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On Perturbative Hardy-Type Inequalities

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Dedicated, with great admiration, to Vladimir A. Marchenko
on the happy occasion of his 100th birthday

Given a three-coefficient Sturm–Liouville differential expression $\tau_0 = r_0^{-1}[-(d/dx)p_0(d/dx) + q_0]$ and its perturbation $\tau_{q_1} = \tau_0 + r_0^{-1}q_1$ on an interval $(a,b) \subseteq \mathbb{R}$, we employ the existence of a strictly positive solution $u_0(\lambda_0,\cdot) > 0$ on (a,b) of $\tau_0 u_0 = \lambda_0 u_0$ to derive a quadratic form inequality for τ_{q_1} that naturally generalizes the well-known Hardy inequality and reduces to it in the particular case $p_0 = r_0 = u_0(0,\cdot) = 1$, $q_0 = \lambda_0 = 0$, $a \in \mathbb{R}$, $b = \infty$.

Key words: Hardy inequality, principal and nonprincipal solutions, oscillation theory, Sturm–Liouville operators

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1. Introduction

We consider a natural generalization of Hardy's inequality for Sturm–Liouville differential expressions: Assuming that p_0, q_0, q_1, and r_0 satisfy Hypothesis 2.1, we compare the Sturm–Liouville equations $\tau_0 u_0 = \lambda_0 u_0$ and $\tau_{q_1} u = \lambda_0 u$ on $(a,b) \subseteq \mathbb{R}$, where τ_0 is of the type

$$\tau_0 = \frac{1}{r_0(x)}\left[-\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x)\right] \text{ for a.e. } x \in (a,b), \tag{1.1}$$

and its perturbation τ_{q_1} is of the form $\tau_{q_1} = \tau_0 + r_0^{-1}q_1$, that is,

$$\tau_{q_1} = \frac{1}{r_0(x)}\left[-\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x) + q_1(x)\right] \text{ for a.e. } x \in (a,b), \tag{1.2}$$

where q_1 is of the form (1.6)

As our principal result we shall prove in Theorem 3.1 a natural generalization (from the point of view of quadratic form perturbations) of Hardy's inequality which, in its well-known original form, is

$$\int_a^\infty dx\,|f'(x)|^2 > \frac{1}{4}\int_a^\infty dx\,\frac{|f(x)|^2}{(x-a)^2}, \quad 0 \neq f \in C_0^\infty((a,\infty)),\ a \in \mathbb{R}. \tag{1.3}$$

In particular, we will derive the following inequality:
Assume $\tau_0 u_0 = \lambda_0 u_0$, $u_0(\lambda_0, x) > 0$ *for* $x \in (a,b)$, *and*

$$\bigg| \int_a dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2} \bigg| < \infty.$$

Then

$$\int_a^b dx\, p_0(x)|f'(x)|^2 > \int_a^b dx \bigg[\lambda_0 r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0, x)^{-4} \\ \times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2} \bigg)^{-2} \bigg] |f(x)|^2, \quad 0 \neq f \in \mathcal{D}_0((a,b)), \tag{1.4}$$

where

$$\mathcal{D}_0((a,b)) = \big\{ g \in L^2((a,b); r_0 dx) \,\big|\, g \in AC_{\text{loc}}((a,b));\ \text{supp}\,(g) \subset (a,b) \text{ compact}; \\ p_0^{1/2} g' \in L^2((a,b); dx) \big\}. \tag{1.5}$$

If, in addition,

$$\bigg| \int^b dx\, p_0(x)^{-1} u_0(\lambda_0, x)^{-2} \bigg| = \infty,$$

then the constant $1/4$ *in* (1.4) *is optimal.*

In this context, the generalized Hardy-type potential q_1 is of the form

$$q_1(x) = q_{1,0,a}(x) \\ := -4^{-1} p_0(x)^{-1} u_0(\lambda_0, x)^{-4} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2} \bigg)^{-2} \\ \text{for a.e. } x \in (a,b). \tag{1.6}$$

In the special case $p_0 = r_0 = u_0(0, \cdot) = 1$, $q_0 = \lambda_0 = 0$, $-\infty < a < b = \infty$, and f smooth and compactly supported in (a, ∞), (1.4) and (1.5) reduce to Hardy's inequality (1.3) and

$$q_1(x) = -4^{-1}(x-a)^{-2}, \quad x \in (a, \infty). \tag{1.7}$$

Finally, we will show how to remove the compact support hypothesis on f in (1.4).

Hardy's inequality (1.3) (see, for instance, [8], [9, Sect. 9.8], [10, Chs. 1, 3, App.]) and its subsequent generalizations (especially, in the multi-dimensional context) has such a rich history that we cannot possibly do it any justice here, but we refer to the very detailed bibliographies in [1, p. 3–5], [2], [3, p. 104–105], [4, 5, 7, 8], [9, p. 240–243], [10, Ch. 3], [11, p. 5–11], [12, 13, 15], [14, Ch. 1], and the references cited therein. Regarding the principal topic at hand, the notion of perturbative Hardy-type inequalities, much less seems to be known and we are only aware of the earlier work [6] in this context.

2. Some background

To set the stage we introduce the following basic assumptions on the the three coefficients p, q, r in the Sturm–Liouville differential expression (2.1) below:

Hypothesis 2.1. *Let $(a,b) \subseteq \mathbb{R}$ and suppose that p, q, r are (Lebesgue) measurable functions on (a,b) such that the following items (i)–(iii) hold:*

(i) $r > 0$ *a.e. on* (a,b), $r \in L^1_{\text{loc}}((a,b); dx)$.
(ii) $p > 0$ *a.e. on* (a,b), $1/p \in L^1_{\text{loc}}((a,b); dx)$.
(iii) q *is real-valued a.e. on* (a,b), $q \in L^1_{\text{loc}}((a,b); dx)$.

Given Hypothesis 2.1, we consider differential expressions τ of the type,

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \quad \text{for a.e. } x \in (a,b) \subseteq \mathbb{R}. \tag{2.1}$$

In the following it will become necessary to refer to the minimal operator $T_{\min}$ in $L^2((a,b); rdx)$ associated with τ. In this context, the *preminimal operator* $\dot{T}_{\min}$ in $L^2((a,b); r\,dx)$ associated with τ is defined by

$$\begin{aligned} &\dot{T}_{\min} f = \tau f, \\ &f \in \operatorname{dom}\big(\dot{T}_{\min}\big) = \big\{g \in L^2((a,b); r\,dx) \,\big|\, g, g^{[1]} \in AC_{loc}((a,b)); \\ &\qquad\qquad \operatorname{supp}(g) \subset (a,b) \text{ is compact; } \tau g \in L^2((a,b); r\,dx)\big\}, \end{aligned} \tag{2.2}$$

where the quasi-derivative $g^{[1]}$ of g is given by

$$g^{[1]}(x) = p(x) g'(x), \quad x \in (a,b).$$

It is well-known that $\dot{T}_{\min}$ is closable, and hence one defines the *minimal operator* $T_{\min}$ as the closure of $\dot{T}_{\min}$,

$$T_{\min} = \overline{\dot{T}_{\min}}.$$

We recall that $\tau - \lambda$, $\lambda \in \mathbb{R}$, is called nonoscillatory at b if and only if $\tau u = \lambda u$ has a real-valued solution $u(\lambda, \cdot)$ which has finitely many zeros near b (in this case all real-valued solutions of $\tau u = \mu u$ with $\mu \leq \lambda$ share this property). Regarding classical oscillation theory, we start with the following celebrated result.

Theorem 2.2 (The Sturm Separation Theorem)**.**

(i) *Assume that p, q_j, r satisfy Hypothesis 2.1 and denote*

$$\tau_j = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q_j(x)\right] \quad \textit{for a.e. } x \in (a,b),\ j = 1,2. \tag{2.3}$$

Suppose $q_2 \geq q_1$ a.e. on (a,b) and, for fixed $\lambda \in \mathbb{R}$, let u_j be a nontrivial real-valued solution of $\tau_j u_j = \lambda u_j$ on $(a,b) \subseteq \mathbb{R}$, $j = 1,2$. If $x_1, x_2 \in (a,b)$ are two consecutive zeros of u_2, then u_1 has at least one zero in $[x_1, x_2]$. In addition, if u_1 does not have a zero in (x_1, x_2), then $q_1 = q_2$ a.e. on $[x_1, x_2]$ and u_1 is a constant multiple of u_2 on $[x_1, x_2]$. If τ is regular at a (respectively, b), then $x_1 = a$ (respectively, $x_2 = b$) is permissible.

Item (i) applies, in particular, to the case $q_1 = q_2 := q$, $\tau_1 = \tau_2 := \tau$, and $\tau u_j = \lambda_j u_j$ with $\lambda_2 \leq \lambda_1$.

(ii) *Assume Hypothesis 2.1, $\lambda \in \mathbb{R}$, and let u_j, $j = 1, 2$, be linearly independent real-valued solutions of $\tau u = \lambda u$. Then the zeros of u_1 and u_2 separate each other (i.e., if $x_1, x_2 \in (a, b)$ are two consecutive zeros of u_2, then u_1 has precisely one zero in (x_1, x_2)).*

Next, we recall the following well-known facts regarding boundedness from below of $T_{\min}$:

Theorem 2.3. *Assume Hypothesis 2.1. Then $T_{\min}$ is bounded from below if and only if there exists $\nu_0 \in \mathbb{R}$ such that for all $\lambda < \nu_0$, $\tau - \lambda$ is nonoscillatory at a and b. Moreover, if $\tau u = \lambda_0 u$ has a strictly positive solution $u(\lambda_0, \cdot) > 0$ on (a, b), then $T_{\min} \geq \lambda_0 I$ and $T_F \geq \lambda_0 I$, where T_F denotes the Friedrichs extension of $T_{\min}$.*

For subsequent purpose we also briefly recall the notion of *(non)principal* solutions of $\tau u = \lambda u$ for some $\lambda \in \mathbb{R}$. In the following the Wronskian of two functions f, g satisfying $f, g \in AC_{loc}((a, b))$, $f^{[1]} g^{[1]} \in C((a, b))$ is defined via

$$W(f, g)(x) = f(x) g^{[1]}(x) - f^{[1]}(x) g(x), \quad x \in (a, b).$$

Theorem 2.4. *Assume Hypothesis 2.1 and let $\lambda \in \mathbb{R}$ be fixed. If $\tau - \lambda$ is nonoscillatory at b, then there exists a real-valued solution $u_b(\lambda, \cdot)$ of $(\tau - \lambda) u = 0$ satisfying the following properties (i)–(iii) in which $\widehat{u}_b(\lambda, \cdot)$ denotes an arbitrary real-valued solution of $(\tau - \lambda) u = 0$ linearly independent of $u_b(\lambda, \cdot)$.*

(i) *$u_b(\lambda, \cdot)$ and $\widehat{u}_b(\lambda, \cdot)$ satisfy the limiting relation*

$$\lim_{x \uparrow b} \frac{u_b(\lambda, x)}{\widehat{u}_b(\lambda, x)} = 0. \tag{2.4}$$

(ii) *$u_b(\lambda, \cdot)$ and $\widehat{u}_b(\lambda, \cdot)$ satisfy*

$$\int^b dx \, |p(x)|^{-1} \widehat{u}_b(\lambda, x)^{-2} < \infty \quad \text{and} \quad \int^b dx \, |p(x)|^{-1} u_b(\lambda, x)^{-2} = \infty. \tag{2.5}$$

It is understood in (2.4) and (2.5) that only x-values beyond the largest zero (if any) of u_b and $\widehat{u}_b$, and only x-values less than the smallest zero (if any) of u_a and $\widehat{u}_a$, are considered.

(iii) *Suppose $x_0 \in (a, b)$ strictly exceeds the largest zero, if any, of $u_b(\lambda, \cdot)$, and $\widehat{u}_b(\lambda, x_0) \neq 0$. If $\widehat{u}_b(\lambda, x_0)/u_b(\lambda, x_0) > 0$, then $\widehat{u}_b(\lambda, \cdot)$ has no (respectively, exactly one) zero in (x_0, b) if $W(u_b(\lambda, \cdot), \widehat{u}_b(\lambda, \cdot)) > 0$ (respectively, $W(u_b(\lambda, \cdot), \widehat{u}_b(\lambda, \cdot)) < 0$). On the other hand, if $\widehat{u}_b(\lambda, x_0)/u_b(\lambda, x_0) < 0$, then $\widehat{u}_b$ has no (respectively, exactly one) zero in (x_0, b) if $W(u_b(\lambda, \cdot), \widehat{u}_b(\lambda, \cdot)) < 0$ (respectively, $W(u_b(\lambda, \cdot), \widehat{u}_b(\lambda, \cdot)) > 0$).*

A result analogous to Theorem 2.4 holds if $\tau - \lambda$ is nonoscillatory at a. That is, one can establish the existence of a distinguished real-valued solution $u_a(\lambda, \cdot)$ of $(\tau - \lambda) u = 0$ which satisfies the following analog to (2.4): If $\widehat{u}_a(\lambda, \cdot)$ is any real-valued solution of $(\tau - \lambda) u = 0$ linearly independent of $u_a(\lambda, \cdot)$, then

$$\lim_{x \downarrow a} \frac{u_a(\lambda, x)}{\widehat{u}_a(\lambda, x)} = 0.$$

Analogs of item (ii) and (iii) of Theorem 2.4 subsequently hold for $u_a(\lambda,\cdot)$ and any real-valued solution $\widehat{u}_a(\lambda,\cdot)$ linearly independent of $u_a(\lambda,\cdot)$.

Definition 2.5. Assume Hypothesis 2.1 and suppose that $\lambda \in \mathbb{R}$. If $\tau - \lambda$ is nonoscillatory at $c \in \{a,b\}$, then a nontrivial real-valued solution $u_c(\lambda,\cdot)$ of $(\tau - \lambda)u = 0$ which satisfies

$$\lim_{\substack{x \to c \\ x \in (a,b)}} \frac{u_c(\lambda,x)}{\widehat{u}_c(\lambda,x)} = 0$$

for any other linearly independent real-valued solution $\widehat{u}_c(\lambda,\cdot)$ of $(\tau - \lambda)u = 0$ is called a *principal solution* of $(\tau - \lambda)u = 0$ at c. A real-valued solution of $(\tau - \lambda)u = 0$ linearly independent of a principal solution at c is called a *nonprincipal solution* of $(\tau - \lambda)u = 0$ at c.

Next, we recall that a straight forward computation yields Jacobi's celebrated factorization identity

$$\begin{aligned} & -(pg')' + h^{-1}(ph')'g = -h^{-1}\big(ph^2(g/h)'\big)', \\ & \quad g,\, pg',\, h,\, ph' \in AC_{loc}((a,b)),\ h > 0 \text{ on } (a,b), \end{aligned} \tag{2.6}$$

or, in a more symmetrical form,

$$\begin{aligned} & -h(pg')' + (ph')'g = -\big(ph^2(g/h)'\big)', \\ & \quad g,\, pg',\, h,\, ph' \in AC_{loc}((a,b)),\ h > 0 \text{ on } (a,b). \end{aligned} \tag{2.7}$$

Remark 2.6. Suppose that for some $\lambda_0 \in \mathbb{R}$, $u(\lambda_0,\cdot)$ is a solution of $\tau u = \lambda_0 u$ on (a,b) such that $u(\lambda_0,\cdot) > 0$ on (a,b). Then, for a.e. $x \in (a,b)$,

$$\tau u(\lambda_0,x) = \lambda_0 u(\lambda_0,x) \text{ is equivalent to } q(x) = \frac{[p(x)u'(\lambda_0,x)]'}{u(\lambda_0,x)} + \lambda_0 r(x).$$

Similarly, for $z \in \mathbb{C}$,

$$\tau v = zv \text{ is equivalent to } -(pv')' + \frac{[pu'(\lambda_0,\cdot)]'}{u(\lambda_0,\cdot)}v = (z-\lambda_0)rv;$$

in particular, a convenient candidate for h in (2.6) is $u(\lambda_0,\cdot)$. Moreover, abbreviating

$$\phi(\lambda_0,x) = \frac{p(x)u'(\lambda_0,x)}{u(\lambda_0,x)} = \frac{u^{[1]}(\lambda_0,x)}{u(\lambda_0,x)}, \quad x \in (a,b),$$

one obtains

$$q(x) = \phi'(\lambda_0,x) + p(x)^{-1}\phi(\lambda_0,x)^2 + \lambda_0 r(x) \text{ for a.e. } x \in (a,b), \tag{2.8}$$

and for the quadratic form associated with $T_{\min}$,

$$(f, T_{\min} f)_{L^2((a,b);rdx)} = \int_a^b r(x)dx\, \overline{f(x)}(\tau f)(x)$$

$$
\begin{aligned}
&= \int_a^b dx\, \overline{f(x)}[-(p(x)f'(x))' + q(x)f(x)] \\
&= \int_a^b dx\, \big[p(x)^{-1}\big|f^{[1]}(x)\big|^2 + q(x)|f(x)|^2\big] \\
&= \int_a^b dx\, \big\{p(x)^{-1}\big|f^{[1]}(x)\big|^2 + \big[\phi'(\lambda_0,x) + p(x)^{-1}\phi(\lambda_0,x)^2 + \lambda_0 r(x)\big]|f(x)|^2\big\} \\
&= \int_a^b dx\, p(x)^{-1}\big|f^{[1]}(x) - \phi(\lambda_0,x)f(x)\big|^2 + \lambda_0 \int_a^b r(x)dx\,|f(x)|^2 \\
&= \int_a^b dx\, p(x)u(\lambda_0,x)^2\left|\left(\frac{f(x)}{u(\lambda_0,x)}\right)'\right|^2 + \lambda_0 \int_a^b r(x)dx\,|f(x)|^2 \\
&> \lambda_0\|f\|^2_{L^2((a,b);rdx)}, \quad 0 \neq f \in \mathrm{dom}(T_{\min}),\ \mathrm{supp}\,(f) \subset (a,b) \text{ compact}, \qquad (2.9)
\end{aligned}
$$

permits the necessary integrations by part with vanishing boundary terms. We also note that equality would hold in (2.9) if and only if for some $C \in \mathbb{C}\backslash\{0\}$, $f(x) = Cu(\lambda_0, x)$, $x \in (a, b)$, which contradicts $\mathrm{supp}\,(f) \subset (a, b)$ compact.

For subsequent use in connection with a perturbative Hardy-type inequality in Theorem 3.1, we isolate one particular quadratic form equality related to (2.9), namely,

$$
\begin{aligned}
&\int_a^b dx\, \big\{p(x)|f'(x)|^2 + [q(x) - \lambda_0 r(x)]|f(x)|^2\big\} \\
&\qquad = \int_a^b dx\, p(x)u(\lambda_0,x)^2\left|\left(\frac{f(x)}{u(\lambda_0,x)}\right)'\right|^2 > 0, \\
&\qquad\qquad 0 \neq f \in \mathrm{dom}(T_{\min}),\ \mathrm{supp}\,(f) \subset (a,b) \text{ compact}. \qquad (2.10)
\end{aligned}
$$

Finally, if $[c, d] \subset (a, b)$ is compact and hence $\tau|_{[c,d]}$ is regular, the computations leading to (2.9) also apply to the Friedrichs extension $T_{F,(c,d)}$ of $T_{\min,(c,d)}$, the minimal operator associated to τ in $L^2((c, d); rdx)$. Here,

$$
\begin{aligned}
T_{F,(c,d)}f = \tau f, \quad f \in \mathrm{dom}(T_{F,(c,d)}) = \big\{g \in L^2((c,d);rdx) \,\big|\, g, g^{[1]} \in AC([c,d]); \\
g(c) = 0 = g(d); \tau g \in L^2((c,d);rdx)\big\}.
\end{aligned}
$$

In fact, they apply to the sesquilinear form $\mathfrak{Q}_{F,(c,d)}$ associated with $T_{F,(c,d)}$, where

$$
\begin{aligned}
\mathfrak{Q}_{F,(c,d)}(f,g) &= \big(|T_{F,(c,d)}|^{1/2}f, \mathrm{sgn}(T_{F,(c,d)})|T_{F,(c,d)}|^{1/2}g\big)_{L^2((c,d);rdx)}, \\
f, g \in \mathrm{dom}(\mathfrak{Q}_{F,(c,d)}) &= \mathrm{dom}\,\big(|T_{F,(c,d)}|^{1/2}\big) \\
&= \big\{h \in L^2((c,d);rdx) \,\big|\, h \in AC([c,d]); \\
&\qquad h(c) = 0 = h(d);\ (pr)^{-1/2}h^{[1]} \in L^2((c,d);rdx)\big\}.
\end{aligned}
$$

Then one obtains in a manner entirely analogous to (2.9),

$$
\mathfrak{Q}_{F,(c,d)}(f,f) = \int_c^d dx\, \big[p(x)^{-1}\big|f^{[1]}(x)\big|^2 + q(x)|f(x)|^2\big]
$$

$$
\begin{aligned}
&= \int_c^d dx\, \big\{p(x)^{-1}\big|f^{[1]}(x)\big|^2 + \big[\phi'(\lambda_0,x) + p(x)^{-1}\phi(\lambda_0,x)^2 + \lambda_0 r(x)\big]|f(x)|^2\big\} \\
&= \int_c^d dx\, p(x)^{-1}\big|f^{[1]}(x) - \phi(\lambda_0,x)f(x)\big|^2 + \lambda_0 \int_a^b r(x)dx\,|f(x)|^2 \\
&= \int_c^d dx\, p(x)u_0(\lambda_0,x)^2\bigg|\bigg(\frac{f(x)}{u(\lambda_0,x)}\bigg)'\bigg|^2 + \lambda_0 \int_c^d r(x)dx\,|f(x)|^2 \\
&\geq \lambda_0\|f\|^2_{L^2((c,d);rdx)}, \quad 0 \neq f \in \operatorname{dom}\big(|T_{F,(c,d)}|^{1/2}\big),
\end{aligned} \tag{2.11}
$$

where equality holds in (2.11) if and only if for some $C \in \mathbb{C}\backslash\{0\}$,

$$
f(x) = Cu(\lambda_0,x), \quad x \in [c,d]. \tag{2.12}
$$

If (2.12) holds, then λ_0 is the lowest eigenvalue of $T_{F,(c,d)}$ and at the same time $u(\lambda_0,\cdot) \in \operatorname{dom}(T_{F,(c,d)})$ is the corresponding eigenfunction of $T_{F,(c,d)}$, unique up to constant multiples, and strictly positive on (c,d).

Hypothesis 2.7. *Let $(a,b) \subseteq \mathbb{R}$ and suppose that p_0, q_0, and r_0 satisfy Hypothesis 2.1. In addition, assume that for some $\lambda_0 \in \mathbb{R}$, $u_0(\lambda_0,\cdot) > 0$ is a strictly positive solution of $\tau_0 u = \lambda_0 u$ on (a,b) (implying $T_{\min} \geq \lambda_0 I$).*

In the following we discuss factorizations of Sturm–Liouville differential expressions and use them to prove a natural generalization of Hardy's inequality (1.3) from the point of view of quadratic form perturbation theory comparing τ_0 and $\tau_{q_1} = \tau_0 + r_0^{-1}q_1$ for appropriate q_1.

Assuming Hypothesis 2.7, we start with a factorization of our comparison differential expression $\tau_{q_{1,\gamma,x_0}}$ on (a,b),

$$
\begin{aligned}
\tau_{q_{1,\gamma,x_0}} &= \tau_0 + r_0(x)^{-1}q_{1,\gamma,x_0}(x) \\
&= \frac{1}{r_0(x)}\bigg[-\frac{d}{dx}p_0(x)\frac{d}{dx} + q_0(x) \\
&\quad + \big[\gamma^2 - (1/4)\big]p_0(x)^{-1}u_0(\lambda_0,x)^{-4}\bigg(\int_{x_0}^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-2}\bigg], \\
&\qquad \gamma \in [0,\infty) \cup i(0,\infty),\ x_0 \in \{a,b\}, \text{ for a.e. } x \in (a,b).
\end{aligned} \tag{2.13}
$$

Introducing the differential expressions

$$
\begin{aligned}
A_{\alpha,x_0} &= \bigg[\frac{p_0(x)}{r_0(x)}\bigg]^{1/2} u_0(\lambda_0,x)\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{\alpha} \\
&\quad \times \frac{d}{dx}u_0(\lambda_0,x)^{-1}\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-\alpha} \\
&= [p_0(x)/r_0(x)]^{1/2}\frac{d}{dx} - [p_0(x)/r_0(x)]^{1/2}u_0(\lambda_0,x)^{-1}u_0'(\lambda_0,x) \\
&\quad - \alpha[p_0(x)r_0(x)]^{-1/2}u_0(\lambda_0,x)^{-2}\bigg(\int_{x_0}^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1},
\end{aligned}
$$

$$\alpha \in \mathbb{C},\ x_0 \in \{a,b\},\ \text{for a.e. } x \in (a,b), \tag{2.14}$$

$$\begin{aligned}
A^+_{\alpha,x_0} &= -\frac{1}{r_0(x)} u_0(\lambda_0,x)^{-1}\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-\alpha} \\
&\quad \times \frac{d}{dx}[p_0(x)r_0(x)]^{1/2} u_0(\lambda_0,x)\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{\alpha} \\
&= -\frac{1}{r_0(x)}\frac{d}{dx}[p_0(x)r_0(x)]^{1/2} - [p_0(x)r_0(x)]^{-1/2} u_0(\lambda_0,x)^{-1} p_0(x) u_0'(\lambda_0,x) \\
&\quad - \alpha[p_0(x)r_0(x)]^{-1/2} u_0(\lambda_0,x)^{-2}\bigg(\int_{x_0}^{x} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-1}, \\
&\qquad \alpha \in \mathbb{C},\ x_0 \in \{a,b\},\ \text{for a.e. } x \in (a,b),
\end{aligned} \tag{2.15}$$

one verifies that

$$\tau_{q_{1,\gamma,x_0}} = A^+_{\alpha,x_0} A_{\alpha,x_0} + \lambda_0 \ \text{ if and only if } \ \gamma^2 - (1/4) = \alpha(\alpha-1). \tag{2.16}$$

Moreover, since $u_0(\lambda_0,\cdot) > 0$ is a solution of $\tau_0 u = \lambda_0 u$ on (a,b), linearly independent solutions $u_{\gamma,\pm}(\lambda_0,\cdot)$ of $\tau_{q_{1,\gamma,x_0}} u = \lambda_0 u$ on (a,b) are explicitly given by

$$\begin{aligned}
u_{\gamma,\pm}(\lambda_0,x) &= u_0(\lambda_0,x)\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{(1/2)\pm\gamma}, \\
&\qquad \gamma \in (0,\infty) \cup i(0,\infty),\ x_0 \in \{a,b\},\ \text{for a.e. } x \in (a,b),
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
u_{0,1}(\lambda_0,x) &= u_0(\lambda_0,x)\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{1/2}, \\
u_{0,2}(\lambda_0,x) &= u_0(\lambda_0,x)\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{1/2} \\
&\quad \times \ln\bigg(\int_{\min(x,x_0)}^{\max(x,x_0)} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg), \\
&\qquad \gamma = 0,\ x_0 \in \{a,b\},\ \text{for a.e. } x \in (a,b).
\end{aligned} \tag{2.18}$$

The factorization (2.16) of $\tau_{q_{1,\gamma,x_0}}$ is possible for $\alpha \in \mathbb{R}$ if and only if $\gamma \geq 0$. In this case A^+_{α,x_0}, $\alpha \in \mathbb{R}$, is the adjoint differential expression of A_{α,x_0} and, as we will show in the next section, this factorization naturally leads to an inequality generalizing Hardy's inequality.

3. Perturbative Hardy-type inequalities

In this section we derive our principal results on perturbative Hardy-type inequalities. We start with the following fundamental result:

Theorem 3.1. *Assume Hypothesis 2.7 and introduce the linear space*

$$\mathcal{D}_0((a,b)) = \big\{g \in L^2((a,b); r_0 dx) \,\big|\, g \in AC_{loc}((a,b));\ \mathrm{supp}\,(g) \subset (a,b) \text{ compact}; \\ p_0^{1/2} g' \in L^2((a,b); dx)\big\}. \tag{3.1}$$

If
$$\bigg|\int_a dx\, p_0(x)^{-1} u_0(\lambda_0,x)^{-2}\bigg| < \infty \tag{3.2}$$
(i.e., $u_0(\lambda_0,\cdot)$ is nonprincipal at a), then for all $0 \neq f \in \mathcal{D}_0((a,b))$,
$$\begin{aligned}\int_a^b dx\, p_0(x)|f'(x)|^2 > \int_a^b dx\, \bigg[&\lambda_0\, r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \\ &\times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2}\bigg]|f(x)|^2. \end{aligned} \tag{3.3}$$
If, in addition,
$$\bigg|\int^b dx\, p_0(x)^{-1} u_0(\lambda_0,x)^{-2}\bigg| = \infty \tag{3.4}$$
(i.e., $u_0(\lambda_0,\cdot)$ is principal at b), then the constant $1/4$ in (3.3) is optimal.

Similarly, if
$$\bigg|\int^b dx\, p_0(x)^{-1} u_0(\lambda_0,x)^{-2}\bigg| < \infty$$
(i.e., $u_0(\lambda_0,\cdot)$ is nonprincipal at b), then for all $0 \neq f \in \mathcal{D}_0((a,b))$,
$$\begin{aligned}\int_a^b dx\, p_0(x)|f'(x)|^2 > \int_a^b dx\, \bigg[&\lambda_0\, r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \\ &\times \bigg(\int_x^b dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2}\bigg]|f(x)|^2. \end{aligned} \tag{3.5}$$
If, in addition,
$$\bigg|\int_a dx\, p_0(x)^{-1} u_0(\lambda_0,x)^{-2}\bigg| = \infty \tag{3.6}$$
(i.e., $u_0(\lambda_0,\cdot)$ is principal at a), then the constant $1/4$ in (3.5) is optimal.

Proof. It suffices to focus on (3.3), the proof of (3.5) being analogous. But inequality (3.3) is an instant consequence of inequality (2.10) upon identifying

$$\begin{aligned}
&\tau \ \text{ with } \ \tau_{q_{1,\gamma,a}}, \quad \gamma \in [0,\infty) \cup i(0,\infty), \\
&u(\lambda_0,\cdot) \ \text{ with } \ u_{\gamma,+}(\lambda_0,\cdot) = u_0(\lambda_0,\cdot)\bigg(\int_a^{\bullet} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{(1/2)+\gamma}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma \in (0,\infty) \cup i(0,\infty), \\
&u(\lambda_0,\cdot) \ \text{ with } \ u_{0,1}(\lambda_0,\cdot) = u_0(\lambda_0,\cdot)\bigg(\int_a^{\bullet} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{1/2}, \quad \gamma = 0, \\
&p \ \text{ with } \ p_0, \\
&q \ \text{ with } \ q_0 + \big[\gamma^2 - (1/4)\big] p_0^{-1} u_0(\lambda_0,\cdot)^{-4}\bigg(\int_a^{\bullet} dt\ p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2},
\end{aligned}$$

$$\gamma \in [0,\infty) \cup i(0,\infty),$$

r with r_0,

which results in

$$\int_a^b dx \Bigg\{ p_0(x)|f'(x)|^2 + \Bigg[q_0(x) - \lambda_0 r_0(x) + \big[\gamma^2 - (1/4)\big] p_0(x)^{-1} u_{0,1}(\lambda_0,x)^{-4} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{-2} \Bigg] |f(x)|^2 \Bigg\}$$
$$= \int_a^b dx\, p_0(x) u_{\gamma,+}(\lambda_0,x)^2 \bigg| \bigg(\frac{f(x)}{u_{\gamma,+}(\lambda_0,x)} \bigg)' \bigg|^2 \geq 0,$$
$$f \in \operatorname{dom}(T_{q_{1,\gamma,a,\min}}), \ \operatorname{supp}(f) \ \text{compact in } (a,b), \ \gamma \in [0,\infty). \tag{3.7}$$

Here we identify $u_{\gamma,+}(\lambda_0,\cdot)$ and $u_{0,1}(\lambda_0,\cdot)$ if $\gamma = 0$ and $T_{q_{1,\gamma,a},\min}$ represents the minimal operator associated with $\tau_{q_{1,\gamma,a}}$.

Since f has compact support in (a,b), and $[c,d] \subset (a,b)$ in (2.11) can be chosen such that $[c,d]$ contains the support of f, one concludes that

$$(3.7) \ \text{extends to } f \in \mathcal{D}_0((a,b)). \tag{3.8}$$

Choosing $\gamma = 0$ yields inequality (3.3).

Since equality in (3.7), and hence equality under the conditions of (3.8), can only hold for

$$f(x) = 0 \ \text{ and } \ f(x) = u_{0,1}(\lambda_0,x) = u_0(\lambda_0,x) \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{1/2}, \quad x \in (a,b), \tag{3.9}$$

the strict perturbative Hardy-type inequality (3.3) results as $f \in \mathcal{D}_0((a,b))$ (but not $u_{0,1}(\lambda_0,\cdot)$) has compact support.

To see that the constant $1/4$ in (3.3) is optimal, suppose by way of contradiction that the $1/4$ in (3.3) can be replaced by $(1/4)+\varepsilon$ for some $\varepsilon \in (0,\infty)$. Writing $\varepsilon = -\gamma^2$ for some $\gamma = \gamma(\varepsilon) \in i(0,\infty)$, then after rearranging terms in (3.7) and a simple integration by parts, one obtains

$$(f, [T_{q_{1,\gamma,a},\min} - \lambda_0 I] f)_{L^2((a,b);r_0dx)} \geq 0,$$
$$f \in \operatorname{dom}(T_{q_{1,\gamma,a},\min}), \ \operatorname{supp}(f) \ \text{compact in } (a,b),$$

so that $T_{q_{1,\gamma,a},\min} \geq \lambda_0 I$. On the other hand, according to (3.4), the solutions $u_{\gamma,\pm}(\lambda_0,\cdot)$ in (2.17), and hence by Theorem 2.2(ii) every solution u of $\tau_{q_{1,\gamma,a}} u = \lambda_0 u$, is oscillatory near b. Thus, $\tau_{q_{1,\gamma,a}} - \lambda_0$ is oscillatory at b and hence $T_{q_{1,\gamma,a},\min} \ngeq \lambda_0 I$, which is a contradiction. □

One notes that the existence of $u_0(\lambda_0,\cdot) > 0$ on (a,b) in Theorem 3.1 implies together with $u_{0,1}(\lambda,\cdot) > 0$ in (2.18) that

$$T_{0,\min} \geq \lambda_0 I, \quad T_{0,F} \geq \lambda_0 I, \quad T_{q_{1,\gamma,a},\min} \geq \lambda_0 I, \quad T_{q_{1,\gamma,a},F} \geq \lambda_0 I. \tag{3.10}$$

Remark 3.2. One can also prove (3.3) for $\gamma = 0$ directly as follows: Choose $c,\, d \in (a,b)$, $c < d$ and $f \in AC_{loc}((a,b))$. Then

$$
\begin{aligned}
&\int_c^d dx\, p_0(x) u_0(\lambda_0,x)^2 \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg) \\
&\qquad \times \bigg| \bigg[u_0(\lambda_0,x)^{-1} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f(x)\bigg]' \bigg|^2 \\
&\quad = \int_c^d dx\, \bigg\{ p_0(x)|f'(x)|^2 + 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \\
&\qquad \times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2} |f(x)|^2 \\
&\qquad + u_0(\lambda_0,x)^{-2} p_0(x) u_0'(\lambda_0,x)^2 |f(x)|^2 + u_0(\lambda_0,x)^{-3} u_0'(\lambda_0,x) \\
&\qquad \times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-1} |f(x)|^2 \\
&\qquad - \Big[\overline{f(x)} f'(x) + f(x) \overline{f'(x)}\Big] \bigg[u_0(\lambda_0,x)^{-1} p_0(x) u_0'(\lambda_0,x) \\
&\qquad + 2^{-1} u_0(\lambda_0,x)^{-2} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-1} \bigg]\bigg\} \\
&\quad = \int_c^d dx\, \bigg\{ p_0(x)|f'(x)|^2 + [q_0(x) - \lambda_0\, r_0(x)] |f(x)|^2 \\
&\qquad - 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2} |f(x)|^2 \bigg\} \\
&\qquad - |f(x)|^2 \bigg[p_0(x) u_0(\lambda_0,x)^{-1} u_0'(\lambda_0,x) \\
&\qquad + 2^{-1} u_0(\lambda_0,x)^{-2} \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-1} \bigg] \bigg|_{x=c}^d \geq 0, \qquad (3.11)
\end{aligned}
$$

integrating by parts once. Choosing $f \in \mathcal{D}_0((a,b))$ and letting $c \downarrow a$, $d \uparrow b$ then yields (3.3) in the case $\gamma = 0$. The argument extends to $\gamma \in (0,\infty)$, but we omit the details.

Remark 3.3. A straightforward application of Theorem 3.1 is the following lower boundedness observation. Assume that the coefficients in the differential expressions $\tau_0 = r_0^{-1}[-(d/dx)p_0(d/dx) + q_0]$ and its perturbation $\tau_{q_1} = \tau_0 + r_0^{-1} q_1$ satisfy Hypothesis 2.1. Introducing the minimal operator $T_{q_1,\min}$ and its Friedrichs extension $T_{q_1,F}$ corresponding to τ_{q_1}, the inequality

$$q_1(x) \geq q_{1,0,x_0}(x) \ \text{ for a.e. } x \in (a,b),$$

where

$$q_{1,0,x_0}(x) = -4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \bigg(\int_{x_0}^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2}\bigg)^{-2}$$

for a.e. $x \in (a,b)$, and $x_0 \in \{a,b\}$,

combined with the inequalities (3.3), (3.5) then yields the lower bounds

$$T_{q_1,\min} \geq \lambda_0 I, \quad T_{q_1,F} \geq \lambda_0 I.$$

Next we show how to obtain a result analogous to Theorem 3.1 with the compact support assumption removed. This requires some notation and preliminary results which we shall introduce next. We begin by introducing, for fixed $\lambda_0 \in \mathbb{R}$, functions W, α and β defined a.e. on (a,b) as follows. First, we introduce the weight

$$W = 4^{-1} p_0^{-1} u_0(\lambda_0,\cdot)^{-4} \bigg(\int_a^{\bullet} dt\; p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{-2} = -q_{1,0,a}, \tag{3.12}$$

and then (with $f_{\pm}(x) = [|f(x)| \pm f(x)]/2$ for a.e. $x \in (a,b)$)

$$\alpha = \begin{cases} q_{0,+}, & \lambda_0 \geq 0, \\ q_{0,+} - \lambda_0 r_0, & \lambda_0 < 0, \end{cases} \quad \beta = \begin{cases} \lambda_0 r_0 + q_{0,-} + W, & \lambda_0 \geq 0, \\ q_{0,-} + W, & \lambda_0 < 0. \end{cases} \tag{3.13}$$

One notes that $\beta > 0$ a.e. on (a,b) for all $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ (respectively, $\alpha \geq 0$) a.e. on (a,b) if $\lambda_0 < 0$ (respectively, $\lambda_0 \geq 0$).

Next, we introduce

$$\dot{H}_{p_0,\alpha}((a,b)) = \bigg\{ f : (a,b) \to \mathbb{C} \,\bigg|\, f \in AC_{\text{loc}}((a,b)),$$
$$\|f\|^2_{p_0,\alpha} := \int_a^b dx\, \big[p_0(x)|f'(x)|^2 + \alpha(x)|f(x)|^2\big] < \infty \bigg\}. \tag{3.14}$$

Remark 3.4. By (3.3), $\|\cdot\|_{p_0,\alpha}$ is a norm on $\mathcal{D}_0((a,b))$. However, $\|\cdot\|_{p_0,\alpha}$ is not necessarily a norm on $\dot{H}_{p_0,\alpha}((a,b))$ since it does not separate points in general (e.g., take $p_0 = r_0 \equiv 1$, $q_0 = 0$, and $\lambda_0 \geq 0$).

Lemma 3.5. *If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$, then there exists a unique $f \in \dot{H}_{p_0,\alpha}((a,b)) \cap L^2((a,b);\beta dx)$ such that*

$$\lim_{n\to\infty} \big[\|f_n - f\|_{p_0,\alpha} + \|f_n - f\|_{L^2((a,b);\beta dx)}\big] = 0. \tag{3.15}$$

Proof. The uniqueness of f follows from the property (3.15), so it suffices to prove existence. Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$, (3.3) implies that $\{f_n'\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2((a,b);p_0 dx)$ and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2((a,b);\alpha dx)$ and in $L^2((a,b);\beta dx)$. Thus, there exist $g_1 \in L^2((a,b);p_0 dx)$ and $g_0 \in L^2((a,b);\alpha dx) \cap L^2((a,b);\beta dx)$ such that

$$\lim_{n\to\infty} \|f_n' - g_1\|_{L^2((a,b);p_0 dx)} = \lim_{n\to\infty} \|f_n - g_0\|_{L^2((a,b);\alpha dx)}$$
$$= \lim_{n\to\infty} \|f_n - g_0\|_{L^2((a,b);\beta dx)} = 0. \tag{3.16}$$

In particular, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a set $N \subset (a,b)$ with $|N| = 0$ (here $|\cdot|$ temporarily abbreviates Lebesgue measure) such that

$$\lim_{k\to\infty} f'_{n_k}(x) = g_1(x), \quad \lim_{k\to\infty} f_{n_k}(x) = g_0(x); \quad x \in (a,b)\backslash N. \tag{3.17}$$

Fix $c \in (a,b)\backslash N$. Then for all $x \in (a,b)\backslash N$,

$$\begin{aligned} g_0(x) &= \lim_{k\to\infty} f_{n_k}(x) \\ &= \lim_{k\to\infty} \bigg[f_{n_k}(c) + \int_c^x dt\, [f'_{n_k}(t) - g_1(t)] + \int_c^x dt\, g_1(t) \bigg], \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} \bigg| \int_c^x dt\, [f'_{n_k}(t) - g_1(t)] \bigg| &\le \int_c^x dt\, |f'_{n_k}(t) - g_1(t)| \\ &= \int_c^x dt\, p_0(t)^{-1/2} p_0(t)^{1/2} |f'_{n_k}(t) - g_1(t)| \\ &\le \bigg(\int_c^x dt\, p_0(t)^{-1} \bigg)^{1/2} \bigg(\int_c^x dt\, p_0(t) |f'_{n_k}(t) - g_1(t)|^2 \bigg)^{1/2} \\ &\le \bigg(\int_c^x dt\, p_0(t)^{-1} \bigg)^{1/2} \|f'_{n_k} - g_1\|_{L^2((a,b);p_0dx)}, \end{aligned} \tag{3.19}$$

and the upper bound in (3.19) converges to 0 as $k \to \infty$ by (3.16). By (3.18) and (3.19),

$$g_0(x) = g_0(c) + \int_c^x dt\, g_1(t), \quad x \in (a,b)\backslash N, \tag{3.20}$$

and it follows that $g_0 \in AC_{loc}((a,b))$ and $g'_0 = g_1$. The conclusion of the lemma now follows from (3.16) by taking $f = g_0$. □

Next, we introduce

$$\begin{aligned} \dot{H}_{p_0,\alpha,0}((a,b)) = \big\{ f \in \dot{H}_{p_0,\alpha}((a,b)) \,\big|\, & \text{there is a Cauchy sequence } \{f_n\}_{n=1}^{\infty} \text{ in} \\ & (\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha}) \text{ such that} \\ & \lim_{n\to\infty} \big[\|f_n - f\|_{p_0,\alpha} + \|f_n - f\|_{L^2((a,b);\beta dx)} \big] = 0 \big\}. \end{aligned} \tag{3.21}$$

$$\begin{aligned} = \big\{ f \in \dot{H}_{p_0,\alpha}((a,b)) \,\big|\, & \text{there is a Cauchy sequence } \{f_n\}_{n=1}^{\infty} \text{ in} \\ & (\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha}) \text{ such that for all } a < c < d < b, \\ & \lim_{n\to\infty} \|f_n - f\|_{L^2((c,d);W dx)} = 0 \big\}. \end{aligned} \tag{3.22}$$

Remark 3.6.

(i) If $f \in \dot{H}_{p_0,\alpha}((a,b))$ and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$ satisfying the limit condition in (3.21), then (3.22) is clearly satisfied. On the other hand, if $f \in \dot{H}_{p_0,\alpha}((a,b))$ and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$ satisfying the limit condition in (3.22), then by Lemma 3.5, there exists $g \in \dot{H}_{p_0,\alpha}((a,b))$ such that

$$\|f_n - g\|_{L^2((a,b);W dx)} \leq \|f_n - g\|_{p_0,\alpha} + \|f_n - g\|_{L^2((a,b);\beta dx)} \to 0 \text{ as } n \to \infty. \tag{3.23}$$

Hence, $f = g$, and thus f satisfies the condition in (3.21).

(ii) Lemma 3.5 implies that $\dot{H}_{p_0,\alpha,0}((a,b))$ can be identified with the completion of $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$.

(iii) If $\lambda_0 \neq 0$, then $\dot{H}_{p_0,\alpha,0}((a,b)) \subseteq L^2((a,b); r_0 dx)$. If $\lambda_0 = 0$, then $\dot{H}_{p_0,\alpha,0}((a,b))$ is not necessarily a subspace of $L^2((a,b); r_0 dx)$, see Remark 3.9 below for an example.

The following result plays a fundamental role in the proof of Corollary 3.14.

Theorem 3.7. *Assume Hypothesis 2.7 and suppose that*

$$\left| \int_a dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \right| < \infty. \tag{3.24}$$

Then for all $f \in \dot{H}_{p_0,\alpha,0}((a,b))$,

$$\int_a^b dx\, p_0(x)|f'(x)|^2 \geq \int_a^b dx \bigg[\lambda_0 r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \\ \times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{-2} \bigg] |f(x)|^2. \tag{3.25}$$

If, in addition,

$$u_0(\lambda_0,\cdot) \bigg(\int_a^{\bullet} dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{1/2} \notin \dot{H}_{p_0,\alpha,0}((a,b)), \tag{3.26}$$

then inequality (3.25) is strict for $0 \neq f \in \dot{H}_{p_0,\alpha,0}((a,b))$.

Proof. Let $f \in \dot{H}_{p_0,\alpha,0}((a,b))$ and $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in the space $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$ satisfying

$$\lim_{n\to\infty} \big[\|f_n - f\|_{p_0,\alpha} + \|f_n - f\|_{L^2((a,b);\beta dx)} \big] = 0. \tag{3.27}$$

Then by (3.27) and (3.3),

$$\int_a^b dx\, \big[p_0(x)|f'(x)|^2 + \alpha(x)|f(x)|^2\big] = \lim_{n\to\infty} \int_a^b dx\, \big[p_0(x)|f_n'(x)|^2 + \alpha(x)|f_n(x)|^2\big] \\ \geq \liminf_{n\to\infty} \int_a^b dx\, \beta(x)|f_n(x)|^2 = \int_a^b dx\, \beta(x)|f(x)|^2, \tag{3.28}$$

which is the desired inequality. One notes that the two equalities in (3.28) follow from (3.27), while the inequality in (3.28) uses (3.3).

Next, suppose, in addition, that (3.26) holds and that $f \neq 0$. By (3.27) there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, and a set $\mathcal{N} \subset (a,b)$ of zero Lebesgue measure, such that for all $x \in (a,b)\backslash\mathcal{N}$,

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \text{ and } \lim_{k\to\infty} f'_{n_k}(x) = f'(x).$$

Then, for all $x \in (a,b)\backslash\mathcal{N}$,

$$\begin{aligned}
&\lim_{k\to\infty} \bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f_{n_k}(x)\bigg]' \\
&\quad = \bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f(x)\bigg]'.
\end{aligned}$$

By (3.26),

$$u_0(\lambda_0,\cdot)^{-1}\bigg(\int_a^{\bullet} dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f(\cdot)$$

is not a constant function on (a,b). Hence, by Fatou's lemma, (3.11), and (3.27),

$$\begin{aligned}
0 &< \int_a^b dx\, p_0(x)u_0(\lambda_0,x)^2 \int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2} \\
&\quad \times \bigg|\bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f(x)\bigg]'\bigg|^2 \\
&\le \liminf_{k\to\infty} \int_a^b dx\, p_0(x)u_0(\lambda_0,x)^2 \int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2} \\
&\qquad \times \bigg|\bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f_{n_k}(x)\bigg]'\bigg|^2 \\
&= \liminf_{k\to\infty} \int_a^b dx\, \big\{p_0(x)|f'_{n_k}(x)|^2 + [q_0(x) - \lambda_0 r_0(x) - W(x)]|f_{n_k}(x)|^2\big\} \\
&= \lim_{k\to\infty} \int_a^b dx\, \big\{p_0(x)|f'_{n_k}(x)|^2 + [q_0(x) - \lambda_0 r_0(x) - W(x)]|f_{n_k}(x)|^2\big\} \\
&= \int_a^b dx\, \big\{p_0(x)|f'(x)|^2 + [q_0(x) - \lambda_0 r_0(x) - W(x)]|f(x)|^2\big\}. \qquad \square
\end{aligned} \tag{3.29}$$

Remark 3.8. Since $\mathcal{D}_0((a,b)) \subset \dot{H}_{p_0,\alpha,0}((a,b))$, Theorem 3.1 implies that the constant $1/4$ in (3.25) is optimal subject to condition (3.4).

Remark 3.9. In the special case $p_0 = r_0 = u_0(0,\cdot) = 1$, $q_0 = \lambda_0 = 0$, and $-\infty < a < b = \infty$, it has been shown in [5, Proposition 3.1 and Theorem 3.4] that

$$\begin{aligned}
\dot{H}_{p_0,\alpha,0}((a,\infty)) &= \big\{f \in AC_{\rm loc}((a,\infty)) \,\big|\, \lim_{x\downarrow a} f(x) = 0;\ f' \in L^2((a,\infty);dx)\big\} \\
&= \big\{f \in AC_{\rm loc}([a,\infty)) \,\big|\, f(a) = 0;\ f' \in L^2((a,\infty);dx)\big\} \\
&= \big\{f \in AC_{\rm loc}((a,\infty)) \,\big|\, (\cdot - a)^{-1}f, f' \in L^2((a,\infty);dx)\big\},
\end{aligned}$$

where

$$AC_{loc}([a,\infty)) = \{f : [a,\infty) \to \mathbb{C} \,|\, f \in AC([a,c]) \text{ for all } c \in (a,\infty)\}.$$

In this case $\dot{H}_{p_0,\alpha,0}((a,\infty)) \not\subset L^2((a,\infty);dx)$ (cf. [5, Remark 3.3]), in particular,

$$\dot{H}_{p_0,\alpha,0}((a,\infty)) \not\subset H_0^2((a,\infty);dx),$$

where, as usual, $H_0^2((a,\infty);dx) = \overline{C_0^\infty((a,\infty))}^{H^2((a,\infty))}$ and

$$\begin{aligned} H^2((a,\infty)) &= \big\{f : (a,\infty) \to \mathbb{C} \,\big|\, f \text{ is weakly differentiable on } (a,\infty); \\ &\qquad \|f\|_{H^2((a,\infty))} < \infty\big\} \\ &= \big\{f \in L^2((a,\infty);dx) \,\big|\, f \in AC([a,b]) \text{ for all } b \in (a,\infty); f' \in L^2((a,\infty);dx)\big\}, \end{aligned}$$

denotes the classical Sobolev space with associated norm

$$\|f\|^2_{H^2((a,\infty))} = \int_a^\infty dx\, \big[|f'(x)|^2 + |f(x)|^2\big], \quad f \in H^2((a,\infty)).$$

These observations show that in the present case, (1.3) extends to all

$$f \in \big\{g \in AC_{\text{loc}}([a,\infty)) \,\big|\, g(a) = 0;\, g' \in L^2((a,\infty);dx)\big\}, \tag{3.30}$$

which is of course well-known.

In analogy with (3.1), let

$$H_{p_0,r_0}((a,b)) = \big\{g \in L^2((a,b);r_0dx) \,\big|\, g \in AC_{\text{loc}}((a,b));\, p_0^{1/2}g' \in L^2((a,b);dx)\big\}$$

and define the norm $\|\cdot\|_{p_0,r_0} : H_{p_0,r_0}((a,b)) \to [0,\infty)$ on $H_{p_0,r_0}((a,b))$ by

$$\|f\|^2_{p_0,r_0} = \int_a^b dx\, \big[p_0(x)|f'(x)|^2 + r_0(x)|f(x)|^2\big], \quad f \in H_{p_0,r_0}((a,b)).$$

One notes that $\mathcal{D}_0((a,b))$ (cf. (3.1)) is a subspace of $H_{p_0,r_0}((a,b))$.

Lemma 3.10. *If $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,r_0})$, then there exists a unique $f \in H_{p_0,r_0}((a,b))$ such that*

$$\lim_{n\to\infty} \|f_n - f\|_{p_0,r_0} = 0.$$

Proof. Uniqueness of the limit is clear, so it suffices to prove the existence claim. Since $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,r_0})$, it follows that $\{f_n'\}_{n=1}^\infty$ and $\{f_n\}_{n=1}^\infty$ are Cauchy sequences in $L^2((a,b);p_0dx)$ and $L^2((a,b);r_0dx)$, respectively. Thus, there exist $g_1 \in L^2((a,b);p_0dx)$ and $g_0 \in L^2((a,b);r_0dx)$ such that

$$\lim_{n\to\infty} \|f_n' - g_1\|_{L^2((a,b);p_0dx)} = \lim_{n\to\infty} \|f_n - g_0\|_{L^2((a,b);r_0dx)} = 0. \tag{3.31}$$

Therefore, there exist $N \subset (a,b)$ with $|N| = 0$ (here $|\cdot|$ temporarily abbreviates Lebesgue measure) and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} f_{n_k}'(x) = g_1(x), \quad \lim_{k\to\infty} f_{n_k}(x) = g_0(x); \quad x \in (a,b)\backslash N. \tag{3.32}$$

Fix $c \in (a,b)\backslash N$. Then

$$\begin{aligned} g_0(x) &= \lim_{k\to\infty} f_{n_k}(x) \\ &= \lim_{k\to\infty} \bigg\{ f_{n_k}(c) + \int_c^x dt\, [f_{n_k}'(t) - g_1(t)] + \int_c^x dt\, g_1(t) \bigg\}, \quad x \in (a,b)\backslash N, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} \bigg| \int_c^x dt\, [f_{n_k}'(t) - g_1(t)] \bigg| &\leq \int_c^x dt\, |f_{n_k}'(t) - g_1(t)| \\ &= \int_c^x dt\, p_0(t)^{-1/2} p_0(t)^{1/2} |f_{n_k}'(t) - g_1(t)| \\ &\leq \bigg[\int_c^x dt\, p_0(t)^{-1} \bigg]^{1/2} \bigg[\int_c^x dt\, p_0(t) |f_{n_k}'(t) - g_1(t)|^2 \bigg]^{1/2} \\ &\leq \bigg[\int_c^x dt\, p_0(t)^{-1} \bigg]^{1/2} \|f_{n_k}' - g_1\|_{L^2((a,b);p_0dx)}, \quad x \in (a,b)\backslash N. \end{aligned} \tag{3.34}$$

The estimate in (3.34) taken together with (3.31) yields

$$\lim_{k\to\infty} \int_c^x dt\, [f_{n_k}'(t) - g_1(t)] = 0, \quad x \in (a,b)\backslash N,$$

so that (3.33) reduces to

$$g_0(x) = g_0(c) + \int_c^x dt\, g_1(t), \quad x \in (a,b)\backslash N. \tag{3.35}$$

In particular, (3.35) implies that $g_0 \in AC_{loc}((a,b))$ and $g_0' = g_1$. Therefore, $f := g_0$ is the desired function. $\square$

Next, introduce the subspace

$$\begin{aligned} H_{p_0,r_0,0}((a,b)) = \big\{ g \in H_{p_0,r_0}((a,b)) \,\big|\, &\text{there exists a Cauchy sequence} \\ &\{g_n\}_{n=1}^{\infty} \text{ in } (\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,r_0}) \text{ such that} \\ &\lim_{n\to\infty} \|g_n - g\|_{p_0,r_0} = 0 \big\} \\ = \big\{ g \in H_{p_0,r_0}((a,b)) \,\big|\, &\text{there exists a Cauchy sequence} \\ &\{g_n\}_{n=1}^{\infty} \text{ in } (\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,r_0}) \text{ such that} \\ &\lim_{n\to\infty} \|g_n - g\|_{L^2((a,b);r_0dx)} = 0 \big\}. \end{aligned} \tag{3.36}$$

Remark 3.11. To justify the second equality in (3.36), it suffices to verify the containment "$\supseteq$" (the containment "$\subseteq$" follows immediately from the definition of $\|\cdot\|_{p_0,r_0}$). In turn, to establish the containment "$\supseteq$," it suffices to note that if $f \in H_{p_0,r_0}((a,b))$ and $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,r_0})$ with $\lim_{n\to\infty} \|f_n - f\|_{L^2((a,b);r_0dx)} = 0$, then by Lemma 3.10, there exists $g \in H_{p_0,r_0}((a,b))$ such that

$$\|f_n - g\|_{L^2((a,b);r_0dx)} \leq \|f_n - g\|_{p_0,r_0} \longrightarrow 0 \ \text{ as } \ n \to \infty.$$

Thus, $g = f$, and it follows that f belongs to the middle set in (3.36).

Define the operator $A : \mathcal{D}_0((a,b)) \to L^2((a,b);r_0dx)$ by

$$Af = p_0^{1/2} r_0^{-1/2} f', \quad f \in \operatorname{dom}(A) = \mathcal{D}_0((a,b)).$$

On $\mathcal{D}_0((a,b))$, $\|\cdot\|_{p_0,r_0}$ coincides with the graph norm of A, so an elementary argument employing Lemma 3.10 yields the following result.

Lemma 3.12. *The operator A is closable and* $\operatorname{dom}\big(\overline{A}\big) = H_{p_0,r_0,0}((a,b))$.

For completeness we recall the following result to be used in the proof of Corollary 3.14 below:

Lemma 3.13. *Suppose $T : \operatorname{dom}(T) \to \mathcal{H}_2$ is a closed operator, $\operatorname{dom}(T) \subseteq \mathcal{H}_1$, and S is a closable operator from $\operatorname{dom}(S) \subseteq \mathcal{H}_1$ to $\mathcal{H}_2$ (or even $\mathcal{H}_3$). If $\operatorname{dom}(S) \supseteq \operatorname{dom}(T)$, then S is T-bounded.*

With these preparations in place, we now state and prove, as a consequence of Theorem 3.7, the following alternative to Theorem 3.1 which avoids the compact support hypothesis on f in (3.3) and improves on [6, Theorem 3.2].

Corollary 3.14. *Assume Hypothesis 2.7 and suppose that*

$$\bigg| \int_a dx\, p_0(x)^{-1} u_0(\lambda_0, x)^{-2} \bigg| < \infty \tag{3.37}$$

(i.e., $u_0(\lambda_0,\cdot)$ is nonprincipal at a). Let B be a closed operator restriction of $\overline{A}$ (i.e., B is closed, $\operatorname{dom}(B) \subseteq \operatorname{dom}\big(\overline{A}\big)$, $Bf = \overline{A}f$ for all $f \in \operatorname{dom}(B)$). Assume that $\mathcal{D}_0((a,b)) \cap \operatorname{dom}(B)$ is a core of B and that

$$\operatorname{dom}(B) \subseteq \operatorname{dom}\big((q_{0,+}/r_0)^{1/2}\big). \tag{3.38}$$

Then for all $f \in \operatorname{dom}(B)$,

$$\begin{aligned} \int_a^b dx\, p_0(x)|f'(x)|^2 \geq \int_a^b dx\, \bigg[&\lambda_0\, r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0,x)^{-4} \\ &\times \bigg(\int_a^x dt\, p_0(t)^{-1} u_0(\lambda_0,t)^{-2} \bigg)^{-2} \bigg] |f(x)|^2. \end{aligned} \tag{3.39}$$

Similarly, if

$$\bigg| \int^{b} dx\, p_0(x)^{-1} u_0(\lambda_0, x)^{-2} \bigg| < \infty$$

(i.e., $u_0(\lambda_0, \cdot)$ is nonprincipal at b), then for all $f \in \operatorname{dom}(B)$,

$$\int_a^b dx\, p_0(x)|f'(x)|^2 \geq \int_a^b dx \bigg[\lambda_0\, r_0(x) - q_0(x) + 4^{-1} p_0(x)^{-1} u_0(\lambda_0, x)^{-4} \times \bigg(\int_x^b dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2}\bigg)^{-2}\bigg] |f(x)|^2. \quad (3.40)$$

If, in addition,

$$u_0(\lambda_0, \cdot)\bigg(\int_a^{\bullet} dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2}\bigg)^{1/2} \notin \operatorname{dom}(B), \quad (3.41)$$

then inequality (3.39) is strict for $0 \neq f \in \operatorname{dom}(B)$. Similarly, if, in addition,

$$u_0(\lambda_0, \cdot)\bigg(\int_{\bullet}^{b} dt\, p_0(t)^{-1} u_0(\lambda_0, t)^{-2}\bigg)^{1/2} \notin \operatorname{dom}(B), \quad (3.42)$$

then inequality (3.40) is strict for $0 \neq f \in \operatorname{dom}(B)$.

Proof. By Theorem 3.7, it suffices to show that $\operatorname{dom}(B) \subseteq \dot{H}_{p_0,\alpha,0}((a,b))$. By Lemma 3.13 and (3.38), there exists $C \in (0,\infty)$ such that

$$\big\|(q_{0,+}/r_0)^{1/2} f\big\|_{L^2((a,b);r_0dx)} \leq C\big[\|Bf\|_{L^2((a,b);r_0dx)} + \|f\|_{L^2((a,b);r_0dx)}\big], \quad f \in \operatorname{dom}(B). \quad (3.43)$$

Let $f \in \operatorname{dom}(B)$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}_0((a,b)) \cap \operatorname{dom}(B)$ such that

$$\lim_{n\to\infty} \|Bf_n - Bf\|_{L^2((a,b);r_0dx)} + \|f_n - f\|_{L^2((a,b);r_0dx)} = 0. \quad (3.44)$$

Then by (3.43),

$$\lim_{n\to\infty} \big\|(q_{0,+}/r_0)^{1/2}(f_n - f)\big\|_{L^2((a,b);r_0dx)} = 0. \quad (3.45)$$

Hence, by (3.44) and (3.45), $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathcal{D}_0((a,b)), \|\cdot\|_{p_0,\alpha})$. Therefore, by Lemma 3.5, there exists a unique $\widetilde{f} \in \dot{H}_{p_0,\alpha}((a,b)) \cap L^2((a,b);\beta dx)$ such that

$$\lim_{n\to\infty} \big[\big\|f_n - \widetilde{f}\big\|_{p_0,\alpha} + \big\|f_n - \widetilde{f}\big\|_{L^2((a,b);\beta dx)}\big] = 0. \quad (3.46)$$

Since $\beta > 0$ a.e. on (a,b), (3.46) implies that there exists $\widetilde{N} \subset (a,b)$ and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $|\widetilde{N}| = 0$ (here $|\cdot|$ temporarily abbreviates Lebesgue measure) and

$$\lim_{k\to\infty} f_{n_k}(x) = \widetilde{f}(x), \quad x \in (a,b)\backslash\widetilde{N}. \quad (3.47)$$

Similarly, since $r_0 > 0$ a.e. on (a,b), (3.44) implies that there exists $N \subset (a,b)$ and a subsequence $\{f_{n_{k_j}}\}_{j=1}^{\infty}$ such that $|N| = 0$ and

$$\lim_{j\to\infty} f_{n_{k_j}}(x) = f(x), \quad x \in (a,b)\backslash N. \tag{3.48}$$

Taken together, relations (3.47) and (3.48) imply $f = \widetilde{f}$. Since $\widetilde{f} \in \dot{H}_{p_0,\alpha,0}((a,b))$ by (3.46) and (3.21), the conclusion of the theorem now follows from an application of Theorem 3.7.

Next, suppose, in addition, that (3.41) holds and that $f \neq 0$. Since $f = \widetilde{f}$, (3.46) implies the existence of a subsequence $\{g_k\}_{k\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$, and a set $\widehat{\mathcal{N}} \subset (a,b)$ of Lebesgue measure zero, such that for all $x \in (a,b)\backslash\widehat{\mathcal{N}}$,

$$\lim_{k\to\infty} g_k(x) = f(x) \;\text{ and }\; \lim_{k\to\infty} g_k'(x) = f'(x).$$

Then, for all $x \in (a,b)\backslash\widehat{\mathcal{N}}$,

$$\begin{aligned}
&\lim_{k\to\infty}\bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} g_k(x)\bigg]'\\
&\quad= \bigg[u_0(\lambda_0,x)^{-1}\bigg(\int_a^x dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\bigg)^{-1/2} f(x)\bigg]'.
\end{aligned}$$

By (3.41), $u_0(\lambda_0,\cdot)^{-1}\Big(\int_a^{\bullet} dt\, p_0(t)^{-1}u_0(\lambda_0,t)^{-2}\Big)^{-1/2} f$ is not a constant function on (a,b). Thus, one can repeat the calculations in (3.29), with f_{n_k} replaced by g_k, and obtain

$$0 < \int_a^b dx\, \big\{p_0(x)|f'(x)|^2 + [q_0(x) - \lambda_0 r_0(x) - W(x)]|f(x)|^2\big\}.$$

As the proof of strict inequality in (3.40) if (3.42) holds is entirely analogous, we omit it. ☐

The case $B = \overline{A}$ is permitted in Corollary 3.14 as long as (3.38) is satisfied.

Once more, in the case $B = \overline{A}$ (so that $D_0((a,b)) \subseteq \mathrm{dom}(B) = \mathrm{dom}(\overline{A})$), the analog of Remark 3.8 applies to optimality of the constant $1/4$ in (3.39) and (3.40).

Finally, returning once more to our starting point, the classical Hardy inequality situation, we offer the following fact:

Remark 3.15. If $p_0 = r_0 = u_0(0,\cdot) = 1$, $q_0 = \lambda_0 = 0$, and $a, b \in \mathbb{R}$, it has been shown in [4, Lemma 3.4] that $\|\cdot\|_{p_0,\alpha}$ and $\|\cdot\|_{p_0,r_0}$ are equivalent norms on $\mathcal{D}_0((a,b))$. Thus, in this case,

$$\dot{H}_{p_0,\alpha,0}((a,b)) = H_{p_0,r_0,0}((a,b)) = H_0^2((a,b)).$$

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Збурені нерівності типу Харді

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Для даного трьохкоефіцієнтного диференціального виразу Штурма–Ліувілля $\tau_0 = r_0^{-1}[-(d/dx)p_0(d/dx)+q_0]$ та його збурення $\tau_{q_1} = \tau_0+r_0^{-1}q_1$ на інтервалі $(a, b) \subseteq \mathbb{R}$, ми використовуємо існування строго додатного розв'язку $u_0(\lambda_0, \cdot) > 0$ на (a, b) для $\tau_0 u_0 = \lambda_0 u_0$ для того, щоб одержати для τ_{q_1} нерівність у квадратичній формі, яка природно узагальнює добре відому нерівність Харді і зводиться до неї в окремому випадку $p_0 = r_0 = u_0(0, \cdot) = 1$, $q_0 = \lambda_0 = 0$, $a \in \mathbb{R}$, $b = \infty$.

Ключові слова: нерівність Харді, головні і неголовні розв'язки, теорія коливань, оператори Штурма–Ліувілля