Journal of Mathematical Physics, Analysis, Geometry
2006, vol. 2, No. 3, pp. 268–277

On Spaces of Boundary Values for Relations Generated by a Formally Selfadjoint Expression and a Nonnegative Operator Function

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Received May 10, 2005

The space of boundary values for a maximal relation generated by a formally selfadjoint differential expression and a nonnegative operator function is constructed.

Key words: symmetric relation, space of boundary values, formally selfadjoint differential expression.
Mathematics Subject Classification 2000: 47A06, 34B05.

1. Introduction

In a well-known paper by F.S. Rofe-Beketov [1], selfadjoint extensions of a minimal operator generated by a formally selfadjoint differential expression are described in the terms of boundary values. The paper stimulated appearance of numerous works generalizing the obtained results [1] in various directions. The papers [2–4] are among them. In [2] dissipative extensions of minimal operators are described, and in [3, 4] abstract spaces of boundary values are introduced.

We construct here a space of boundary values in the context of [3, 4] for a linear relation generated by a formally selfadjoint differential expression and a nonnegative operator function. These relations were defined and explored in [5] for the finite-dimensional case and in [6] for the infinite-dimensional case. The situation is more complicated in the infinite-dimensional case. The domain of maximal relation contains functions the values of which do not belong to the source space. So, there are some ordered pairs belonging to the maximal relation such that the Lagrange formula is not valid. In [6] it is assumed that the domain does not contain these functions. We do not assume this condition here.

2. Notations and Auxiliary Statements

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, then $A(t)$ be an operator function strongly measurable on the compact interval $[a,b]$; the values of $A(t)$ be bounded operators in H such that for all $x \in H$ the scalar product $(A(t)x,x) \geqslant 0$ almost everywhere. Suppose the norm $\|A(t)\|$ to be integrable on the interval $[a,b]$.

By l we denote the differential expression of order r ($r = 2n$ or $r = 2n+1$):

$$l[y] = \begin{cases} \sum_{k=1}^{n}(-1)^k\{(p_{n-k}(t)y^{(k)})^{(k)} - i[(q_{n-k}(t)y^{(k)})^{k-1} + (q_{n-k}(t)y^{(k-1)})^{(k)}]\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + p_n(t)y, \\ \sum_{k=0}^{n}(-1)^k\{i[(q_{n-k}(t)y^{(k)})^{(k+1)} + (q_{n-k}(t)y^{(k+1)})^{k}] + (p_{n-k}(t)y^{(k)})^{(k)}\}. \end{cases}$$

Coefficients of l are bounded selfadjoint operators in H. The leading coefficients, $p_0(t)$ in the case $r = 2n$ and $q_0(t)$ in the case $r = 2n+1$, have the bounded inverse operator almost everywhere. The functions $p_{n-k}(t)$ are strongly differentiable k times and the functions $q_{n-k}(t)$ are strongly differentiable k times in the case $r = 2n$, as well as $k+1$ times in the case $r = 2n+1$. In general, we do not assume the coefficients of the expression l to be smooth as we have just stated. According to [7] we treat l as a quasidifferential expression. The quasiderivatives for the expression l are defined in [7]. Suppose that the functions $p_j(t)$, $q_m(t)$ are strongly measurable, the function $q_0(t)$ in the case $r = 2n+1$ is strongly differentiable, and the norms of functions

$$p_0^{-1}(t),\ p_0^{-1}(t)q_0(t),\ q_0(t)p_0^{-1}(t)q_0(t),\ p_1(t),\dots,p_n(t),\ q_0(t),\dots,q_{n-1}(t) \quad \text{(in case } r = 2n),$$

$$q_0'(t),\ q_1(t),\dots,q_n(t),\ p_0(t),\dots,p_n(t) \quad \text{(in case } r = 2n+1)$$

are integrable on the interval $[a,b]$. Thus we consider the case when the expression l is regular on the interval $[a,b]$.

We define the scalar product

$$\langle y,z\rangle = \int_a^b (A(t)y(t),z(t))dt,$$

where $y(t)$, $z(t)$ are H-valued functions that are continuous on $[a,b]$. We identify with zero the functions y such that $\langle y,y\rangle = 0$ and, having made the completion, obtain Hilbert space denoted by $\mathrm{B} = L_2(H,A(t);a,b)$. Let $\tilde{y}$ be some element belonging to B, i.e., $\tilde{y}$ is a corresponding class of functions. If $y_1, y_2 \in \tilde{y}$, then y_1, y_2 are identified with respect to the norm generated by the scalar product $\langle\cdot,\cdot\rangle$. By $\tilde{y}$ denote the class of functions containing y. Suppose $y \in \tilde{y}$. Without loss of generality, we will say often that $y(t)$ belongs to B.

Let $G(t)$ be the set of elements $x \in H$ such that $A(t)x = 0$, $H(t)$ be the orthogonal complement of $G(t)$ in H, $H = H(t) \oplus G(t)$, and $A_0(t)$ be the restriction of $A(t)$ to $H(t)$. Suppose that $H_\tau(t)$ $(-\infty < \tau < \infty)$ is the Hilbert scale of spaces generated by the operator $A_0^{-1}(t)$. For fixed t, the operator $A_0^{1/2}(t)$ is the continuous one-to-one mapping of $H(t) = H_0(t)$ onto $H_{1/2}(t)$. By $\hat{A}_0^{1/2}(t)$ denote the operator adjoint to $A_0^{1/2}(t)$. The operator $\hat{A}_0^{1/2}(t)$ is the continuous one-to-one mapping of $H_{-1/2}(t)$ onto $H(t)$ and $\hat{A}_0^{1/2}(t)$ is an extension of $A_0^{1/2}(t)$. Let $\tilde{A}_0(t) = A_0^{1/2}(t)\hat{A}_0^{1/2}(t)$. The operator $\tilde{A}_0(t)$ is the continuous one-to-one mapping of $H_{-1/2}(t)$ onto $H_{1/2}(t)$ and $\tilde{A}_0(t)$ is an extension of $A_0(t)$. By $\tilde{A}(t)$ denote the operator defined on $H_{-1/2}(t) \oplus G(t)$ such that $\tilde{A}(t)$ is equal to $\tilde{A}_0(t)$ on $H_{-1/2}(t)$ and $\tilde{A}(t)$ is equal to zero on $G(t)$. The operator $\tilde{A}(t)$ is an extension of $A(t)$.

In [6] it is proved that spaces $H_{-1/2}(t)$ are measurable with respect to the parameter t [8, Ch. 1] whenever we take the functions of the form $\tilde{A}_0^{-1}(t)A^{1/2}(t)h(t)$ in place of the measurable functions, where $h(t)$ is a measurable function with the values in H. The space B is the measurable sum of the spaces $H_{-1/2}(t)$.The space B consists of elements (i.e., classes of functions) with the representatives of the form $\tilde{A}_0^{-1}(t)A^{1/2}(t)h(t)$, where $h(t) \in L_2(H; a, b)$, i.e., $\int_a^b \|h(t)\|^2\, dt < \infty$.

We define the minimal and maximal relations generated by the expression l and the function $A(t)$ in the following way. Let D' be the set of functions y satisfying the conditions: a) the quasiderivatives $y^{[0]}, \ldots, y^{[r]}$ of function y exist, they are absolutely continuous up to the order $r-1$; b) $l[y](t) \in H_{1/2}(t)$ almost everywhere; c) the function $\tilde{A}_0^{-1}(t)l[y]$ belongs to B. To each class of functions identified in B with $y \in D'$ we assign the class of functions identified in B with $\tilde{A}_0^{-1}(t)l[y]$. This correspondence may be not an operator because it can be occurred that some function y is identified with zero in B and $\tilde{A}_0^{-1}(t)l[y]$ isn't equal to zero. So, we get the linear relation L' in the space B. The closure of L' we denote by L. The relation L is called maximal. By L_0 denote the restriction of L to the set of elements $\tilde{y} \in$ B with representatives $y \in D'$ such that $y^{[k]}(a) = y^{[k]}(b) = 0$ $(k = 0, 1, \ldots, r-1)$. The relation L_0 is called minimal.

Terminology concerning linear relations can be found in the monographs [9–11]. In what further we will use the following notations: R is the range of values; $\{\cdot,\cdot\}$ is the ordered pair; $\ker L$ is the set of elements z such that $\{z, 0\} \in L$.

We consider the differential equation $l[y] = \lambda A(t)y$, where λ is a complex number. Let $W_j(t,\lambda)$ be the operator solution of this equation satisfying the initial conditions: $W_j^{[k-1]}(a,\lambda) = \delta_{jk}E$ (E is the identity operator, δ_{jk} is the

Kronecker symbol, $j, k = 1, \ldots, r$). By $W(t,\lambda)$ we denote the one-row operator matrix $(W_1(t,\lambda), \ldots, W_r(t,\lambda))$. The operator $W(t,\lambda)$ maps continuously H^r into H for fixed t, λ. The adjoint operator $W^*(t,\lambda)$ maps continuously H into H^r. If $l[y]$ exists for the function y, then we denote $\hat{y} = (y, y^{[1]}, \ldots, y^{[r-1]})$ (we treat $\hat{y}$ as a one-columned matrix). Let $z = (z_1, \ldots, z_m)$ be some system of functions such that $l[z_j]$ exists for each j. By $\hat{z}$ we denote the matrix $(\hat{z}_1, \ldots, \hat{z}_m)$. The analogous notations are used for the operator functions.

We consider the operator matrices of orders $2n$ and $2n+1$ for the expression l in the cases $r = 2n$ and $r = 2n+1$ respectively:

$$J_{2n}(t) = \begin{pmatrix} & & & & -E \\ & & & \cdots & \\ & & -E & & \\ & E & & & \\ \cdots & & & & \\ E & & & & \end{pmatrix},$$

$$J_{2n+1}(t) = \begin{pmatrix} & & & & & -E \\ & & & & \cdots & \\ & & & -E & & \\ & & 2iq_0^{-1}(t) & & & \\ & E & & & & \\ \cdots & & & & & \\ E & & & & & \end{pmatrix},$$

where all unmarked elements are equal to zero. (In the matrix $J_{2n+1}(t)$ the element $2iq_0^{-1}(t)$ stands on the intersection of the row $n+1$ and the column $n+1$.) Suppose $y, z \in D'$, then the Lagrange formula takes in these notations the following form:

$$\int_\alpha^\beta (l[y], z)dt - \int_\alpha^\beta (y, l[z])dt = (J_r(t)\hat{y}(t), \hat{z}(t))|_\alpha^\beta, \quad a \leqslant \alpha < \beta \leqslant b. \tag{1}$$

It follows from "the method of the variation of arbitrary constants" that the general solution of the equation $l[y] - \lambda\tilde{A}(t) = \tilde{A}(t)f(t)$ is represented in the form:

$$y(t, f, \lambda) = W(t,\lambda)\left(x + \int_a^t V(s,\lambda)\tilde{A}(s)f(s)ds\right), \tag{2}$$

where $x \in H^r$, $V(t,\lambda)$ is the operator of H to H^r for fixed t, λ, and $V(t,\lambda)$

satisfies the condition

$$\hat{W}(t,\lambda)V(t,\lambda) = \begin{pmatrix} 0 \\ \dots \\ 0 \\ -E \end{pmatrix}. \tag{3}$$

It follows from (2), (3) that

$$\hat{y}(t,f,\lambda) = \hat{W}(t,\lambda)\left(x + \int\limits_a^t V(s,\lambda)\tilde{A}(s)f(s)ds\right). \tag{4}$$

Using (1), we obtain

$$\hat{W}^*(t,\bar{\lambda})J_r(t)\hat{W}(t,\lambda) = J_r(a). \tag{5}$$

It follows from (3), (5) that

$$V(t,\lambda) = J_r^{-1}(a)W^*(t,\bar{\lambda}). \tag{6}$$

Let Q_0 be the set of elements $x \in H^r$ such that the function $W(t,0)$ is identified with zero in the space B, i.e., $\int_a^b \left\|A^{1/2}(s)W(s,0)x\right\|^2 ds = 0$. It follows from the equalities

$$W(t,\lambda)x = W(t,0)\left(x + \lambda\int\limits_a^t V(s,0)\tilde{A}(s)W(s,\lambda)xds\right), \tag{7}$$

$$W(t,0)x = W(t,\lambda)\left(x - \lambda\int\limits_a^t V(s,\lambda)\tilde{A}(s)W(s,0)xds\right) \tag{8}$$

that the function $W(t,\lambda)x$ is identified with zero in the space B if and only if $x \in Q_0$ (in the finite-dimensional case this fact was obtained in [7]).

Let Q be the orthogonal complement of Q_0 in H^r, $H^r = Q \oplus Q_0$. We define the norm

$$\|x\|_- = \left(\int\limits_a^b \left\|A^{1/2}(s)W(s,0)x\right\|^2\right)^{1/2} \leqslant c\,\|x\|, \quad x \in Q, \tag{9}$$

in the space Q. By Q_- denote the completion of Q with respect to this norm. It follows from equalities (7), (8) that we obtain the same set Q_- with the equivalent norm whenever we replace $W(s,0)$ by $W(s,\lambda)$ in (9).

Suppose the sequence $\{x_n\}$ ($x_n \in Q$) converges to $x_0 \in Q_-$ in Q and suppose $\tilde{W}(t,\lambda)x_n$ is a class of functions with the representative $W(t,\lambda)x_n$. Then $\{\tilde{W}(t,\lambda)x_n\}$ is the fundamental sequence in B. Hence $\{\tilde{W}(t,\lambda)x_n\}$ converges to some element belonging to B. We denote this element by $\tilde{W}(t,\lambda)x_0$. If the sequence $\{\tilde{W}(t,\lambda)x_n\}$ ($x_n \in Q$) converges in B, then there exists a unique element $x_0 \in Q_-$ such that $\tilde{y} = \tilde{W}(t,\lambda)x_0$. Indeed, it follows from (9) that the sequence $\{x_n\}$ is fundamental in Q_-, and we can take $\lim\limits_{n\to\infty} x_n$ in place of x_0. Hence $\tilde{W}(t,\lambda)x_0$ belongs to $\ker(L-\lambda E)$ for every $x_0 \in Q_-$ and $\tilde{W}(t,\lambda)x_0 \neq 0$ in B for $x_0 \in Q_-$, $x_0 \neq 0$.

The space Q_- can be treated as a negative one with respect to Q. By Q_+ we denote the corresponding space with the positive norm (see [9, Ch. 2]). Let $W_0(\lambda)$ denote the operator $x \to \tilde{W}(t,\lambda)x$ and let $W_1(\lambda)$ denote the operator $\tilde{f} \to \int_a^b W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds$. Then

$$W_0^*(\lambda) = W_1(\bar{\lambda}). \tag{10}$$

The operator $W_0(\lambda)$ is the continuous one-to-one mapping of Q_- into B for every fixed λ and the range of $W_0(\lambda)$ is closed. This implies that $W_1(\lambda)$ maps continuously B onto Q_+. So, $R(W_1(\lambda)) = Q_+$ for every λ.

Lemma 1. *The relation L consists of all ordered pairs $\{\tilde{y},\tilde{f}\} \in$ B $\oplus$ B such that*

$$\tilde{y} = \tilde{W}(t,0)x + \tilde{F}, \tag{11}$$

where $x \in Q_-$, $\tilde{F}$ is the class of functions identified in B *with the function*

$$F(t) = W(t,0)\int_a^t V(s,0)\tilde{A}(s)f(s)ds. \tag{12}$$

P r o o f. Assume that the ordered pair $\{\tilde{y},\tilde{f}\} \in$ B $\oplus$ B satisfies (11), (12). It follows from the reasoning before Lemma 1 that $\{\tilde{y},\tilde{f}\} \in L$. Also we note that $\{\tilde{y},\tilde{f}\} \in L'$ whenever $x \in Q$ in (11).

Now we assume that the ordered pair $\{\tilde{y},\tilde{f}\} \in L$. Then there exists a sequence of the ordered pair $\{\tilde{y}_n,\tilde{f}_n\} \in L'$ such that $\{\tilde{y}_n,\tilde{f}_n\}$ converges to $\{\tilde{y},\tilde{f}\}$ in B $\oplus$ B as $n \to \infty$ and the function y_n is represented in the form

$$y_n(t) = W(t,0)\left(x_n + \int_a^t V(s,0)\tilde{A}(s)f_n(s)ds\right), \tag{13}$$

where $x_n \in Q$. Since the sequence $\{\tilde{y}_n,\tilde{f}_n\}$ converges in B $\oplus$ B, we see that the sequence $\{\tilde{W}(t,0)x_n\}$ converges in B. By taking the limit $n \to \infty$ in (13), we obtain (11). The lemma is proved.

R e m a r k 1. Since $\left\|V(s,0)A^{1/2}(s)\right\| \in L_2(a,b)$, $\left\|A^{1/2}(s)f(s)\right\| \in L_2(a,b)$, we see that the integral in (12) exists and this integral is continuous with respect to t.

R e m a r k 2. For any ordered pair $\{\tilde{y},\tilde{f}\} \in L$ there exists a unique element $x \in Q_-$ and a unique function $F(t)$ of form (12) such that equality (11) is true. Indeed, if f_1, f_2 are identified in B, then $\tilde{A}(t)f_1 = \tilde{A}(t)f_2$. To conclude the proof, it remains to note that $\tilde{W}(t,0)x \neq 0$ in B for $x \in Q_-$ and $x \neq 0$.

R e m a r k 3. The relation $L - \lambda E$ consists of ordered pairs $\{\tilde{y},\tilde{f}\}$ of the form (11), (12), where $W(t,0)$ and $V(t,0)$ are replaced by $W(t,\lambda)$ and $V(t,\lambda)$ respectively. Hence the operator $x \to \tilde{W}(t,\lambda)x$ is the continuous one-to-one mapping of Q_- onto $\ker(L-\lambda E)$.

Lemma 2. *L_0 is the closed symmetric relation.*

P r o o f. Suppose the sequence of ordered pairs $\{\tilde{y}_n,\tilde{f}_n\} \in L_0$ converges to $\{\tilde{y},\tilde{f}\}$ in B$\oplus$B. It follows from the definition of L_0 that we can take representatives y_n, f_n of classes of functions $\tilde{y}_n$, $\tilde{f}_n$ such that (13) is true and $\hat{y}_n(a) = \hat{y}_n(b) = 0$. Using (4), we get in (13) $x_n = \int_a^b V(s,0)\tilde{A}(s)f_n(s)ds = 0$. Calculating the limit $n \to \infty$ in this equality and in (13), we obtain that $x = 0$ in (11), (12) and

$$\int_a^b V(s,0)\tilde{A}(s)f(s)ds = 0. \tag{14}$$

It follows from (4) and (14)that $y \in D'$ and $\hat{y}(a) = \hat{y}(b) = 0$. This implies that $\{\tilde{y},\tilde{f}\} \in L_0$. Using the Lagrange formula (1), we get that L_0 is the symmetric relation. The lemma is proved.

R e m a r k 4. It follows from the proof Lemma 2 that the range $R(L_0)$ of the relation L_0 consists of all elements $\tilde{f} \in$ B such that (14) true. Therefore $R(L_0)$ is closed and L_0^{-1} is the bounded operator on $R(L_0)$. Substituting $V(s,\lambda)$ for $V(s,0)$, we obtain the analogous statements for the relation $L_0 - \lambda E$. Then (6) implies that the range $R(L_0-\lambda E)$ of the relation $L_0 - \lambda E$ consists of all elements $\tilde{f} \in$ B such that

$$\int_a^b W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds = 0.$$

Thus, $R(L_0 - \lambda E) = \ker W_1(\lambda)$.

By N_λ we denote the defect subspace of the relation L_0, i.e., N_λ is the orthogonal complement of the range $R(L_0-\lambda E)$ of the relation $L_0-\lambda E$ in the space B.

Lemma 3. $N_\lambda = \ker(L - \bar{\lambda}E)$.

P r o o f. It follows from Remarks 3, 4 and equality (10) that orthogonal complement of $\ker W_1(\lambda) = R(L_0 - \lambda E)$ in B coincides with $R(W_0(\bar\lambda)) = \ker(L - \bar\lambda E)$. This completes the proof of Lemma 3.

Lemma 4. $L_0^* = L$.

P r o o f. Using the Lagrange formula (1), we get $L' \subset L_0^*$. Consequently, $L \subset L_0^*$. It is known (see [11]) that L_0^* is the direct sum of the subspaces L_0, $\tilde N_\lambda$, $\tilde N_{\bar\lambda}$: $L_0^* = L_0 \dot{+} \tilde N_\lambda \dot{+} \tilde N_{\bar\lambda}$, where $\tilde N_\lambda$ is the set of ordered pairs of the form $\{z, \bar\lambda z\}$, $z \in N_\lambda$. Since $L_0 \subset L$, $\tilde N_\lambda \subset L$, $\tilde N_{\bar\lambda} \subset L$, we obtain $L_0^* \subset L$. The lemma is proved.

3. The Main Result

In what further, we denote $W(t,0) = W(t)$ for the simplification of notations. Let $\{\tilde y, \tilde f\} \in L$. It follows from Lemma 1 and formulae (6), (12) that $\tilde y = \tilde W(t)c + \tilde F$, where $c \in Q_-$,

$$F(t) = W(t) J_r^{-1}(a) \int\limits_a^t W^*(s)\tilde A(s) f(s) ds.$$

We denote

$$W_1(0)\tilde f = \int_a^b W^*(s)\tilde A(s) f(s) ds = d \in Q_+$$

and define a pair of boundary values $\{Y, Y'\} \in Q_- \oplus Q_+$ for the pair $\{\tilde y, \tilde f\} \in L$ by the formulae

$$Y = \Gamma_1\{\tilde y, \tilde f\} = c + (1/2) J_r^{-1}(a) d \in Q_-, \quad Y' = \Gamma_2\{\tilde y, \tilde f\} = d \in Q_+. \tag{15}$$

It follows from Remark 2 that a pair of boundary values $\{Y, Y'\}$ is uniquely determined by the pair $\{\tilde y, \tilde f\} \in L$. We denote by Γ the operator taking $\{\tilde y, \tilde f\} \in L$ to $\{Y, Y'\} \in Q_- \oplus Q_+$ by formulae (15).

Theorem. *The range $R(\Gamma)$ of the operator Γ coincides with $Q_- \oplus Q_+$ and "the Green formula" is valid:*

$$\langle \tilde f_1, \tilde y_2 \rangle - \langle \tilde y_1, \tilde f_2 \rangle = (Y_1', Y_2) - (Y_1, Y_2'), \tag{16}$$

where $\{\tilde y_1, \tilde f_1\}, \{\tilde y_2, \tilde f_2\} \in L$, $\Gamma\{\tilde y_1, \tilde f_1\} = \{Y_1, Y_1'\}$, $\Gamma\{\tilde y_2, \tilde f_2\} = \{Y_2, Y_2'\}$.

P r o o f. Using Lemma 1 and the equality $R(W_1(0)) = Q_+$, we obtain the first part of the theorem. Now we shall prove (16). It follows from Lemma 1 that

$\tilde{y}_i = \tilde{z}_i + \tilde{F}_i$, where $\tilde{z}_i = \tilde{W}(t)c_i$ $(c_i \in Q_-)$, $\tilde{F}_i$ is the class of functions with the representative

$$F_i(t) = W(t)J_r^{-1}(a)\int_a^t W^*(s)\tilde{A}(s)f_i(s)ds, \quad i = 1, 2.$$

We denote $d_i = \Gamma_2\{\tilde{y}_i, \tilde{f}_i\}$. Then $c_i = \Gamma_1\{\tilde{y}_i, \tilde{f}_i\} - (1/2)J_r^{-1}(a)d_i$. The expression l is defined on the functions $F_i(t)$ and $l[F_i] = \tilde{A}(t)f_i$. Using the Lagrange formula (1) and equalities (4), (5), we get

$$\begin{aligned}\langle \tilde{f}_1, \tilde{F}_2\rangle - \langle \tilde{F}_1, \tilde{f}_2\rangle &= (J_r(b)\hat{W}(b)J_r^{-1}(a)d_1, \hat{W}(b)J_r^{-1}(a)d_2)\\ &= (\hat{W}^*(b)J_r(b)\hat{W}(b)J_r^{-1}(a)d_1, J_r^{-1}(a)d_2)\\ &= (J_r(a)J_r^{-1}(a)d_1, J_r^{-1}(a)d_2) = (d_1, J_r^{-1}(a)d_2). \qquad (17)\end{aligned}$$

We take two sequences $\{c_{i,n}\}$ $(c_{i,n} \in Q,\ i = 1, 2)$ such that $c_{i,n}$ converges to c_i in Q_- as $n \to \infty$. We denote $z_{i,n} = W(t)c_{i,n}$. The sequence $\{\tilde{z}_{i,n}\}$ converges to $\tilde{z}_i$, $i = 1, 2$, in B. The expression l is defined on the functions $z_{i,n}(t)$ and $l[z_{i,n}] = 0$. It follows from the Lagrange formula (1) and equalities (4), (5) that

$$\begin{aligned}\langle \tilde{f}_1, \tilde{z}_{2.n}\rangle &= \langle \tilde{f}_1, \tilde{z}_{2n}\rangle - \langle \tilde{F}_1, 0\rangle = (J_r(b)\hat{W}(b)J_r^{-1}(a)d_1, \hat{W}(b)c_{2,n})\\ &= (\hat{W}^*(b)J_r(b)\hat{W}(b)J_r^{-1}(a)d_1, c_{2,n}) = (J_r(a)J_r^{-1}(a)d_1, c_{2,n}) = (d_1, c_{2,n}).\end{aligned}$$

We calculate the limit $n \to \infty$ in this equality. Using $d_1 \in Q_+$, we obtain

$$\langle \tilde{f}_1, \tilde{z}_2\rangle = (d_1, c_2), \qquad (18)$$

and analogously

$$\langle \tilde{z}_1, \tilde{f}_2\rangle = (c_1, d_2). \qquad (19)$$

It follows from the equality $(J_r^{-1}(a)d_1, d_2) = -(d_1, J_r^{-1}(a)d_2)$ and (17)–(19) that

$$\begin{aligned}\langle \tilde{f}_1, \tilde{y}_2\rangle - \langle \tilde{y}_1, \tilde{f}_2\rangle &= \langle \tilde{f}_1, \tilde{z}_2\rangle + \langle \tilde{f}_1, \tilde{F}_2\rangle - \langle \tilde{z}_1, \tilde{f}_2\rangle - \langle \tilde{F}_1, \tilde{f}_2\rangle\\ &= (d_1, J_r^{-1}(a)d_2) + (d_1, c_2) - (c_1, d_2)\\ &= (d_1, c_2 + (1/2)J_r^{-1}(a)d_2) - (c_1 + (1/2)J_r^{-1}(a)d_1, d_2) = (Y_1', Y_2) - (Y_1, Y_2').\end{aligned}$$

The theorem is proved.

Thus it follows from the theorem that the ordered four $(Q_-, Q_+, \Gamma_1, \Gamma_2)$ are the space of boundary values for the relation L in the sense of [3, 4, 6]. In [3, 4] an abstract space of boundary values is introduced for operators and in [6] it is introduced for relations. The case $Q_+ = Q_-$ is considered in these papers.

The detailed bibliography related to the topics of our paper is represented in the monographs [9, 10].

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