Journal of Mathematical Physics, Analysis, Geometry
2006, vol. 2, No. 1, pp. 3–8

Compact Spacelike Surfaces in the 3-Dimensional de Sitter Space

A.A. Borisenko

Department of Mechanics and Mathematics, V.N. Karazin Kharkov National University
4 Svobody Sq., Kharkov, 61077, Ukraine
E-mail:borisenk@univer.kharkov.ua

Received September 14, 2005

We establish several sufficient conditions for a compact spacelike surface in the 3-dimensional de Sitter space to be totally geodesic or spherical.

Key words: de Sitter space, compact spacelike surface, second fundamental form, Gaussian curvature; totally umbilical round sphere.
Mathematics Subject Classification 2000: 53C42 (primary); 53B30, 53C45 (secondary).

Let E_1^4 be a 4-dimensional Lorentz–Minkowski space, that is, the space E_1^4 endowed with the Lorentzian metric tensor $\langle\,,\rangle$ given by

$$\langle,\rangle = (dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_0)^2,$$

where (x_1, x_2, x_3, x_0) are the canonical coordinates of E_1^4. The 3-dimensional unitary de Sitter space is defined as the following hyperquadric of E_1^4:

$$S_1^3 = \{x \in R^4 : \langle x, x\rangle = 1\}.$$

As it is well known, S_1^3 inherits from E_1^4 a time-orientable Lorentzian metric which makes it the standard model of a Lorentzian space of constant sectional curvature one. A smooth immersion $\psi\colon F \to S_1^3 \subset E_1^4$ of a 2-dimensional connected manifold M is said to be a spacelike surface if the induced metric via ψ is a Riemannian metric on M, which, as usual, is also denoted by $\langle\,,\rangle$. The time-orientation of S_1^3 allows us to define a (global) unique timelike unit normal field n on F, tangent to S_1^3, and hence we may assume that F oriented by n. We will refer to n as the Gauss map of F.

The work supported by research grant DFFD of Ukrainian Ministry of Education and Science, No. 01. 07/ 00132.

We note that Lobachevsky space L^3 is the set of points

$$L^3 = \{x \in E^4_1 : \langle x, x\rangle = -1, x_0 > 0\}.$$

It is well known that a compact spacelike surface in the 3-dimensional de Sitter space S^3_1 is diffeomorphic to a sphere S^2. Thus, it is interesting to look for additional assumptions for such a surface to be totally geodesic or totally umbilical round sphere.

There are two possible kinds of geometric assumptions: extrinsic, that is relative to the second fundamental form, and intrinsic, namely, concerning to the Gaussian curvature of the induced metric. As regards to the extrinsic approach, J. Ramanathan [10] proved that every compact spacelike surface in S^3_1 of constant mean curvature is totally umbilical. This result was generalized to hypersurface of any dimension by S. Montiel [9]. J. Aledo and A. Romero characterize the compact spacelike surfaces in S^3_1 whose second fundamental form defines a Riemannian metric. They studied the case of constant Gaussian curvature K_{II} of the second fundamental form, proving that the totally umbilical round spheres are the only compact spacelike surfaces in S^3_1 with $K < 1$ and constant K_{II} [2]. With respect to the intrinsic approach H. Li [8] obtained that compact spacelike surface of constant Gaussian curvature is totally umbilical. And he proved there is no complete spacelike surface in S^3_1 with constant Gaussian curvature $K > 1$. J. Aledo and A. Romero proved the same result without condition that Gaussian curvature is constant [2]. But it is true more general result.

Theorem 1. *Let F be a $\mathcal{C}^2$-regular complete spacelike surface in de Sitter space S^3_1. If Gaussian curvature $K \geqslant 1$ the surface F is totally geodesic great sphere with Gaussian curvature $K = 1$.*

S.N. Bernshtein proved that an explicitly given saddle surface over a whole plane in the Euclidean space E^3 with slower than linear growth at infinity must be a cylinder. He proved this theorem for surfaces of class $\mathcal{C}^2$ [4], and it was generalized to the nonregular case in [1].

A surface F^2 of smoothness class C^1 in S^3 may be projected univalently into a great sphere S^2_0 if the great spheres tangent to F^2 do not pass through points Q_1, Q_2 polar to S^2_0.

The surface F^2 in S^3 is called a saddle surface if any closed rectifiable contour $\mathcal{L}$, that is in the intersection of F^2 with an arbitrary great sphere S^2 in S^3, lies in an open hemisphere, and is deformable to a point in the surface can be spanned by a two-dimensional simply connected surface Q contained in $F^2 \cap S^2$. In other words, from the surface it is impossible to cut off a crust by a great sphere S^2, that is, on F^2 there do not exist domains with boundary that lie in an open great hemisphere of S^2 and are wholly in one of the great hemispheres of S^3

into which it is divided by the great sphere S^2. In this case when F is a regular surface of class $\mathcal{C}^2$, the saddle condition is equivalent to the condition that the Gaussian curvature of F^2 does not exceed one. We have the following result.

Theorem 2 ([5–7]). *Let F be an explicitly given compact saddle surface of smoothness class $\mathcal{C}^1$ in the spherical space S^3. Then F is a totally geodesic great sphere.*

This theorem is a generalization of a theorem of Bernshtein to a spherical space. For regular space we obtain the following corollary.

Theorem 3 ([5, 6]). *Let F be an explicitly given compact surface that is regular of class $\mathcal{C}^2$ in the spherical space S^3. If the Gaussian curvature K of F satisfies $K \leqslant 1$ then F is a totally geodesic great sphere.*

This theorem was stated in [6]. Really Theorems 2 and 3 had been proved in [7] but were formulated there for a centrally symmetric surfaces. The final version was in [5].

It seems to us that the following conjecture must hold under a restriction on the Gaussian curvature of the surface. Suppose that F is an embedded compact surface, regular of class $\mathcal{C}^2$, in the spherical space S^3. If the Gaussian curvature K of F satisfies $0 < K \leqslant 1$, then F is a totally geodesic great sphere.

A.D. Aleksandrov [3] had proved that an analytical surface in Euclidean space E^3 homeomorphic to a sphere is a standard sphere if principal curvatures satisfy the inequality

$$(k_1 + c)(k_2 + c) \leqslant 0. \tag{1}$$

This result had been generalized for analytic surfaces in spherical space S^3 and Lobachevsky space L^3 [7]:

a) in S^3 with additional hypothesis of positive Gaussian curvature;

b) in L^3 under additional assumptions that principal curvatures k_1, k_2 satisfy $|k_1|, |k_2| > c_0 > 1$.

But in Lobachevsky space the result is true under weaker analytic restriction.

Theorem 4. *Let F be a $\mathcal{C}^3$ regular surface homeomorphic to the sphere in the Lobachevsky space L^3. If $|k_1|, |k_2| > c_0 > 1$ and principle curvatures k_1 and k_2 satisfy* (1)*, then the surface is an umbilical round sphere in L^3.*

Analogical result it is true for surfaces in the de Sitter space S_1^3.

Theorem 5. *Let F be a C^3 regular compact spacelike surface in the de Sitter space S^3_1. If $|k_1|, |k_2| < 1$ and principal curvatures satisfy* (1)*, then the surface is an umbilical round sphere in S^3_1.*

Let S^3_1 be a simply-connected pseudo-Riemannian space of curvature 1 and signature $(+,+,-)$. It can be isometrically embedded in the pseudo-Euclidean space E^4_1 of signature $(+,+,+,-)$ as the hypersurface given by the equation $x_1^2+x_2^2+x_3^2-x_0^2=1$. Together with E^4_1 we consider the superimposed Euclidean space E^4 with unit sphere S^3 given by the equation $x_1^2+x_2^2+x_3^2-x_0^2=1$. We specify a mapping of S^3_1 into S^3. To the point P of S^3_1 with position vector r we assign the point $\tilde{P}$ with position vector $\tilde{r} = r/\sqrt{1+2x_0^2}$. Under the mapping, to a surface $F \subset S^3_1$ corresponds a surface $\tilde{F} \subset S^3$. Let b_{ij} and $\tilde{b}_{ij}$ be the coefficients of the second quadratic forms of F and $\tilde{F}$, and $n = (n_1, n_2, n_3, n_0)$ be a normal vector field on F.

Lemma 1 ([7]). $\tilde{b}_{ij} = b_{ij}/\sqrt{1+2x_0^2}\sqrt{1+2n_0^2}$.

P r o o f o f T h e o r e m 1. From the condition $K \geqslant 1$ it follows that F is a compact spacelike surface in the de Sitter space S^3_1. Locally a spacelike surface is explicitly given over totally geodesic great sphere $S^2_0 \subset S^3_1$ and the orthogonal projection $p\colon F \to S^2_0$ in S^3_1 is covering. Indeed, p is a local diffeomorphism. The compactness of F and the simply connectedness of S^2_0 imply that p is a global diffeomorphism F on S^2_0 and the surface F is globally explicitly given over S^2_0.

We map from a surface F in S^3_1 to a surface $\tilde{F}$ in S^3. If F has a definite metric and Gaussian curvature $K \geqslant 1$, then $\tilde{F}$ has Gaussian curvature not greater than 1. This follows immediately from Lemma 1, Gauss's formula and the fact that $\langle n, n\rangle = -1$ for normals to F. In a pseudo-Euclidean space, the analogous correspondence between surfaces and their curvatures was used by Sokolov [11].

The surface $\tilde{F}$ satisfies the conditions of Theorem 3. It follows that $\tilde{F}$ is a totally geodesic great sphere. By Lemma 1 the ranks of the second quadratic forms of $\tilde{F}$ and F coincide and we obtain that the surface F is a totally geodesic surface in S^3_1.

P r o o f o f T h e o r e m s 4 a n d 5. The normal $n(u_1, u_2)$ to F is chosen so that the principal curvature satisfy (1). In a neighborhood of an arbitrary nonumbilical point P we choose coordinate curves consisting of the lines of curvature, and an arbitrary orthogonal net in the case of umbilical point. At P the coefficients of the first quadratic form are $e = g = 1, f = 0$. Let F_1 be the surface with radius vector $\rho = (r - cn)/\sqrt{|c^2-1|}$.

In both cases the surface F_1 lies in S^3_1. Moreover,

$$\rho_{u_1} = \frac{(1+ck_1)}{\sqrt{|c^2-1|}} r_{u_1}, \quad \rho_{u_2} = \frac{(1+ck_2)}{\sqrt{|c^2-1|}} r_{u_2}.$$

The unit normal $n_1 = \dfrac{cr - n}{\sqrt{|c^2 - 1|}}$. From the conditions on the principal curvatures of F in Theorems 4, 5 it follows that

$$\langle \rho_{u_1}, \rho_{u_1} \rangle > 0, \quad \langle \rho_{u_2}, \rho_{u_2} \rangle > 0$$

and F_1 is a spacelike surface in S_1^3. The coefficients of the second quadratic form of the surface F_1 are

$$L_1 = \frac{(1 + ck_1)(k_1 + c)}{\sqrt{c^2 - 1}}, \quad N_1 = \frac{(1 + ck_2)(k_2 + c)}{\sqrt{c^2 - 1}}.$$

The Gaussian curvature of F_1 at the point P_1 is equal to

$$K = 1 - \frac{(k_1 + c)(k_2 + c)|c^2 - 1|}{(1 + k_1 c)^2 (1 + k_2 c)^2} \geqslant 1.$$

The same is true in umbilical points too. The surface F_1 satisfies the conditions of Theorem 1. It follows that the surface F_1 is a totally geodesic great sphere in S_1^3 and F is an umbilical surface in L^3 or S_1^3.

References

[1] *G.M. Adelson-Vel'sky*, The generalization of one geometrical theorem of S.N. Bernshtein. — *Dokl. Akad. Nauk USSR* **49** (1945), 6. (Russian)

[2] *J.A. Aledo and A. Romero*, Compact spacelike surfaces in the 3-dimensional de Sitter space with nondegenerate second fundamental form. — *Diff. Geom. and its Appl.* **19** (2003), 97–111.

[3] *A.D. Aleksanrdov*, On the curvature of surfaces. — *Vestnik Leningr. Univ. Ser. Mat. Mech. Astrs.* **19** (1966) No. 4, 5–11. (Russian)

[4] *S.N. Bernshtein* , Amplification of the theorem of surfaces with negative curvature. (Sobr. Soch. V. 3.) Publ. House Akad. Nauk USSR, Moscow, 1960. (Russian)

[5] *A.A. Borisenko*, On explicitly given saddle surfaces in a spherical space. — *Usp. Mat. Nauk* **54** (1999), No. 5, 151–152; Engl. transl.: *Russian Math. Surveys* **54** (1999), No. 5, 1021–1022.

[6] *A.A. Borisenko*, Complete l-dimensional surfaces of nonpositive extrinsic curvature in a Riemannian space. — *Mat. Sb.* **104** (1977), 559–576; Engl. transl.: *Mat. Sb.* **33** (1977), 485–499.

[7] *A.A. Borisenko*, Surfaces of nonpositive extrinsic curvature in spaces of constant curvature. — *Mat. Sb.* **114** (1981), 336–354; Engl. transl.: *Mat. Sb.* **42** (1982), 297–310.

[8] *H. Li*, Global rigidity theorems of hypersurfaces. — *Ark. Mat.* **35** (1997), 327–351.

[9] *S. Montiel,* An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature. — *Indiana Univ. Math. J.* **37** (1988), 909–917.

[10] *J. Ramanathan*, Complete spacelike hypersurfaces of constant mean curvature in de Sitter space. — *Indiana Univ. Math. J.* **36** (1987), 349–359.

[11] *D.D. Sokolov*, The structure of the limit cone of a convex surface in pseudo-Euclidean space. — *Usp. Mat. Nauk* **30** (1975), No. 1(181), 261–262. (Russian)