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Topological Properties of the Set of Admissible Transformations of Measures

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Suppose a topological semigroup G acts on a topological space X. A transformation $g \in G$ is called an admissible (partially admissible, singular, equivalent, invariant) transformation for μ relative to ν if $\mu_g \ll \nu$ (accordingly: $\mu_g \not\ll \nu$, $\mu_g \perp \nu$, $\mu_g \sim \nu$, $\mu_g = c \cdot \nu$), where $\mu_g(E) := \mu(g^{-1}E)$. We denote its collection by $A(\mu|\nu)$ (accordingly: $AP(\mu|\nu)$, $S(\mu|\nu)$, $E(\mu|\nu)$, $I(\mu|\nu)$). It is shown that all these sets are Borel subsets of very bounded types. In particular, $A(\mu|\nu)$ is a $G_{\delta\sigma\delta}$-subset of G. If G is a Polish group, then $A(\mu|\nu)$, $E(\mu|\nu)$ and $I(\mu|\nu)$ admit a Polish topology.

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Let X be a measurable space and μ a probability measure on X. A transformation $g : X \to X$ is called admissible for μ if $\mu_g \ll \mu$ where $\mu_g = \mu g^{-1}$. Such transformations are important for the study of measures. Let X be a topological group. The simplest transformations of X are translations. Denote by $A(\mu)$ the set of admissible translations of μ. For example, admissible translations arise naturally in the theory of stochastic processes. T.S. Pitcher [14] has done the general definition of an admissible translation and the simplest properties of $A(\mu)$ for measures which correspond to stochastic processes. In detail some algebraic and topological properties for admissible translations of measures were considered by A.V. Skorohod [18] for a Hilbert space and by Y. Okazaki [12] for a separable metric group.

It turned out that the structure of μ depends on the "volume" of $A(\mu)$ substantially. In the case $X = \mathbb{R}$ and $[0; \infty) \subset A(\mu)$, A.V. Skorohod [17] has proved that μ is absolutely continuous relative to the Lebesgue measure and its support is of the form $[a; \infty)$. P.L. Brockett [4] has generalized this fact to the case of locally compact σ-compact groups. Moreover, the famous Mackey–Weil theorem [11] asserts that if X is a standard Borel group and $A(\mu) = X$, then X admits a locally compact topology and μ is mutually absolutely continuous with respect to Haar measure.

Transformations which take μ to its equivalent (to oneself), constitute an important special case. We denote the set of such transformations by $E(\mu)$ ($I(\mu)$). Restricting our considerations to such transformations only, we obtain a classical object of study in Ergodic theory. Let X be a locally compact group. There exists a measure (the Haar measure) such that $I(\mu) = X$. This fact plays a key role in Harmonic analysis. At any case, $I(\mu)$ (the group of invariance) is compact and plays an important role in arithmetic of probability measures (see history and details in [8]).

Let $X = \mathbb{T}$ be the circle group and μ a probability measure on $\mathbb{T}$. $E(\mu)$ can be viewed as a group of eigenvalues for a nonsingular dynamical system and belongs to the class of so-called "saturated" subgroups [1, 9]. This approach demonstrates an interesting interplay between Harmonic analysis and Ergodic theory.

These results constitute the basement of study of $A(\mu)$ and similar sets in more detail. In particular, some algebraic, measure theoretical properties, together with a Lebegsue-type decomposition of this set have been considered (see [7]). In this article we study topological properties of the set of admissible transformations.

Let $X = G$ be a separable metric group. Y. Okazaki [12] has shown that the sets $E(\mu)$ and $A(\mu)$ are Borel. Let us consider the group $E(\mu)$. Two methods of proving that $E(\mu)$ is Borel are known. The first introduces the strong operator topology on $E(\mu)$ and shows that this topolgy is Polish (see [18, 1, 13]). In the first part of the article we generalize this fact to all Polish G-spaces. Note that in the general case (see Remark 2.3), $E(\mu)$ is not complete in the strong operator topology and it is necessary to amplify it by the initial topology. At the end of the first part we present some applications of our results to t-ergodic measures, which generalize corresponding facts for Abelian groups ([5, 9, 13]). The second method was used by Y. Okazaki [12]. In the second part of the article we use the de Possel theorem to prove much more (see Theorem 3.1). In particular, for $X = \mathbb{T}$ we establish that $E(\mu)$ is the set of the type $G_{\delta\sigma\delta}$. This give a restriction on $E(\mu)$ as well as saturation.

1. Preliminaries and basic definitions

Let $(X, \mathcal{B})$ be a Borel space. We can assume without loss of generality that X is separable, i.e.: if $x, y \in X$ and $x \neq y$, then there exists $E \in \mathcal{B}$ such that $x \in E \not\ni y$.

Definition 1.1. *A pair (G, X) is called a (semi)group of transformations if:*
1) G is a (semi)group and X is a Borel space;

2) the mapping $g : (X, \mathcal{B}) \to (X, \mathcal{B})$, $x \mapsto g \cdot x$, is Borel for all $g \in G$; $(gh) \cdot x = g \cdot (h \cdot x)$, and if e is the unit in G then $e \cdot x = x$, $\forall g, h \in G$, $\forall x \in X$.

Let G be a topological (semi)group and X a topological space. Then the (semi)group of transformations (G, X) is said to be topological if the mapping $(g, x) \mapsto g \cdot x$ is continuous.

Definition 1.2. *Let (G, X) and (H, Y) be a (semi)group. A pair (p, τ), where $p : G \to H$ is a homomorphism and $\tau : X \to Y$ a Borel mapping, is called a morphism from (G, X) to (H, Y) if p and τ acts as follows*

$$\tau(g \cdot x) = p(g) \cdot \tau(x) \ , \quad \forall g \in G, \ \forall x \in X.$$

If (G, X) and (H, Y) are topological (semi)groups of transformations then p and τ are supposed to be continuous.

Hence the set of [topological] (semi)groups of transformations form a category.

Let $E \in \mathcal{B}$. The image and the inverse image of E is denoted by $g \cdot E$ and $g^{-1}E$ respectively. If G is a group, then $g \cdot E$ is denoted simply by gE.

Let $M(X)$ be the set of all finite Borel measures on X. The subset of all positive measures is denoted by $M^+(X)$. A measure $\mu \in M^+(X)$ is called probabilistic if $\mu(X) = 1$. The Dirac mass at a point x is denoted by δ_x. Let $\mu, \nu \in M(X)$. We write $\mu \ll \nu$ if $|\mu|$ is absolutely continuous relative to $|\nu|$, and $\mu \perp \nu$ if $|\mu|$ and $|\nu|$ are mutually singular. Equivalence $\mu \sim \nu$ means that $\mu \ll \nu$ and $\nu \ll \mu$. If $\mu = \mu_1 + \mu_2$ whith $\mu_1 \perp \mu_2$, then μ_1 and μ_2 are called parts of μ.

Let $\mu \in M(X)$ and $g \in G$. Denote by μ_g the measure on $(X, \mathcal{B})$ determined by the relation

$$\mu_g(E) = \mu(g^{-1}E), E \in \mathcal{B}.$$

Then $(\mu_g)_h(E) = \mu_g(h^{-1}E) = \mu(g^{-1}h^{-1}E) = \mu_{hg}(E)$, i.e., $(\mu_g)_h = \mu_{hg}$.

Let μ, $\nu \in M(X)$. One can represent them in the form

$$\mu = \mu^1 + \mu^2, \ \nu = \nu^1 + \nu^2, \ \text{ where } \mu^1 \sim \nu^1, \ \mu^2 \perp \nu, \ \nu^2 \perp \mu.$$

This decomposition is called the Lebesgue decomposition of measures μ and ν. Denote by $\frac{d\mu}{d\nu}$ the derivative of μ with respect to ν. Then

$$\frac{d\mu}{d\nu} = \frac{d\mu^1}{d\nu^1}, \ \ \nu^1 - \text{a.e. ; } \quad \text{and} \quad \frac{d\mu}{d\nu} = 0, \ \ (\nu^2 + \mu^2) - \text{a.e.}$$

Denote by m_G the left Haar measure of a locally compact group G.

For a function $f(x)$ we put: $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{f(x), 0\}$. Then $f(x) = f^+(x) - f^-(x)$.

This article is devoted to the study of topological properties of the sets which are determined in the following definitions.

Definition 1.3. *Let $\mu \in M(X)$. A transformation $g \in G$ is called an admissible (partially admissible, singular, equivalent, invariant) transformation for μ if $\mu_g \ll \mu$(respectively: $\mu_g \not\perp \mu$, $\mu_g \perp \mu$, $\mu_g \sim \mu$, $\mu_g = \mu$). Their set denoted by $A(\mu)$ (respectively: $AP(\mu)$, $S(\mu)$, $E(\mu)$, $I(\mu)$).*

Obviously

$$I(\mu) \subset E(\mu) \subset A(\mu) \subset AP(\mu),\ AP(\mu) \cap S(\mu) = \emptyset,\ AP(\mu) \cup S(\mu) = G.$$

It is clear that, if G has a unit e, then $e \in I(\mu)$.

The following definition is a natural generalization of the previous one.

Definition 1.4. *Let $\mu,\ \nu \in M(X)$. A transformation $g \in G$ is called an admissible (partially admissible, singular, equivalent, invariant) transformation for μ relative to ν if $\mu_g \ll \nu$ (respectively: $\mu_g \not\perp \nu$, $\mu_g \perp \nu$, $\mu_g \sim \nu$, $\mu_g = c \cdot \nu$, where $c = \|\nu\|/\|\mu\|$). Their set denoted by $A(\mu|\nu)$ (respectively: $AP(\mu|\nu)$, $S(\mu|\nu)$, $E(\mu|\nu)$, $I(\mu|\nu)$).*

Evidently, the corresponding inclusions are true for these sets:

$$I(\mu|\nu) \subset E(\mu|\nu) \subset A(\mu|\nu) \subset AP(\mu|\nu),$$
$$AP(\mu|\nu) \cap S(\mu|\nu) = \emptyset,\ AP(\mu|\nu) \cup S(\mu|\nu) = G.$$

Clearly, if G is a group, then $E(\mu|\nu) = E(|\mu|||\nu|)$, $A(\mu|\nu) = A(|\mu|||\nu|)$, $AP(\mu|\nu) = AP(|\mu|||\nu|)$, $S(\mu|\nu) = S(|\mu|||\nu|)$. Thus we will often restrict our considerations to probability measures only.

The case when $X = G$ is a group makes a special interest. The following operators arise naturally

$$L_g(x) = gx\,, \quad R_g(x) = xg^{-1}\,, \quad C_g(x) = gxg^{-1} = L_g R_g(x), \quad \forall x, g \in X.$$

These operators determine the left, right and conjugate actions of G on X. By default, the action of G on X is left, i.e. $g \cdot x = gx$.

Definition 1.5. *Let $G = X$ be a group. The sets $AP(\mu|\nu)$, $S(\mu|\nu)$, $A(\mu|\nu)$, $E(\mu|\nu)$, $I(\mu|\nu)$ relative to the left (right, conjugate) action of the group on itself is denoted with the subindex l (respectively r, c), i.e.,*

$$AP_l(\mu|\nu), S_l(\mu|\nu), A_r(\mu|\nu), E_c(\mu|\nu) \ \ etc.$$

Put $A_t(\mu|\nu) = A_l(\mu|\nu) \cap [A_r(\mu|\nu)]^{-1}$, $\ A_t(\mu) = A_l(\mu) \cap [A_r(\mu)]^{-1}$ etc.

We remark that for noncommutative groups, P.L. Brockett [4] and Y. Okazaki [12] used the term admissible translations for the elements of $A_t(\mu)$.

Set $G^* = G \cup \{id_X\}$. Then G^* is a semigroup with unit. Corresponding sets relative to G^* are denoted by $AP^*(\mu|\nu)$, $A^*(\mu|\nu)$ etc. A set E is called G-invariant if $g^{-1}(E) = E$ for all $g \in G$.

Definition 1.6. *A measure μ is called t-ergodic, if for all its nonzero parts α and β, there exist g, $h \in G^*$ such that $\alpha_g \not\perp \beta$ and $\alpha \not\perp \beta_h$.*

Evidently, μ is t-ergodic if and only if $|\mu|$ is t-ergodic.

2. The strong topology on $AP(\mu)$

Let X be a separable metric space. Let G be a separable metric (semi)group which acts continuously on X. Let μ and ν be probability measures on X.

For $\alpha \in L^1(\mu)$ let $\alpha_g = \alpha^1 + \alpha^2$ be the Lebesgue decomposition of α_g relative to ν, where $\alpha^1 \ll \nu$ and $\alpha^2 \perp \nu$. Put

$$T_{\nu,g}(\alpha) = \alpha^1.$$

Then $T_{\nu,g}$ is a linear contractive operator from $L^1(\mu)$ to $L^1(\nu)$.

Now we define the strong operator topology (strong topology, for short) on $AP(\mu|\nu)$ (compare with [18, 1, 13]).

Definition 2.1. *A sequence $g_n \in AP(\mu|\nu)$ is called convergent to $g \in AP(\mu|\nu)$ in the strong topology if*

$$\lim_{n\to\infty} \|T_{\nu,g_n}(\alpha) - T_{\nu,g}(\alpha)\| = 0, \quad \forall \alpha \in L^1(\mu), \tag{2.1}$$

for the semigroup case, and, additionally to (2.1),

$$\lim_{n\to\infty} \|T_{\mu,g_n^{-1}}(\beta) - T_{\mu,g^{-1}}(\beta)\| = 0, \quad \forall \beta \in L^1(\nu), \tag{2.2}$$

for the group case.

Definition 2.2. *Let g, $h \in AP(\mu|\nu)$. Let $\{\mu_n\}$ and $\{\nu_n\}$ be a countable dense subset in $L^1(\mu)$ and $L^1(\nu)$ respectively. If G is a semigroup we put*

$$d(h,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|T_{\nu,h}(\mu_n) - T_{\nu,g}(\mu_n)\|}{1 + \|T_{\nu,h}(\mu_n) - T_{\nu,g}(\mu_n)\|},$$

and if G is a group we put

$$d(h,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{\|T_{\nu,h}(\mu_n) - T_{\nu,g}(\mu_n)\|}{1 + \|T_{\nu,h}(\mu_n) - T_{\nu,g}(\mu_n)\|} + \frac{\|T_{\mu,h^{-1}}(\nu_n) - T_{\mu,g^{-1}}(\nu_n)\|}{1 + \|T_{\mu,h^{-1}}(\nu_n) - T_{\mu,g^{-1}}(\nu_n)\|} \right).$$

Notice some simple properties.

Proposition 2.1. *1. $d(g,h)$ is a pseudometric on $AP(\mu|\nu)$.*

2. The topology which determined by $d(g,h)$ and the strong topology are coincide.

3. Let $I(g,h)=\{x:\ g\cdot x=h\cdot x\}$. Put

$$\mu_{g,h}(E):=\mu(E\cap(X\setminus I(g,h)))\ ,\quad \nu_{g,h}(E):=\mu(E\cap(X\setminus I(g,h))).$$

Then $d(g,h)$ is a metric if and only if the set $AP(\mu_{g,h}|\nu)\cap\{g,h\}$ (in the case if G is a group, one of the sets $AP(\mu_{g,h}|\nu)\cap\{g,h\}$ or $AP(\nu_{g,h}|\mu)\cap\{g^{-1},h^{-1}\}$) is not empty for all $g,h\in AP(\mu|\nu), g\neq h$.

4. If $d(g,h)$ is a metric, then $AP(\mu|\nu)$ is a separable metric space.

5. If G is a group, then the mapping $j:AP(\mu|\nu)\to AP(\nu|\mu)$, $j(g)=g^{-1}$, is a homeomorphism.

6. If h and t are invertible, then the mapping $i(g)=tgh^{-1}$ is homeomorphism from $B(\mu|\nu)$ to $B(\mu_h|\nu_t)$, where $B(.)$ is one of the sets $AP(.)$, $A(.)$, $E(.)$, $I(.)$.

P r o o f. We give the proof for the case when G is a group.

1., 2. Evidently.

3. Let $g\in AP(\mu_{g,h}|\nu)$. Let $(\mu_{g,h})_g=\mu^1+\mu^2$ with $\mu^1\ll\nu$, $\mu^2\perp\nu$, and α be the part of $\mu_{g,h}$ such that $\alpha_g=\mu^1$. By hypothesis, we can chose $x_0\in\operatorname{supp}\alpha$ and a neighborhood U of x_0 such that $g\cdot U\cap h\cdot U=\emptyset$. Put $\beta=\alpha|_U=(\mu_{g,h})|_U$ and chose μ_n such that $\|\beta-\mu_n\|<0,1\|\beta\|$. Then

$$\begin{aligned}(1+\|T_{\nu,g}(\mu_n)-T_{\nu,h}(\mu_n)\|)2^n d(g,h)&\geq\|T_{\nu,g}(\mu_n)-T_{\nu,h}(\mu_n)\|\\ &=\|T_{\nu,g}(\beta)-T_{\nu,h}(\beta)+T_{\nu,g}(\mu_n-\beta)-T_{\nu,h}(\mu_n-\beta)\|\\ \geq\|T_{\nu,g}(\beta)-T_{\nu,h}(\beta)\|-2\|\mu_n-\beta\|&\geq\|T_{\nu,g}(\beta)\|-0,2\|\beta\|=0,8\|\beta\|>0.\end{aligned}$$

Hence $d(g,h)>0$. Analogically, if $g^{-1}\in AP(\nu_{g,h}|\mu)$, then $d(g,h)>0$.

Conversely. Let d be a metric and $g\neq h$. Then $d(g,h)>0$. Let $\mu_n=\mu_n^1+\mu_n^2$ with $\mu_n^1\ll\mu_{g,h}$, $\mu_n^2\perp\mu_{g,h}$. Then μ_n^2 is concentrated on $I(g,h)$ and therefore $(\mu_n^2)_g=(\mu_n^2)_h$. If $g,h\in S(\mu_{g,h}|\nu)$, then

$$\|T_{\nu,g}(\mu_n)-T_{\nu,h}(\mu_n)\|=\|T_{\nu,g}(\mu_n^1)-T_{\nu,h}(\mu_n^1)+T_{\nu,g}(\mu_n^2)-T_{\nu,h}(\mu_n^2)\|=0.$$

Similarly, if $g^{-1},h^{-1}\in S(\nu_{g,h}|\mu)$, then $\|T_{\mu,g^{-1}}(\nu_n)-T_{\mu,h^{-1}}(\nu_n)\|=0$. Thus $d(g,h)=0$. This contradiction concludes the proof.

4. For $k, l \in \mathbb{N}$ and $\mathbf{m} = (m_1, \dots, m_l), \mathbf{q} = (q_1, \dots, q_l) \in \mathbb{N}^l$, we put

$$U_{k,l,\mathbf{m},\mathbf{q}} =$$

$$\left\{ g \in AP(\mu|\nu) : \|T_{\nu,g}(\mu_n) - \nu_{m_n}\| < \frac{1}{k}, \|T_{\mu,g^{-1}}(\nu_n) - \mu_{q_n}\| < \frac{1}{k}, n = 1, \dots, l \right\}.$$

Select one element in every nonempty set $U_{k,l,\mathbf{m},\mathbf{q}}$. Then we get at most countable set R. It is easy to show that R is dense in $AP(\mu|\nu)$.

5. By Theorem 4.2 [7], the mapping j is a bijection. Denote by d_1 a pseudometric on $AP(\mu|\nu)$ and the corresponding function on $AP(\nu|\mu)$ denoted by d_2. It follows from our construction that

$$d_2(g^{-1}, h^{-1}) = d_1(g, h).$$

Hence d_2 is a pseudometric too, and the mapping j is a homeomorphism (of the spaces with pseudometrics). Notice that if $d_1(g, h)$ is a metric, then $d_2(g, h)$ is a metric too and j is a metric isomorphism.

6. By Theorem 4.2 [7], the mapping i is a bijection. It is clear that $\{(\mu_n)_h\}$ and $\{(\nu_n)_t\}$ forms a dense subset in $L^1(\mu_h)$ and $L^1(\nu_t)$ respectively. Denote the corresponding pseudometric on $AP(\mu_h|\nu_t)$ by d_1. Since $T_{\nu_t, tgh^{-1}}((\mu_n)_h) = T_\nu((\mu_n)_g) = T_{\nu,g}(\mu_n)$ and $T_{\mu_h, hg^{-1}t^{-1}}((\nu_n)_t) = T_{\mu_h}((\nu_n)_{hg^{-1}}) = T_\mu((\nu_n)_{g^{-1}}) = T_{\mu,g^{-1}}(\nu_n)$, then

$$d_1(tg_1h^{-1}, tg_2h^{-1}) = d(g_1, g_2).$$

Hence i is a homeomorphism. In particular, if d is a metric, then d_1 is a metric too, and i is an isometrics. ∎

In the following proposition we study continuity of algebraic operations and elementary topological properties of the sets $A(\mu|\nu), E(\mu|\nu)$ and $I(\mu|\nu)$ in the strong topology.

Proposition 2.2. *Let g, $h \in AP(\mu|\nu)$*

1. *If $g_n \in A(\mu|\nu)$ and $d(g_n, g) \to 0$, then $g \in A(\mu|\nu)$.*

   If $g_n, h_n \in A(\mu), d(g_n, g) \to 0$ and $d(h_n, h) \to 0$, then $d(g_n h_n, gh) \to 0$.

   Let G be a group (this condition is essentially), then

2. *If $g_n \in E(\mu|\nu)[I(\mu|\nu)]$ and $d(g_n, g) \to 0$, then $g \in E(\mu|\nu)[I(\mu|\nu)]$.*

3. *If $g_n, h_n \in E(\mu)[I(\mu)], d(g_n, g) \to 0$ and $d(h_n, h) \to 0$, then $d(g_n h_n, gh) \to 0$.*

4. *If $g_n, g \in E(\mu)[I(\mu)]$ and $d(g_n, g) \to 0$, then $d(g_n^{-1}, g^{-1}) \to 0$.*

In particular, if $d(g,h)$ is a metric, then $A(\mu|\nu)$ is closed in $AP(\mu|\nu)$ in the strong topology. If, in addition, G is a group, then $E(\mu|\nu)$ and $I(\mu|\nu)$ is closed in $AP(\mu|\nu)$ with respect to the strong topology; $A(\mu)$ is a closed topological semigroup; $E(\mu)$ and $I(\mu)$ are closed topological groups.

P r o o f. We prove the proposition assuming that G is a group.

1. Let $g_n \in A(\mu|\nu)$ and $g_n \to g$ in the strong topology. Let $\mu_g = \mu^1 + \mu^2$ with $\mu^1 \ll \nu$, $\mu^2 \perp \nu$. It is necessary to show that $\mu^2 = 0$. In the converse case, let $\gamma > 0$ be a part of $|\mu|$ such that $\gamma_g \ll \mu^2$. Then

$$T_{\nu,g}(\gamma) = 0 \text{ and } \|T_{\nu,g_k}(\gamma) - T_{\nu,g}(\gamma)\| = \|T_{\nu,g_k}(\gamma)\| = \|\gamma\| \not\to 0,$$

which is a contradiction.

Let $g_n \to g$ and $h_n \to h$ with respect to the strong topology. Then $g, h \in A(\mu)$. Let $\varepsilon > 0$ and $\alpha \in L^1(\mu)$. Then $\alpha_h \in L^1(\mu)$. Choose N such that

$$\|T_{\mu,h_n}(\alpha) - T_{\mu,h}(\alpha)\| = \|\alpha_{h_n} - \alpha_h\| < \frac{1}{2}\varepsilon$$

and

$$\|T_{\mu,g_n h}(\alpha) - T_{\mu,gh}(\alpha)\| = \|(\alpha_h)_{g_n} - (\alpha_h)_g\| < \frac{1}{2}\varepsilon$$

for all $n > N$. Then

$$\begin{gathered}\|T_{\mu,g_n h_n}(\alpha) - T_{\mu,gh}(\alpha)\| \\ = \|\alpha_{g_n h_n} - \alpha_{gh}\| \le \|\alpha_{g_n h_n} - \alpha_{g_n h}\| + \|\alpha_{g_n h} - \alpha_{gh}\| < \varepsilon\,, \ \forall n > N.\end{gathered}$$

Hence

$$\lim_{n\to\infty} \|T_{\mu,g_n h_n}(\alpha) - T_{\mu,gh}(\alpha)\| = 0. \tag{2.3}$$

Further, put $\alpha_{g_n^{-1}} - \alpha_{g^{-1}} = \beta_n^1 + \beta_n^2$ with $\beta_n^1 \ll \mu, \beta_n^2 \perp \mu$. By the hypothesis

$$\|T_{\mu,g_n^{-1}}(\alpha) - T_{\mu,g^{-1}}(\alpha)\| = \|\beta_n^1\| \to 0.$$

Assume that $(\beta_n^2)_{h_n^{-1}} = \gamma_n^1 + \gamma_n^2$ with $\gamma_n^1 \ll \mu, \gamma_n^2 \perp \mu$. Let δ_n be the part of β_n^2 such that $(\delta_n)_{h_n^{-1}} = \gamma_n^1$. Then $\delta_n = (\gamma_n^1)_{h_n} \ll \mu$. This contradicts to our choice of β_n^2. Whence $\gamma_n^1 = 0$ and

$$\begin{aligned}\|T_{\mu,h_n^{-1}g_n^{-1}}(\alpha) - T_{\mu,h_n^{-1}g^{-1}}(\alpha)\| &= \|T_\mu((\alpha_{g_n^{-1}} - \alpha_{g^{-1}})_{h_n^{-1}})\| \\ &= \|T_\mu((\beta_n^1)_{h_n^{-1}})\| \le \|\beta_n^1\| \to 0.\end{aligned}$$

Let $\alpha_{g^{-1}} = \alpha^1 + \alpha^2$ with $\alpha^1 \ll \mu, \alpha^2 \perp \mu$. Analogously, we can prove that $\alpha^2_{h_n^{-1}}$ and $\alpha^2_{h^{-1}}$ are mutually singular with μ. Since α^1 is not depended on n, then

$$\|T_{\mu,h_n^{-1}g^{-1}}(\alpha) - T_{\mu,h^{-1}g^{-1}}(\alpha)\| = \|T_{\mu,h_n^{-1}}(\alpha^1) - T_{\mu,h^{-1}}(\alpha^1)\| \to 0.$$

Thus

$$\|T_{\mu,h_n^{-1}g_n^{-1}}(\alpha) - T_{\mu,h^{-1}g^{-1}}(\alpha)\|$$
$$\leq \|T_{\mu,h_n^{-1}g_n^{-1}}(\alpha) - T_{\mu,h_n^{-1}g^{-1}}(\alpha)\| + \|T_{\mu,h_n^{-1}g^{-1}}(\alpha) - T_{\mu,h^{-1}g^{-1}}(\alpha)\| \to 0.$$

By the above and (2.3), we see that $g_n h_n \to gh$ in the strong topology.

2. Let $g_n \in E(\mu|\nu)$ and $g_n \to g$ in the strong topology. Then $g \in A(\mu|\nu)$. Moreover, by Proposition 2.1, g_n^{-1} tends to g^{-1} with respect to the strong topology on $AP(\nu|\mu)$. Since $g_n^{-1} \in E(\nu|\mu)$, then $g^{-1} \in A(\nu|\mu)$. Thus $g \in E(\mu|\nu)$.

Let $g_n \in I(\mu|\nu)$ and $g_n \to g$ in the strong topology. It is proved that $g \in E(\mu|\nu)$. Since

$$\lim_{n\to\infty} \|T_{\nu,g_k}(\mu) - T_{\nu,g}(\mu)\| = \lim_{n\to\infty} \|c \cdot \nu - \mu_g\| = 0,$$

then $\mu_g = c \cdot \nu$ and $g \in I(\mu|\nu)$.

3., 4. If $g_n \to g$ and $h_n \to h$ in the strong topology, then $g,\ h \in E(\mu)[I(\mu)]$ by item 2.

By Proposition 2.1 (5) and item 1, $g_n^{-1} \to g^{-1}$ and $g_n h_n \to gh$ with respect to the strong topology on $E(\mu)$. ■

To prove the main theorem of this section we need three lemmas as follows.

Lemma 2.1. *Let $\{g_n\} \subset AP(\mu|\nu)$ be a fundamental sequence in the strong topology, g_n tends to g with respect to the original topology, $\alpha \ll \mu$ and the following condition is fulfilled*

$$(i)\ \lim_{n\to\infty} \|T_{\nu,g_n}(\alpha)\| = \|T_{\nu,g}(\alpha)\|.$$

Then

$$\lim_{n\to\infty} \|T_{\nu,g_n}(\alpha) - T_{\nu,g}(\alpha)\| = 0. \tag{2.4}$$

P r o o f. We can assume without loss of generality that $\alpha > 0$. Represent α in the form

$$\alpha = \beta^n + \gamma^n = \beta + \gamma, \text{ with } \beta^n_{g_n} \ll \nu, \gamma^n_{g_n} \perp \nu, \beta_g \ll \nu, \gamma_g \perp \nu.$$

Let $\beta^n_{g_n} = f_n\nu$ and $\beta_g = f\nu$. Evidently, $\{\beta^n_{g_n}\}$ is fundamental if and only if $\{f_n\}$ is fundamental in $L^1(\nu)$. Hence there exist the limit $a := \lim_{n\to\infty} \|T_{\nu,g_n}(\alpha)\|$ and a subsequence f_{n_k} which converges to $F(x)$ ν-a.e. Then $a = \|F\| = \|T_{\nu,g}(\alpha)\| = \|f\|$. Clearly, it is enough to prove (2.4) for a subsequence. Thus we can assume without loss of generality that $n_k = k$. It is necessary to prove that $F = f\nu$-a.e. Since $\|F\| = \|f\|$, it is enough to prove that

$$F(x) \leq f(x)\ \nu - \text{a.e.} \tag{2.5}$$

In the converse case, there exists a compact K such that: $\nu(K) > 0$; $F(x) > f(x)$ on $K; \gamma_g(K) = 0$ and $f_n(x)$ converges uniformly to $F(x)$ on K. Since $\alpha_g(K) = \beta_g(K)$, then for some $\varepsilon > 0$ we can find a neighborhood $W \supset g^{-1}K$ such that

$$\alpha(W) < \beta(g^{-1}K) + \varepsilon = \int\limits_K f(x)d\nu + \varepsilon < \int\limits_K F(x)d\nu - \varepsilon.$$

Since $g_n \to g$, we can find N such that

$$g_n^{-1}K \subset W \text{ and } \beta^n(g_n^{-1}K) > \int\limits_K F(x)d\nu - \varepsilon, \ \forall n > N.$$

Then

$$\int\limits_K F(x)d\nu - \varepsilon < \beta^n(g_n^{-1}K) \leq \alpha(W) < \int\limits_K F(x)d\nu - \varepsilon, \ \forall n > N.$$

This contradiction concludes the proof. ∎

R e m a r k 2.1. Condition (i) is important. Really, let $G = X = \mathbb{R}$ and $\mu = \nu = \frac{1}{3}(\delta_0 + \delta_2 + m|_{[-1;1]})$. Then $g_n = 2 - \frac{1}{n} \in AP(\mu|\nu)$, g_n converges to $g = 2 \in AP(\mu|\nu)$ in the original topology. It is easily be checked that $\{g_n\}$ is fundamental in the strong topology and does not satisfy condition (i). It is obvious that (2.4) is fails.

Lemma 2.2. *Let G be a semigroup [group] and $\{g_n\} \subset A(\mu|\nu)[E(\mu|\nu)]$ a fundamental sequence in the strong topology. If g_n converges to g with respect to the initial topology, then $g \in A(\mu|\nu)[E(\mu|\nu)]$.*

P r o o f. Assume the converse and $g \notin A(\mu|\nu)$. Then there exists a part α with a compact support K of the measure μ such that $\alpha_g \perp \nu$. For every natural number n we choose an open set W_n such that $g \cdot K \subset W_n$ and $\alpha_{g_n}(W_n) < 0,1\|\alpha\|$. Put $K_n = K \setminus g_n^{-1}W_n \subset K$. Then K_n is compact and

$$\alpha(K_n) \geq \alpha(K) - \alpha(g_n^{-1}W_n) > 0,9\|\alpha\|.$$

Since the maps $x \mapsto g_k \cdot x$ are continuous on K and converge to the map $x \mapsto g \cdot x$, then there exists $m > n$ such that $g_m \cdot K_n \subset W_n$. In particular, $g_m \cdot K_n \cap g_n \cdot K_n = \emptyset$. Then

$$\|T_{\nu,g_m}(\alpha) - T_{\nu,g_n}(\alpha)\|$$
$$= \|\alpha_{g_m} - \alpha_{g_n}\| \geq (\alpha_{g_m} - \alpha_{g_n})(g_m \cdot K_n) \geq \alpha(K_n) - \alpha_{g_n}(W_n) > 0,8\|\alpha\|.$$

This contradicts to the fact that $\{g_n\}$ is fundamental.

If $\{g_n\} \subset E(\mu|\nu)$, then we have proved that $g \in A(\mu|\nu)$. By Propositions 2.1, 2.2 and Theorem 4.2 [7], $\{g_n^{-1}\} \subset E(\nu|\mu)$ is fundamental in the strong topology on $E(\nu|\mu)$. Thus $g^{-1} \in A(\nu|\mu)$. Hence $g \in E(\mu|\nu)$. ■

Lemma 2.3. *Let G be a semigroup [group], $\{g_n\} \subset A(\mu|\nu)[E(\mu|\nu)]$ a fundamental sequence in the strong topology, and suppose that g_n converges to g with respect to the original topology. Then $g \in A(\mu|\nu)[E(\mu|\nu)]$ and g_n converges to g in the strong topology.*

P r o o f. By Lemma 2.2, $g \in A(\mu|\nu)[E(\mu|\nu)]$. Then $\|T_{\nu,g_n}(\alpha)\| = \|\alpha_{g_n}\| = \|\alpha\| = \|\alpha_g\|, \forall \alpha \ll \mu$. By Lemma 2.1, we have

$$\lim_{n\to\infty} \|T_{\nu,g_n}(\alpha) - T_{\nu,g}(\alpha)\| = 0\,, \quad \forall \alpha \ll \mu,$$

and the lemma is proved for the semigroup case. Let G be a group. It is remain to prove that (2.2) is true for all measures β, $0 < \beta \ll \nu$, i.e. $\lim_{n\to\infty} \|T_{\mu,g_n^{-1}}(\beta) - T_{\mu,g^{-1}}(\beta)\| = 0$,

Represent β in the form

$$\beta = \beta^n + \gamma^n = \beta^0 + \gamma^0 \text{ with } \beta^n_{g_n^{-1}} \ll \mu, \gamma^n_{g_n^{-1}} \perp \mu, \beta^0_{g^{-1}} \ll \mu, \gamma^0_{g^{-1}} \perp \mu.$$

By hypothesis, $\beta^n_{g_n^{-1}}$ is fundamental and thus converges to some measure $\alpha \ll \mu$. Hence

$$\beta^n = \alpha_{g_n} + (\beta^n - \alpha_{g_n}),$$

where $\|\beta^n - \alpha_{g_n}\| = \|\beta^n_{g_n^{-1}} - \alpha\| \to 0$ and $\alpha_{g_n} \to \alpha_g$ (this follows from 2.1, since $\alpha \ll \mu$ and $g_n \in A(\mu|\nu)$). In particular, $\beta^n \to \alpha_g$. Since β^n is the part of β, then $\beta^n = \xi_n\beta$, where ξ_n takes part two values 0 and 1. Then ξ_n converges to a function ξ on some subsequence. The function ξ takes part only values 0 and 1 too. Clearly that $\alpha_g = \xi\beta$. Thus α_g is a part of β. From the proof of Lemma 2.1 (see (2.5)) it follows that $\beta^0 = \eta + \alpha_g$, where η is a part of β and $\eta \perp \alpha_g$. It is necessary to prove that $\eta = 0$, since, in this case, condition (i) of Lemma 2.1 is true.

Let $\eta \neq 0$. Then $\eta_{g^{-1}} \ll \mu$. Thus, by Lemmas 2.1 and 2.2, $(\eta_{g^{-1}})_{g_n} \to (\eta_{g^{-1}})_g = \eta$. Choose and fix a natural number n so much that

$$\|\beta^n - \alpha_g\| < 0,1\|\eta\|\ , \quad \|\eta_{g_n g^{-1}} - \eta\| < 0,1\|\eta\|. \tag{2.6}$$

Choose E_1 and E_2 such that

$$\beta^n|_{E_1} = \alpha_g|_{E_1},\ \beta^n(E_2) < 0,1\|\eta\|,\ \alpha_g(E_2) = 0 \text{ and } \beta^n(X \setminus (E_1 \cup E_2)) = 0.$$

If $\eta = \theta + \theta_1$ with $\theta \ll \eta_{g_n g^{-1}}, \theta_1 \perp \eta_{g_n g^{-1}}$, then (2.6) implies $\|\theta_1\| < 0,1\|\eta\|$. Since θ and β^n are parts of β, then, by our choice of E_2, we have $\beta|_{E_2} = \beta^n|_{E_2}$ and $\theta(E_2) \leq \beta^n(E_2)$. Thus

$$\theta(X \setminus (E_1 \cup E_2)) = \theta(X) - (\theta(E_1) + \theta(E_2)) = \theta(X) - \theta(E_2) \geq \theta(X) - \beta^n(E_2)$$

$$= \eta(X) - \theta_1(X) - \beta^n(E_2) > \|\eta\| - 0,2\|\eta\| = 0,8\|\eta\|.$$

By the above there exists a part λ of η, and hence of β, such that $\lambda \perp \beta^n$ and $\lambda \ll \eta_{g_n g^{-1}}$. But $\lambda_{g_n^{-1}} \ll \eta_{g^{-1}} \ll \mu$ and $\lambda_{g_n^{-1}} \perp \beta^n_{g_n^{-1}}$. This contradicts to our choice of β^n. ■

The following theorem is the main result of this section.

Theorem 2.1. *Let G be a Polish semigroup [group] and let X be a Polish G-space. Let μ and ν be measures on X. Denote by r the metric on G and set d is the pseudometric from definition 2.2. Then $AP(\mu|\nu)$ relative to the metric*

$$\rho(g,h) = \max\{d(g,h), r(g,h)\}$$

is a separable metric space. $A(\mu|\nu)$ [$E(\mu|\nu)$ and $I(\mu|\nu)$] is closed in $AP(\mu|\nu)$ and complete in this metric. If $\mu = \nu$, then $A(\mu)$ is a Polish semigroup [$E(\mu)$ and $I(\mu)$ are Polish groups].

P r o o f. Clearly that $\rho(g,h)$ determines a metric on $AP(\mu|\nu)$. Let $U_{k,l,\mathbf{m},\mathbf{q}}$ are defined in Proposition 2.1 (4). Let $U_{k,l,\mathbf{m},\mathbf{q}}$ be a nonempty set. We can choose a countable set which is dense in $U_{k,l,\mathbf{m},\mathbf{q}}$ in the metric r. Let Q be the union of such sets. Let us show that Q is dense in $AP(\mu|\nu)$.

Let $\varepsilon > 0, g \in AP(\mu|\nu)$. It is easily shown that we can find $U_{k,l,\mathbf{m},\mathbf{q}}$ such that

$$g \in U_{k,l,\mathbf{m},\mathbf{q}} \text{ and } d(g,h) < \varepsilon, \forall h \in U_{k,l,\mathbf{m},\mathbf{q}}.$$

Let $t \in Q \cap U_{k,l,\mathbf{m},\mathbf{q}}$ such that $r(g,h) < \varepsilon$. Then $\rho(g,t) < \varepsilon$.

If $g_n \in A(\mu|\nu)[E(\mu|\nu)]$ and $\rho(g_n, g) \to 0$, then g_n tends to g in the initial topology and $\{g_n\}$ is a fundamental sequence in the strong topology. By Lemma 2.2, $g \in A(\mu|\nu)[E(\mu|\nu)]$. Thus these sets are closed.

If $\{g_n\}$ is a fundamental sequence in the topology which is determined by metric ρ, then $\{g_n\}$ is fundamental in the strong and the initial topologies. Thus $\{g_n\}$ tends to an element $g \in G$. Then $g \in A(\mu|\nu)[E(\mu|\nu)]$ by Lemma 2.3. Hence these sets are complete.

The continuity of the group operations follows from Proposition 2.2.

Assertions for $I(\mu|\nu)$ and $I(\mu)$ are true since these sets are closed in the strong topology. The theorem is proved. ■

R e m a r k 2.2. The condition that G is a group is important. Really. Let $X = [0; +\infty)$. Set

$$T(x) = \begin{cases} x, & x \in [0;1) \\ x-1, & x \in [1;+\infty) \end{cases}, \quad S(x) = \{x\},$$

where $\{x\}$ is the fractional part of x. Put μ is equivalent to Lebesgue measure on X and let G be the semigroup generated by T and S. Since $S^2 = S, T^k S = S$, then $G = \{ST^k, T^k, k = 0, 1, 2 \dots\}$. Thus G with the pointwise topology is a Polish semigroup and has one limit point S. Moreover, the strong topology and the initial one are coincide. Evidently that $T \in E(\mu)$, $S \in A(\mu)$ and T^n converges to S strongly. Hence $E(\mu)$ is not closed.

Later on of this section we will consider $AP(\mu|\nu)$ with the topology generated by metric ρ. In particular, the mapping $g \mapsto T_{\nu,g}(\alpha)$ and the function $g \mapsto T_{\nu,g}(\alpha)(E)$ are continuous, where $\alpha \in L^1(\mu), E \in \mathcal{B}(X)$.

Theorem 2.2. *Let G be a subgroup of a Polish group $H = X$, μ and ν be measures on H. Then $AP(\mu|\nu)$ is a separable metric space with respect to the strong topology. Moreover, $A(\mu|\nu)$, $E(\mu|\nu)$ and $I(\mu|\nu)$ are closed in this topology on $AP(\mu|\nu)$.*

If G is closed, then $A(\mu|\nu)$, $E(\mu|\nu)$ and $I(\mu|\nu)$ are complete. The semigroup $A(\mu)$ and the groups $E(\mu)$ and $I(\mu)$ are Polish. Moreover, the strong topology on $A(\mu|\nu)$ is stronger then the topology induced from H.

P r o o f. Since $I(g,h) = \emptyset$, then $d(g,h)$ is a metric. $A(\mu|\nu)$, $E(\mu|\nu)$ and $I(\mu|\nu)$ are closed by Proposition 2.2.

Let $\{g_n\} \subset A(\mu|\nu)$ $(E(\mu|\nu), I(\mu|\nu))$ be a fundamental sequence with respect to the strong topology. Let us prove that $\{g_n\}$ is fundamental in the initial topology and hence converges to an element $g \in H$. In fact, let $x \in \operatorname{supp} \mu$. Then for every neighborhood U of x the following equality is true:

$$\lim_{m,n\to\infty} \|T_{\nu,g_n}(\mu|_U) - T_{\nu,g_m}(\mu|_U)\| = \lim_{m,n\to\infty} \|(\mu|_U)_{g_n} - (\mu|_U)_{g_m}\| = 0.$$

But this is possible iff $g_n \cdot x$ is fundamental in H. Hence $g_n \cdot x$ converges to an element $y \in H$. Thus $g_n \to yx^{-1} := g$. If G is closed, then $g \in G$. By Lemma 2.3, g_n converges to g with respect to the strong topology. Therefore $A(\mu|\nu)$, $E(\mu|\nu)$ and $I(\mu|\nu)$ are complete.

Now we prove the last assertion. Assume that g_n converges to g in the strong topology. By the above, g_n converges to some element h with respect to the initial topology. Hence, by Lemma 2.3, $g_n \to h$ in the strong topology. Thus $g = h$ and $g_n \to g$ with respect to the strong topology. ∎

R e m a r k 2.3. If G is not closed, then, in general, the strong and initial topologies may be not comparable. For example, let $G = \mathbb{R}, H = \mathbb{T}^2$. Let $p : \mathbb{R} \to \mathbb{T}^2$ be an embedding with the dense image. If $\nu = \mu = m_{\mathbb{T}^2}$, then the strong topology on $\mathbb{T}^2$ coincide with the initial one. Thus the strong topology on $\mathbb{R}$ is induced from $\mathbb{T}^2$. Clearly that one is weaker then the initial topology and is not complete.

The following proposition shows that $AP(\mu|\nu)$ has some local algebraic structure.

Proposition 2.3. *Let G be a group. For all $g \in AP(\mu|\nu)$ there exists a neighborhood V of g such that*

$$h_1 h_2^{-1} g \in AP(\mu|\nu), \quad \forall h_1, h_2 \in V.$$

P r o o f. Let $g \in AP(\mu|\nu)$ and $\mu = \alpha + \gamma$, where $\alpha_g \ll \nu, \gamma_g \perp \nu$. Put $a = \|\alpha\|$. Then the set

$$V = \{h \in AP(\mu|\nu) : \|T_{\nu,h}(\mu) - T_{\nu,g}(\mu)\| < 0,1a, \; \|T_{\mu,h^{-1}}(\alpha_g) - T_{\mu,g^{-1}}(\alpha_g)\| < 0,1a\}$$

is open. Let $h_1, h_2 \in V$. Then α_g can be represented in the form

$$\alpha_g = \alpha_g^1 + \gamma_g^1 \text{ with } \alpha^1_{h_2^{-1}g} \ll \mu, \gamma^1_{h_2^{-1}g} \perp \mu \, (\alpha = \alpha^1 + \gamma^1, \alpha^1 \perp \gamma^1).$$

By the choice of V we have

$$\|\alpha^1_{h_2^{-1}g} - \alpha\| < 0,1a \text{ end } \|\alpha^1_{h_2^{-1}g}\| = \|\alpha^1\| \geq 0,9a. \tag{2.7}$$

Hence

$$\|T_{\nu,h_1h_2^{-1}g}(\alpha^1)\| \geq \|T_{\nu,g}(\alpha^1)\| - \|T_{\nu,h_1h_2^{-1}g}(\alpha^1) - T_{\nu,g}(\alpha^1)\|$$

$$\geq \|T_{\nu,g}(\alpha^1)\| - (\|T_{\nu,h_1}(\alpha^1_{h_2^{-1}g}) - T_{\nu,g}(\alpha^1_{h_2^{-1}g})\| + \|T_{\nu,g}(\alpha^1_{h_2^{-1}g} - \alpha^1)\|) > 0,7a$$

(we take into account the following facts: $\alpha_g^1 \ll \gamma, \|T\| \leq 1, \alpha^1_{h_2^{-1}g} \ll \mu$, (2.7) and our choice of V). Thus $h_1 h_2^{-1} g \in AP(\mu|\nu)$. ■

Our nearest goal is to prove that the topology generated by product of two measures is the product topology. First, we prove the following lemma.

Lemma 2.4. *Let nonzero measures α, $\alpha_1, \dots \in M^+(X)$ and β, $\beta_1, \dots \in M^+(Y)$ be norm restricted. Then the following assertions are equivalent:*

1. *$\alpha_n \times \beta_n \to \alpha \times \beta$ (by the norm);*

2. There exist $0 < a \le k_n \le b < \infty$ *such that*

$$k_n\alpha_n \to \alpha \quad \textit{and} \quad \frac{\beta_n}{k_n} \to \beta.$$

P r o o f. **1.⇒ 2.** Let μ and ν be normalized convex linear hulls of measures $\alpha, \alpha_1, \dots$ and $\beta, \beta_1, \dots$, respectively. Let $\alpha = F\mu, \alpha_n = F_n\mu, \beta = G\nu, \beta_n = G_n\nu$. By hypothesis

$$\int\limits_Y \int\limits_X |F_n(x)G_n(y) - F(x)G(y)| d\mu(x) d\nu(y) \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

Let $S_n(y) = \int_X |F_n(x)G_n(y) - F(x)G(y)| d\mu(x)$. Then they are ν-measurable. Choose $c > 1$ such that $\nu(A) \ne 0$, where $A = \{y : 1/c \le G(y) \le c\}$. By the Chebyshev inequality, there exists constant N (not depending on n) such that the following inequality is true:

$$\nu(S_n \ge N\|S_n\|) \le \frac{1}{2}\nu(A).$$

Thus there exists $y_n \in A$ such that $S_n(y_n) < N\|S_n\|$. Hence, putting $k_n = G_n(y_n)/G(y_n)$, we receive

$$\int\limits_X |F_n(x) \cdot k_n - F(x)| d\mu(x) \le cN\|S_n\|.$$

Then (2.8) implies $F_n(x) \cdot k_n \to F(x)$ or $k_n\alpha_n \to \alpha$.

Analogously, there exist constants d_n such that $d_n \cdot \beta_n \to \beta$. Therefore

$$\alpha_n \times \beta_n = (k_n\alpha_n \times d_n\beta_n) \cdot \frac{1}{k_n d_n} \to \alpha \times \beta \ne 0.$$

Hence $k_n d_n \to 1$. Substituting k_n on $\frac{k_n}{\sqrt{k_n d_n}}$ and d_n on $\frac{d_n}{\sqrt{k_n d_n}}$, we receive a desired sequence (since if $k_{n_l} \to 0$ $(\to \infty)$, then $\alpha = 0$ $(\beta = 0)$ by the norm boundedness of the sequence $\alpha_n(\beta_n)$).

2.⇒ 1. It is followed from the inequality

$$\|\alpha_n \times \beta_n - \alpha \times \beta\| \le \|(k_n\alpha_n - \alpha) \times \frac{\beta_n}{k_n}\| + \|\alpha \times \left(\frac{\beta_n}{k_n} - \beta\right)\|.$$

The lemma is proved. ■

Proposition 2.4. *If* $G = G_1 \times G_2, X = X_1 \times X_2$ *and* $\mu_1 \times \mu_2$, *then the topology on* $AP(\mu) = AP(\mu_1) \times AP(\mu_2)$ *is the product topology.*

P r o o f. By Proposition 5.1 [7] we have $AP(\mu) = AP(\mu_1) \times AP(\mu_2)$.

Let $(g_n, h_n) \to (g, h)$ in $AP(\mu)$. Let us show that $g_n \to g$ in the topology on $AP(\mu_1)$. Clearly $r_1(g_n, g) \to 0$. It is remain to prove that $d_1(g_n, g) \to 0$. Let $0 < \alpha \in L^1(\mu_1)$, $0 < \beta \in L^1(\mu_2)$. Then for every $(g, h) \in AP(\mu)$ the following equality is true:

$$T_{\mu,(g,h)}(\alpha \times \beta) = T_\mu((\alpha \times \beta)_{(g,h)}) = T_\mu(\alpha_g \times \beta_h) = T_{\mu_1}(\alpha_g) \times T_{\mu_2}(\beta_h). \quad (2.9)$$

By Lemma 2.4, there exist $0 < a \le k_n \le b < \infty$ such that

$$k_n T_{\mu_1}(\alpha_{g_n}) \to T_{\mu_1}(\alpha_g).$$

Hence it is enough to prove that $k_n \to 1$. By symmetry with respect to α and β, it is enough to prove that $\overline{\lim} k_n \le 1$. In the converse case, we can assume without loss of generality that $k_n \to c > 1$. Then, by Lemmas 2.4 and (2.9), we have

$$\frac{1}{k_n} T_{\mu_2}(\beta_{h_n}) \to T_{\mu_2}(\beta_h)$$

for every part β of μ_2. Choosing a part β such that $\beta_h \ll \mu_2$, we see that

$$\|\beta\| \leftarrow \|\frac{1}{k_n} T_{\mu_2}(\beta_{h_n})\| \le \frac{1}{k_n}\|\beta\| \to \frac{1}{c}\|\beta\|,$$

which is a contradiction. By symmetry of the definition of the strong topology, it is proved that $d_1(g_n, g) \to 0$. Hence $\rho_1(g_n, g) \to 0$ too.

Conversely, let $\rho_1(g_n, g) \to 0$ and $\rho_2(h_n, h) \to 0$. Clearly that $r((g_n, h_n), (g, h)) \to 0$. Suppose $\gamma \in L^1(\mu)$ has the form $\gamma = \alpha \times \beta$, where $\alpha \in L^1(\mu_1)$, $\beta \in L^1(\mu_2)$. Then (2.9) implies

$$T_{\mu,(g_n,h_n)}(\gamma) = T_{\mu,(g_n,h_n)}(\alpha \times \beta) \to T_{\mu,(g,h)}(\alpha \times \beta) = T_{\mu,(g,h)}(\gamma). \quad (2.10)$$

Since every measure $\gamma \in L^1(\mu)$ admits an approximation by finite sums of measures of the form $\alpha \times \beta$, (2.10) is true in the general case. Thus $\rho((g_n, h_n), (g, h)) \to 0$. ■

R e m a r k 2.4. For a countable product $\mu = \mu_1 \times \mu_2 \times \dots$ the analogical proposition is not true (for example, if μ is a right Gaussian measure on $\mathbb{R}^\infty$). But on finite products $AP(\mu_1) \times \cdots \times AP(\mu_n)$ (which naturally identifies with the closed subsets in $AP(\mu)$ of the forms $AP(\mu_1) \times \cdots \times AP(\mu_n) \times \{e\} \times \dots$) the induced topology from $AP(\mu)$ is coincide with the product topology.

Let us consider some properties of t-ergodic measures.

Proposition 2.5. *Let (p, τ) be a morphism from (G, X) to (H, Y) and $\mu \in M(X)$ t-ergodic. Then $\tau(\mu)$ is t-ergodic.*

P r o o f. Let ν_1 and ν_2 be pats of $\tau(\mu)$. Put α and β are the parts of μ such that $\tau(\alpha) = \nu_1, \tau(\beta) = \nu_2$. Then there exists $g \in G$ such that $\alpha_g \not\perp \beta$. Thus $(\nu_1)_{p(g)} = \tau(\alpha_g) \not\perp \tau(\beta) = \nu_2$. ■

The constructed topology on $AP(\mu|\nu)$ gives an another characterization of t-ergodic measures which explains the word "ergodic". The following proposition is an analog of Proposition 1 [13] (see Proposition 5.5 [9] too).

Proposition 2.6. *Let G be a group and $\mu \in M^+(X)$. Then the following properties are equivalent:*

1. *μ is t-ergodic.*
2. *μ is D-ergodic for every countable subgroup $D \subset G$ such that $D \cap AP(\mu)$ is dense in $AP(\mu)$.*
3. *There exists a countable subgroup $D \subset G$ such that μ is D-ergodic.*

P r o o f. **1. ⇒ 2.** Let $D \cap AP(\mu)$ be dense in $AP(\mu)$. Let B and B' be two disjoint D-invariant subsets of positive measure. Then $T_{\mu,h}(1_{B'}\mu)$ is concentrated on B' for all $h \in D$. Since $D \cap AP(\mu)$ is dense, then $T_{\mu,h}(1_{B'}\mu)$ is concentrated on B' for all $h \in AP(\mu)$ too. Hence $1_B\mu \perp (1_{B'}\mu)_h$ for all $h \in AP(\mu)$ and therefore for all $h \in G$, which is a contradiction.

3. ⇒ 1. Suppose there exist $0 < \alpha \ll \mu$ and $0 < \beta \ll \mu$ such that $\alpha \perp \beta_g$ for every $g \in D$. Let B_g be a Borel set such that α is concentrated on B_g and $\beta_g(B_g) = 0$. Put $B_0 = \cap_g B_g$, $B = \cup_g g^{-1}B_0$. Then B is a D-invariant subset of positive measure and $\beta(B) \le \sum_g \beta_g(B_0) = 0$. This contradiction concludes the proof. ■

Proposition 2.7. *For a finite product of groups of transformations a finite product of t-ergodic measures is t-ergodic.*

P r o o f. Consider the product of two probability measures. Let D_i be countable subgroups which are generated by countable dense subsets of $AP(\mu_i), i = 1, 2$. Then μ_i is D_i-ergodic. Let us consider the countable subgroup $D = D_1 \times D_2$. By Proposition 2.4, $D \cap AP(\mu)$ is dense in $AP(\mu)$. By Proposition 2.6, it is enough to prove that μ is D-ergodic. Let B be a Borel D-invariant set and $\mu(B) > 0$. Let $x_2 \in X_2$. Put $B_{x_2} := \{x_1 : (x_1; x_2) \in B\}$. Let $g_1 \in D_1$. Since $(g_1, e)^{-1}B = B$, then $g_1^{-1}B_{x_2} = B_{x_2}$. Thus we have either $\mu_1(B_{x_2}) = 0$ or $\mu_1(B_{x_2}) = 1$. Set $B_2 := \{x_2 : \mu_1(B_{x_2}) = 1\}$. By the Fubini theorem, we have

$$0 < \mu(B) = \int\limits_{X_2} \mu_1(B_{x_2}) d\mu_2 = \mu_2(B_2).$$

It is remains to show that the set B_2 is D_2-invariant. Let $g_2 \in D_2$. Fixed x_2. Since $(e, g_2)^{-1}B = B$, then $pr_{X_1}B = B_{x_2} = B_{g_2 \cdot x_2}$. Thus $B_2 = g_2^{-1}B_2$. ■

The following proposition is an analog of Lemma 1 [5] (the condition of commutativity is important).

Proposition 2.8. *Let G be Abelian. Let $\alpha \in M^+(G)$ and $\mu \in M^+(X)$ be t-ergodic. Then $\alpha * \mu$ is t-ergodic too.*

P r o o f. By Proposition 2.7, $\alpha \times \mu$ is t-ergodic. Put $\tau(g, x) = g \cdot x$ and $p(g, h) = gh$. Then (p, τ) is a morphism from $(G \times G, G \times X)$ to (G, X). By Lemma 6.1 [7] and Proposition 2.5, $\alpha * \mu = \tau(\alpha \times \mu)$ is t-ergodic too. ■

Notice the following proposition.

Proposition 2.9. *Let G be a group and μ t-ergodic. Let D be a countable subgroup of G such that $D \cap AP(\mu)$ is dense in $AP(\mu)$. If $\nu \in M^+(X)$ and $D \subset E(\nu)$, then we have either $\mu \ll \nu$ or $\mu \perp \nu$.*

P r o o f. Let E be a Borel set such that $|\mu|(E) > 0$ and $\nu(E) = 0$. Then, by the hypothesis, we have $\nu_g(E) = 0, \forall g \in D$. Thus the set $E_0 = \cup_{g \in D} g^{-1}E$ is D-invariant and $\nu(E_0) = 0$. Since μ is t-ergodic, then $\mu(E_0) = 1$. Therefore $|\mu| \perp \nu$. ■

3. Borel type of $AP(\mu|\nu)$

The following theorem is the main result of this section.

Theorem 3.1. *Let (G, X) be a topological semigroup of transformations of a separable metric space X. Let μ and ν be regular probability measures on X. Then*

1. *There exists a Borel function $\rho(x, g) : X \times G \to [0; \infty)$ such that for every fixed G, $\rho(\cdot, g)$ is a density of the absolutely continuous part of μ_g relative to ν.*

2. *The sets $AP(\mu|\nu)$, $A(\mu|\nu)$, $E(\mu|\nu)$, $I(\mu|\nu)$ and $S(\mu|\nu)$ are Borel subsets of G of very bounded types, namely: $AP(\mu|\nu)$ is a $G_{\delta\sigma\delta\sigma}$-set; $A(\mu|\nu)$ is a $G_{\delta\sigma\delta}$-set; $E(\mu|\nu)$ is a $G_{\delta\sigma\delta\sigma\delta}$-set (if G is a group, then $E(\mu|\nu)$ is a $G_{\delta\sigma\delta}$-set); $I(\mu|\nu)$ is intersection of a $G_{\delta\sigma\delta}$-set and a $F_{\sigma\delta\sigma\delta}$-set; $S(\mu|\nu)$ is a $F_{\sigma\delta\sigma\delta}$-set.*

To prove of Theorem 3.1 we use the de Possel theorem. Let us recall it. Set $B^{(i)}$ to be B if $i = 1$ and $X \setminus B$ if $i = 0$. Let $\{B_n\}$ be a base of the topology on X. Put

$$\mathcal{N} = \cup_n \mathcal{N}_n, \text{ where } \mathcal{N}_n = \left\{ B_1^{(i_1)} \cap B_2^{(i_2)} \cap \cdots \cap B_n^{(i_n)},\ \ i_j = 0, 1,\ \ j = 1, \ldots, n \right\}.$$

Denote all distinct nonempty sets from $\mathcal{N}_n$ by $\{A_n^k\}$. Then $\mathcal{N}_n = \{A_n^k\}$ is a *finite* family of G_δ-subsets of the separable metric space X.

Let μ and ν be finite regular measures on X. Then $\mathcal{N}$ is a net for ν. Consider the next sequence of Borel functions

$$f_n(x) = \sum_{k:\nu(A_n^k)\neq 0} \frac{\mu(A_n^k)}{\nu(A_n^k)} \cdot \chi_{A_n^k}(x)\ , \tag{3.1}$$

$$\overline{D}(x) = \overline{\lim}_{n\to\infty} f_n(x)\ , \tag{3.2}$$

where χ_E is the characteristic function of a set E. De Possel theorem ([15, Ch. IV, §10]) asserts that:

if $\rho(x)$ is a density of the absolutely continuous part of μ relative to ν, then we have

$$\lim_{n\to\infty} f_n(x) = \overline{D}(x) = \rho(x) \quad \nu\text{-a.e.} \tag{3.3}$$

Denote by $f_n(x,g)$ and $\overline{D}(x,g)$ the corresponding functions for couple of measures μ_g and ν. Put $I_m = \{k : \nu(A_m^k) \neq 0\}$, then

$$f_m(x,g) = \begin{cases} \frac{\mu_g(A_m^k)}{\nu(A_m^k)}, & \text{for } (x,g) \in A_m^k \times G, \quad \text{if } k \in I_m. \\ 0, & \text{for } (x,g) \in A_m^k \times G, \quad \text{if } k \notin I_m. \end{cases}$$

To prove Theorem 3.1 we need two propositions.

Proposition 3.1. *Let (G, X) be a topological semigroup of transformations of a separable metric space X. Let μ and ν be probability measures on X and $\mathcal{B}(G)$ a σ-algebra of subsets of G. Assume that the following condition is true:*

(i) there is a net $\mathcal{N} = \bigcup_{n=1}^\infty \mathcal{N}_n = \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty \{A_n^k\}$ for ν in $\mathcal{B}$ such that the functions $\mu_g(A_n^k)$ are Borel for every $n, k \in \mathbb{N}$.

Then

1. *There exists a finite nonnegative $\mathcal{B} \times \mathcal{B}(G)$-measurable function $\rho(x,g)$ of two variables such that:*

   1) *For every fixed g the function $\rho(\cdot, g)$ is a density of the absolutely continuous part of μ_g relative to ν.*

2) For all $c \in \mathbb{R}$ the functions

$$h_c^+(g) = \int_X \rho_c^+(x,g)d\nu(x),$$

$$h_c^-(g) = \int_X \rho_c^-(x,g)d\nu(x),$$

$$h(g) = \int_X \rho(x,g)d\nu(x),$$

are Borel, where $\rho(x,g) - c = \rho_c^+(x,g) - \rho_c^-(x,g)$.

3) If $E = \{(x,g) \in X \times G : \ \rho(x,g) > 0\}$, then the function

$$Q(g) = \int_X \chi_E(x,g)d\nu(x)$$

is Borel.

2. The sets $I(\mu|\nu), E(\mu|\nu), A(\mu|\nu), AP(\mu|\nu), S(\mu|\nu)$ are Borel.

P r o o f. We start with proving Statement 1 of our theorem.

1. If we prove that the function $\overline{D}(x,g)$ is $\mathcal{B} \times \mathcal{B}(G)$-Borel and satisfies conditions 1–3, then we receive a required function putting $\rho(x,g) = \overline{D}(x,g)$ if $\overline{D}(x,g)$ is finite and $\rho(x,g) = 1$ if $\overline{D}(x,g)$ is infinite.

First we shall show that $\overline{D}(x,g)$ is Borel.

Since $\overline{D}(x,g) \geq 0$, $\forall (x,g) \in X \times G$, then it is enough to prove that the sets $Q = \{(x,g) : \ \overline{D}(x,g) \geq c\}$ are Borel for all $c > 0$. Since the functions $\mu_g(A_n^k)$ are Borel then the sets

$$L(n,k,\varepsilon,c) = \{g : \ \mu_g(A_n^k) > (c-\varepsilon)\nu(A_n^k)\}$$

are Borel for every $n,k \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}$ (if c is fixed we shall write simple $L(n,k,\varepsilon)$). Thus the sets $A_n^k \times L(n,k,\varepsilon)$ lie in $\mathcal{B} \times \mathcal{B}(G)$. Put

$$Q_\varepsilon = \cap_{n=1}^\infty \cup_{m=n}^\infty \cup_{I_m} A_m^k \times L(m,k,\varepsilon) \quad \text{ and } Q_0 = \cap_{p=1}^\infty Q_{1/p}. \tag{3.4}$$

Clearly that $Q_0 \in \mathcal{B} \times \mathcal{B}(G)$. Let us show that $Q = Q_0$.

Let $(x,g) \in Q_0$. Then $(x,g) \in Q_{1/p}$, $\forall p \in \mathbb{N}$. Thus for every $n \in \mathbb{N}$ there exist $m_n \geq n$ and $k_n \in I_{m_n}$ such that $(x,g) \in A_{m_n}^{k_n} \times L(m_n,k_n,1/p)$, i.e., $x \in A_{m_n}^{k_n}$ and $g \in L(m_n,k_n,1/p)$. This is equivalent to the following inequality

$$f_{m_n}(x,g) = \frac{\mu_g(A_{m_n}^{k_n})}{\nu(A_{m_n}^{k_n})} > c - \frac{1}{p}.$$

Therefore $\overline{D}(x,g) \geq c - \frac{1}{p}$. Since p is arbitrary, then $\overline{D}(x,g) \geq c$. Hence $Q_0 \subset Q$.

Conversely, let $(x,g) \in Q$. Fixed $p \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ there exist $m_n \geq n$ and $k_n \in \mathbb{N}$ such that $x \in A^{k_n}_{m_n}$ and $f_{m_n}(x,g) > c - \frac{1}{p}$. This equivalent to the following inclusion

$$(x,g) \in A^{k_n}_{m_n} \times L(m_n, k_n, 1/p) \subset \cup_{m=n}^{\infty} \cup_{I_m} A^k_m \times L(m,k,1/p).$$

Since n is arbitrary, then $(x,g) \in Q_{1/p}$. Since p is arbitrary too, then $(x,g) \in \cap_p Q_{1/p} = Q_0$. Hence $\overline{D}(x,g)$ is a Borel function on $X \times G$.

It been demonstrated above that for a fixed g the function $\overline{D}(x,g)$ is the density of the absolutely continuous part of μ_g relative to ν. Thus item 1) is proved.

Let us prove item 2). Fixed $c \geq 0$ and put $\overline{D}(x,g) - c = \overline{D}^+_c(x,g) - \overline{D}^-_c(x,g)$. It is enough to prove that the function $h^+_c(g) = \int_X \overline{D}^+_c(x,g)d\nu(x)$ is Borel, since, in this case, the functions $h(g) = h^+_0(g)$ and $h^-_c(g) = h^+_c(g) - h(g) + c$ will be Borel too.

Let $f_n(x,g) - c = f^+_n(x,g) - f^-_n(x,g)$. Since $f_n(x,g) \to \overline{D}(x,g)$, then $f^+_n(x,g) \to \overline{D}^+_c(x,g)$ ν-a.e. for a fixed g. For $\lambda \geq 0$ we put

$$\Lambda_n(\lambda) = \{k \ : \ f_n(x,g) - c \geq \lambda\} = \{k \ : \ \mu_g(A^k_n) \geq (\lambda + c)\nu(A^k_n)\} \cap I_n\},$$

where the last equality is true for $c + \lambda > 0$ (remark that $\Lambda_n(\lambda)$ depends on g). Then

$$\sum_{k \in \Lambda_n(\lambda)} \nu(A^k_n) \leq \frac{1}{\lambda + c} \sum_{k \in \Lambda_n(\lambda)} \mu_g(A^k_n) \leq \frac{1}{\lambda + c} \tag{a}$$

and this inequality is true for all $n \in \mathbb{N}$, $g \in G$, $c + \lambda > 0$.

Thus for $p > 1$ and $c + \lambda > 0$, an application of the Hölder inequality yields

$$\int\limits_{\{x:\ \sqrt[p]{f^+_n(x,g)} \geq \lambda\}} \sqrt[p]{f^+_n(x,g)}d\nu(x) = \sum_{k \in \Lambda_n(\lambda^p)} \sqrt[p]{\frac{\mu_g(A^k_n)}{\nu(A^k_n)} - c} \cdot \nu(A^k_n)$$

$$= \sum_{k \in \Lambda_n(\lambda^p)} \sqrt[p]{\mu_g(A^k_n) - c\nu(A^k_n)} \cdot \sqrt[p]{(\nu(A^k_n))^{p-1}}$$

$$\leq \sqrt[p]{\sum_{k \in \Lambda_n(\lambda^p)} (\mu_g(A^k_n) - c\nu(A^k_n))} \cdot \sqrt[p]{\left(\sum_{k \in \Lambda_n(\lambda^p)} \nu(A^k_n)\right)^{p-1}} \leq \sqrt[p]{\left(\frac{1}{\lambda^p + c}\right)^{p-1}}, \tag{b}$$

where the last inequality is true since the first factor is evidently not larger 1 and for the evaluation of second one we use (a). But for a fixed $p > 1$ the inequality

(b) means that the functions $\sqrt[p]{f_n^+(x,g)}$ are ν-uniformly integrable relative to n ($g \in G$ is fixed too). Therefore, setting $p = (2l+3)/(2l+1)$, we get

$$H_l(g) := \int_X \sqrt[p]{\overline{D}_c^+(x,g)} d\nu(x) = \lim_{n\to\infty} \int_X \sqrt[p]{f_n^+(x,g)} d\nu(x)$$

$$= \lim_{n\to\infty} \sum_{k\in\Lambda_n(0)} \sqrt[p]{\mu_g(A_n^k) - c\nu(A_n^k)} \cdot \sqrt[p]{(\nu(A_n^k))^{p-1}}. \qquad (c)$$

Now we consider the Borel functions

$$R_{nk}^l(g,c) = \max\{\left[\mu_g(A_n^k) - c\cdot\nu(A_n^k)\right]^{\frac{2l+1}{2l+3}} \cdot \left[\nu(A_n^k)\right]^{\frac{2}{2l+3}} ;0\}. \qquad (3.5)$$

Then the sum standing under the limit in (c) is

$$\sum_k R_{nk}^l(g,c) \quad \text{and} \quad H_l(g) = \lim_{n\to\infty} \sum_k R_{nk}^l(g,c). \qquad (d)$$

Thus the function $H_l(g)$ is Borel. Put $\psi(x,g) = \sqrt{\overline{D}_c^+(x,g)} + \overline{D}_c^+(x,g)$. Then: 1) $\psi(x,g)$ is Borel ν-integrable for every fixed $g \in G$ (the function $\sqrt{\overline{D}_c^+(x,g)} \in L^2(\nu)$, and by the Cauchy inequality, it is integrable since ν is finite), 2) $\sqrt[p]{\overline{D}_c^+(x,g)} \le \psi(x,g)$ for every $l \in \mathbb{N}$. Since $\sqrt[p]{\overline{D}_c^+(x,g)} \to \overline{D}_c^+(x,g)$ as $l \to \infty$, then the Lebesgue theorem yields

$$h_c^+(g) = \int_X \overline{D}_c^+(x,g) d\nu(x) = \lim_{l\to\infty} \int_X \sqrt[p]{\overline{D}_c^+(x,g)} d\nu(x) = \lim_{l\to\infty} H_l(g), \quad \forall g \in G.$$

By (d) this equality can be rewrote in the form

$$h_c^+(g) = \lim_{l\to\infty} \lim_{n\to\infty} \sum_k R_{nk}^l(g,c). \qquad (3.6)$$

Thus the function $h_c^+(g)$ is Borel and item 2) is proved.

Set $c = 1$. Clearly that

$$\overline{D}(x,g) > 0 \Leftrightarrow (x,g) \in E = \cup_{l=1}^\infty \cap_{n=1}^\infty \cup_{m=n}^\infty A(m,l),$$

where $A(m,l) = \{(x,g) : \ f_m(x,g) > 1/l\} = \cup_{I_m} A_m^k \times L(m,k,1-1/l)$ is the union of disjoint rectangles and hence is a Borel set.

Evidently, $\chi_E(x,g)$ is the limit of nondecreasing Borel functions $\chi_{E_l}(x,g)$ where $E_l = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A(m,l)$. Since $0 \le \chi_{E_l}(x,g) \le 1$, then apply the Lebesgue theorem to obtain

$$\lim_{l\to\infty} \int\limits_X \chi_{E_l}(x,g)d\nu(x) = \int\limits_X \chi_E(x,g)d\nu(x)$$

for every g. Thus it is enough to show that the functions standing under the limit are Borel. Since $\chi_{E_l}(x,g)$ is the limit of nondecreasing sequence of the bounded functions $\chi_{E_l^n}(x,g)$ where $E_l^n = \cup_{m=n}^{\infty} A(m,l)$, then, by Lebesgue theorem, it is enough to prove that the function

$$\int\limits_X \chi_{E_l^n}(x,g)d\nu(x), \text{ where } E_l^n = \cup_{m=n}^{\infty} A(m,l),$$

is Borel for all l and n.

But $A(m,l) = \cup_{I_m} A_m^k \times L(m,k,1-1/l)$. Therefore E_l^n is a countable union of rectangles. Thus, by the same argument, it is enough to show that the function $\int_X \chi_A(x,g)d\nu(x)$ is Borel, when A is a finite union of rectangles. But such union can be represented as a finite union of disjoint rectangles of the form $A_i = Q_i \times P_i$, $i = 1,\ldots,q$, where $Q_i \in \mathcal{B}$, $P_i \in \mathcal{B}(G)$. Hence $\chi_A(x,g) = \chi_{Q_1}(x)\chi_{P_1}(g) + \cdots + \chi_{Q_q}(x)\chi_{P_q}(g)$. Therefore the function

$$\int\limits_X \chi_A(x,g)d\nu(x) = \chi_{P_1}(g)\int\limits_X \chi_{Q_1}(x)d\nu(x) + \cdots + \chi_{P_q}(g)\int\limits_X \chi_{Q_q}(x)d\nu(x)$$

is Borel. This completes the proof. In particular we have

$$Q(g) = \int\limits_X \chi_E(x,g)d\nu(x) = \lim_{l\to\infty}\lim_{n\to\infty}\int\limits_X \chi_{E_l^n}(x,g)d\nu(x). \tag{3.7}$$

2. Obviously $g \in AP(\mu|\nu)$ if and only if $h(g) > 0$. Thus the set $AP(\mu|\nu)$ is Borel. Hence $S(\mu|\nu) = G \setminus AP(\mu|\nu)$ is Borel too.

Since $g \in A(\mu|\nu)$ iff $h(g) = 1$, then $A(\mu|\nu)$ is Borel.

Clearly $E(\mu|\nu) = A(\mu|\nu) \cap \{g: \ Q(g) = 1\}$. Whence the set $E(\mu|\nu)$ is Borel.

Evidently, $g \in I(\mu|\nu)$ if and only if $\rho(x,g) = 1, \nu$-a.e. This equality is equivalent to the following equalities $h_1^+(g) = h_1^-(g) = 0$ (since $h_1^+(g) > 0$ or $h_1^-(g) > 0$ iff $\rho(x,g) > 1$ or $\rho(x,g) < 1$ on a set of positive measure, but the last is equivalent to the condition $g \notin I(\mu|\nu)$). Therefore the set $I(\mu|\nu)$ is Borel. The proposition is proved. ■

In the following proposition the proofs of items 1 and 2 are standard (see [12, Lemma 3] or [3, Prop. 11, Ch. 9, §2, item 6]), though they are the particular cases of item 3. Our proof of item 4 is more informative then its group analog (see [12, Lemma 4]).

Proposition 3.2. *Suppose a topological semigroup G acts continuously on a topological space X with a regular Borel probability measure μ. Then*

1. *If a set U is open, then the function $\mu_g(U) = \mu(g^{-1}U)$ is lower semicontinuous.*

2. *If a set K is closed, then the function $\mu_g(K) = \mu(g^{-1}K)$ is upper semicontinuous.*

3. *If a set W is open in $X \times G$, then the function*

$$R(g) \equiv \int\limits_X \chi_W(x,g)d\mu(x) = \mu(pr_X(W \cap X \times \{g\}))$$

   i s lower semicontinuous.

4. *The function $\mu_g(E)$ is Borel for every Borel set E.*

P r o o f. **3.** Let $R(g_0) = a > 0$ and $\varepsilon > 0$. Let K be a compact in $pr_X(W \cap X \times \{g_0\})$ such that $\mu(K) > a - \varepsilon$. Since W is open, then for every $x \in K$ there exist neighborhoods $U(x)$ of x and $V(x)$ of g_0 such that $U(x) \times V(x) \subset W$. Since K is compact, then it is covered by some sets $U(x_i), i = 1, \dots, n$. Set $U = \cup_{i=1}^n U(x_i)$ and $V = \cap_{i=1}^n V(x_i)$. Then U is a neighborhood of K, V is a neighborhood of g_0 and $U \times V \subset W$. Thus

$$R(g) \geq \mu(pr_X(U \times V \cap X \times \{g\})) = \mu(U) \geq \mu(K) > a - \varepsilon$$

for every $g \in V$. Note that if we set $\tau(x,g) = g \cdot x, W = \tau^{-1}(E)$ and $E \in \mathcal{B}$, then $pr_X(W \cap X \times \{g\}) = g^{-1}E$. Therefore if we put $E = U$, then item 1 follows from item 3.

4. It is obvious that this assertion is a corollary of the following trivial lemma (putting L is the family of open sets). This lemma gives a structure of the σ-algebra $\mathcal{B}(\mathcal{L})$ generated by the family $\mathcal{L}$ (a structure of a σ-ring see [5, §5, Ex. 9]) and shows that it can be introduced the hierarchy on $\mathcal{B}(\mathcal{L})$ which is analogous to the hierarchy of Borel sets. Put $\omega_0 = \text{Card}\,\mathbb{N}$.

Lemma 3.1. *Let it be done an infinite family $\mathcal{L}$ of subsets of a set X. Suppose the family L is formed by finite intersections and finite unions of elements from $\mathcal{L}$. Set $\mathcal{A}_0 = \{A \setminus C = A \cap (X \setminus C);$ where $A, C \in L\}$. Suppose the family $\mathcal{A}_\alpha$ consists of all countable unions (intersections) of sets from $\cup_{\xi<\alpha}\mathcal{A}_\xi$ for every odd (even) ordinal numbers $\alpha < \omega_1$. Then*

$$\mathcal{B}(\mathcal{L}) = \cup_{\alpha<\omega_1}\mathcal{A}_\alpha \text{ and } Card\mathcal{B}(\mathcal{L}) \leq (Card\mathcal{L})^{\omega_0}. \qquad \blacksquare$$

R e m a r k 3.1. Since every measure μ can be represented in the form $\mu = \mu^+ - \mu^-$ where $\mu^-, \mu^+ \in M^+(X)$, then the function $\mu_g(E) = \mu_g^+(E) - \mu_g^-(E)$ is Borel for all Borel E.

P r o o f o f T h e o r e m 2.1. The first part follows from Proposition 3.1 immediately. Let us prove the second part. Later on all notations are taken from Proposition 3.1. Moreover, we use the next elementary equalities. If $f_n(g)$ converges to $f(g)$ at every point, then

$$\{g : f(g) > d\} = \bigcup_{c=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ g : f_m(g) \overset{(\geq)}{>} d + \frac{1}{c} \right\},$$

$$\{g : f(g) \geq d\} = \bigcap_{c=1}^{\infty} \bigcup_{n=c}^{\infty} \left\{ g : f_n(g) > d - \frac{1}{c} \right\}.$$

Set U_{nk}^r forms a decreasing system of open subsets of X such that $\cap_r U_{nk}^r = A_n^k$. Put

$$F_{nk}^{lr}(g, c) = \max\{ \left[\mu_g(U_{nk}^r) - c \cdot \nu(A_n^k) \right]^{\frac{2l+1}{2l+3}} \cdot \left[\nu(A_n^k) \right]^{\frac{2}{2l+3}} ; 0\}.$$

By Proposition 3.2, the function $F_{nk}^{lr}(g,c)$ is lower semicontinuous. Then $\{F_{nk}^{lr}(g,c)\}$ is a decreasing sequence which converges to $R_{nk}^l(g,c)$. For a fixed n the index k takes a *finite* set of values only. Thus the function $Z_n^{lr}(g,c) := \sum_k F_{nk}^{lr}(g,c)$ is correctly defined and lower semicontinuous. Moreover, $\{Z_n^{lr}(g,c)\}$ is decreasing by r and converges to the function $\sum_k R_{nk}^l(g,c)$. Then (3.6) implies

$$h_c^+(g) = \lim_{l\to\infty} \lim_{n\to\infty} \lim_{r\to\infty} Z_n^{lr}(g,c).$$

Since $h(g) + h_c^-(g) = h_c^+(g) + c$, then $\{h(g) > \frac{1}{a}\} \subset \left\{ h_{\frac{1}{2a}}^+(g) > \frac{1}{2a} \right\}$. Therefore $\{h(g) > 0\} \subset \bigcup_{a=1}^{\infty} \left\{ h_{\frac{1}{a}}^+(g) > \frac{1}{a} \right\}$. Conversely, if $h_{\frac{1}{a}}^+(g) > \frac{1}{a}$, then $\rho_{\frac{1}{a}}^+(x,g) > 0$ on a set of ν-positive measure. All the more $\rho(x,g) > \frac{1}{a}$ on this set and $h(g) > 0$. Thus the following equality is true:

$$\{h(g) > 0\} = \bigcup_{a=1}^{\infty} \left\{ h_{\frac{1}{a}}^+(g) > \frac{1}{a} \right\}.$$

By this equality and nonincreasing $Z_n^{lr}(g,c)$ with respect to r, we get

$$AP(\mu|\nu) = \{g : \ h(g) > 0\} = \bigcup_{a=1}^{\infty} \left\{ h_{\frac{1}{a}}^+(g) > \frac{1}{a} \right\}$$

$$= \bigcup_{a=1}^{\infty}\left\{\bigcup_{b=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{q=l}^{\infty}\left\{g:\ \lim_{n\to\infty}\lim_{r\to\infty} Z_n^{qr}\left(g,\frac{1}{a}\right) > \frac{1}{a}+\frac{1}{b}\right\}\right\}$$

$$= \bigcup_{a=1}^{\infty}\left\{\bigcup_{b=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{q=l}^{\infty}\left[\bigcup_{c=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left\{g:\ \lim_{r\to\infty} Z_m^{qr}\left(g,\frac{1}{a}\right) > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}\right]\right\}$$

$$= \bigcup_{a=1}^{\infty}\left\{\bigcup_{b=1}^{\infty}\bigcup_{l=1}^{\infty}\bigcap_{q=l}^{\infty}\left[\bigcup_{c=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left[\bigcap_{r=1}^{\infty}\left\{g:\ Z_m^{qr}\left(g,\frac{1}{a}\right) > \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}\right]\right]\right\}.$$

Hence $AP(\mu|\nu)$ is a $G_{\delta\sigma\delta\sigma}$-set and $S(\mu|\nu)$ is a $F_{\sigma\delta\sigma\delta}$-set.

Now we prove the equality

$$\bigcap_{a=1}^{\infty}\left\{g:\ h(g) > 1-\frac{1}{a}\right\} = \bigcap_{a=1}^{\infty}\left\{g:\ h^+_{\frac{1}{a}}(g) \geq 1-\frac{1}{2a}\right\}.$$

If $h(g) > 1-\frac{1}{a}$, then $h^+_{\frac{1}{a}}(g)+\frac{1}{a} > 1-\frac{1}{a}$ and $h^+_{\frac{1}{a}}(g) > 1-\frac{1}{2a}$. This proves the inclusion "$\subset$". Conversely, let $h^+_{\frac{1}{a}}(g) \geq 1-\frac{1}{2a}$ and $\rho^+_{\frac{1}{a}}(x,g) > 0$ on a set E. Then

$$h(g) \geq \int_E \rho(x,g)d\nu = \int_E \left(\rho^+_{\frac{1}{a}}(x,g)+\frac{1}{a}\right)d\nu \geq 1-\frac{1}{2a}+\frac{1}{a}\nu(E) > 1-\frac{1}{2a}.$$

This proves the inclusion "$\supset$".

Further, since $h(g)\leq 1$ and $Z_m^{lr}(g,c)$ is nonincreasing with respect to r, then

$$A(\mu|\nu) = \{g:\ h(g)=1\} = \bigcap_{c=1}^{\infty}\left\{g:\ h(g) > 1-\frac{1}{c}\right\} = \bigcap_{c=1}^{\infty}\left\{g:\ h^+_{\frac{1}{c}}(g) \geq 1-\frac{1}{2c}\right\}$$

$$= \bigcap_{c=1}^{\infty}\left\{\bigcap_{a=1}^{\infty}\bigcup_{l=a}^{\infty}\left\{g:\ \lim_{n\to\infty}\lim_{r\to\infty} Z_n^{lr}\left(g,\frac{1}{c}\right) > 1-\frac{1}{2c}-\frac{1}{a}\right\}\right\}$$

$$= \bigcap_{c=1}^{\infty}\left\{\bigcap_{a=1}^{\infty}\bigcup_{l=a}^{\infty}\left[\bigcup_{b=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}\left[\bigcap_{r=1}^{\infty}\left\{g:\ Z_m^{lr}\left(g,\frac{1}{c}\right) > 1-\frac{1}{2c}-\frac{1}{a}+\frac{1}{b}\right\}\right]\right]\right\}.$$

Thus $A(\mu|\nu)$ is a $G_{\delta\sigma\delta}$-set.

Since $E(\mu|\nu) = A(\mu|\nu)\cap\{g:\ Q(g)=1\}$, it is enough to prove that the set $H := \{g:\ Q(g)=1\}$ is a $G_{\delta\sigma\delta\sigma\delta}$-set. Since χ_{E_l} is nondecreasing, then by (3.7) and the Lebesgue theorem, we get

$$H = \bigcap_{a=4}^{\infty}\bigcup_{l=1}^{\infty}\left\{g:\ \lim_{n\to\infty}\int_X \chi_{E_l^n}(x,g)d\nu(x) > 1-\frac{1}{a}\right\}$$

$$= \bigcap_{a=3}^{\infty} \bigcup_{l=1}^{\infty} \left[\bigcap_{n=1}^{\infty} \left\{ g: \int_X \chi_{E_l^n}(x,g) d\nu(x) > 1 - \frac{1}{a} \right\} \right],$$

(since $\chi_{E_l^n}(x,g)$ is nonincreasing) where (recall that $c = 1$)

$$E_l^n = \cup_{m=n}^{\infty} A(m,l) = \cup_{m=n}^{\infty} \cup_{k \in I_m} A_m^k \times L(m,k,1-1/l). \tag{3.8}$$

Since $A_m^k = \cap_r U_{nk}^r$ and

$$L(m,k,1-1/l) = \left\{ g: \ \mu_g(A_m^k) > \frac{1}{l} \nu(A_m^k) \right\}$$
$$= \bigcup_{t=1}^{\infty} \bigcap_{r=1}^{\infty} \left\{ g: \ \mu_g(U_{mk}^r) > \left(\frac{1}{l} + \frac{1}{t} \right) \nu(A_m^k) \right\},$$

then

$$A_m^k \times L(m,k,1-1/l) = \bigcup_{t=1}^{\infty} \bigcap_{r=1}^{\infty} \left(U_{mk}^r \times \left\{ g: \ \mu_g(U_{mk}^r) > \left(\frac{1}{l} + \frac{1}{t} \right) \nu(A_m^k) \right\} \right).$$

Denote the set standing in the round brackets by V_{mk}^{rt}. By Proposition 3.2, this set is open. Taking into account (3.8), we get

$$E_l^n = \bigcup_{m=n}^{\infty} \bigcup_{k \in I_m} \bigcup_{t=1}^{\infty} \left[\bigcap_{r=1}^{\infty} V_{mk}^{rt} \right] = \bigcup_{q=1}^{\infty} E_l^{nq},$$

where $E_l^{nq} = \bigcup_{m=n}^{n+q} \bigcup_{k \in I_m} \bigcup_{t=1}^{q} \left[\bigcap_{r=1}^{\infty} V_{mk}^{rt} \right]$ — form an increasing sequence of G_δ-sets. Let $E_l^{nq} = \cap_{s=1}^{\infty} W_{sl}^{nq}$, where W_{sl}^{nq} is a decreasing sequence of open subsets of $X \times G$. Then

$$\chi_{E_l^n}(x,g) = \lim_{q \to \infty} \lim_{s \to \infty} \chi_{W_{sl}^{nq}}(x,g).$$

Taking into account of the character of convergence and the Lebesgue theorem, we get

$$H = \bigcap_{a=2}^{\infty} \bigcup_{l=1}^{\infty} \left[\bigcap_{n=1}^{\infty} \left[\bigcup_{q=1}^{\infty} \left[\bigcap_{s=1}^{\infty} \left\{ g: \int\limits_X \chi_{W_{sl}^{nq}}(x,g) d\nu(x) > 1 - \frac{1}{a} \right\} \right] \right] \right].$$

By Proposition 3.2, the set standing in the figure brackets is open. Thus H is a set of the type $G_{\delta\sigma\delta\sigma\delta}$.

If G is a group, then $E(\mu|\nu) = A(\mu|\nu) \cap [A(\mu|\nu)]^{-1}$ [7, Theorem 1.2]. Therefore $E(\mu|\nu)$ is a $G_{\delta\sigma\delta}$-set.

Clearly that

$$g \in I(\mu|\nu) \Leftrightarrow \rho(x,g) = 1 \ \nu - \text{a.e.} \quad \Leftrightarrow g \in \left\{ g: \ h_1^-(g) = 0 \right\} \bigcap \left\{ g: \ h_1^+(g) = 0 \right\}.$$

If $h_1^+(g)=0$, then $h_1^-(g)=h_1^+(g)-h(g)+1$ equal to zero if and only if $h(g)=1$. Hence $I(\mu|\nu)=A(\mu|\nu)\cap\{g:\ h_1^+(g)=0\}$. Since $\{g:\ h_1^+(g)>0\}$ is a $G_{\delta\sigma\delta\sigma}$-set (see the proof for $AP(\mu|\nu)$), then $I(\mu|\nu)$ is the intersection of the $G_{\delta\sigma\delta}$-set and the $F_{\sigma\delta\sigma\delta}$-set. Theorem is proved. ■

R e m a r k 3.2. If G is a separable metric semigroup, then $\rho(x,g)$ is a function of the $G_{\delta\sigma\delta\sigma}$-type, i.e., the inverse image of an open set is the $G_{\delta\sigma\delta\sigma}$-set.

Really. Let U_{nk}^r be open sets in X such that $\cap_r U_{nk}^r=A_n^k$. Then

$$L(n,k,\varepsilon)=\cup_{b=1}^{\infty}\cap_{r=1}^{\infty}\left\{g:\ \mu_g(U_{nk}^r)>\left(c-\varepsilon+\frac{1}{b}\right)\nu(A_n^k)\right\}.$$

By Proposition 3.2, this set is a $G_{\delta\sigma}$-set. Thus $A_m^k\times L(m,k,\varepsilon)$ is a $G_{\delta\sigma}$-set and its complement is a $F_{\sigma\delta}$-set.

Formula (3.4) follows

$$\{(x,g):\ \overline{D}(x,g)>c\}=\cup_{q=1}^{\infty}\cap_{p=1}^{\infty}\cap_{n=1}^{\infty}\cup_{m=n}^{\infty}\cup_{I_m}A_m^k\times L\left(m,k,\frac{1}{p},c+\frac{1}{q}\right),$$

$$\{(x,g):\ \overline{D}(x,g)<c\}=\cup_{p=1}^{\infty}\cup_{n=1}^{\infty}\cap_{m=n}^{\infty}\cap_{I_m}(X\times G)\setminus\left(A_m^k\times L\left(m,k,\frac{1}{p},c\right)\right),$$

$$\{(x,g):\ \overline{D}(x,g)=\infty\}=\cap_{c=1}^{\infty}\cap_{p=1}^{\infty}\cap_{n=1}^{\infty}\cup_{m=n}^{\infty}\cup_{I_m}A_m^k\times L\left(m,k,\frac{1}{p},c\right).$$

Thus, taking into account that G is separable and metric, all these sets and their intersections are $G_{\delta\sigma\delta\sigma}$-sets.

Let U be open. Then $U=\cup_a(c_1^a;c_2^a)$, where $c_2^a\le c_1^{a+1}$. Hence (if $1\in U$)

$$\rho^{-1}(U)=\cup_a\{(x,g):\ c_1^a<\overline{D}(x,g)<c_2^a\}\ (\cup\{(x,g):\ \overline{D}(x,g)=\infty\})$$

is a $G_{\delta\sigma\delta\sigma}$-set.

R e m a r k 3.3. Let G be a separable metric group, $\mu=\nu$ and $I=\{(x,g):\ g\cdot x=x\}$. Then we may assume that $\rho(x,g)$ satisfies the following conditions:

1. $\rho(x,g)$ is the function of at most $G_{\delta\sigma\delta\sigma\delta}$-type.
2. $\rho(x,g)=1,\ \forall(x,g)\in I$.
3. $\rho(x,g)>0,\ \forall g\in E(\mu)$.

In particular, if $G=E(\mu)$, then $\ln\rho(x,g^{-1})$ is at most $G_{\delta\sigma\delta\sigma\delta}$-type cocycle.

Really. Since I is closed in $X\times G$, then $I^g=\{x:\ (x,g)\in I\}\in\mathcal{B}(X)$. Evidently, for all $E\subset I^g$ we have $g^{-1}E=E$. Thus $\mu_g|_{I^g}=\mu|_{I^g}$. Hence $\rho(x,g)=1,\mu$-a.e. on I^g. By Remark 3.2 and Theorem 3.1, the set

$$A=\rho^{-1}(0)\cap X\times E(\mu)$$

is a $F_{\sigma\delta\sigma\delta}$-set. Put

$$\tilde{\rho}(x,g) = \rho(x,g) \text{ for } (x,g) \notin I \cup A, \text{ and } \tilde{\rho}(x,g) = 1, \text{ for } (x,g) \in I \cup A.$$

Thus, if $1 \notin U$ (if $1 \in U$), then

$$\tilde{\rho}^{-1}(U) = \rho^{-1}(U)\left\{(X \times G) \setminus (I \cup A)\right\} \quad \left[\tilde{\rho}^{-1}(U) = \rho^{-1}(U) \cup I \cup A\right]$$

is a $G_{\delta\sigma\delta\sigma\delta}$-set. The cocycle inequality follows from proposition 3.2 [7]. ■

For the set $I(\mu|\nu)$ Theorem 3.1 may be improved.

Proposition 3.3. *Let G and X be separable metric spaces. Then the set $I(\mu|\nu)$ is closed in G.*

P r o o f. By Proposition 3.2, the function $\mu_g(K)$ is upper semicontinuous. Thus, if g_n tends to g, then

$$\overline{\lim}_{n\to\infty}\mu_{g_n}(K) \le \mu_g(K).$$

Therefore, by Theorem 2.1 [2], for every bounded real continuous function $f(x)$ (and since $\mu_{g_n} = c\nu$) we get

$$c\int f d\nu = \int f d\mu_{g_n} \to \int f d\mu_g.$$

Since every measure is determined by its values on such functions completely, then $\mu_g = c\nu$ and $g \in I(\mu|\nu)$. ■

Moreover, we can repeat the proof of Theorem 1.2.4 [8] word for word and prove the following proposition.

Proposition 3.4. *If $X = G$ is a separable metric group, then the subgroups $I_l(\mu)$, $I_r(\mu)$, $I_t(\mu)$ are compact.*

R e m a r k 3.4. In the general case this proposition is not true. In fact, let $G = \mathbb{R}, H = \mathbb{T}^2$ and $p : \mathbb{R} \to \mathbb{T}^2$ be an embedding with the dense image. Evidently, if $\mu = m_{\mathbb{T}^2}$, then $I_l(\mu) = \mathbb{R}$.

R e m a r k 3.5. Note that, if $X = G$, then the representation of $E(\mu)$ in $\mathbf{U}(L^2(\mu))$ determined by the equality

$$S_g(f)(x) = \sqrt{\frac{d\mu_g}{d\mu}(x)} f\left(g^{-1}\cdot x\right) = \sqrt{\rho(x,g)} f\left(g^{-1}\cdot x\right),$$

is exact.

Let G be a group, then the group $E(\mu)$ plays special role among considerable sets. It is naturally to raise the question of description such groups. According to Theorem 3.1, $E(\mu)$ is a set of very bounded type. Now we give some open questions.

1. *For which subgroups $H \subset G$ there exists $\mu \in M^+(G)$ such that $H = E(\mu)$?*

2. *What is the Borel class of $E(\mu)$ exactly?*

For example, let ω be a right Gauss measure on $\mathbb{R}^\infty$. Then $E(\omega) = l^2$ [16, § 5, Th. 1]. Let us prove that l^2 is a $F_\sigma \setminus G_\delta$-set.

Really, let $f_n(\mathbf{x}) = x_1^2 + \cdots + x_n^2, \mathbf{x} = (x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$, be continuous functions on $\mathbb{R}^\infty$. Since

$$l^2 = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{\mathbf{x} : f_n(\mathbf{x}) \le k\},$$

then l^2 is a F_σ-set. In the other side, l^2 is not G_δ-set by the results of [10, Ch. VI, § 34].

3. *What are the classes for which there exists a group of quasiinvariance of a probability measure?*

For Polish groups the Mackey–Weil theorem [11] may be formulated the following way.

Theorem. *Let $X = G$ be a Polish group. Then the following propositions are equivalent:*

1. There exists a measure μ such that $E(\mu)$ is open.

2. G is local compact.

All examples of $E(\mu)$ which are known to the author are sets of F_σ-type.

4. *How are connected properties of G with Borel classes of all $E(\mu)$?*

Proposition 3.5. *Let $X = G$ be a Polish group. Then the following propositions are equivalent*

1. $E(\mu)$ is a set of G_δ-type for all probability measures μ.

2. G is discrete.

P r o o f. Obviously, it is enough to prove the sufficiency. Let H be a countable dense subgroup of G. Set $\mu = \sum_{h\in H} \alpha_h \delta_h$, $\sum_{h\in H} \alpha_h = 1$, $\alpha_h > 0$. Then $E(\mu) = H$. By [10, § 34, Th. 3 and § 9, Th. 4], H is a G_δ-set only if $G = H$. ■

Naturally, this questions are considered for the most important cases when G is either local compact or Abelian or $X = G$.

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