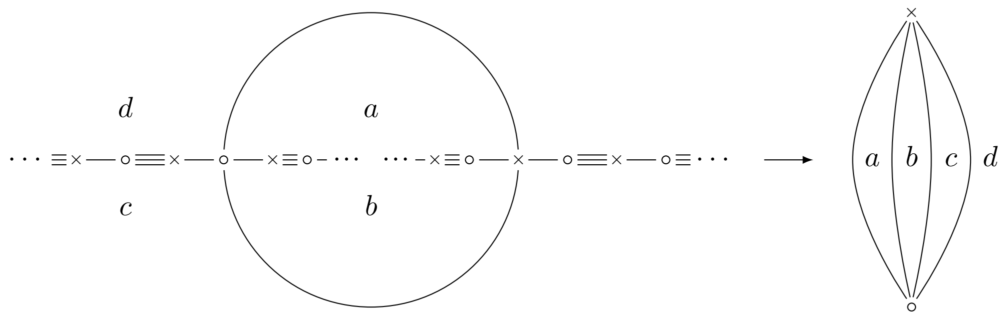

Fig. 4.1: Line complex of G.

and $\circ$, four edges and four faces, each face containing exactly one point of the set $\{a, b, c, d\}$. We label the faces by the base points they contain, and denote them by D_a, D_b, D_c, D_d. If f is a local homeomorphism $\mathbf{C} \to \overline{\mathbf{C}}$ whose asymptotic values are contained in the set $\{a, b, c, d\}$ then the preimage $X = f^{-1}(Y)$ is a partition of the plane into vertices, edges and faces. We label the vertices and faces of this partition by the same labels as their images.

Two such partitions are considered equivalent if they can be mapped to each other by a homeomorphism of the plane. Two local homeomorphisms f_1 and f_2, whose asymptotic values are contained in the set of base points, with equivalent partitions, satisfy $f_1 = f_2 \circ \phi$ where ϕ is a homeomorphism. A partition X is completely determined by its 1-skeleton which is called the *line complex*. This is a bipartite graph embedded in the plane whose vertices have the same degree, equal to the number of base points.

The same construction can be made for a local homeomorphism from $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ or $\overline{\mathbf{C}}\backslash\{0\}$ to $\overline{\mathbf{C}}$. When drawing a line complex, we usually do not draw the true preimage $f^{-1}(Y)$, but an equivalent graph.

We suppose for simplicity that $\{a, b, c, d\} \subset \mathbf{C}$, and consider the function

$$g_1(z) = \frac{b\exp(z) - a}{\exp(z) - 1}$$

which is a universal covering of $\overline{\mathbf{C}}\backslash\{a, b\}$ by $\mathbf{C}$. Its line complex consists of a chain infinite in both directions of the form

$$\cdots - \circ \equiv \times - \circ \equiv \times - \cdots .$$

Let B be the region which is the union of D_c and D_d and the edge between them. This region has infinitely many bounded preimages under g_1. We choose one of them and call it B_1.

Similarly, the function

$$g_2(z) = \frac{d\exp(1/z) - c}{\exp(1/z) - 1}$$

performs a universal covering of $\overline{\mathbf{C}}\backslash\{c, d\}$ by the punctured sphere $\overline{\mathbf{C}}\backslash\{0\}$. Its line complex is similar to that of g_1, and we choose a component B_2 of the preimage $g_2^{-1}(\overline{\mathbf{C}}\backslash B)$.

Since g_1 and g_2 map ∂B_1 and $\partial B_2\backslash\{0\}$ homeomorphically on the same curve ∂B (with opposite orientations), we can glue the restriction of g_1 on $\mathbf{C}\backslash B_1$ with the restriction of g_1 on $\mathbf{C}\backslash(\{0\} \cup B_2)$, along a homeomorphism ψ between the boundary circles of these punctured disks, such that $g_1 = g_2 \circ \psi$ on ∂B_1. Since the homeomorphism ψ is smooth, it has a quasiconformal extension to a quasiconformal homeomorphism $B_1 \to \overline{\mathbf{C}}\backslash B_2$, which we denote by the same letter. We can arrange that $\psi(0) = 0$. Thus we obtain a quasiregular local homeomorphism

$$g(z) = \begin{cases} g_1(z), & z \in \mathbf{C}\backslash B_1, \\ g_2(\psi(z)), & z \in B_1\backslash\{0\} \end{cases}.$$

The line complex of g in $\mathbf{C}^*$ is shown in Fig. 4.1 (left). Since g is quasiregular, there is a homeomorphism ϕ such that $G = g \circ \phi$ is meromorphic in $\mathbf{C}^*$.

Next we consider the entire function $F(z) = G(\exp(iz))$. The line complex of F is shown in Fig. 4.2.

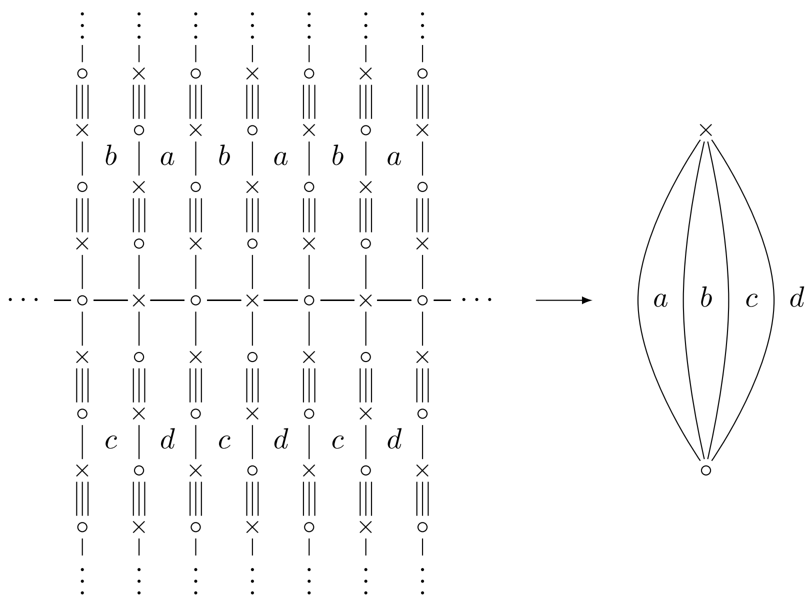

Fig. 4.2: Line complex of F.

It remains to show that the pull-back of the spherical metric via F has infinite diameter. To do this we consider two simple curves in $\overline{\mathbf{C}}$:

A curve A from a to d which is contained in the union of faces D_a and D_d of Y with their common boundary edge, and a curve B from b to c which is contained in the union of faces D_b and D_c of Y with their common boundary edge.

Evidently these curves have disjoint closures in $\overline{\mathbf{C}}$. Each of these curves has infinitely many disjoint F-preimages which are curves beginning and ending at ∞. Let us call these preimages α_j and β_j, $j \in \mathbf{Z}$, and assume that they are enumerated in the natural order, so that α_k separates all α_j, β_j with $j < k$ from β_k and all α_j, β_j with $j > k$.

Now consider two points p and q in $\mathbf{C}$ which are separated by $2N$ curves α_j and β_j. Let γ be any curve with endpoints p and q. Then the image $F(\gamma)$ must hit A and B alternately, at least N times each, so the length of this image and of γ itself is at least $(2N-1)\delta$ where δ is the distance between A and B.

Remark 4.7. One can obtain an explicit representation of the function F. It can be shown that F is a ratio of two solutions of the Mathieu equation

$$w'' + (\cos(z/2) + \lambda)w = 0,$$

where λ is subject to the condition that this ratio has period 2π.

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Про конформні метрики додатної кривини в площині

Walter Bergweiler, Alexandre Eremenko, and James Langley

Доведено три теореми про розв'язки рівняння $\Delta u + e^{2u} = 0$ в площині. Перші дві явно описують усі увігнуті розв'язки. Третя теорема стверджує, що діаметр площини з метрикою з лінійним елементом $e^u|dz|$ не менше ніж $4\pi/3$, за винятком двох явно описаних сімей розв'язків u.

Ключові слова: рівняння Ліувілля, додатна кривина, мероморфна функція, сферична похідна