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On isometric dilations of commutative systems of linear operators

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The isometric dilation of two parameter semigroup $T(n) = T_1^{n_1}T_2^{n_2}$, where $n = (n_1, n_2) \in \mathbb{Z}_+^2$, for a commutative system $\{T_1, T_2\}$ of linear bounded operators, one of which is a contraction, $\|T_1\| \le 1$, is constructed. The building of the dilation is based on characteristic qualities of the commutative isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_{s=1}^2$, which was given in the previous work by the author [8]. The isometric dilations $U(n)$ and $\overset{+}{U}(n)$ of the semigroups $T(n)$ and $T^*(n)$ are shown to be unitarily linked.

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The functional model of the contractive linear operator T is commonly considered as an analogue of the spectral decomposition for the nonunitary operator T, [4, 9]. The construction of the functional models is based on the study of the basic properties of the unitary dilation U of the operator T, [4].

In this work, the isometric dilation $U(n)$ for the two-parameter semigroup $T(n) = T_1^{n_1}T_2^{n_2}$ where $n = (n_1, n_2) \in \mathbb{Z}_+^2$ is constructed using the construction of the commutative isometric expansion $\left\{V_s, \overset{+}{V}_s\right\}_1^2$ for the commutative operator system $\{T_1, T_2\}$ such that $\|T_1\| \le 1$ (which was presented in the work [8]). The construction of the dilation $U(n)$ is based on consistency conditions for systems of equations that are corresponding to the expansions $\{V_1, V_2\}$. Similarly, the isometric dilation $\{V_1, V_2\}$, $n \in \mathbb{Z}_+^2$, is constructed using corresponding consistency conditions for equations that are corresponding to the expansions $\left\{\overset{+}{V}_1, \overset{+}{V}_2\right\}$. It turns

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out that the dilations $U(n)$ and $\overset{+}{U}(n)$ are acting in the separate Hilbert spaces $\mathcal{H}_{N,\Gamma}$ and $\mathcal{H}_{N^*,\Gamma^*}$, besides, the spaces $\mathcal{H}_{N,\Gamma}$ and $\mathcal{H}_{N^*,\Gamma^*}$ are intersecting and their intersection $\mathcal{H} = \mathcal{H}_{N,\Gamma} \cap \mathcal{H}_{N^*,\Gamma^*}$ has such property that $U^*(n_1,0)f = \overset{+}{U}(n_1,0)\, f$, where $f \in \mathcal{H}$ and $n_1 \in \mathbb{Z}_+$. Moreover, the restriction of the dilation $U(n_1,0)$ on $\mathcal{H}$ is a unitary operator such that $P_H U(n_1,0)|_H = T_1^{n_1}$, $n_1 \in \mathbb{Z}_+$.

I. Consider the commutative system of linear bounded operators $\{T_1, T_2\}$, $[T_1, T_2] = T_1T_2 - T_2T_1 = 0$, in the separable Hilbert space H. Hereinafter, we will suppose that one of the operators of the system $\{T_1, T_2\}$, e.g., T_1, is a contraction, $\|T_1\| \le 1$. Following [6, 8], define the commutative unitary expansion for the system $\{T_1, T_2\}$.

Definition 1. *Let the commutative system of linear bounded operators $\{T_1, T_2\}$ be given in Hilbert space H where T_1 is a contraction, $\|T_1\| \le 1$. The set of mappings*

$$\begin{aligned} V_1 &= \begin{bmatrix} T_1 & \Phi \\ \Psi & K \end{bmatrix}; \quad V_2 = \begin{bmatrix} T_2 & \Phi N \\ \Psi & K \end{bmatrix}: \quad H \oplus E \to H \oplus \tilde{E}; \\ \overset{+}{V}_1 &= \begin{bmatrix} T_1^* & \Psi^* \\ \Phi^* & K^* \end{bmatrix}; \quad \overset{+}{V}_2 = \begin{bmatrix} T_2^* & \Psi^*\tilde{N}^* \\ \Phi^* & K^* \end{bmatrix}: \quad H \oplus \tilde{E} \to H \oplus E, \end{aligned} \tag{1}$$

where E and $\tilde{E}$ are Hilbert spaces, is called the commutative unitary expansion of the commutative system of operators T_1, T_2 in H, $[T_1, T_2] = 0$, if there are such operators σ, τ, N, Γ and $\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{N}$, $\tilde{\Gamma}$ in the Hilbert spaces E and $\tilde{E}$, where σ, τ, $\tilde{\sigma}$, $\tilde{\tau}$ are selfadjoint, that the following relations are taking place:

$$\begin{aligned} &1)\quad \overset{+}{V}_1 V_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \qquad V_1 \overset{+}{V}_1 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}; \\ &2)\quad V_2^* \begin{bmatrix} I & 0 \\ 0 & \tilde{\sigma} \end{bmatrix} V_2 = \begin{bmatrix} I & 0 \\ 0 & \sigma \end{bmatrix}; \quad \overset{+}{V}_2^* \begin{bmatrix} I & 0 \\ 0 & \tau \end{bmatrix} \overset{+}{V}_2 = \begin{bmatrix} I & 0 \\ 0 & \tilde{\tau} \end{bmatrix}; \\ &3)\quad T_2\Phi - T_1\Phi N = \Phi\Gamma; \qquad \Psi T_2 - \tilde{N}\Psi T_1 = \tilde{\Gamma}\Psi; \\ &4)\quad \tilde{N}\Psi\Phi - \Psi\Phi N = K\Gamma - \tilde{\Gamma}K; \\ &5)\quad \tilde{N}K = KN. \end{aligned} \tag{2}$$

Consider the following class of commutative systems of linear operators $\{T_1, T_2\}$.

Definition 2. *The commutative system of operators T_1, T_2 is attributed to the class $C(T_1)$ and is called the contracting T_1 operator system if:*

$$\begin{aligned}
&1)\ T_1 \text{ is a contraction, } \|T_1\| \le 1;\\
&2)\ E = \overline{\tilde{D}_1 H} \supseteq \overline{\tilde{D}_2 H}; \quad \tilde{E} = \overline{D_1 H} \supseteq \overline{D_2 H};\\
&3)\ \dim \overline{T_2 \tilde{D}_1 H} = \dim E; \quad \dim \overline{D_1 T_2 H} = \dim \tilde{E};\\
&4)\ \text{operators } \ D_1|_{\tilde{E}}, \quad \tilde{D}_1 T_2^*\Big|_{\overline{T_2 \tilde{D}_1 H}}, \quad \tilde{D}_1\Big|_E, \quad T_2^* D_1|_{\overline{D_1 T_2 H}}\\
&\text{are boundedly invertible, where } D_s = T_s^* T_s - I,\\
&\tilde{D}_s = T_s T_s^* - I,\ s = 1, 2.
\end{aligned} \tag{3}$$

It is easy to see that if $\{T_1, T_2\} \in C(T_1)$ then unitary expansion (1) always exists, [6, 8]. Indeed, let

$$\Psi = \sqrt{\tilde{\sigma}_1} = \sqrt{-D_1}; \quad \Phi = \tilde{D}_1 T_2^* \sqrt{\sigma_1^{-1}}; \quad K = \sqrt{\tilde{\sigma}_1} T_1^* T_2^* \sqrt{\sigma_1^{-1}};$$

$$N = -\sqrt{\sigma_1^{-1}} T_2 \tilde{D}_2 T_1^* \sqrt{\sigma_1^{-1}}; \quad \tilde{N} = -\sqrt{\tilde{\sigma}_1^{-1}} T_1^* \tilde{D}_2 T_1^{*-1} \sqrt{\tilde{\sigma}_1^{-1}};$$

$$\Gamma = \sqrt{\sigma_1^{-1}} T_2 \left(\tilde{D}_2 - \tilde{D}_1\right) \sqrt{\sigma_1^{-1}}; \quad \tilde{\Gamma} = \sqrt{\tilde{\sigma}_1^{-1}} T_2^{*-1} (D_2 - D_1) \sqrt{\tilde{\sigma}_1^{-1}};$$

$$\sigma = -\sqrt{\sigma_1^{-1}} T_1 \tilde{D}_2 T_1^* \sqrt{\sigma_1^{-1}}; \quad \tilde{\sigma} = -\sqrt{\tilde{\sigma}_1^{-1}} D_2 \sqrt{\tilde{\sigma}_1^{-1}} :$$

$$\tau = -\sqrt{\sigma_1^{-1}} T_2 \tilde{D}_2 T_2^* \sqrt{\sigma_1^{-1}}; \quad \tilde{\tau} = -\sqrt{\tilde{\sigma}_1^{-1}} T_2^{*-1} T_1^* D_2 T_1 T_2^{-1} \sqrt{\tilde{\sigma}_1^{-1}},$$

taking into account (3).

Then it is easy to see that relations 1)–5) (2) are true [8].

II. Following the work [8], define the vector-functions of discrete argument $h_n \in H$, $u_n \in E$, $v_n \in \tilde{E}$ at the points of integer-valued grid $n = (n_1; n_2) \in \mathbb{Z}_+^2$ ($n_k \ge 0$; $k = 1, 2$; $n_k \in \mathbb{Z}$). Consider [8] the system of equations

$$\left\{ \begin{array}{ll} \partial_1 h_n = T_1 h_n + \Phi u_n; & h_{(0,0)} = h_0;\\ \partial_2 h_n = T_2 h_n + \Phi N u_n; & n \in \mathbb{Z}_+^2;\\ v_n = \Psi h_n + K u_n; & \end{array} \right. \quad V_s \begin{bmatrix} h_n \\ u_n \end{bmatrix} = \begin{bmatrix} \partial_s h_n \\ v_n \end{bmatrix}, \ s = 1, 2, \tag{4}$$

where $\partial_1 h_n = h_{(n_1+1;n_2)}$, $\partial_2 h_n = h_{(n_1;n_2+1)}$ are the corresponding shifts by different variables. The next theorems are dedicated to the study of consistency conditions for the discrete system of equations (4).

Theorem 1. *The system (4) is consistent only if the vector-function u_n is a solution of the equation*

$$\{N\partial_1 - \partial_2 + \Gamma\} u_n = 0. \tag{5}$$

The proof of the theorem follows from the equality of the mixed shifts $\partial_1\partial_2 h_n = \partial_2\partial_1 h_n$ taking into account condition 3) (2), [8].

Theorem 2. *Suppose that u_n is a solution of equation (5) and the vector-functions h_n and v_n are given by relations (4). Then v_n satisfies the following equation*

$$\left\{\tilde{N}\partial_1 - \partial_2 + \tilde{\Gamma}\right\} v_n = 0. \tag{6}$$

The proof of the Theorem 2 is given in [8].

The following conservation laws

$$\begin{array}{ll} 1) & \begin{array}{l} \|\partial_1 h_n\|^2 + \|v_n\|^2 = \|h_n\|^2 + \|u_n\|^2\,; \\ \|\partial_2 h_n\|^2 + \langle\tilde{\sigma} v_n, v_n\rangle = \|h_n\|^2 + \langle\sigma u_n, u_n\rangle\,; \end{array} \\ 2) & \langle(\tilde{\sigma}_1 - \tilde{\sigma}_2)\, v_n, v_n\rangle + \langle\tilde{\sigma}_2\partial_1 v_n, \partial_1 v_n\rangle - \langle\tilde{\sigma}_1\partial_2 v_n, \partial_2 v_n\rangle \\ & = \langle(\sigma_1 - \sigma_2)\, u_n, u_n\rangle + \langle\sigma_2\partial_1 u_n, \partial_1 u_n\rangle - \langle\sigma_1\partial_2 u_n, \partial_2 u_n\rangle \end{array} \tag{7}$$

are true for the discrete system of equations (4). Obviously, the relations 1) (7) are a simple corollary of 1), 2) (2), while the equality 2) (7) follows from the coincidence of the norms $\|\partial_1\partial_2 h_n\|^2 = \|\partial_2\partial_1 h_n\|^2$ and plays an important role hereinafter.

Similarly to (4), consider (see [8]) the vector-functions $\tilde{h}_n \in H$, $\tilde{u}_n \in E$, $\tilde{v}_n \in \tilde{E}$ at the integer-valued grid points $n = (n_1; n_2) \in \mathbb{Z}_-^2$ ($n_k < 0$; $k = 1,\ 2$; $n_k \in \mathbb{Z}$). Define the two-variable dual type of system of equations (4)

$$\left\{\begin{array}{ll} \tilde{\partial}_1\tilde{h}_n = T_1^*\tilde{h}_n + \Psi^*\tilde{v}_n; & \tilde{h}_{(-1;-1)} = \tilde{h}_{-1}; \\ \tilde{\partial}_2\tilde{h}_n = T_2^*\tilde{h}_n + \Psi^*\tilde{N}^*\tilde{v}_n; & n \in \mathbb{Z}_-^2; \\ \tilde{u}_n = \Phi^*\tilde{h}_n + K^*\tilde{v}_n; & \end{array}\right. \quad \overset{+}{V}_s \begin{bmatrix} \tilde{h}_n \\ \tilde{v}_n \end{bmatrix} = \begin{bmatrix} \tilde{\partial}_s\tilde{h}_n \\ \tilde{u}_n \end{bmatrix},\ s = 1, 2, \tag{8}$$

where $\tilde{\partial}_1\tilde{h}_n = \tilde{h}_{(n_1-1;n_2)}$, $\tilde{\partial}_2\tilde{h}_n = \tilde{h}_{(n_1;n_2-1)}$ are shifts by different variables formally adjoint to ∂_1 and ∂_2, so that $\tilde{\partial}_s = \partial_s^*$, $s = 1, 2$, in the metric of the space l^2. Statements similar to the Theorems 1 and 2 are true for the system (8).

Theorem 3. *Consistency of the system of equations (8) takes place only if $\tilde{v}_n$ is the solution of the equation*

$$\left\{\tilde{N}^*\tilde{\partial}_1 - \tilde{\partial}_2 + \tilde{\Gamma}^*\right\} \tilde{v}_n = 0. \tag{9}$$

Theorem 4. *Vector-function $\tilde{u}_n$ (8) satisfies the following equation*

$$\{N^*\partial_1 - \partial_2 + \Gamma^*\}\, \tilde{u}_n = 0 \tag{10}$$

under the conditions that $\tilde{v}_n$ is the solution of (9) and $\tilde{h}_n$ are given by relations (8).

Similarly to (7), the following conservation laws

$$\begin{array}{ll} 1) & \left\|\tilde{\partial}_1\tilde{h}_n\right\|^2 + \|\tilde{u}_n\|^2 = \left\|\tilde{h}_n\right\|^2 + \|\tilde{v}_n\|^2\,; \\ & \left\|\tilde{\partial}_2\tilde{h}_n\right\|^2 + \langle\tau\tilde{u}_n,\tilde{u}_n\rangle = \left\|\tilde{h}_n\right\|^2 + \langle\tilde{\tau}\tilde{v}_n,\tilde{v}_n\rangle\,; \\ 2) & \langle(\tau_1-\tau_2)\,\tilde{u}_n,\tilde{u}_n\rangle + \left\langle\tau_2\tilde{\partial}_1\tilde{u}_n,\tilde{\partial}_1\tilde{u}_n\right\rangle - \left\langle\tau_1\tilde{\partial}_2\tilde{u}_n,\tilde{\partial}_2\tilde{u}_n\right\rangle \\ & = \langle(\tilde{\tau}_1-\tilde{\tau}_2)\,\tilde{v}_n,\tilde{v}_n\rangle + \left\langle\tilde{\tau}_2\tilde{\partial}_1\tilde{v}_n,\tilde{\partial}_1\tilde{v}_n\right\rangle - \left\langle\tilde{\tau}_1\tilde{\partial}_2\tilde{v}_n,\tilde{\partial}_2\tilde{v}_n\right\rangle \end{array} \tag{11}$$

are true for the dual system (8) in view of 2) (2).

III. Turn to the construction of the dilation for the operator systems $\{T_1,T_2\}$ of the class $C(T_1)$ (3). First of all, construct the unitary dilation [4, 6, 9] for the contraction T_1. As usually [6, 8], we will denote by $l^2_M(G)$ the Hilbert space of G-valued functions $u_k \in G$, where $k \in M$ pnd $M \subseteq \mathbb{Z}$ are such that $\sum\limits_{k\in M}\|u_k\|^2 < \infty$. Let $\mathcal{H}$ be the Hilbert space of the following type

$$\mathcal{H} = D_- \oplus H \oplus D_+, \tag{12}$$

where $D_- = l^2_{\mathbb{Z}_-}(E)$ and $D_+ = l^2_{\mathbb{Z}_+}(\tilde{E})$. Specify the dilation U on the vector-functions $f = (u_k, h, v_k)$ from $\mathcal{H}$ (12) in the following way:

$$Uf = \left(P_{D_-}u_{k-1}, \tilde{h}, \tilde{v}_k\right), \tag{13}$$

where $\tilde{h} = T_1h + \Phi u_{-1}$, $\tilde{v}_0 = \Psi h + K u_{-1}$, $\tilde{v}_k = v_{k-1}$ $(k = 1, 2\ \ldots,)$ and P_{D_-} is the operator of contraction on D_-. The unitary property of U (13) in $\mathcal{H}$ follows from 1) (2). Take advantage now of equations (5) and (6) as a way to continue the incoming D_- and outgoing D_+ subspaces

$$D_- = l^2_{\mathbb{Z}_-}(E); \quad D_+ = l^2_{\mathbb{Z}_+}(\tilde{E}) \tag{14}$$

by the second variable "n_2". At first, continue functions $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$ from the semiaxis $\mathbb{Z}_-$ into the domain

$$\tilde{\mathbb{Z}}^2_- = \mathbb{Z}_- \times (\mathbb{Z}_- \cup \{0\}) = \left\{n = (n_1; n_2) \in \mathbb{Z}^2 : n_1 < 0; n_2 \le 0\right\}, \tag{15}$$

using the following Cauchy problem

$$\begin{cases} \tilde{\partial}_2 u_n = \left(N\tilde{\partial}_1 + \Gamma\right)u_n; & n = (n_1, n_2) \in \tilde{\mathbb{Z}}^2_-; \\ u_n|_{n_2=0} = u_{n_1} \in l^2_{\mathbb{Z}_-}(E). \end{cases} \tag{16}$$

As a result, we obtain the Hilbert space $D_-(N,\Gamma)$ which is formed by u_n, the solutions of (16), at the same time the norm in $D_-(N,\Gamma)$ is induced by the norm of initial data $\|u_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$.

N o t e 1. Note that the formal continuation of the function $u_{n_1} \in l^2_{\mathbb{Z}_-}(E)$ from the semiaxis $\mathbb{Z}_-$ using the Cauchy problem (16) has wider domain of existence then $\tilde{\mathbb{Z}}^2_-$ (15). Really, if we continue u_{n_1} with nulls on $\mathbb{Z}_+$ then using recurrent relation, we obtain u_n that is given in the cone $\mathcal{K}_-$:

$$\mathcal{K}_- = \left\{n = (n_1, n_2) \in \mathbb{Z}^2 : n_2 \le 0; n_1 + n_2 < 0\right\}. \tag{17}$$

Similarly, continue functions $v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E})$ from the semiaxis $\mathbb{Z}_+$ into the domain $\mathbb{Z}^2_+ = \mathbb{Z}_+ \times \mathbb{Z}_+$ using the Cauchy problem

$$\left\{\begin{array}{l} \tilde{\partial}_2 v_n = \left(\tilde{N}\tilde{\partial}_1 + \tilde{\Gamma}\right) v_n; \qquad n = (n_1, n_2) \in \mathbb{Z}^2_+; \\ v_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(E). \end{array}\right. \tag{18}$$

Thus, we obtain Hilbert space $D_+(\tilde{N},\tilde{\Gamma})$ that is made of solutions v_n (18), besides $\|v_n\| = \|v_{n_1}\|_{l^2_{\mathbb{Z}_+}(\tilde{E})}$. Unlike the evident recurrent scheme (16) of the layer-to-layer calculation of $n_2 \to n_2 - 1$ for u_n, in this case, while constructing v_n in $\mathbb{Z}^2_+$, we are dealing with the implicit linear system of equations for layer-to-layer calculation of $n_2 \to n_2 + 1$ for the function v_n. Therefore it is necessary to study solvability and uniqueness of Cauchy problem (18). First, study reversibility of linear pencils of operators $Nz + \Gamma$ and $\tilde{N}z + \tilde{\Gamma}$.

Lemma 1. *Suppose the commutative unitary expansion V_s, $\overset{+}{V}_s$ (1) is such that*

$$\operatorname{Ker}\Phi = \operatorname{Ker}\Psi^* = \{0\} \tag{19}.$$

Then $\operatorname{Ker}N \cap \operatorname{Ker}\Gamma = \{0\}$ *given* $\operatorname{Ker}K^* = \{0\}$*, and respectively* $\operatorname{Ker}\tilde{N}^* \cap \operatorname{Ker}\tilde{\Gamma}^* = 0$ *given* $\operatorname{Ker}K = \{0\}$.

P r o o f. Let $G = \operatorname{Ker}N \cap \operatorname{Ker}\Gamma$ then it follows from the equality $T_2\Phi = T_1\Phi N + \Phi\Gamma$ that the subspace $L = \operatorname{span}\left\{T_1^k \Phi g : g \in G; k \in \mathbb{Z}_+\right\}$ from H has properties $T_1 L \subset L$, $T_2 L = 0$. It follows from the equality $T_2^* T_2 + \Psi^*\tilde{\sigma}\Psi = I$ that $h = \Psi^*\tilde{\sigma}\Psi h$ takes place for all $h \in L$, therefore $\Phi g = \Psi^*\tilde{g} = \Psi^*\tilde{\sigma}\Psi\Phi g$ and so $\tilde{g} = \tilde{\sigma}\Psi\Phi g$, in view of $\operatorname{Ker}\Psi^* = 0$ (19). Since $T_2^*\Phi N + \Psi^*\tilde{\sigma}K = 0$ then $K^*\tilde{g} = K^*\tilde{\sigma}\Psi\Phi g = -N^*\Phi^* T_2\Phi g = 0$; then $\tilde{g} = 0$ because of $\operatorname{Ker}K^* = 0$. So $\Phi g = \Psi^*\tilde{g} = 0$ and thus $g = 0$ in view of $\operatorname{Ker}\Phi = 0$ (19). Similarly, one proves the second statement of the lemma. ∎

N o t e 2. Note that if the suppositions of the Lemma 1 are true and the spaces E and $\tilde{E}$ are finite dimensional, then the linear pencils $Nz+\Gamma$ and $\tilde{N}^*z+\tilde{\Gamma}^*$

are reversible operators for all $z \in \mathbb{C}$, except for the finite number of points that are zeroes of polynomials $\det(Nz+\Gamma)=0$ and $\det\left(\tilde{N}^*z+\tilde{\Gamma}^*\right)=0$ respectively. Since reversibility of $\tilde{N}z+\tilde{\Gamma}$ and of the adjoint to it operator $\tilde{N}^*\bar{z}+\tilde{\Gamma}^*$ are equivalent in the finite dimensional space $\tilde{E}$, then reversibility of $\tilde{N}z+\tilde{\Gamma}$ follows from the Lemma 2 when $\dim\tilde{E}<\infty$.

Turn to the solvability of Cauchy problem (18).

Statement 1. *Let $\dim\tilde{E}<\infty$ and the assumptions of the Lemma 1 be true, then the solution v_n of Cauchy problem (18) exists and is unique in the domain $\mathbb{Z}_+^2$ for all initial data v_{n_1} from $l^2_{\mathbb{Z}_+}(\tilde{E})$.*

P r o o f. First, consider the case of the finite initial data v_{n_1}, i.e., let $v_{n_1}=0$ when $n_1>n$, where $n\in\mathbb{Z}_+$. Show that the vector-function $v\,(n_1,1)$ which is a solution of problem (18) that also turns to zero when $n_1>n$, is uniquely defined by initial data v_{n_1}. It is necessary to prove that the homogeneous linear system of equations generated by (18) has only trivial solution. It follows from (18) when $v_{n_1}=0$, that function $v\,(n_1,1)$ satisfies the system of equations

$$\begin{cases} \tilde{\Gamma}v(0,1)=0; \\ \tilde{N}v(0,1)+\tilde{\Gamma}v(1,1)=0; \\ \dots \\ \tilde{N}v(n-1,1)+\tilde{\Gamma}v(n,1)=0; \\ \tilde{N}v(n,1)=0. \end{cases} \tag{20}$$

Multiply the second equality in (20) by z, the third one — by z^2, and so on, finally, the last one — by z^{n+1} $(z\in\mathbb{C})$; then after summation we obtain that

$$(\tilde{N}z+\tilde{\Gamma})\,\{v(0,1)+zv(1,1)+\cdots+z^nv(n,1)\}=0.$$

It follows from the Note 2, in view of reversibility of $\tilde{N}z+\tilde{\Gamma}$, that

$$\sum_{k=0}^{n} z^k v(k,1)=0$$

for all $z\in\mathbb{C}$ except for a finite number of points. Therefore $v(k,1)=0$ for all k, $0\le k\le n$. Thus, the first layer $v(k,1)$ is defined from equations (18) by the initial data v_k, $0\le k\le n$ unambiguously. Realizing in that way layer-to-layer reconstruction of $v(k,p+1)$ by $v(k,p)$, we will obtain the unique solution of the Cauchy problem (18) in the domain $\mathbb{Z}_+^2$. The general case follows from the considered case of the finite initial data as a result of natural approximation. ∎

N o t e 3. It is not difficult to establish (similarly to the Note 1) that the solution of Cauchy problem (18) exists in the conic domain $\mathcal{K}_+$:

$$\mathcal{K}_+ = \left\{ n = (n_1, n_2) \in \mathbb{Z}^2 : n_2 \geq 0; n_1 + n_2 \geq 0 \right\}. \tag{21}$$

Consider now the operator-function of discrete argument

$$\tilde{\sigma}_\Delta = \begin{cases} I: & \Delta = (1;0); \\ \tilde{\sigma}; & \Delta = (0;1). \end{cases} \tag{22}$$

Let L_0^n be the nonincreasing broken line that connects points $O = (0,0)$ and $n = (n_1, n_2) \in \mathbb{Z}_+^2$ and linear segments of which are parallel to the axes OX ($n_2 = 0$) and OY ($n_1 = 0$). Denote by $\{P_k\}_0^N$ all integer-valued points from $\mathbb{Z}_+^2$, $P_k \in \mathbb{Z}_+^2$ ($N = n_1 + n_2$) that lay on L_0^n, beginning with $(0,0)$ and finishing with the point (n_1, n_2), that are numbered in nondescending order (of one of the coordinates of P_k). Assuming that $P_{-1} = (-1, 0)$, establish the quadratic form

$$\langle \tilde{\sigma} v_k \rangle^2_{L_0^n} = \sum_{k=0}^{N} \left\langle \tilde{\sigma}_{P_k - P_{k-1}} v_{P_k}, v_{P_k} \right\rangle, \tag{23}$$

on the vector-functions $v_k \in D_+(\tilde{N}, \tilde{\Gamma})$.

Similarly, consider the nondecreasing broken line L_m^{-1} in $\tilde{\mathbb{Z}}_-^2$ (15) that connects points $m = (m_1, m_2) \in \tilde{\mathbb{Z}}_-^2$ and $(-1, 0)$, the straight segments of which are parallel to OX and OY. Let $\{Q_s\}_M^{-1}$ ($M = m_1 + m_2$) be all integer-valued points on L_m^{-1}, beginning with $m = (m_1, m_2)$ and finishing with $(-1, 0)$, that are numbered in nondescending order (of one of the coordinates of Q_s). Define the metric in $D_-(N, \Gamma)$,

$$\langle \sigma u_k \rangle^2_{L_m^{-1}} = \sum_{s=M}^{-1} \left\langle \sigma_{Q_s - Q_{s-1}} u_{Q_s}, u_{Q_s} \right\rangle, \tag{24}$$

besides $Q_M - Q_{M-1} = (1, 0)$, and the operator-function σ_Δ is defined similarly to $\tilde{\sigma}_\Delta$ (22). Denote by $\tilde{L}_{-n}^{-1}$ the broken line in $\tilde{\mathbb{Z}}_-^2$ that is obtained from the curve L_0^n in $\mathbb{Z}_+^2$ ($n \in \mathbb{Z}_+^2$) using the shift by "n":

$$\tilde{L}_{-n}^{-1} = \left\{ Q_s = (l_1, l_2) \in \tilde{\mathbb{Z}}_-^2 : (l_1 + n_1 + 1, l_2 + n_2) = P_k \in L_0^n \right\}. \tag{25}$$

IV. Having now the Hilbert space $D_-(N, \Gamma)$, that is formed by the solutions of Cauchy problem (16), and space $D_+(\tilde{N}, \tilde{\Gamma})$, that is formed by the solutions of (18) respectively, we can define Hilbert space

$$\mathcal{H}_{N,\Gamma} = D_-(N, \Gamma) \oplus H \oplus D_+(\tilde{N}, \tilde{\Gamma}), \tag{26}$$

the norm in which is defined by the norm of the initial space $\mathcal{H} = D_- \oplus H \oplus D_+$ (12). Denote by $\hat{\mathbb{Z}}_+^2$ the subset in $\mathbb{Z}_+^2$,

$$\hat{\mathbb{Z}}_+^2 = \mathbb{Z}_+^2 \backslash (\{0\} \times \mathbb{N}) = \{(0,0)\} \cup (\mathbb{N} \times \mathbb{Z}_+), \tag{27}$$

that obviously is an addition semigroup.

For every $n \in \hat{\mathbb{Z}}_+^2$ (27), define an operator-function $U(n)$ that acts on the vectors $f = (u_k, h, v_k) \in \mathcal{H}_{n,\Gamma}$ (26)in the following way:

$$U(n)f = f(n) = (u_k(n), h(n), v_k(n)), \tag{28}$$

where $u_k(n) = P_{D_-(N,\Gamma)} u_{k-n}$ ($P_{D_-(N,\Gamma)}$ is an orthoprojector that corresponds with the restriction on $D_-(N,\Gamma)$); $h(n) = y_0$, besides $y_k \in H$ ($k \in \mathbb{Z}_+^2$) is a solution of the Cauchy problem

$$\begin{cases} \tilde{\partial}_1 y_k = T_1 y_k + \Phi u_{\tilde{k}}; \\ \tilde{\partial}_2 y_k = T_2 y_k + \Phi N u_{\tilde{k}}; \\ y_n = h; \quad k = (k_1, k_2) \in \mathbb{Z}_+^2 \quad 0 \leq k_1 \leq n_1 - 1, \quad 0 \leq k_2 \leq n_2; \end{cases} \tag{29}$$

at the same time $\tilde{k} = k - n$, when $0 \leq k_1 \leq n_1 - 1$, $0 \leq k_2 \leq n_2$, and finally

$$v_k(n) = \hat{v}_k + v_{k-n} \tag{30}$$

and $\hat{v}_k = K u_{\tilde{k}} + \Psi y_k$, where y_k is a solution of the Cauchy problem (29).

The vector-function $u_{\tilde{k}}$, that is obtained as a result of the shift by "n", automatically satisfies the consistency equation (5), since, according to the construction, u_k is a solution of the Cauchy problem (16). And it follows from the equation (6) that $v_k(n)$ (30) continues uniquely into the whole domain $\mathbb{Z}_+^2$ as a solution of the equation (18), that is always possible in the context of the suppositions of the Statement 1.

The following facts justify that $U(n)$ (28) is defined if $n \in \hat{\mathbb{Z}}_+^2$ (27): first, $\{T_1, T_2\} \in C(T_1)$ (3); second, the choice of the metric (23), and third, the construction of the space $D_+(\tilde{N}, \tilde{\Gamma})$ that is generated by the Cauchy problem (18) with the initial data from the semiaxis $\mathbb{Z}_+$.

Thus, the operator-function $U(n)$ (28) maps the space $\mathcal{H}_{N,\Gamma}$ (26) into itself for all $n \in \hat{\mathbb{Z}}_+^2$ (27).

Theorem 5. *Suppose* $\dim \tilde{E} < \infty$ *and the suppositions of Lemma 1 are taking place, then the following conservation law is true for the vector-function* $f(n) = U(n)f$ *(28):*

$$\|h(n)\|^2 + \langle \tilde{\sigma} v_k(n) \rangle^2_{L_0^{\hat{n}}} = \|h\|^2 + \langle \sigma u_k \rangle^2_{\tilde{L}^{-1}_{-n}} \tag{31}$$

for all $n \in \hat{\mathbb{Z}}_+^2$ *(27) and for all nondecreasing broken lines* $\hat{L}_0^{\hat{n}}$ *that connect points* $O = (0,0)$ *and* $\hat{n} = (n_1 - 1, n_2) \in \mathbb{Z}_+^2$*, where* $\tilde{L}_{-\hat{n}}^{-1}$ *is a broken line that is obtained from* L_0^n *by the shift (25) by "n", at the same time the corresponding σ-forms in (31) have the appearance of (23) and (24). The operator-function* $U(n)$ *(28) is a semigroup,* $U(n) \cdot U(m) = U(n+m)$*, for all* n*,* $m \in \hat{\mathbb{Z}}_+^2$ *(27).*

P r o o f. The equality (31) easily follows from the isometric correspondence of the operators V_1, V_2 (1) in accordance with 1) and 2) (2). The fact that the operator-function $U(n)$ (28) is a semigroup when $n \in \hat{\mathbb{Z}}_+^2$ (27) follows from the elementary calculations taking into account the continuation of the function $v_k(n)$ (30) into the domain $\mathbb{Z}_+^2$ by the equation (18). ■

It follows from (31) that it is natural to define in the space $\mathcal{H}_{N,\Gamma}$ (26) the indefinite, generally speaking, metric

$$\langle f \rangle_\sigma^2 = \langle \sigma u_k \rangle_{L_{-\infty}^{-1}}^2 + \|h\|^2 + \langle \tilde{\sigma} v_k \rangle_{L_0^\infty}^2 , \tag{32}$$

where L_0^∞ and $L_{-\infty}^{-1}$ are nondecreasing broken lines in $\mathbb{Z}_+^2$ and in $\hat{\mathbb{Z}}_-^2$ (15) connecting point $O = (0,0)$ with $\infty = (\infty, \infty)$ and point $-\infty = (-\infty, -\infty)$ with $(-1; 0)$ respectively, straight segments of these broken lines are parallel to the axes OX and OY.

Consider the subspace $\mathcal{K}$ from $\mathbb{Z}_+^2$ that contains $O = (0,0)$ and is an addition semigroup. $T(n)$ denotes the semigroup of linear operators over $\mathcal{K}$,

$$T(n) = T_1^{n_1} T_2^{n_2}, \quad n = (n_1, n_2) \in \mathcal{K}, \tag{33}$$

assuming that the commutative system of linear operators $\{T_1, T_2\}$ belongs to the class $C\,(T_1)$ (3).

Definition 3. [4] *Semigroup of operators* $U(n)$*;* $U(n)U(m) = U(n+m)$*;* $\forall n$*,* $m \in \mathcal{K}$*, that is given in the Hilbert space* $\mathcal{H}$ *such that*

$$\mathcal{H} \supseteq H; \quad P_H U(n)|_H = T(n), \quad n \in \mathcal{K}, \tag{34}$$

where P_H *is an orthoprojector on* H*, is called the dilation of a discrete operator semigroup* $T(n)$ *(33) that is acting in the Hilbert space* H*. If for every* $n \in \mathcal{K}$ *the operator-function* $U(n)$ *is an isometric or unitary operator in* $\mathcal{H}$ *then* $U(n)$ *is called isometric or unitary dilation* $T(n)$*.*

Consider the family of one-parameter semigroup $G_+(p)$ in $\mathbb{Z}_+^2$,

$$G_+(p) = \left\{ np : p \in \hat{\mathbb{Z}}_+^2, n \in \mathbb{Z}_+ \right\}, \tag{35}$$

besides the point $p = (p_1, p_2) \in \hat{\mathbb{Z}}^2_+$ is such that numbers p_1 and p_2 are coprime. In particular, if $p_1 = (1, 0)$ then it is obvious that $G_+(p) = \mathbb{Z}_+$. Narrow now the semigroup $T(n)$ (33) on $G_+(p)$ (35), i.e., for the given $p = (p_1, p_2) \in \hat{\mathbb{Z}}^2_+$ consider the one-parameter semigroup $T_n(p) = (T_1^{p_1} T_2^{p_2})^n$ from $n \in \mathbb{Z}_+$, which looks like $T_n(p_1) = T_1^n$ when $p = p_1 = (1, 0)$. Choose now fixed broken line L_0^p with linear segments that are parallel to the axes OX and OY, which connects points O and $p \in \hat{\mathbb{Z}}^2_+$; and then make its group shift in $\mathbb{Z}^2_+$,

$$L_0^\infty(p) = \{n + kp : n \in L_0^p, k \in \mathbb{Z}_+\} \tag{36}$$

and similarly shift L_0^p in $\tilde{\mathbb{Z}}^2_-$,

$$L_{-\infty}^{-1}(p) = \{n + k(p_1 + 1, p_2) : n \in L_0^\infty, k \in \mathbb{Z}_-\}. \tag{37}$$

In accordance with (32), specify the quadratic form in $\mathcal{H}_{N,\Gamma}$ (26) that is associated with the semigroup $G_+(p)$ (35),

$$\langle f\rangle^2_{\sigma,p} = \langle \sigma u_k\rangle^2_{L^{-1}_{-\infty}(p)} + \|h\|^2 + \langle \tilde{\sigma} v_k\rangle^2_{L_0^\infty(p)}. \tag{38}$$

The next statement follows from the Theorem 5.

Theorem 6. *Suppose $\{T_1, T_2\} \in C(T_1)$ (3), $\dim \tilde{E} < \infty$ and the suppositions of the Lemma 1 are true, then for every $p \in \hat{\mathbb{Z}}^2_+$ (27) the operator semigroup $T_n(p) = T(np)$ that is narrowed on $G_+(p)$ (35) has the isometric (in metric$\langle f\rangle^2_{\sigma,p}$ (38)) dilation $U_n(p) = U(np)$ (28) that acts in the Hilbert space $\mathcal{H}_{N,\Gamma}$ (26).*

N o t e 4. Using the semigroup property of dilation $U_n(p)$ (28) by parameter $n \in \mathbb{Z}_+$ and isometric property of $U_n(p)$ in metric (38), we obtain that

$$\left\langle U_n(p)h, U_m(p)h'\right\rangle_{\sigma,p} = \left\langle T_{n-m}(p)h, h'\right\rangle, \tag{39}$$

when $n \geq m$ ($n,m \in \mathbb{Z}_+$) and for all h, $h' \in H$. Thus, the subspace

$$\operatorname{span}\left\{U_n(p)H : n \in \mathbb{Z}_+, p \in \hat{\mathbb{Z}}^2_+\right\}$$

in $\mathcal{H}_{N,\Gamma}$ (26) is defined by the initial commutative operator system $\{T_1, T_2\} \in C(T_1)$ (3).

V. Similarly to the stated in the Paragraph III method of continuation of subspaces D_+ and D_- (14) from the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ by the second variable "n_2", consider the dual situation corresponding to equations (9) and (10). Denote by $D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$ the Hilbert space generated by solutions $\tilde{v}_n$ of Cauchy problem

$$\begin{cases} \partial_2 \tilde{v}_n = \left(\tilde{N}^* \partial_1 + \tilde{\Gamma}^*\right) \tilde{v}_n; \quad n = (n_1, n_2) \in \mathbb{Z}^2_+; \\ \tilde{v}_n|_{n_2=0} = v_{n_1} \in l^2_{\mathbb{Z}_+}(\tilde{E}). \end{cases} \tag{40}$$

the norm in which is induced by the norm of the initial data

$$\left\{\begin{array}{l} \partial_2\tilde{v}_n = \left(\tilde{N}^*\partial_1 + \tilde{\Gamma}^*\right)\tilde{v}_n; \quad n=(n_1,n_2)\in\mathbb{Z}^2_+; \\ \tilde{v}_n|_{n_2=0} = v_{n_1}\in l^2_{\mathbb{Z}_+}(\tilde{E}). \end{array}\right. \tag{40}$$

Continuing the function $v_{n_1}\in l^2_{\mathbb{Z}_+}(\tilde{E})$ by null on the left semiaxis, as in the case of (18), it is easy to establish that the solution of the Cauchy problem (40) exists in the cone $\mathcal{K}_+$ (21).

Continue now every function $u_{n_1}\in l^2_{\mathbb{Z}_-}(E)$ into the domain $\tilde{\mathbb{Z}}^2_-$ (15) using the Cauchy problem

$$\left\{\begin{array}{l} \partial_2\tilde{u}_n = \left(N^*\partial_1 + \Gamma^*\right)\tilde{u}_n; \quad n=(n_1,n_2)\in\tilde{\mathbb{Z}}^2_-; \\ \tilde{u}_n|_{n_2=0} = u_{n_1}\in l^2_{\mathbb{Z}_-}(E). \end{array}\right. \tag{41}$$

As a result, we obtain the Hilbert space $D_-(N^*,\Gamma^*)$ generated by $\tilde{u}_n$, solutions of (41), besides $\|\tilde{u}_n\| = \|u_{n_1}\|_{l^2_{\mathbb{Z}_-}(E)}$. Constructing the solutions $\tilde{u}_n$ of the Cauchy problem (41), we have the implicit scheme of layer-to-layer calculation of $n_2\to n_2-1$ solutions $\tilde{u}_n$. Using now the Lemma 1 and Note 2, we can formulate an analogue of the Statement 1.

Statement 2. *Let* $\dim E<\infty$ *and the suppositions of Lemma 1 be true, then the solution* $\tilde{u}_n$ *of the Cauchy problem (41) exists and is unique in the domain* $\tilde{\mathbb{Z}}^2_-$ *(15) for all initial data* $u_{n_1}\in l^2_{\mathbb{Z}_-}(E)$.

Note that, as in the case of the problem (40), solutions of the Cauchy problem (41) have wider domain of existence and uniqueness, namely, $\mathcal{K}_-$ (17).

N o t e 5. The sufficient condition for the simultaneous existence of solutions of Cauchy problems (18) and (41), in view of the reversibility of operators K and K^*, according to the Lemma 1, is following: all the requirements of the Lemma 1 are met and $\dim E=\dim\tilde{E}<\infty$.

Hence we come to the Hilbert space

$$\mathcal{H}_{N^*,\Gamma^*} = D_-(N^*,\Gamma^*)\oplus H\oplus D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right), \tag{42}$$

the metric in which is induced by the norm of the initial space $\mathcal{H}=D_-\oplus H\oplus D_+$ (12). Note the dual features of the spaces $\mathcal{H}_{N,\Gamma}$ (26) and $\mathcal{H}_{N^*,\Gamma^*}$ (32), which consist in that that differential operators of Cauchy problems (16) and (41) and operators (18) and (40) also, are adjoint with each other correspondingly in the metric l^2.

Define now in the space $\mathcal{H}_{N^*,\Gamma^*}$ (42) the operator-function $\overset{+}{U}(n)$ for $n\in\hat{\mathbb{Z}}^2_+$ (27), which acts on $\tilde{f}=\left(\tilde{u}_k,\tilde{h},\tilde{v}_k\right)\in\mathcal{H}_{N^*,\Gamma^*}$ in the following way:

$$\overset{+}{U}(n)\tilde{f} = \tilde{f}(n) = \left(\tilde{u}_k(n),\tilde{h}(n),\tilde{v}(n)\right), \tag{43}$$

where $\tilde{v}_k(n) = P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)}\tilde{v}_{k+n}$ ($P_{D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)}$ is an orthoprojector on $D_+\left(\tilde{N}^*,\tilde{\Gamma}^*\right)$); $\tilde{h}(n) = \tilde{y}_{(-1;0)}$, besides $\tilde{y}_k$ ($k \in \tilde{\mathbb{Z}}^2_-$) satisfies the Cauchy problem

$$\left\{ \begin{array}{l} \partial_1 \tilde{y}_k = T_1^* \tilde{y}_k + \Psi^* \tilde{v}_{\tilde{k}}; \\ \partial_2 \tilde{y}_k = T_2^* \tilde{y}_k + \Psi^* N^* \tilde{v}_{\tilde{k}}; \\ \tilde{y}_{(-n_1;-n_2)} = h;\ k = (k_1;k_2) \in \tilde{\mathbb{Z}}^2_-\ (-n_1 \le k_1 \le -1;\ -n_2 \le k_2 \le 0); \end{array} \right. \tag{44}$$

besides $\tilde{k} = k + n$ ш $(-n_1 \le k_1 \le -1;\ -n_2 \le k_2 \le 0)$; and finally

$$\tilde{u}_k(n) = \hat{u}_k + \tilde{u}_{k+n}, \tag{45}$$

and $\hat{u}_k = K^* \tilde{v}_{\tilde{k}} + \Phi^* \tilde{y}_k$, where $\tilde{y}_k$ is a solution of the system (44).

As in the case of the mapping $U(n)$ (28), the function $\tilde{v}_{\tilde{k}}$ is obtained after the shift by "$-n$" and automatically satisfies the consistency condition for (10) and the function $\tilde{u}_k(n)$ (45) has natural continuation into the whole domain $\tilde{\mathbb{Z}}^2_-$ (15) on account of the equation (41).

Similarly to (22), define the operator-function

$$\tau_\Delta = \left\{ \begin{array}{ll} I; & \Delta = (-1,0); \\ \tau; & \Delta = (0,-1). \end{array} \right. \tag{46}$$

Denote by L_m^{-1} the nondecreasing broken line in $\tilde{\mathbb{Z}}^2_-$ (15) with linear segments that are parallel to the axes OX and OY which connects points $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$ and $(-1,0)$. Choose now all the points $\{Q_s\}_M^{-1}$ ($M = m_1 + m_2$) on L_m^{-1} that are numerated in the nonascending order (of one of the coordinates Q_s) beginning with the point $(-1,0)$ and finishing with $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$. Define in the space $D_-\left(N^*, \Gamma^*\right)$ the quadratic form

$$\langle \tau \tilde{u}_k \rangle^2_{L_m^{-1}} = \sum_{s=M}^{-1} \left\langle \tau_{Q_s - Q_{s+1}} \tilde{u}_{Q_s}, \tilde{u}_{Q_s} \right\rangle, \tag{47}$$

where $Q_0 = (0,0)$. For the broken line L_0^n in $\mathbb{Z}^2_+$, $n = (n_1, n_2) \in \mathbb{Z}^2_+$, of the similar type with points $\{P_k\}_0^N$ ($N = n_1 + n_2$) on L_0^n which are also chosen in the nonascending order, define the quadratic form for the functions $\tilde{v}_k \in D_+\left(\tilde{N}^*, \tilde{\Gamma}^*\right)$

$$\langle \tilde{\tau} \tilde{v}_k \rangle^2_{L_0^n} = \sum_{k=0}^{N} \left\langle \tilde{\tau}_{P_k - P_{k+1}} \tilde{v}_{P_k}, \tilde{v}_{P_k} \right\rangle, \tag{48}$$

where $P_N - P_{N+1} = (-1,0)$ and $\tilde{\tau}_\Delta$ is defined similarly to τ_Δ (48). Denote by $\tilde{L}_0^m$ the broken line in $\mathbb{Z}^2_+$ obtained from the curve L_m^{-1} from $\tilde{\mathbb{Z}}^2_-$ using the shift by "m"

$$\tilde{L}_0^m = \left\{ P_k = (l_1, l_2) \in \mathbb{Z}^2_+ : (l_1 + m_1, l_2 + m_2) = Q_s \in L_m^{-1} \right\}, \tag{49}$$

where $m = (m_1, m_2) \in \tilde{\mathbb{Z}}^2_-$. Similarly to the Theorem 5, the following statement takes place.

Theorem 7. *Suppose* $\dim E < \infty$ *and the requirements of the Lemma 1 are met, then for the vector-function* $\tilde{f}(n) = \overset{+}{U}(n)\tilde{f}$ *(43) the equality*

$$\|\tilde{h}(n)\|^2 + \langle \tau \tilde{u}_k(n) \rangle^2_{L^{-1}_{-n}} = \|h\|^2 + \langle \tilde{\tau} \tilde{v}_k \rangle_{\tilde{L}^{-n}_0} \tag{50}$$

takes place for all $n \in \hat{\mathbb{Z}}^2_+$ *(27) and for all broken lines* L^{-1}_{-n} *connecting points* $-n = (-n_1, -n_2) \in \tilde{\mathbb{Z}}^2_-$ *and* $(-1, 0)$ *where* $\tilde{L}^{-n}_0$ *is a curve in* $\mathbb{Z}^2_+$ *obtained from* L^{-1}_{-n} *using the shift (49) by "$-n$" and corresponding* τ*-forms in (50) have the appearance of (47) and (48). The operator-function* $\overset{+}{U}(n)$ *(43) has the semigroup property,* $\overset{+}{U}(n)\,\overset{+}{U}(m) = \overset{+}{U}(n+m)$ *for all* n, $m \in \hat{\mathbb{Z}}^2_+$ *(27).*

As in the case of the theorem 5, the proof is reduced to the use of the isometric property of $\overset{+}{V}_1$ and $\overset{+}{V}_2$ (1) in view of 1) and 2) (2). The check of the semigroup property of the operator-function $\overset{+}{U}(n)$ (43) is quite simple as in the proof of the Theorem 5.

Define in $\mathcal{H}_{N^*,\Gamma^*}$ (43) the quadratic form

$$\langle \tilde{f} \rangle^2_\tau = \langle \tau \tilde{u}_k \rangle^2_{L^{-1}_{-\infty}} + \|\tilde{h}\|^2 + \langle \tilde{\tau} \tilde{v}_k \rangle^2_{L^\infty_0}, \tag{51}$$

where $L^{-1}_{-\infty}$ and L^∞_0 are nondecreasing allowable broken lines in $\tilde{\mathbb{Z}}^2_-$ and $\mathbb{Z}^2_+$ (with segments parallel to the axes OX and OY) connecting points $-\infty = (-\infty, -\infty)$ with $(-1, 0)$ and $(0, 0)$ with $\infty = (\infty, \infty)$ respectively.

Further, consider the family of one-parameter semigroup in $\tilde{\mathbb{Z}}^2_- \cup (0, 0)$

$$G_-(q) = \left\{ nq : q = (q_1, q_2) \in \tilde{\mathbb{Z}}^2_-; n \in \mathbb{Z}_+ \right\}, \tag{52}$$

where numbers q_1 and q_2 are coprime ideals and, moreover, $(-q_1 - 1, -q_2) \in \hat{\mathbb{Z}}^2_+$ (27). Choose fixed allowable broken line $L^{-1}_{\tilde{q}}$ in $\tilde{\mathbb{Z}}^2_-$ connecting points $\tilde{q} = (q_1, q_2) \in \tilde{\mathbb{Z}}^2_-$ (where $(-q_1 - 1, -q_2) \in \hat{\mathbb{Z}}^2_+$) and $(-1, 0)$ and make its group shift in $\tilde{\mathbb{Z}}^2_-$,

$$L^{-1}_{-\infty}(q) = \left\{ n + kq : n \in L^{-1}_{\tilde{q}}; k \in \mathbb{Z}_+ \right\}, \tag{53}$$

and in $\mathbb{Z}^2_+$,

$$L^\infty_0(q) = \left\{ n + kq : n \in L^{-1}_q; k \in \mathbb{Z}_- \right\}, \tag{54}$$

respectively. Similarly to (38), define the metric along $G_-(p)$ (52) in the space $\mathcal{H}_{N^*,\Gamma^*}$

$$\langle f \rangle^2_{\tau,q} = \langle \tau \tilde{u}_k \rangle^2_{L^{-1}_{-\infty}(q)} + \|\tilde{h}\|^2 + \langle \tilde{\tau} \tilde{v}_k \rangle^2_{L^\infty_0(q)}, \tag{55}$$

where broken lines $L_{-\infty}^{-1}(q)$ and $L_0^{\infty}(q)$ have the appearance of (53) and (54).

Theorem 8. *Suppose* $\{T_1, T_2\} \in C(T_1)$ *(3),* $\dim E < \infty$, *and the conditions of lemma 1 are met, then for all* $q \in \tilde{\mathbb{Z}}_-^2$ *(15) such that* $(-q_1 - 1, -q_2) \in \hat{\mathbb{Z}}_+^2$ *(27), the operator semigroup* $T_n^*(q) = T^*(-nq)$ *(33) from* $n \in \mathbb{Z}_+$, *narrowed on* $G_-(q)$ *(50), always has an isometric (in the metric* $\langle \tilde{f} \rangle_{\tau,q}^2$ *(55)) dilation* $\overset{+}{U}_n(|q|) = \overset{+}{U}(-nq)$ *(43) which acts in the space* $\mathcal{H}_{N^*,\Gamma^*}$ *(42).*

N o t e 6. For the dual dilation $\overset{+}{U}(|q|) = \overset{+}{U}(-nq)$ (43), as well as for $U_n(p) = U(np)$ (28), the relation

$$\left\langle \overset{+}{U}_n(|q|)h, \overset{+}{U}_m(|q|)h' \right\rangle_{\tau,q} = \langle T_{n-m}^*(q)h, h' \rangle \tag{56}$$

is true when $n \geq m$ and for all h, $h' \in H$. Hence, the subspace

$$\operatorname{span}\left\{ \overset{+}{U}_n(|q|)h : h \in H, n \in \mathbb{Z}_+, (-q_1 - 1, -q_2) \in \hat{\mathbb{Z}}_+^2 \right\}$$

in $\mathcal{H}_{N^*,\Gamma^*}$ (42) is defined by the initial operator system $\{T_1, T_2\}$ from $C(T_1)$ (3).

V. Note that Hilbert spaces $\mathcal{H}_{N,\Gamma}$ (26) and $\mathcal{H}_{N^*,\Gamma^*}$ (42) have the common part, namely the space $\mathcal{H}$ (12) which per se defines them in view of the corresponding Cauchy problems (16), (18) and (40), (41). Moreover, narrowings of the dilations $U(n_1; 0)$ (28) and $\overset{+}{U}(n_1; 0)$ (43) on the invariant subspace $\mathcal{H}$ are unitary operators, besides $U^*(n_1; 0) = \overset{+}{U}(n_1; 0)$ $\forall n_1 \in \mathbb{Z}_+$. It follows from the Note 4 that the dilation $U(n)$ (28) has the "+ minimality" property, that means the "observability" of the system (4), and it follows from the Note 6 respectively that dilation $\overset{+}{U}(n)$ (43) satisfies "$-$ minimality" condition, that corresponds with "controllability" of the open system (8), [2, 7, 9]. The next definition follows from notes made earlier.

Definition 4. *Consider the operator semigroup* $T(n)$, *defined when* $n \in \hat{\mathbb{Z}}_+^2$ *(27), that corresponds to the commutative operator system* $\{T_1, T_2\}$ *from the class* $C(T_1)$ *(3). Let* $U(n)$ *be the isometric dilation (in terms of the Definition 3 of the semigroup* $T(n)$*) that acts in the space* $\mathcal{H}_+$ *and the operator-function* $\overset{+}{U}(n)$, *defined in* $\mathcal{H}_-$, *be the isometric dilation of the adjoint semigroup* $T^*(n)$. *The pair of dilations* $U(n)$ *and* $\overset{+}{U}(n)$ *is called minimally-unitarily connected if the following conditions are met.*

1) The Hilbert space $\mathcal{H}_0 = \mathcal{H}_+ \cap \mathcal{H}_-$ *is invariant with regard to the operator-functions* $U(n_1; 0)$ *and* $\overset{+}{U}(n_1; 0)$ $\forall n_1 \in \mathbb{Z}_+$, *besides restrictions of* $U(n_1; 0)$ *and*

$\overset{+}{U}(n_1;0)$ *on* $\mathcal{H}_0$ *are unitary operators and, moreover,* $U^*(n_1;0) = \overset{+}{U}(n_1;0)$ $\forall n_1 \in \mathbb{Z}_+$.

2) Restriction of the semigroup $U(n_1;0)$ *on* $\mathcal{H}_0$ *is the minimal [4, 9] unitary dilation of the semigroup* $T_1^{n_1}$ *when* $n_1 \in \mathbb{Z}_+$,

$$\mathcal{H}_0 = \operatorname{span}\{U(n_1;0)\,h : h \in H; n \in \mathbb{Z}\}.$$

3) The equalities

$$\mathcal{H}_+ = \operatorname{span}\left\{U(n)\mathcal{H}_0 : n \in \hat{\mathbb{Z}}_+^2\right\};$$

$$\mathcal{H}_- = \operatorname{span}\left\{\overset{+}{U}(n)\mathcal{H}_0 : n \in \hat{\mathbb{Z}}_+^2\right\}$$

are taking place.

Note that Point 3) of the Definition 4 means that there are no adduction subspaces in $\mathcal{H}_+$ and $\mathcal{H}_-$ for the operators $U(n)$ and $\overset{+}{U}(n)$ on which $U(n)$ and $\overset{+}{U}(n)$ are unitary and which are not connected with the initial system $\{T_1, T_2\}$.

It is easy to see that minimally-unitary connected dilations $U(n)$ in $\mathcal{H}_+$ and $\overset{+}{U}(n)$ in $\mathcal{H}_-$ are defined up to isomorphism. As is well known [4, 9], the minimal unitary dilation $U(n_1;0)$ of the contraction semigroup $T_1^{n_1}$ $(n_1 \in \mathbb{Z}_+)$ in $\mathcal{H}_0$ is defined uniquely (up to isomorphism). And from the point 3) of the Definition 4 follows that corresponding isomorphism between $U(n)$ in $\mathcal{H}_+$ and $U'(n)$ in $\mathcal{H}'_+$ (for example) could be defined in the following way: $U(n)f \to U'(n)f$ where $f \in \mathcal{H}_0$, though this correspondence not necessarily is a unitary operator. Note that from the constructions of the dilations $U(n)$ (28) in $\mathcal{H}_{N,\Gamma}$ (26) and $\overset{+}{U}(n)$ (43) in $\mathcal{H}_{N^*,\Gamma^*}$ (42), it follows that the pair $U(n)$ and $\overset{+}{U}(n)$ is defined "unambiguously" by the initial operator system $\{T_1, T_2\}$ from $C(T_1)$ (3) in accordance with (49) and (55).

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