Journal of Mathematical Physics, Analysis, Geometry
2006, vol. 2, No. 4, pp. 347–357

On the Solvability of a Class of Operator Differential Equations of the Second Order on the Real Axis

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Received December 15, 2005

Solvability of the operator differential equation of the second order with variable coefficients on the real axis in a certain weight space is studied. The main part of the equation is an abstract elliptic equation in Hilbert space. We note that sufficient conditions on operator coefficients of the perturbed part, preserving ellipticity of the equation, are found in the paper, and estimations of the norms of intermediate derivative operators via the main part of the operator differential equation in a certain weight space are also obtained.

Key words: operator differential equation, selfadjoint operator, discontinuous coefficient, regular solution, Hilbert space.
Mathematics Subject Classification 2000: 47E05, 34B05.

Let A be a selfadjoint positive definite operator in the separable Hilbert space H.

We denote by $L_{2,\kappa}(R;H)$, $R=(-\infty;+\infty)$ Hilbert space of H-valued functions defined in R with the norm

$$\|u\|_{L_{2,\kappa}(R;H)} = \left(\int_{-\infty}^{+\infty} \|u(t)\|_H^2 \, e^{-\kappa t} dt \right)^{1/2}, \quad \kappa \in R.$$

We denote by $W_{2,\kappa}^2(R;H)$ the space of H-valued functions, such that

$$\frac{d^2u(t)}{dt^2} \in L_{2,\kappa}(R;H), \ A^2u(t) \in L_{2,\kappa}(R;H),$$

with the norm

$$\|u\|_{W^2_{2,\kappa}(R;H)} = \left(\left\| \frac{d^2u}{dt^2} \right\|^2_{L_{2,\kappa}(R;H)} + \|A^2u\|^2_{L_{2,\kappa}(R;H)} \right)^{1/2}, \quad \kappa \in R.$$

It is obvious that for $\kappa = 0$ we have the spaces $L_{2,0}(R;H) = L_2(R;H)$, $W^2_{2,0}(R;H) = W^2_2(R;H)$ described in details in [1, Ch. 1].

Here and below derivatives are considered in the sense of the theory of generalized functions.

We introduce the following notations: $L(X,Y)$ is the set of linear bounded operators acting from Hilbert space X to another Hilbert space Y, and $L_\infty(R;B)$ is the set of B-valued essentially bounded operator functions in R, where B is a Banach space.

Now let us formulate the statement of the problem to be studied. Consider the operator differential equation of the second order of the form

$$-\frac{d^2u(t)}{dt^2} + A^2u(t) + A_1(t)\frac{du(t)}{dt} + A_2(t)u(t) = f(t), \quad t \in R, \tag{1}$$

where $f(t) \in L_{2,\kappa}(R;H)$, $u(t) \in W^2_{2,\kappa}(R;H)$, $A_1(t)$ and $A_2(t)$ are, in general, linear unbounded operators, defined for almost all $t \in R$.

Definition 1. *A vector function $u \in W^2_{2,\kappa}(R;H)$, satisfying equation (1) almost everywhere for a given $f(t) \in L_{2,\kappa}(R;H)$, is called a regular solution of equation (1).*

Definition 2. *If a regular solution of equation (1) exists for any $f(t) \in L_{2,\kappa}(R;H)$, and the inequality*

$$\|u\|_{W^2_{2,\kappa}(R;H)} \le const\,\|f\|_{L_{2,\kappa}(R;H)}$$

holds, then equation (1) is called regularly solvable.

In this paper we suggest conditions on the operator coefficients of equation (1) to provide its regular solvability. We note that the main part of the studied equation (1) is the abstract elliptic equation [2] in the Hilbert space. Taking into account the above, the aim of this paper is to find conditions on the operator coefficients of the perturbed part of equation (1) that preserve ellipticity of this equation.

We should note that equation (1) was considered in [3–5], where sufficient conditions of its solvability in Hilbert spaces without weight were obtained. Analogous problems in the weight space, when the operators of the perturbed part of the equation were the powers of the operator A, multiplied by the complex numbers, were investigated in [2].

Consider the operator differential equation

$$-\frac{d^2u(t)}{dt^2} + A^2u(t) = f(t), \quad t \in R, \tag{2}$$

where $f(t) \in L_{2,\kappa}(R;H)$, $u(t) \in W^2_{2,\kappa}(R;H)$.

Denote by Γ_0 the operator acting from the space $W^2_{2,\kappa}(R;H)$ to $L_{2,\kappa}(R;H)$ in the following way:

$$\Gamma_0 u(t) = -\frac{d^2u(t)}{dt^2} + A^2u(t), \quad u(t) \in W^2_{2,\kappa}(R;H).$$

Then the following theorem showing a link of solvability of equation (2)* with the lower bound of the spectrum of operator A is valid.

Theorem 1. *Let A be a selfadjoint positive definite operator with the lower bound λ_0 of the spectrum, i.e., $A = A^* \geq \lambda_0 E$ $(\lambda_0 > 0)$, E is a unit operator and the number $\kappa \in R$ satisfies the condition $|\kappa| < 2\lambda_0$. Then Γ_0 is an isomorphism between $W^2_{2,\kappa}(R;H)$ and $L_{2,\kappa}(R;H)$.*

P r o o f. In the equation $\Gamma_0 u(t) = f(t)$, $u(t) \in W^2_{2,\kappa}(R;H)$, $f(t) \in L_{2,\kappa}(R;H)$ we substitute $u(t) = \vartheta(t)e^{\frac{\kappa}{2}t}$, then $\vartheta(t) = u(t)e^{-\frac{\kappa}{2}t} \in L_2(R;H)$. Since

$$-\frac{d^2u(t)}{dt^2} + A^2u(t) = -\left(\frac{d}{dt} + \frac{\kappa}{2}\right)^2 \vartheta(t)e^{\frac{\kappa}{2}t} + A^2\vartheta(t)e^{\frac{\kappa}{2}t} = f(t),$$

we obtain

$$-\left(\frac{d}{dt} + \frac{\kappa}{2}\right)^2 \vartheta(t) + A^2\vartheta(t) = f(t)e^{-\frac{\kappa}{2}t}. \tag{3}$$

Having $g(t) = f(t)e^{-\frac{\kappa}{2}t} \in L_2(R;H)$, we can write (3) in the form

$$-\left(\frac{d}{dt} + \frac{\kappa}{2}\right)^2 \vartheta(t) + A^2\vartheta(t) = g(t) \tag{4}$$

in the space $L_2(R;H)$, i.e., $\vartheta(t) \in W^2_2(R;H)$, $g(t) \in L_2(R;H)$.

*Note that equation (2) is of the type considered in [2, p. 17], however our theorem does not follow from Th. 5.5, since in our paper we provide a certain weight index and the problem is considered on the axis R, while in [2] the problem is studied on the semi-axis R_+. Besides, the conditions on the right-hand side of the equation and the proof are different.

Now let us denote by

$$\Gamma_{0,\kappa}\vartheta(t) = -\left(\frac{d}{dt}+\frac{\kappa}{2}\right)^2 \vartheta(t) + A^2\vartheta(t), \quad \vartheta(t) \in W_2^2(R;H).$$

In this case equation (4) can be rewritten in the form $\Gamma_{0,\kappa}\vartheta(t) = g(t)$ where $\vartheta(t) \in W_2^2(R;H)$, $g(t) \in L_2(R;H)$. To solve this equation we do the Fourier transformation in (4)

$$\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)\widehat{\vartheta}(\xi) = \widehat{g}(\xi), \tag{5}$$

where $\widehat{\vartheta}(\xi)$, $\widehat{g}(\xi)$ are Fourier transforms of the vector functions $\vartheta(t)$, $g(t)$ respectively. Let us prove that for $|\kappa| < 2\lambda_0$ the operator pencil

$$\Gamma_{0,\kappa}(-i\xi;A) = -\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2 \tag{6}$$

is invertible. Really, let $\lambda \in \sigma(A)$ $(\lambda \geq \lambda_0)$, then characteristic polynomial of (6) has the following form:

$$\Gamma_{0,\kappa}(-i\xi;\lambda) = -\left(-i\xi+\frac{\kappa}{2}\right)^2 + \lambda^2 = \xi^2 + i\xi\kappa - \frac{\kappa^2}{4} + \lambda^2.$$

Hence we obtain that

$$|\Gamma_{0,\kappa}(-i\xi;\lambda)| = \left|\xi^2 + i\xi\kappa - \frac{\kappa^2}{4} + \lambda^2\right|$$

$$= \left[\left(\xi^2 - \frac{\kappa^2}{4} + \lambda^2\right)^2 + \xi^2\kappa^2\right]^{1/2} \geq \xi^2 - \frac{\kappa^2}{4} + \lambda^2$$

$$\geq \lambda^2 - \frac{\kappa^2}{4} \geq \lambda_0^2 - \frac{\kappa^2}{4} > 0,$$

i.e., it follows from the spectral decomposition of the operator A that the operator pencil $\Gamma_{0,\kappa}(-i\xi;A)$ is invertible for $|\kappa| < 2\lambda_0$. So we can find $\widehat{\vartheta}(\xi)$ from (5):

$$\widehat{\vartheta}(\xi) = \left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi). \tag{7}$$

We obtain

$$\vartheta(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi)e^{i\xi t}d\xi.$$

It is obvious that $\vartheta(t)$ satisfies equation (4) almost everywhere. Now we prove that $\vartheta(t) \in W_2^2(R;H)$. Really, by the Plancherel theorem it suffices to show that $A^2\widehat{\vartheta}(\xi) \in L_2(R;H)$ and $\xi^2\widehat{\vartheta}(\xi) \in L_2(R;H)$. It is clear that

$$\|\vartheta\|^2_{W_2^2(R;H)} = \left\|\frac{d^2\vartheta}{dt^2}\right\|^2_{L_2(R;H)} + \left\|A^2\vartheta\right\|^2_{L_2(R;H)}$$

$$= \left\|\xi^2\widehat{\vartheta}(\xi)\right\|^2_{L_2(R;H)} + \left\|A^2\widehat{\vartheta}(\xi)\right\|^2_{L_2(R;H)}.$$

Since

$$\left\|A^2\widehat{\vartheta}(\xi)\right\|_{L_2(R;H)} = \left\|A^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi)\right\|_{L_2(R;H)}$$

$$\le \sup_{\xi\in R}\left\|A^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\right\| \left\|\widehat{g}(\xi)\right\|_{L_2(R;H)},$$

we can estimate the norm $\left\|A^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\right\|$ for $\xi \in R$. It follows from the spectral theory of selfadjoint operators that

$$\left\|A^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\right\| = \sup_{\lambda\in\sigma(A)}\left|\lambda^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 + \lambda^2\right)^{-1}\right|$$

$$= \sup_{\lambda\in\sigma(A)} \frac{\lambda^2}{\left[\left(\xi^2-\frac{\kappa^2}{4}+\lambda^2\right)^2+\xi^2\kappa^2\right]^{1/2}}$$

$$\le \sup_{\lambda\in\sigma(A)} \frac{\lambda^2}{\xi^2-\frac{\kappa^2}{4}+\lambda^2} \le \sup_{\lambda\in\sigma(A)} \frac{\lambda^2}{\lambda^2-\frac{\kappa^2}{4}} \le \frac{\lambda_0^2}{\lambda_0^2-\frac{\kappa^2}{4}}.$$

Hence,

$$\left\|A^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi)\right\|_{L_2(R;H)} \le b_0(\kappa)\left\|\widehat{g}(\xi)\right\|_{L_2(R;H)},$$

where

$$b_0(\kappa) = \frac{4\lambda_0^2}{4\lambda_0^2-\kappa^2}. \tag{8}$$

Similarly, we have

$$\left\|\xi^2\widehat{\vartheta}(\xi)\right\|_{L_2(R;H)} = \left\|\xi^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi)\right\|_{L_2(R;H)}$$

$$\leq \sup_{\xi\in R}\left\|\xi^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E+A^2\right)^{-1}\right\| \|\widehat{g}(\xi)\|_{L_2(R;H)}\,.$$

Then for $\xi \in R$ and $|\kappa| < 2\lambda_0$ we obtain that

$$\left\|\xi^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 E+A^2\right)^{-1}\right\| = \sup_{\lambda\in\sigma(A)}\left|\xi^2\left(-\left(-i\xi+\frac{\kappa}{2}\right)^2 +\lambda^2\right)^{-1}\right|$$

$$\leq \sup_{\lambda\in\sigma(A)}\frac{\xi^2}{\xi^2-\frac{\kappa^2}{4}+\lambda^2}\leq\frac{\xi^2}{\xi^2+\lambda_0^2-\frac{\kappa^2}{4}}\leq 1.$$

We conclude

$$\left\|\xi^2\widehat{\vartheta}(\xi)\right\|_{L_2(R;H)} \leq \|\widehat{g}(\xi)\|_{L_2(R;H)}\,.$$

Consequently, we have found $\vartheta(t) \in W_2^2(R;H)$. It is obvious that the vector function $\vartheta(t)e^{\frac{\kappa}{2}t} \in L_{2,\kappa}(R;H)$ and it is a regular solution of equation (2).

On the other hand, the operator Γ_0 is bounded from $W_{2,\kappa}^2(R;H)$ into the space $L_{2,\kappa}(R;H)$. Really,

$$\|\Gamma_0 u\|^2_{L_{2,\kappa}(R;H)} = \left\|-\frac{d^2u}{dt^2}+A^2u\right\|^2_{L_{2,\kappa}(R;H)} \leq 2\,\|u\|^2_{W^2_{2,\kappa}(R;H)}\,.$$

Hence, the operator $\Gamma_0 : W_{2,\kappa}^2(R;H) \to L_{2,\kappa}(R;H)$ is one-to-one and bounded. Then it follows from the Banach theorem on the inverse operator that the operator $\Gamma_0^{-1} : L_{2,\kappa}(R;H) \to W_{2,\kappa}^2(R;H)$ is bounded. Therefore Γ_0 is an isomorphism between the spaces $W_{2,\kappa}^2(R;H)$ and $L_{2,\kappa}(R;H)$. Theorem 1 is proved.

The theorem shows that the norm $\|\Gamma_0 u\|_{L_{2,\kappa}(R;H)}$ is equivalent to the norm $\|u\|_{W^2_{2,\kappa}(R;H)}$ in the space $W_{2,\kappa}^2(R;H)$ and, since it is known that the operators of intermediate derivatives

$$A^{2-j}\frac{d^j}{dt^j} : W_{2,\kappa}^2(R;H) \to L_{2,\kappa}(R;H), \quad j=0,1,$$

are continuous, the norms of these operators can be estimated via $\|\Gamma_0 u\|_{L_{2,\kappa}(R;H)}$.

The following theorem is true.

Theorem 2. *The following inequalities are valid for any* $u(t) \in W_{2,\kappa}^2(R;H)$ *and* $|\kappa| < 2\lambda_0$:

$$\left\|A^2u\right\|_{L_{2,\kappa}(R;H)} \leq b_0(\kappa)\,\|\Gamma_0 u\|_{L_{2,\kappa}(R;H)}\,, \tag{9}$$

$$\left\|A\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} \leq b_1(\kappa)\,\|\Gamma_0 u\|_{L_{2,\kappa}(R;H)}\,, \tag{10}$$

where $b_0(\kappa)$ *is defined in (8), and*

$$b_1(\kappa) = \begin{cases} \dfrac{\lambda_0}{2^{1/2}(2\lambda_0^2 - \kappa^2)^{1/2}}, & \text{if} \quad 0 \le \dfrac{\kappa^2}{4\lambda_0^2} < \dfrac{1}{3}, \\ \dfrac{2\lambda_0|\kappa|}{4\lambda_0^2 - \kappa^2}, & \text{if} \quad \dfrac{1}{3} \le \dfrac{\kappa^2}{4\lambda_0^2} < 1. \end{cases} \tag{11}$$

P r o o f. If we denote by $\vartheta(t) = u(t)e^{-\frac{\kappa}{2}t}$, then inequality (9) has the following form:

$$\left\|A^2\vartheta\right\|_{L_2(R;H)} \le b_0(\kappa) \left\|\Gamma_{0,\kappa}\vartheta\right\|_{L_2(R;H)}. \tag{12}$$

It follows from equality (7) that $(\widehat{g}(\xi) = \Gamma_{0,\kappa}\widehat{\vartheta}(\xi))$ and

$$\left\|A^2\widehat{\vartheta}(\xi)\right\|_{L_2(R;H)} \le b_0(\kappa) \left\|\Gamma_{0,\kappa}\widehat{\vartheta}(\xi)\right\|_{L_2(R;H)},$$

which is equivalent to inequality (12). That's why inequality (9) is true. Let us prove inequality (10). Substituting $u(t) = \vartheta(t)e^{\frac{\kappa}{2}t}$, inequality (10) can be rewritten in the equivalent form

$$\left\|A\left(\frac{d}{dt} + \frac{\kappa}{2}\right)\vartheta\right\|_{L_2(R;H)} \le b_1(\kappa) \left\|\Gamma_{0,\kappa}\vartheta\right\|_{L_2(R;H)}. \tag{13}$$

We show that (13) is true. Substituting $\Gamma_{0,\kappa}\vartheta(t) = g(t)$ and applying the Fourier transformation, we obtain

$$\left\|A\left(-i\xi + \frac{\kappa}{2}\right)\Gamma_{0,\kappa}^{-1}(-i\xi; A)\widehat{g}(\xi)\right\|_{L_2(R;H)}$$

$$= \left\|A\left(-i\xi + \frac{\kappa}{2}\right)\left(-\left(-i\xi + \frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\widehat{g}(\xi)\right\|_{L_2(R;H)}$$

$$\le \sup_{\xi \in R}\left\|A\left(-i\xi + \frac{\kappa}{2}\right)\left(-\left(-i\xi + \frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\right\| \left\|\widehat{g}(\xi)\right\|_{L_2(R;H)}. \tag{14}$$

Thus we estimate the following norm for $\xi \in R$

$$\left\|A\left(-i\xi + \frac{\kappa}{2}\right)\left(-\left(-i\xi + \frac{\kappa}{2}\right)^2 E + A^2\right)^{-1}\right\|$$

$$= \sup_{\lambda \in \sigma(A)}\left|\lambda\left(-i\xi + \frac{\kappa}{2}\right)\left(-\left(-i\xi + \frac{\kappa}{2}\right)^2 + \lambda^2\right)^{-1}\right|$$

$$= \sup_{\lambda\in\sigma(A)} \frac{\lambda\left(\xi^2+\frac{\kappa^2}{4}\right)^{1/2}}{\left(\left(\xi^2+\lambda^2-\frac{\kappa^2}{4}\right)^2+\xi^2\kappa^2\right)^{1/2}}$$

$$\le \sup_{\lambda\in\sigma(A)} \frac{\lambda\left(\xi^2+\frac{\kappa^2}{4}\right)^{1/2}}{\xi^2+\lambda^2-\frac{\kappa^2}{4}} \le \sup_{\lambda\in\sigma(A)} \frac{\left(\frac{\xi^2}{\lambda^2}+\frac{\kappa^2}{4\lambda_0^2}\right)^{1/2}}{\frac{\xi^2}{\lambda^2}+1-\frac{\kappa^2}{4\lambda_0^2}} \le \sup_{r\ge 0} \frac{\left(r+\frac{\kappa^2}{4\lambda_0^2}\right)^{1/2}}{r+1-\frac{\kappa^2}{4\lambda_0^2}}.$$

We denote

$$\theta(r) = \frac{\left(r+\frac{\kappa^2}{4\lambda_0^2}\right)^{1/2}}{r+1-\frac{\kappa^2}{4\lambda_0^2}}, r\ge 0,\ |\kappa|<2\lambda_0.$$

Then,

$$\theta'(r) = \frac{\frac{1}{2}\left(r+\frac{\kappa^2}{4\lambda_0^2}\right)^{-1/2}\left(1-r-\frac{3\kappa^2}{4\lambda_0^2}\right)}{\left(r+1-\frac{\kappa^2}{4\lambda_0^2}\right)^2}.$$

It is clear that if

$$\frac{1}{3} \le \frac{\kappa^2}{4\lambda_0^2} < 1,$$

then $\theta'(r) < 0$, because $\theta(r)$ attains its maximum at zero, i.e., if

$$\frac{1}{3} \le \frac{\kappa^2}{4\lambda_0^2} < 1,$$

then

$$\max \theta(r) = \theta(0) = \frac{2\lambda_0|\kappa|}{4\lambda_0^2-\kappa^2}.$$

If $0 \le \frac{\kappa^2}{4\lambda_0^2} < \frac{1}{3}$, then the function $\theta(r)$ attains its maximum at $r_0 = 1 - \frac{3\kappa^2}{4\lambda_0^2}$. In this case

$$\max \theta(r) = \theta(r_0) = \frac{\lambda_0}{2^{1/2}(2\lambda_0^2-\kappa^2)^{1/2}}.$$

Taking into account the expressions in (14), we obtain

$$\left\| A\left(-i\xi+\frac{\kappa}{2}\right)\Gamma_{0,\kappa}^{-1}(-i\xi;A)\widehat{g}(\xi)\right\|_{L_2(R;H)} \le b_1(\kappa)\left\|\widehat{g}(\xi)\right\|_{L_2(R;H)},$$

which is equivalent to inequality (13) and which, in its turn, is equivalent to the inequality

$$\left\| A\frac{du}{dt}\right\|_{L_{2,\kappa}(R;H)} \le b_1(\kappa)\left\|\Gamma_0 u\right\|_{L_{2,\kappa}(R;H)},$$

where

$$b_1(\kappa) = \begin{cases} \dfrac{\lambda_0}{2^{1/2}(2\lambda_0^2 - \kappa^2)^{1/2}} & \text{for} \quad 0 \le \dfrac{\kappa^2}{4\lambda_0^2} < \dfrac{1}{3}, \\ \dfrac{2\lambda_0|\kappa|}{4\lambda_0^2 - \kappa^2} & \text{for} \quad \dfrac{1}{3} \le \dfrac{\kappa^2}{4\lambda_0^2} < 1. \end{cases}$$

Theorem 2 is proved.

Remark 1. *The operator Γ_0 is not invertible for $\kappa = \pm 2\lambda_0$.*

Before obtaining the exact conditions of regular solvability for the operator differential equation (1), we formulate the following lemma.

Lemma 1. *Suppose that the conditions of Th. 1 are satisfied and the operators $S_j(t) = A_j(t)A^{-j} \in L_\infty(R; L(H,H))$, $j = 1,2$. Then the operator Γ, corresponding to the left-hand side of equation (1), is a continuous map from the space $W_{2,\kappa}^2(R;H)$ into $L_{2,\kappa}(R;H)$.*

P r o o f. Since for any $u(t) \in W_{2,\kappa}^2(R;H)$

$$\|\Gamma u\|_{L_{2,\kappa}(R;H)} \le \|\Gamma_0 u\|_{L_{2,\kappa}(R;H)} + \left\| \sum_{j=1}^{2} A_j(t) \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R;H)}$$

$$\le \|\Gamma_0 u\|_{L_{2,\kappa}(R;H)} + \sum_{j=1}^{2} \sup_t \|S_j(t)\|_{H\to H} \left\| A^j \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R;H)},$$

then, taking into account Th. 1 and the theorem on the intermediate derivatives [1, Ch. 1], we have from the last inequality

$$\|\Gamma u\|_{L_{2,\kappa}(R;H)} \le const \, \|u\|_{W_{2,\kappa}^2(R;H)}.$$

Lemma is proved.

Now we prove the main theorem of the paper, i.e., the theorem on the regular solvability of equation (1). It should be noted that conditions of solvability are expressed only in terms of the operator coefficients of (1).

Theorem 3. *Suppose that the conditions of Lem. 1 are satisfied, and*

$$\sum_{j=1}^{2} c_j(\kappa) \sup_t \|S_j(t)\| < 1,$$

where the numbers $c_j(\kappa) = b_{2-j}(\kappa)$, $j = 1,2$, are defined in Th. 2. Then equation (1) is regularly solvable.

P r o o f. We present equation (1) in the form of the following operator equation

$$\Gamma_0 u(t) + (\Gamma - \Gamma_0) u(t) = f(t), \tag{15}$$

where $f(t) \in L_{2,\kappa}(R;H)$, $u(t) \in W^2_{2,\kappa}(R;H)$.

The equation $\Gamma_0 u(t) = f(t)$ is regularly solvable by Th. 1. After substitution $\Gamma_0 u(t) = w(t)$, equation (15) can be written in the form $(E + (\Gamma - \Gamma_0)\Gamma_0^{-1})w(t) = f(t)$. Then for any $w(t) \in L_{2,\kappa}(R;H)$ we have by Th. 2:

$$\left\|(\Gamma - \Gamma_0)\Gamma_0^{-1} w\right\|_{L_{2,\kappa}(R;H)} = \|(\Gamma - \Gamma_0)u\|_{L_{2,\kappa}(R;H)}$$

$$= \left\| \sum_{j=1}^{2} A_j(t) \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R;H)} \le \sum_{j=1}^{2} \left\| A_j(t) \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R;H)}$$

$$\le \sum_{j=1}^{2} \sup_t \|S_j(t)\|_{H\to H} \left\| A^j \frac{d^{2-j}u}{dt^{2-j}} \right\|_{L_{2,\kappa}(R;H)}$$

$$\le \sum_{j=1}^{2} \sup_t \|S_j(t)\|_{H\to H}\, c_j(\kappa)\, \|\Gamma_0 u\|_{L_{2,\kappa}(R;H)}$$

$$= \sum_{j=1}^{2} c_j(\kappa) \sup_t \|S_j(t)\|_{H\to H} \|w\|_{L_{2,\kappa}(R;H)}\,.$$

Since $\sum\limits_{j=1}^{2} c_j(\kappa) \sup\limits_t \|S_j(t)\|_{H\to H} < 1$, the operator $E + (\Gamma - \Gamma_0)\Gamma_0^{-1}$ is invertible in the space $L_{2,\kappa}(R;H)$. Then, $u(t)$ can be defined by the following formula

$$u(t) = \Gamma_0^{-1}(E + (\Gamma - \Gamma_0)\Gamma_0^{-1})^{-1} f(t).$$

It follows that

$$\|u\|_{W^2_{2,\kappa}(R;H)} \le const\, \|f\|_{L_{2,\kappa}(R;H)}\,.$$

Theorem 3 is proved.

Corollary 1. *Assume that* $\kappa = 0$ *and the inequality*

$$\frac{1}{2} \sup_t \left\|A_1(t) A^{-1}\right\| + \sup_t \left\|A_2(t) A^{-2}\right\| < 1$$

holds. Then the operator Γ *is an isomorphism between the spaces* $W_2^2(R;H)$ *and* $L_2(R;H)$ *(see [4, 5]).*

The Author is thankful to Acad. of NAS of Azerbaijan M.G. Gasimov and Prof. S.S. Mirzoyev for discussion of the paper.

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