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Massiveness of Exceptional Sets of Multi-Term Asymptotic Representations of Subharmonic Functions in the Plane

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It was investigated the massiveness of exceptional sets that arise in multi-term asymptotic representations of subharmonic functions. It was shown that such exceptional sets be any $C_{0,1+\gamma}$-sets where $\gamma \in [0,1]$. This fact distinguishes the case of n-term asymptotic from the case of functions of completely regular growth.

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In the recent decades a number of new effects has been found in the studies of connection between the behavior of a subharmonic function at infinity and the growth of the distribution function of its Riesz measure in the terms of multi-term asymptotic representations. These facts show the essential differences between the multi-term asymptotics and the classical case of the single-term asymptotic representation (the functions of completely regular growth of Levin–Pflüger).

As known [6], the exceptional sets, that appeared in the main theorem of the function theory of completely regular growth, can be reduced up to $C_{0,0}$-sets.

Recall that a set $E \subset C$ is the $C_{0,\alpha}$-set, $\alpha > 0$, if it can be covered by the disks $\{z : |z - z_j| < r_j\}$ such that

$$\lim_{R\to\infty} \frac{1}{R^\alpha} \sum_{|z_j|\le R} r_j^\alpha = 0. \tag{1}$$

If limit (1) is zero for every positive α, then E is a $C_{0,0}$-set.

It was shown in [3] and [5] that the exceptional sets arising in investigations of n-term asymptotics cannot in general be less than $C_{0,1}$ even if the Riesz masses are concentrated on a finite system of rays.

If the Riesz masses are distributed on the plane arbitrarily, then the existence of the asymptotics of a desired form can be guaranteed only outside $C_{0,2}$-set ([4, 5]).

Then there arises a question about connection between the distribution of the Riesz masses in the plane and the massiveness of the exceptional sets. The question was put by V. Logvinenko after the author's papers ([1, 2]) on the asymptotic behavior of a subharmonic function with the Riesz measure in a parabolic domain had been published.

In this article we prove the existence of exceptional $C_{0,\alpha}$-sets, $\alpha \in [0,1]$, for the case of multi-term asymptotic representations. This provides one more difference of the cases of a single- and multi-term asymptotic representations. Thus a complete description of the massiveness of exceptional sets appeared in multi-terms asymptotic representations of subharmonic functions in the plane is given. This result is obtained by using the reasoning from [4]. Note that we omit the calculations similar to those in [4]. We will assume that the reader is familiar with the articles [3, 4], and we will point out the aspects of proofs containing the differences.

Before formulation of results let us give the notations and the definitions which will be used below.

We say that a function $f(t), t > 0$ has multi-term (n-term) asymptotics as $t \to \infty$, if f can be represented in the form

$$f(t) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + \ldots + \Delta_n t^{\rho_n} + \varphi(t),$$

where Δ_j, $j = 1, 2, \ldots, n$, are real constants; $0 < [\rho_1]^* < \rho_n < \ldots < \rho_1$, and the function $\varphi(t)$ is small in a certain sense in comparison with the previous term. Similarly, we understand the expression "polynomial asymptotics of a function $f(z)$, $z \to \infty$". In the later case the coefficients Δ_j, $j = 1, \ldots, n$, are functions only of $\theta = \arg z$ and $t = |z|$.

Let $u(z)$ be a subharmonic function; μ — its measure of Riesz, $\mu(t, \theta) = \mu(\{|z| < t, 0 < \arg z \leq \theta\})$.

Put $G_\omega := \{z : z = re^{i\theta}, 2^k \leq r < 2^{k+1}, |\theta| \leq \operatorname{arctg} 2^{k(\omega-1)}, k = 0, 1, \ldots\}$ where $\omega \in (0, 1)$.

We denote various constants by C and the Lebesgue measure in the plane z by λ_z.

*As usual, $[a]$ is the integral part of a number a.

The technique used in this paper allows us to generalize all the obtained results for the multi-term asymptotic representations. Thus without loss of generality here we consider only the case of two-term asymptotics.

Theorem 1. *Let $u(z)$ be a subharmonic function of noninteger order in the plane C with the support of its Riesz measure concentrated in the domain G_ω. Assume that the following relation for the measure μ is valid*

$$\mu(t,\theta) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + \varphi(t,\theta), \quad \theta \in (0, 2\pi]. \tag{2}$$

Here $p = [\rho_1] < \rho_2 < \rho_1; \Delta_1 > 0; \mu(t,\theta) = 0$ for all θ and $t \le t_0$, and the function φ satisfies the following estimate for some $q \ge 1$

$$\int_R^{2R} \sup_{\theta\in[0,2\pi]} |\varphi(t,\theta)|^q dt = o(R^{\rho_2 q+1}), \;\; R \to \infty. \tag{3}$$

Then the order of the function u is equal to ρ_1 and

$$u(re^{i\theta}) = \Delta_1 \frac{\pi r^{\rho_1}}{\sin \pi\rho_1} \cos \rho_1(\theta - \pi) + \Delta_2 \frac{\pi r^{\rho_2}}{\sin \pi\rho_2} \cos \rho_2(\theta - \pi) + \psi(re^{i\theta}),$$

where $\psi(re^{i\theta}) = o(r^{\rho_2})$, $r \to \infty$, uniformly for $\theta \in [0, 2\pi]$, if the point $z = re^{i\theta}$ does not belong to certain $C_{0,1+\omega}$-set.

If in (3) the number $q > 1$, then

$$\int_{G_\omega \bigcap \{R\le|z|\le 2R\}} |\psi(z)|^q d\lambda_z = o(R^{\rho_2 q+1+\omega}), \;\; R \to \infty.$$

P r o o f. Without loss of generality we can assume that measure μ has infinitely smooth density and its support lies inside G_ω. This follows from the condition $\operatorname{supp}\mu \subseteq G_\omega$ and the proof of [4, Th. 4]. Indeed, the proof of Th. 4 in [4] is based on the fragmentation of the plane into the "collars" (parts of the plane). In these "collars" the density of some measure, "close" in a certain sense to μ, is defined. The density of this new measure is equal to zero in the part of the "collar" adjoining to its boundary.

According to the Riesz–Brelo theorem [7] and conditions (2) and (3), the order of the function u is equal to ρ_1. Moreover, by this theorem the function u represented as represented in the form of $u = J + P$, where

$$J(z) = \operatorname{Re} \int_{\mathbb{C}} \left[\ln\left(1 - \frac{z}{\zeta}\right) + \sum_{k=1}^{p} \frac{1}{k} \left(\frac{z}{\zeta}\right)^k \right] d\mu(\zeta)$$

is the canonical potential of the measure μ, and P is a harmonic polynomial of the degree not greater than p. Since u is the function of noninteger order, then, without loss of generality, we can assume that $P(z) \equiv 0$, i.e., $u \equiv J$ or

$$u(re^{i\theta}) = \lim_{R\to\infty} \int\limits_{\{\zeta:|\zeta|<R\}} \left[\ln\left(1 - \frac{re^{i\theta}}{\zeta}\right) + \sum_{k=1}^{p} \frac{1}{k}\left(\frac{re^{i\theta}}{\zeta}\right)^k \right] d\mu(\zeta)$$

$$= \mathrm{Re} \lim_{R\to\infty} \int\limits_{\theta-2\pi}^{\theta} d\alpha \int\limits_{0}^{R} \left[\ln\left(1 - \frac{re^{i\theta}}{te^{i\alpha}}\right) + \sum_{k=1}^{p} \frac{1}{k}\left(\frac{re^{i\theta}}{te^{i\alpha}}\right)^k \right] \frac{\partial^2 \mu}{\partial t \partial \alpha} dt$$

$$= \mathrm{Re} \lim_{R\to\infty} \left\{ \int\limits_{\theta-2\pi}^{\theta} \left[\ln\left(1 - \frac{re^{i\theta}}{te^{i\alpha}}\right) + \sum_{k=1}^{p} \frac{1}{k}\left(\frac{re^{i\theta}}{te^{i\alpha}}\right)^k \right] \frac{\partial \mu(t,\alpha)}{\partial \alpha}\bigg|_0^R d\alpha \right.$$

$$\left. - \int\limits_0^R \frac{(re^{i\theta})^{p+1}}{t^{p+1}} dt \int\limits_{\theta-2\pi}^{\theta} \frac{e^{-ip\alpha}}{te^{i\alpha} - re^{i\theta}} \cdot \frac{\partial \mu(t,\alpha)}{\partial \alpha} d\alpha \right\}$$

$$= \mathrm{Re} \lim_{R\to\infty} (A(R, re^{i\theta}) + B(R, re^{i\theta})).$$

By integrating by parts the expression $A(R,z)$ and using the condition that $\mu(t,\theta) \equiv 0$ in some neighborhood of the origin, we obtain $A(R, re^{i\theta}) \to 0$ as $R \to \infty$ uniformly for each compact set in z-plane, $z = re^{i\theta}$.

So

$$u(re^{i\theta}) = -\mathrm{Re} \left\{ (re^{i\theta})^{p+1} \int\limits_0^{\infty} \frac{dt}{t^{p+1}} \int\limits_{\theta-2\pi}^{\theta} \frac{e^{-ip\alpha}}{te^{i\alpha} - re^{i\theta}} \cdot \frac{\partial \mu(t,\alpha)}{\partial \alpha} \right\}$$

$$= -\mathrm{Re} \left\{ r^{p+1} \int\limits_0^{\infty} \frac{\mu(t, 2\pi)}{t^{p+1}(t-r)} dt \right.$$

$$\left. + i(re^{i\theta})^{p+1} \int\limits_{\theta-2\pi}^{\theta} e^{-ip\alpha} d\alpha \int\limits_0^{\infty} \frac{(p+1)te^{i\alpha} - pre^{i\theta}}{(te^{i\alpha} - re^{i\theta})^2 t^{p+1}} \mu(t,\alpha) dt \right\}.$$

Let us substitute here expression (2) and take into account that $\mathrm{supp}\, \mu \subseteq G_\omega$. It is not hard to see that the principal terms of the asymptotics u are

$$\Delta_j \frac{\pi r^{\rho_j}}{\sin \pi \rho_j} \cos \rho_j (\theta - \pi), \; j = 1, 2.$$

Thus we have only to investigate the behavior of the remainder term ψ:

$$\psi(re^{i\theta}) = -\operatorname{Re}\left\{ r^{p+1}\int\limits_0^\infty \frac{\varphi(t,2\pi)}{t^{p+1}(t-r)}dt \right.$$

$$\left. +i(re^{i\theta})^{p+1}\int\limits_{\theta-2\pi}^{\theta} e^{-ip\alpha}d\alpha \int\limits_0^\infty \frac{(p+1)te^{i\alpha}-pre^{i\theta}}{(te^{i\alpha}-re^{i\theta})^2 t^{p+1}}\varphi(t,\alpha)dt \right\}$$

$$= -\operatorname{Re}\{\psi_1(re^{i\theta})+\psi_2(re^{i\theta})\}.$$

Let us estimate the function $\psi_1(re^{i\theta})$. By the Hardy–Littlewood theorem [8] on the bound of the Hilbert transform and the reasoning from [3, Th. 1], we conclude that for $q \in (1,\infty)$

$$\left\{ \int\limits_R^{2R} |\psi_1(re^{i\theta})|^q dr \right\}^{\frac{1}{q}}$$

$$= \left\{ \int\limits_R^{2R} \left| r^{p+1}\int\limits_0^\infty \frac{\varphi(t,2\pi)dt}{t^{p+1}(t-r)} \right|^q dr \right\}^{\frac{1}{q}} = o(R^{\rho_2+\frac{1}{q}}),\ R\to\infty. \tag{4}$$

If $q \in [1,\infty)$ then the set

$$e_1 = \left\{ r\in[0,\infty) : \left| r^{p+1}\int\limits_0^\infty \frac{\varphi(t,2\pi)dt}{t^{p+1}(t-r)} \right| > \tau(r)r^{\rho_2} \right\},$$

where the function $\tau(r)$ tends to zero sufficiently slowly, has the zero relative Lebesgue measure. Since $\operatorname{supp}\mu \subseteq G_\omega$, then the estimate

$$|\psi_1(z)| \le \tau(|z|)|z|^{\rho_2}$$

is valid outside the set $E_1 = \{z : |z| \in e_1, z \in G_\omega\}$. It is easy to see that E_1 is $C_{0,1+\omega}$-set.

To estimate the function ψ_2 we split the ray $[1,\infty)$ into semi-intervals $[2^k, 2^{k+1})$, $k = 0,1,\ldots$. For $r \in [2^k, 2^{k+1})$ we have

$$\psi_2(re^{i\theta}) = i\left\{ (re^{i\theta})^{p+1}\left(\int\limits_0^{2^{k-1}} + \int\limits_{2^{k-1}}^{2^{k+2}} + \int\limits_{2^{k+2}}^{\infty} \right) \frac{dt}{t^{p+1}} \right.$$

$$\times \int\limits_{\theta-2\pi}^{\theta} \frac{(p+1)te^{i\alpha} - pre^{i\theta}}{e^{ip\alpha}(te^{i\alpha} - re^{i\theta})^2} \varphi(t,\alpha)d\alpha \Bigg\}$$

$$= i(J_1^{(k)} + J_2^{(k)} + J_3^{(k)}).$$

From (3) it is easy to conclude that $|z|^{-\rho_2}(|J_1^{(k)}| + |J_3^{(k)}|) \to 0$ uniformly for $\theta = \arg z \in [0, 2\pi]$ as $|z| \to \infty$.

Let us represent the integral $J_2^{(k)}$ as a sum of two integrals estimated similarly. So we consider only one of them, namely:

$$\tilde{\psi}_2(re^{i\theta}) = (re^{i\theta})^{p+1} \int\limits_{2^{k-1}}^{2^{k+2}} \frac{dt}{t^p} \int\limits_{\theta-2\pi}^{\theta} \frac{p+1}{e^{i(p-1)\alpha}(te^{i\alpha} - re^{i\theta})^2} \varphi(t,\alpha)d\alpha.$$

Put

$$\Phi_k(\zeta) = (p+1)e^{2i\arg\zeta}\zeta^{-(p+1)}\varphi(\zeta)\chi_k(\zeta),$$

where $\chi_k(\zeta)$ is a characteristic function of the ring $\{\zeta : 2^{k-1} \le |\zeta| \le 2^{k+2}\}$. As it follows from the definition of the domain G_ω and estimate (3),

$$\left(\int\limits_{G_\omega \bigcap \{z: 2^k \le |\zeta| \le 2^{k+1}\}} |\Phi_k(\zeta)|^q d\lambda_\zeta \right)^{\frac{1}{q}} = o\left(2^{k(\rho_2 + \frac{1+\omega}{q} - (p+1))}\right), \ k \to \infty. \tag{5}$$

In these notations we have

$$\tilde{\psi}_2(z) = z^{p+1} \int\limits_{|\zeta| \in \mathbb{C}} \frac{\Phi_k(\zeta)}{(z-\zeta)^2} d\lambda_\zeta, \ z = re^{i\theta}. \tag{6}$$

To estimate this function we use the following fact*, which is a special case of [8, Th. 4, p. 56].

Theorem A. *Let $f \in L^p(R^2), 1 \le p < \infty$. Then a transformation*

$$T_\varepsilon(f)(z) = \int\limits_{|\zeta| \ge \varepsilon} \frac{f(\zeta)}{(z-\zeta)^2} d\lambda_\zeta, \quad \varepsilon > 0,$$

has the following properties:

a) $\lim\limits_{\varepsilon \to 0} T_\varepsilon(f)(z)$ *exists for almost all* z.

*The reference to this result is missing in [4].

b) Let $T^*(f)(z) = \sup_{\varepsilon>0} |T_\varepsilon(f)(z)|$. *If* $f \in L^1(R^2)$ *then*

$$\text{mes}\{z : |T^*(f)(z)| > \eta\} \ \le \frac{C||f||_{L^1}}{\eta},\ \eta > 0,$$

where the constant C *does not depend on* f *and* η.

c) If $1 < q < \infty$ *then* $||T^*f||_q \le C_q||f||_q$, *where the constant* C_q *depends only on the number* q.

By Theorem A it follows from (5) that for $q \in (1, \infty)$

$$||z^{p+1}\tilde{\psi}_2(z)||_{L^q(G_\omega \bigcap\{z:2^k\le|z|\le 2^{k+1}\})} = o\left(2^{k(\rho_2+\frac{1+\omega}{q})}\right),\ k \to \infty, \tag{7}$$

and for $q \in [1, \infty)$ the measure of the set

$$E_k^{(2)} = \left\{z : |z| \in G_\omega \bigcap \{z : 2^k \le |z| \le 2^{k+1}\} : |z^{p+1}\tilde{\psi}_2(z)| > \eta_k|z|^{\rho_2}\right\}$$

satisfies the estimate

$$\text{mes}\, E_k^{(2)} = \eta_k^{-q} o(2^{k(1+\omega)}),\ k \to \infty. \tag{8}$$

If the sequence of numbers $\{\eta_k\}$ tends to zero sufficiently slowly, then it follows from (8) that

$$E_2 = \bigcup_{k=1}^{\infty} E_k^{(2)}$$

is a $C_{0,1+\omega}$-set. Hence $E = E_1 + E_2$ is a $C_{0,1+\omega}$-set.

If $q \in (1, \infty)$ then we obtain by virtue of (7) and (4)

$$\int\limits_{G_\omega \bigcap\{z:R\le|z|\le 2R\}} |\psi(z)|^q d\lambda_z = o(R^{\rho_2 q+1+\omega}),\ R \to \infty.$$

The theorem is proved.

R e m a r k. In [9, p. 5] the transformation (6), that defined the function $\tilde{\psi}_2(z)$ by the function $\Phi_k(\zeta)$, is called the Berling transformation of the function $\Phi_k(\zeta)$.

Now we will show that the massiveness of exceptional sets established in Th. 1 cannot be reduced. Namely, the following fact holds:

Theorem 2. *Let $\omega \in (0,1)$ be some fixed number. Then for any numbers ρ_1 and ρ_2, $\rho_1 > \rho_2$,[*] and a number $\kappa \in (0,\omega)$ there exists a subharmonic function u such that* $\operatorname{supp}\mu \subseteq G_\omega$;

$$\mu(t,\theta) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + \varphi(t,\theta),$$

where $\Delta_1 > 0$ and

$$\int_T^{2T} \sup_{\theta\in[0,2\pi]} |\varphi(t,\theta)|^q dt = o(T^{\rho_2 q+1}),\ T\to\infty, q\geq 1,$$

but the asymptotic representation

$$u(re^{i\theta}) = \sum_{j=1}^{2} \frac{\pi r^{\rho_j}\Delta_j}{\sin\pi\rho_j}\cos\rho_j(\theta-\pi) + \psi(re^{i\theta}),$$

$$\psi(re^{i\theta}) = o(r^{\rho_2}),\ r\to\infty,$$

does not take place uniformly for $\theta\in[0,2\pi]$ on some exceptional set E that is not a $C_{0,1+\kappa}$-set.

P r o o f. We follow the same scheme as in [4, Th. 3].

Let us introduce the following notations: a) $R = 2^{\frac{1}{\rho_1-\rho_2}}$; b) $\delta_k = \frac{R-1}{3\cdot 2^k}$; $\gamma_k = \frac{2\pi}{2^k}$; c) a sequence $\{\eta_k\}$, $k=1,2,\ldots$, tends to zero sufficiently slowly.

We proceed in several steps.

1) We begin with the construction of the Riesz measure μ of the function to be determined. Let the distribution function of the measure μ be

$$\mu(t,\theta) = \Delta_1 t^{\rho_1} + \Delta_2 t^{\rho_2} + \varphi(t,\theta),$$

where $\Delta_1 = 1$ and $\Delta_2 = 0$. We represent the function φ as a sum of two terms φ_1 and φ_2.

Let N be some positive integer which will be chosen later on. Put

$$\varphi_1(t,\theta) = \varphi_2(t,\theta) = 0\ \forall t < R^N,\ \forall\theta\in[0,2\pi].$$

Let $k \geq N$ and $R^k \leq t < R^{k+1}$. Define the function $\varphi_1(t,0)$ in the following way: $\varphi_1(t,0) = \eta_k R^{k\rho_2}$ on the segments $[R^k(1+3\cdot 2^N j\delta_k), R^k(1+(3\cdot 2^N j+1)\delta_k)]$, $j = 0,1,\ldots,2^{k-N}-1$; $\varphi_1(t,0) = 0$ on semi-intervals $[R^k(1+(3\cdot 2^N j+2)\delta_k), R^k(1+3\cdot 2^N(j+1)\delta_k))$, and we define $\varphi_1(t,0)$ as a linear function on the rest of the ray. Next we require the function $\varphi_1(t,\theta)$ to be independent of θ in the angles

[*] This condition is missing in [4] and is taken for granted.

$j\gamma_{k-N} < \theta \leq (j+1)\gamma_{k-N}$, $j = 0, 1, \ldots, \left[\frac{\operatorname{arctg} 2^{k(\omega-1)}}{\gamma_{k-N}}\right] - 1$. On the rays $\theta = j\gamma_{k-N}$ we define this function in the same way as on the ray $\theta = 0$.

We define φ_2 as

$$\varphi_2(t,\theta) = -\frac{\varphi_1(t,0)}{\gamma_{k-N}}\theta, \ 0 \leq \theta < \left[\frac{\operatorname{arctg} 2^{k(\omega-1)}}{\gamma_{k-N}}\right]\gamma_{k-N}.$$

Extend the function $\varphi = \varphi_1 + \varphi_2$ as constant in θ on the rest of the interval $[0, 2\pi)$.

Show now that the function

$$\mu(t,\theta) = t^{\rho_1} + \varphi(t,\theta)$$

is a distribution function of the Riesz measure for some subharmonic function u on the plane and this measure is concentrated in the set G_ω.

To prove this fact it is sufficient to investigate the behavior of the expression

$$S(t_1, t_2; \theta_1, \theta_2) = \mu(t_2, \theta_2) - \mu(t_2, \theta_1) - \mu(t_1, \theta_2) + \mu(t_1, \theta_1)$$

for any $0 < t_1 < t_2; 0 \leq \theta_1 < \theta_2 < 2\pi$.

Let $R^k(1 + 3 \cdot 2^N j\delta_k)) \leq t_1 < t_2 < R^k(1 + 3 \cdot 2^N(j+1)\delta_k))$ and $j\gamma_{k-N} < \theta_1 < \theta_2 \leq (j+1)\gamma_{k-N}$, $j = 0, 1, \ldots, \left[\frac{\operatorname{arctg} 2^{k(\omega-1)}}{\gamma_{k-N}}\right] - 1$, $k \geq N$. Then

$$\begin{aligned} S(t_1, t_2; \theta_1, \theta_2) &= \varphi(t_2, \theta_2) - \varphi(t_2, \theta_1) - \varphi(t_1, \theta_2) + \varphi(t_1, \theta_1) \\ &= \frac{1}{\gamma_{k-N}}(\varphi_1(t_1, 0) - \varphi_1(t_2, 0))(\theta_2 - \theta_1) \geq 0. \end{aligned} \quad (9)$$

If $R^k(1 + 3 \cdot 2^N j\delta_k)) \leq t_1 < t_2 < R^k(1 + 3 \cdot 2^N(j+1)\delta_k))$ and points $\theta_1, \theta_2 > (j+1)\gamma_{k-N}$, $j = 0, 1, \ldots, \left[\frac{\operatorname{arctg} 2^{k(\omega-1)}}{\gamma_{k-N}}\right] - 1$, $k \geq N$, then

$$S(t_1, t_2; \theta_1, \theta_2) \equiv 0 \quad (10)$$

by the construction of φ.

If $t_1, t_2 < R^N$, then it is clear that for any θ_1, θ_2

$$S(t_1, t_2; \theta_1, \theta_2) \equiv 0. \quad (11)$$

Obviously, (9), (10) and (11) are sufficient for the confirmation that the constructed function $\mu(t,\theta)$ is the distribution function of the measure of a subharmonic function on the plane and the support of this measure is concentrated in the set G_ω.

2) It follows from the construction of φ that

$$\varphi(t,\theta) = o(t^{\rho_2}),\ t \to \infty,$$

uniformly for $\theta \in [0, 2\pi]$. Hence in view of Th. 1 the function u has a multi-term asymptotics of the form:

$$u(re^{i\theta}) = \frac{\pi r^{\rho_1}}{\sin \pi\rho_1} \cos \rho_1(\theta - \pi) + \psi(re^{i\theta}).$$

Here $\psi(re^{i\theta}) = o(r^{\rho_2})$ uniformly for $\theta \in [0, 2\pi]$ as $r \to \infty$, if the point $z = re^{i\theta}$ does not belong to some $C_{0,1+\omega}$-set.

Consider the set

$$E_{0,k} = \Big\{ z : |z - R^k(1 + 3 \cdot 2^N m\delta_k)| \le \delta_k^\alpha R^k,$$
$$m \in \left[\left[\frac{2^{k-N}-1}{3}\right] + 1, 2\left[\frac{2^{k-N}-1}{3}\right]\right]\Big\},$$

where $k \ge N$ and the number $\alpha > 1$ will be chosen later on. The estimate of the remainder term $\psi(re^{i\theta})$ on the set $E_{0,k}$ is carried out in the same way as in the proof [4, Th. 3], where it is shown that on the set $E_{0,k}$ the inequality

$$-\psi(z) \ge C(\alpha - 1)\eta_k |z|^{\rho_2} \ln|z| - C_1 2^{-N} \eta_k |z|^{\rho_2} \ln|z|$$

is valid. Here C is a universal constant and C_1 does not depend on z, N and k. It is easy to see that for any $\alpha > 1$ there exists $N \in \mathbb{N}$ such that

$$-\psi(z) \ge \Delta \eta_k |z|^{\rho_2} \ln|z|$$

with $\Delta > 0$.

It is clear that the analogous estimate is valid also on the sets

$$E_{j,k} = \Big\{ z : |z - R^k(1 + 3 \cdot 2^N m\delta_k) e^{ij\gamma_{k-N}}| \le \delta_k^\alpha R^k,$$
$$m \in \left[\left[\frac{2^{k-N}-1}{3}\right] + 1, 2\left[\frac{2^{k-N}-1}{3}\right]\right]\Big\},\ j = 0, 1, 2, \ldots, \left[\frac{\operatorname{arctg} 2^{k(\omega-1)}}{\gamma_{k-N}}\right] - 1.$$

So for such choice of N and for all $z \in E = \bigcup\limits_{j,k} E_{j,k}$ the relation

$$\psi(z) \le -\Delta \eta_k |z|^{\rho_2} \ln|z|$$

holds. Therefore, if η_k decreases sufficiently slowly, then we obtain the inequality

$$\psi(z) < -|z|^{\rho_2}$$

for all great by modulus $z \in E$.

3) We estimate now a relative $(1+\kappa)$-measure of the set E. We have

$$\operatorname{rel}\ \underset{1+\kappa}{\operatorname{mes}}(E\bigcap\{z:|z|\le R^{k+1}\})$$
$$\ge \frac{1}{R^{(k+1)(1+\kappa)}}2^{2k}2^{k(\omega-1)}(\delta_k^\alpha R^k)^{1+\kappa}$$
$$= C2^{k(1+\omega-\alpha(1+\kappa))}.$$

Hence the relative $(1+\kappa)$-measure of the set E is equal to ∞ if $\alpha(1+\kappa)<1+\omega$. It is easy to see that for any $0\le\kappa<\omega$ there exists $\alpha>1$ such that the relative $(1+\kappa)$-measure of the set E is infinity.

This completes the proof of the theorem.

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