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A new proof of Frank–Weissenborn inequality

Anatoly Gol'dberg

Department of Mathematics & Statistics, Bar-Ilan University
Ramat-Gan, 52900, Israel

E-mail:misgo@bezeqint.net

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A new proof of the Frank–Weissenborn inequality is given. This proof uses the theory of algebroid functions.

Let f be a transcendental meromorphic in $\mathbb{C}$ function and all the poles of f be simple. We use the standard notation of the value distribution theory [1]. We also denote by $Q(r, f)$ any quantity, satisfying $Q(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside some system of intervals that have a finite common length in the case of a function f of infinite order.

In [2] the following remarkable inequality was proved:

Lemma 1. *Let* $\epsilon > 0$. *Then*

$$N(r, f) \le (1 + \epsilon)N(r, 1/f'') + Q(r, f). \tag{1}$$

We give a new proof of the inequality (1). This proof uses elements of the theory of algebroid functions. We prove by the way that (1) holds with $\epsilon = 0$.

Denote

$$A_f(z) := \left(\frac{f'''}{f''}\right)^2 - \frac{3}{4}\frac{f^{(4)}}{f''}.$$

Let z_0 be a simple pole of f, i.e., $f(z) = c(z - z_0)^{-1} + h(z)$, where h is an analytic function at z_0. One can suppose, without loss of generality, that $c = 1$. We have

$$f^{(n)}(z) = \frac{(-1)^n n!}{(z - z_0)^{n+1}} + h^{(n)}(z), \; n = 1, 2, 3, \dots .$$

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Further

$$\frac{f'''(z)}{f''(z)} = \frac{-6(z-z_0)^{-4}+h'''}{2(z-z_0)^{-3}+h''} = -\frac{3}{z-z_0}(1+O((z-z_0)^3))$$
$$= -\frac{3}{z-z_0} + O((z-z_0)^2);$$
$$\frac{f^{(4)}(z)}{f''(z)} = \frac{24(z-z_0)^{-5}+h^{(4)}}{2(z-z_0)^{-3}+h''} = \frac{12}{(z-z_0)^2}(1+O((z-z_0)^3))$$
$$= \frac{12}{(z-z_0)^2} + O(z-z_0);$$
$$A_f(z) = O(z-z_0).$$

Hence $A_f(z_0) = 0$ and

$$n(r, 1/A_f) \geq n(r, f). \tag{2}$$

Further

$$\overline{n}(r, A_f) \leq n(r, 1/f''). \tag{3}$$

Now we will exploit the standart notions of the algebroid functions theory and some its basic results [3, Ch. 1, §7; Ch. 3, §7]; [4, 5].

Let us consider the algebroid function

$$B_f(z) := \sqrt{A_f(z)}.$$

Since all the poles of A_f are of the second order, then all the poles of $B_f(z)$ are of the first order.

Recall ([4, §1]) that $B_f(z)$ can be represented as

$$B_f(z) = (z-z_0)^{\tau/2} g((z-z_0)^{1/2})$$

in some heigborhood of its zero z_0, where $g(z)$ is holomorphic at $z=0$ and $\tau \in \mathbb{N}$ is the order of z_0.

Thus from (2) we have $n(r, 1/B_f) \geq n(r, f)$ and hence

$$N(r, 1/B_f) \geq N(r, f). \tag{4}$$

Inequality (3) implies $n(r, B_f) \leq n(r, 1/f'')$ and hence

$$N(r, B_f) \leq N(r, 1/f''). \tag{5}$$

By Logarithmic Derivative Lemma [5, 6] $m(r, B_f) = Q(r, f)$. By the First Main Theorem [3, 4]

$$T(r, B_f) = m(r, B_f) + N(r, B_f) = Q(r, f) + N(r, B_f)$$

$$T(r, B_f) \geq Q(r, f) + N(r, 1/B_f),$$

and thus

$$N(r, B_f) \geq N(r, 1/B_f) + Q(r, f). \qquad (6)$$

From (4)–(6) we obtain

$$N(r, f) \leq N(r, 1/f'') + Q(r, f),$$

i.e., (1) with $\epsilon = 0$.

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