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On the Discrete Spectrum of Complex Banded Matrices

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The discrete spectrum of complex banded matrices that are compact perturbations of the standard banded matrix of order p is under consideration. The rate of stabilization for the matrix entries sharp in the sense of order which provides finiteness of the discrete spectrum is found. The p-banded matrix with the discrete spectrum having exactly p limit points on the interval $(-2,2)$ is constructed. The results are applied to study the discrete spectrum of asymptotically periodic Jacobi matrices.

Key words: banded matrices, discrete spectrum, asymptotically periodic Jacobi matrices.
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1. Introduction

In the recent papers [1, 2] I. Egorova and L. Golinskii studied the discrete spectrum of complex Jacobi matrices such that the operators in $\ell^2(\mathbb{N})$, $\mathbb{N} := \{1, 2, \ldots\}$ generated by these matrices are compact perturbations of the discrete laplacian. In turn, these papers are the discrete version of the known Pavlov theorems ([3, 4]) for the differential operators of the second order on the semiaxis. The sufficient conditions for the spectrum to be finite and empty, the domains containing the discrete spectrum and the conditions for the limit sets of the discrete spectrum were found. The goal of this work is to extend the results to the case of operators, generated by banded matrices.

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Let us remind that an infinite matrix $D = \|d_{ij}\|_{i,j=1}^{\infty}$ is called the *banded matrix of order* p or just p-banded if

$$d_{ij} = 0, \quad |i-j| > p, \qquad d_{ij} \neq 0, \quad |i-j| = p, \qquad d_{ij} \in \mathbb{C}. \tag{1.1}$$

According to this definition, the Jacobi matrices are banded matrices of order $p = 1$. Throughout the whole paper we assume that

$$\lim_{i\to\infty} d_{i,i\pm p} = 1, \qquad \lim_{i\to\infty} d_{i,i\pm r} = 0, \quad |r| < p, \tag{1.2}$$

and so the operators in $\ell^2 = \ell^2(\mathbb{N})$ generated by matrices (1.1)–(1.2) are compact perturbation of the standard banded operator

$$D_0 : \qquad d_{i,i\pm p} = 1, \qquad d_{ij} = 0, \quad |i-j| \neq p, \qquad D_0 = S^p + (S^*)^p, \tag{1.3}$$

where S is the one-sided shift operator in ℓ^2. It is well known that the spectrum $\sigma(D_0)$ of D_0 is the closed interval $[-2, 2]$. According to the Weyl theorem (see, e.g., [5]) the spectrum of the perturbed operator $\sigma(D) = [-2, 2] \bigcup \sigma_{\mathrm{d}}(D)$, where the discrete spectrum $\sigma_{\mathrm{d}}(D)$ is at most denumerable set of points of the complex plane, which are eigenvalues of finite algebraic multiplicity. All its accumulation points belong to the interval $[-2, 2]$. Let us denote by E_D the limit set for the set $\sigma_{\mathrm{d}}(D)$. So, $E_D = \emptyset$ means that the discrete spectrum is finite.

Remind that the *convergence exponent* or *Taylor–Besicovitch index* of a closed point set $F \subset [-2, 2]$ is the value

$$\tau(F) := \inf\{\varepsilon > 0 : \sum_{j=1}^{\infty} |l_j|^{\varepsilon} < \infty\},$$

where $\{l_j\}$ are the adjacent intervals of F.

Definition 1.1. *We say that the matrix* D (1.1) *belongs to the class* $\mathcal{P}_p(\beta)$*,* $0 < \beta < 1$*, if*

$$q_n := |d_{n,n-p} - 1| + \sum_{r=-p+1}^{p-1} |d_{n,n+r}| + |d_{n,n+p} - 1| \le C_1 \exp(-C_2\, n^{\beta}), \tag{1.4}$$

$n \in \mathbb{N}$*, with the constants* C_1*,* $C_2 > 0$*, depending on* D*.*

The main result of the present paper is the following.

Theorem 1.2. *Let* $D \in \mathcal{P}_p(\beta)$ *where* $0 < \beta < \frac{1}{2}$*. Then* E_D *is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies*

$$\dim E_D \le \tau(E_D) \le \frac{1-2\beta}{1-\beta}\,, \tag{1.5}$$

where $\dim E_D$ *is the Hausdorff dimension of* E_D*. Moreover, if* $D \in \mathcal{P}(\frac{1}{2})$ *then* $E_D = \emptyset$*, i.e., the discrete spectrum is finite.*

It turns out that the exponent 1/2 in Th. 1.2 is sharp in the following sense.

Theorem 1.3. *For arbitrary* $\varepsilon > 0$ *and arbitrary points* $\nu_1, \nu_2, \dots, \nu_p \in (-2, 2)$ *there exists an operator* $D \in \mathcal{P}_p(\frac{1}{2} - \varepsilon)$ *such that its discrete spectrum* $\sigma_{\mathrm{d}}(J)$ *is infinite and, moreover,*

$$E_D = \{\nu_1, \nu_2, \dots, \nu_p\}.$$

The Theorems 1 and 2 are proved in Sect. 4, where the domains containing $\sigma_{\mathrm{d}}(D)$ are also found (under the different assumptions than (1.4)). In Sect. 2 the connection is established between the discrete spectrum and zeros of the determinant constructed of p linearly independent solutions of the linear difference equations for the eigenvector. In Sect. 3 the properties of the Jost matrix solutions are studied. Finally, in the last Sects. 5 and 6 the main results are applied to study the spectrum of doubly-infinite complex banded matrices and the spectrum of the doubly-infinite asymptotically p-periodic complex Jacobi matrices.

2. The Determinants of Independent Solutions and the Eigenvalues

We start out with the equation

$$D\vec{y} = \lambda \vec{y} \tag{2.1}$$

for generalized eigenvectors $\vec{y} = \{y_n\}_{n \ge 1}$ in the coordinate form:

$$\begin{cases} d_{11}\, y_1 + d_{12}\, y_2 + \ldots + d_{1,p+1}\, y_{p+1} = \lambda y_1, \\ d_{21}\, y_1 + d_{22}\, y_2 + \ldots + d_{2,p+2}\, y_{p+2} = \lambda y_2, \\ \dots \\ d_{p,1}\, y_1 + d_{p,2}\, y_2 + \ldots + d_{p,2p}\, y_{2p} = \lambda y_p, \\ d_{n,n-p} y_{n-p} + d_{n,n-p+1} y_{n-p+1} + \ldots + d_{n,n+p} y_{n+p} = \lambda y_n, n = p+1, p+2, \dots . \end{cases} \tag{2.2}$$

It is advisable to define coefficients $d_{i,j}$ for the indices with $\min(i, j) \le 0$ as follows:

$$d_{ij} = 1, \quad |i - j| = p, \qquad d_{ij} = 0, \quad |i - j| \ne p, \tag{2.3}$$

and so system (2.2) is equivalent to

$$d_{n,n-p}\, y_{n-p} + d_{n,n-p+1}\, y_{n-p+1} + \ldots + d_{n,n+p}\, y_{n+p} = \lambda y_n, \quad n \in \mathbb{N}, \tag{2.4}$$

with the initial conditions

$$y_{1-p} = y_{2-p} = \ldots = y_0 = 0. \tag{2.5}$$

Thus, the vector $\vec{y} = \{y_n\}_{n \geq 1-p} \in \ell^2$ is the eigenvector of the operator D corresponding to the eigenvalue λ if and only if $\{y_n\}$ satisfies (2.4), (2.5).

It seems natural to analyze equation (2.4) within the framework of the general theory of linear difference equations. The equation

$$x(n+k) + a_1(n)x(n+k-1) + \ldots + a_k(n)y(n) = 0 \tag{2.6}$$

is said to belong to the Poincaré class if $a_k(n) \neq 0$ and there exist limits (in $\mathbb{C}$)

$$b_j = \lim_{n\to\infty} a_j(n), \quad j = 1, 2, \ldots, k.$$

Denote by $\{w_j\}_{j=1}^k$ all the roots (counting the multiplicity) of the characteristic equation

$$w^k + b_1 w^{k-1} + \ldots + b_k = 0. \tag{2.7}$$

One of the cornerstones of the theory of linear difference equations is the following result due to Perron.

Theorem ([6, Satz 3]). *Let the roots $\{w_j\}$ of* (2.7) *lie on the circles $\Gamma_l = \{|w| = \rho_l\}$, $l = 1, 2, \ldots, m$, $\rho_j \neq \rho_k$, and exactly $v_l \geq 1$ of them* (*counted according to their multiplicity*) *belongs to each circle Γ_l, so $\nu_1 + \ldots + \nu_m = k$. Then* (2.6) *has a fundamental system of solutions*

$$S = \{y_1, \ldots, y_k\} = \bigcup_{l=1}^m S_l\,,$$

the sets $\{S_l\}$ are disjoint, $|S_l| = \nu_l$, and for any nontrivial linear combination $y(n)$ of the solutions from S_l

$$\limsup_{n\to\infty} \sqrt[n]{|y(n)|} = \rho_l, \qquad l = 1, 2, \ldots, m, \tag{2.8}$$

holds.

Proposition 2.1. *For any $\lambda \in \mathbb{C}\backslash[-2,2]$ the dimension of the space of ℓ^2-solutions of* (2.4) *equals p.*

P r o o f. Note that equation (2.4) has order $k = 2p$ (after dividing through by the leading coefficient $d_{n,n+p}$) and belongs to the Poincaré class by assumption (1.2). Its characteristic equation (2.7) has now the form

$$w^{2p} - \lambda w^p + 1 = (w^p - z)(w^p - z^{-1}) = 0\,: \quad \lambda = z + z^{-1},\ z < 1\,.$$

For its roots we have

$$|w_1| = \ldots = |w_p| = |z| < 1 < |z|^{-1} = |w_{p+1}| = \ldots = |w_{2p}|.$$

By the Perron theorem, there exists the fundamental system S of the solutions of (2.4)

$$S = \{y_1, \dots, y_p; y_{p+1}, \dots, y_{2p}\} = S_1 \cup S_2\,; \quad \dim \operatorname{span} S_1 = \dim \operatorname{span} S_2 = p\,,$$

and each solution $y \in \operatorname{span} S_1$ is in ℓ^2 (and even decreases exponentially fast). Let now y be any solution of (2.4) from ℓ^2,

$$y = \sum_{j=1}^{p} c_j y_j + \sum_{j=p+1}^{2p} c_j y_j = y' + y''\,.$$

But $y' \in \ell^2$, and so $y'' \in \ell^2$ which by (2.8) and $|z|^{-1} > 1$ is possible only when $c_j = 0$ for $j = p+1, \dots, 2p$, as needed. ■

Proposition 2.2. *Let $\{y_n^{(i)}\}_{n\ge 1-p}$, $i = 1, 2, \dots, p$, be linearly independent solutions of* (2.4) *from ℓ^2. The number λ is an eigenvalue of the operator D if and only if*

$$\det Y_0(\lambda) = \begin{vmatrix} y_{1-p}^{(1)} & y_{2-p}^{(1)} & \dots & y_0^{(1)} \\ \dots & \dots & \dots & \dots \\ y_{1-p}^{(p)} & y_{2-p}^{(p)} & \dots & y_0^{(p)} \end{vmatrix} = 0. \tag{2.9}$$

P r o o f. Suppose that $\det Y_0(\lambda) = 0$. Then there are numbers $\alpha^{(1)}, \dots, \alpha^{(p)}$, which do not vanish simultaneously, such that

$$\begin{cases} \alpha^{(1)} y_{1-p}^{(1)} & + \dots & + \alpha^{(p)} y_{1-p}^{(p)} = 0\,, \\ \dots & \dots & \dots \\ \alpha^{(1)} y_0^{(1)} & + \dots & + \alpha^{(p)} y_0^{(p)} = 0\,. \end{cases}$$

Hence the linear combination

$$y_n = \alpha^{(1)} y_n^{(1)} + \dots + \alpha^{(p)} y_n^{(p)}, \qquad n \ge 1-p, \tag{2.10}$$

belongs to ℓ^2 and satisfies (2.4), (2.5), i.e., λ is an eigenvalue of the operator D.

Conversely, let λ be an eigenvalue and $y = \{y_n\}_{n\ge 1}$ a corresponding eigenvector. Then $\{y_n\}_{n\ge 1-p}$ is an ℓ^2-solution of (2.4) with the initial conditions (2.5). By Prop. 2.1 (2.10) holds with coefficients $\alpha^{(1)}, \dots, \alpha^{(p)}$ which do not vanish simultaneously. Then (2.9) follows immediately from (2.5). ■

3. The Matrix-Valued Jost Solution

The goal of this section is to establish the existence of matrix-valued analogue of the Jost solution for the banded matrix D. Once we have the Jost matrix solution at our disposal, we will be able to construct p linearly independent square-summable solutions of (2.4) and, in view of Prop. 2.2, to reduce the study of the location of the discrete spectrum for the matrix D to the location of the zeros for determinant (2.9), composed of these p solutions.

It is convenient to rewrite the initial equation in the form of a three-term recurrence matrix relation, by looking at D as a block-Jacobi matrix. Along this way we can extend the standard techniques of proving the existence of the Jost solution for the Jacobi matrices to the case of banded matrices.

Define the following $p \times p$-matrices:

$$A_k = \begin{pmatrix} d_{(k-1)p+1,\,(k-2)p+1} & \dots & d_{(k-1)p+1,\,(k-1)p} \\ \vdots & & \vdots \\ d_{kp,\,(k-2)p+1} & \dots & d_{kp,\,(k-1)p} \end{pmatrix},$$
$$B_k = \begin{pmatrix} d_{(k-1)p+1,\,(k-1)p+1} & \dots & d_{(k-1)p+1,\,kp} \\ \vdots & & \vdots \\ d_{kp,\,(k-1)p+1} & \dots & d_{kp,\,kp} \end{pmatrix}, \tag{3.1}$$
$$C_k = \begin{pmatrix} d_{(k-1)p+1,\,kp+1} & \dots & d_{(k-1)p+1,\,(k+1)p} \\ \vdots & & \vdots \\ d_{kp,\,kp+1} & \dots & d_{kp,\,(k+1)p} \end{pmatrix}.$$

Then the matrix D can be represented in the form

$$D = \begin{pmatrix} B_1 & C_1 & 0 & 0 & \dots \\ A_2 & B_2 & C_2 & 0 & \dots \\ 0 & A_3 & B_3 & C_3 & \dots \\ 0 & 0 & A_4 & B_4 & \dots \\ \dots & \dots & \dots & \ddots & \ddots \end{pmatrix}, \tag{3.2}$$

with the upper triangular matrices A_k and the lower triangular C_k which are invertible due to (1.1). To be consistent with (2.3) we put $A_1 = C_0 = I$ a unit $p \times p$ matrix, $B_0 = 0$.

Having p solutions $\{\varphi_j^{(l)}\}_{j \ge 1-p}$, $l = 1, 2, \dots, p$ of equation (2.4) at hand, we can make up $p \times p$ matrices

$$\Psi_j = \begin{pmatrix} \varphi^{(1)}_{(j-1)p+1} & \dots & \varphi^{(p)}_{(j-1)p+1} \\ \vdots & & \vdots \\ \varphi^{(1)}_{jp} & \dots & \varphi^{(p)}_{jp} \end{pmatrix}, \quad j \in \mathbb{Z}_+ := 0, 1, \dots . \tag{3.3}$$

and write (2.4) in the matrix form

$$A_k\Psi_{k-1} + B_k\Psi_k + C_k\Psi_{k+1} = \lambda\Psi_k, \quad k \in \mathbb{N}. \tag{3.4}$$

It will be convenient to modify equation (3.4), getting rid of the coefficients A_k's. Suppose that

$$\sum_{k=1}^{\infty} \|I - A_k\| < \infty\,, \tag{3.5}$$

where $\|\cdot\|$ is any norm in the space of matrices. It is well known, that there exists an infinite product (from the right to the left)

$$A := \prod_{j=1}^{\infty} A_j = \lim_{n\to\infty} (A_k A_{k-1} \dots A_1)\,,$$

and all the matrices A and A_n are invertible. Denote

$$L_j := \prod_{i=j+1}^{\infty} A_j, \qquad L_j A_j = L_{j-1}, \qquad \lim_{j\to\infty} L_j = I. \tag{3.6}$$

The multiplication of (3.4) from the left by L_k gives

$$L_{k-1}\Psi_{k-1} + L_k B_k\Psi_k + L_k C_k\Psi_{k+1} = \lambda L_k\Psi_k\,,$$

$$L_{k-1}\Psi_{k-1} + L_k B_k L_k^{-1}\cdot L_k\Psi_k + L_k C_k L_{k+1}^{-1} L_{k+1}\Psi_{k+1} = \lambda L_k\Psi_k\,.$$

Hence the matrices

$$\Phi_k := L_k\Psi_k \tag{3.7}$$

satisfy

$$\Phi_{k-1} + \tilde{B}_k\Phi_k + \tilde{C}_k\Phi_{k+1} = \lambda\Phi_k\,, \qquad k \in \mathbb{N}, \tag{3.8}$$

with

$$\tilde{B}_k = L_k B_k L_k^{-1}, \qquad \tilde{C}_k := L_k C_k L_{k+1}^{-1}\,. \tag{3.9}$$

For the definiteness sake we choose the "row norm"

$$\|T\| := \max_{1\le k\le p} \sum_{j=1}^{p} |t_{kj}|, \qquad T = \{t_{kj}\}_{k,j=1}^{p}.$$

Then by (3.1)

$$\max\bigl(\|A_k - I\|, \|B_k\|, \|C_k - I\|\bigr) \le \hat{q}_k := \max_{1\le j\le p} (q_{(k-1)p+j}), \quad k \in \mathbb{N}, \tag{3.10}$$

q_k are defined in (1.4). In accordance with (2.3) $q_{1-p} = \ldots = q_0 = 0$, so we put $\hat{q}_0 = 0$. It is clear that

$$\sum_{k=1}^{\infty} \hat{q}_k \le \sum_{k=1}^{\infty} q_k \le p \sum_{k=1}^{\infty} \hat{q}_k \tag{3.11}$$

so (3.5) holds whenever $\{q_n\} \in \ell^1$.

We will use the complex parameter z related to the spectral parameter λ by the Zhukovsky transform:

$$\lambda = z + z^{-1}; \quad |z| < 1\,.$$

Denote by g the Green kernel

$$g(n,k,z) = \begin{cases} \dfrac{z^{k-n} - z^{n-k}}{z - z^{-1}}, & k > n, \\ 0, & k \le n, \end{cases} \qquad n,k \in \mathbb{Z}_+ := \{0,1,\ldots\},\ z \ne 0. \tag{3.12}$$

It is clear that $g(n,k,z)$ satisfies the recurrence relations

$$g(n,k+1,z) + g(n,k-1,z) - (z+z^{-1})g(n,k,z) \ = \delta(n,k), \tag{3.13}$$
$$g(n-1,k,z) + g(n+1,k,z) - (z+z^{-1})g(n,k,z) \ = \delta(n,k), \tag{3.14}$$

where $\delta(n,k)$ is the Kronecker symbol.

We proceed with the following conditional result.

Proposition 3.1. *Suppose that equation* (3.8) *has a solution* V_n *with the asymptotic behavior at infinity*

$$\lim_{n\to\infty} V_n(z) z^{-n} = I \tag{3.15}$$

for $z \in \mathbb{D}$. *Then* V_n *satisfies the discrete integral equation*

$$V_n(z) = z^n I + \sum_{k=n+1}^{\infty} J(n,k,z)\, V_k(z), \quad n \in \mathbb{N}, \tag{3.16}$$

with

$$J(n,k,z) = -g(n,k,z)\tilde{B}_k + g(n,k-1,z)\left(I - \tilde{C}_{k-1}\right). \tag{3.17}$$

P r o o f. Let us multiply (3.13) by V_k, (3.8) for V_k by $g(n,k)$, and subtract the latter from the former

$$g(n,k+1)V_k + g(n,k-1)V_k - g(n,k)V_{k-1} - g(n,k)\tilde{B}_k V_k - g(n,k)\tilde{C}_k V_{k+1} = \delta(n,k)V_k.$$

Summing up over k from n to N gives

$$V_n = \sum_{k=n+1}^{N} \left\{-g(n,k)\tilde{B}_k + g(n,k-1)\left(I - \tilde{C}_{k-1}\right)\right\} V_k$$
$$+ g(n,N+1)V_N - g(n,N+1)\tilde{C}_N V_{N+1}.$$

For $|z| < 1$ we have by (3.12) and (3.15)

$$\lim_{N\to\infty} \left(g(n,N+1)V_N - g(n,N)\tilde{C}_N V_{N+1}\right) = z^n I,$$

which along with $J(n,n) = 0$ leads to (3.16), as needed. ■

The converse statement is equally simple.

Proposition 3.2. *Each solution* $\{V_n(z)\}_{n\ge 0}$, $z \in \overline{\mathbb{D}}$, *of equation* (3.16) *with* $n \in \mathbb{Z}_+$ *satisfies the three-term recurrence relation* (3.8).

P r o o f. Write for $n \ge 1$

$$V_{n-1} + V_{n+1} = (z^{n-1} + z^{n+1})I + \sum_{k=n}^{\infty} J(n-1,k)V_k + \sum_{k=n+2}^{\infty} J(n+1,k)V_k$$
$$= (z + z^{-1})z^n I + J(n-1,n)V_n + J(n-1,n+1)V_{n+1}$$
$$+ \sum_{k=n+2}^{\infty} \{J(n-1,k) + J(n+1,k)\}V_k.$$

By (3.12), (3.17) and (3.14)

$$J(n-1,n) = -\tilde{B}_n, \quad J(n-1,n+1) = -(z+z^{-1})\tilde{B}_{n+1} + I - \tilde{C}_n$$

and

$$J(n-1,k) + J(n+1,k) = (z+z^{-1})J(n,k), \quad k \ge n+2.$$

Hence

$$V_{n-1} + V_{n+1} + \tilde{B}_n V_n - (I - \tilde{C}_n)V_{n+1} = (z+z^{-1})z^n I - \tilde{B}_n V_n - (z+z^{-1})\tilde{B}_n V_{n+1}$$
$$+ (I - \tilde{C}_n)V_{n+1} + \sum_{k=n+2}^{\infty} \left\{(z + z^{-1}J(n,k)\right\} V_k + \tilde{B}_n V_n - (I - \tilde{C}_n)V_{n+1}$$
$$= (z+z^{-1})\left(z^n + \sum_{k=n+1}^{\infty} J(n,k)v_k\right) = (z+z^{-1})V_n,$$

which is exactly (3.8). ■

The Jost Solution. To analyze equation (3.16) we introduce new variables

$$\tilde{V}_n(z) := z^{-n} V_n, \quad \tilde{J}(n,k,z) := z^{k-n} J(n,k,z), \tag{3.18}$$

so that, instead of (3.16), we have

$$\tilde{V}_n(z) = I + \sum_{k=n+1}^{\infty} \tilde{J}(n,k,z)\tilde{V}_k(z), \quad n \in \mathbb{Z}_+. \tag{3.19}$$

Now $\tilde{J}(n,m,\cdot)$ is a polynomial with matrix coefficients. Since

$$|g(n,k,z)z^{k-n}| = \frac{|z^{2(k-n)}-1|}{|z-z^{-1}|} \le |z| \min\left\{|k-n|, \frac{2}{|z^2-1|}\right\},$$

the kernel $\tilde{J}$ is bounded by

$$\|\tilde{J}(n,k,z)\| \le |z| \min\left\{|k-n|, \frac{2}{|z^2-1|}\right\} h_k, \quad z \in \overline{\mathbb{D}}, \tag{3.20}$$

where

$$h_k := \|\tilde{B}_k\| + \|I - \tilde{C}_{k-1}\| = \|L_k B_k L_k^{-1}\| + \|I - L_{k-1} C_{k-1} L_k^{-1}\|, \quad k \in \mathbb{N}, \tag{3.21}$$

(see (3.9)). We have

$$h_k \le \|L_k\| \|L_k^{-1}\| \left(\|B_k\| + \|I - A_k\| + \|A_k\| \|I - C_{k-1}\|\right). \tag{3.22}$$

Since $\|A_k\| \le C(D)$ and by (3.6) $\|L_k\| \cdot \|L_k^{-1}\| \le C(D)$ (throughout the rest of the paper $C = C(D)$ stands for various positive constants which depend only on p and the original matrix D), we see from (3.10) that

$$h_k \le C(D)(\hat{q}_{k-1} + \hat{q}_k). \tag{3.23}$$

The existence of the Jost solutions for equation (3.8) will be proved under the assumption

$$\sum_i q_i = \sum_i \left(|d_{i,\,i-p} - 1| + |d_{i,\,i+p} - 1| + \sum_{r=1-p}^{p-1} |d_{i,\,i+r}| \right) < \infty,$$

which by (3.10), (3.11) and (3.23) implies

$$\sum_k (\|I - A_k\| + \|B_k\| + \|I - C_k\|) < \infty, \qquad \sum_k h_k < \infty. \tag{3.24}$$

The main result concerning equation (3.16) is the following.

Theorem 3.3. *(i) Suppose that*

$$\sum_{k=1}^{\infty} q_k < \infty, \tag{3.25}$$

q_k are defined in (1.4). *Then equation* (3.16) *has a unique solution V_n, which is analytic in $\mathbb{D}$, continuous on $\mathbb{D}_1 := \overline{\mathbb{D}} \setminus \{\pm 1\}$ and* *

$$\|V_n - z^n I\| \le C|z|^n \left\{ \frac{|z|}{|z^2-1|} \sum_{k=n}^{\infty} q_k \right\} \exp\left\{ \frac{C|z|}{|z^2-1|} \sum_{k=n}^{\infty} q_k \right\} \tag{3.26}$$

for $z \in \mathbb{D}_1$, $n \in \mathbb{Z}_+$.

(ii) Suppose that

$$\sum_{k=1}^{\infty} k q_k < \infty. \tag{3.27}$$

Then V_n is analytic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and

$$\|V_n - z^n I\| \le C|z|^n \left\{ \sum_{k=n}^{\infty} k q_k \right\} \exp\left\{ C \sum_{k=n}^{\infty} k q_k \right\}, \quad z \in \mathbb{D}, \quad n \in \mathbb{Z}_+. \tag{3.28}$$

P r o o f. We apply the method of successive approximations. Write (3.19) as

$$F_n(z) = G_n(z) + \sum_{k=n+1}^{\infty} \tilde{J}(n,k,z) F_k(z) \tag{3.29}$$

with

$$F_k(z) := \tilde{V}_k(z) - 1, \quad G_n(z) := \sum_{k=n+1}^{\infty} \tilde{J}(n,k,z). \tag{3.30}$$

(i) By (3.20)

$$\|\tilde{J}(n,k,z)\| \le \phi(z)\, h_k, \quad z \in \mathbb{D}_1, \qquad \phi(z) := 2|z||z^2-1|^{-1}. \tag{3.31}$$

The series in (3.30) converges uniformly on compact subsets of $\mathbb{D}_1$ by (3.25), (3.24), and so G_n is analytic in $\mathbb{D}$ and continuous on $\mathbb{D}_1$. As a starting point for the method of successive approximation, we put $F_{n,1} = G_n$ and denote

$$F_{n,j+1}(z) := \sum_{k=n+1}^{\infty} \tilde{J}(n,k,z) F_{k,j}(z).$$

*Following the terminology of selfadjoint case for Jacobi matrices, we call this solution the *Jost solution*. The function V_0 is the matrix analogue of the *Jost function*.

Let $\sigma_0(n) := \sum_{k=n+1}^{\infty} h_k$. By induction on j we prove that

$$\|F_{n,j}(z)\| \leq \frac{(\phi(z)\sigma_0(n))^j}{(j-1)!}. \tag{3.32}$$

Indeed, for $j = 1$ we have $F_{n,1} = G_n$ and the result holds by the definition of σ_0 and (3.31). Next, let (3.32) be true. Then

$$|F_{n,j+1}(z)| \leq \phi(z) \sum_{k=n+1}^{\infty} h_k \|F_{k,j}(z)\| \leq \frac{(\phi(z))^{j+1}}{(j-1)!} \sum_{k=n+1}^{\infty} h_k \sigma_0^j(k).$$

An elementary inequality $(a+b)^{j+1} - a^{j+1} \geq (j+1)ba^j$ gives

$$\sum_{k=n+1}^{\infty} h_k \sigma_0^j(k) \leq \frac{1}{j} \sum_{k=n+1}^{\infty} \{\sigma_0^{j+1}(k-1) - \sigma_0^{j+1}(k)\} = \frac{\sigma_0^{j+1}(n)}{j},$$

which proves (3.32) for $F_{n,j+1}$. Thereby the series

$$F_n(z) = \sum_{j=1}^{\infty} F_{n,j}(z)$$

converges uniformly on compact subsets of $\mathbb{D}_1$ and solves (3.29), being analytic in $\mathbb{D}$ and continuous on $\mathbb{D}_1$. It is also clear from (3.32) that

$$\|F_n(z)\| = \|\tilde{V}_n(z) - I\| \leq \sum_{j=1}^{\infty} \|F_{n,j}(z)\| \leq \phi(z)\sigma_0(n) \exp\{\phi(z)\sigma_0(n)\}. \tag{3.33}$$

To reach (3.26) it remains only to note that by (3.23)

$$\sigma_0(n) \leq C \sum_{k=n+1}^{\infty} (\hat{q}_{k-1} + \hat{q}_k) \leq 2C \sum_{k=n}^{\infty} \hat{q}_k \leq 2C \sum_{j=n}^{\infty} q_j$$

(the latter inequality easily follows from the definition of $\hat{q}_k$).

To prove the uniqueness suppose that there are two solutions F_n and $\tilde{F}_n$ of (3.29). Take the difference and apply (3.31):

$$\begin{gathered} |F_n - \tilde{F}_n(z)| = |\sum_{k=n+1}^{\infty} \tilde{J}(n,k,z) \left(F_k(z) - \tilde{F}_k(z)\right)|, \\ s_n \leq \sum_{m=n+1}^{\infty} \phi(z) s_m h_m = r_n, \end{gathered} \tag{3.34}$$

where $s_n := \|F_n(z) - \tilde{F}_n(z)\|$.

Clearly, $r_n \to 0$ as $n \to \infty$ and if $r_m = 0$ for some m, then by (3.34) we have $s_n \equiv 0$. If $r_n > 0$, then

$$\frac{r_{n-1} - r_n}{r_n} = \frac{s_n \phi(z) h_n}{r_n} \le \phi(z) h_n, \quad r_k \le \prod_{j=k+1}^{M} (1 + \phi(z) h_j)\ r_M \tag{3.35}$$

which leads to $r_m = 0$ and again $s_n \equiv 0$. So the uniqueness is proved.
(ii) The same sort of reasoning is applicable with

$$\|\tilde{J}(n,k,z)\| \le |z||k-n|h_k \le k h_k$$

and

$$\|F_{n,j}(z)\| \le \frac{\sigma_1^j(n)}{(j-1)!}, \quad \sigma_1(n) := \sum_{k=n+1}^{\infty} k h_k \tag{3.36}$$

instead of (3.31) and (3.32), respectively. We have

$$\|F_n(z)\| = \|\tilde{V}_n(z) - I\| \le \sum_{j=1}^{\infty} \|F_{n,j}(z)\| \le \sigma_1(n) \exp\{\sigma_1(n)\} \tag{3.37}$$

and

$$\sigma_1(n) \le C \sum_{k=n+1}^{\infty} k(\hat{q}_{k-1} + \hat{q}_k) \le 2C \sum_{k=n}^{\infty} k\hat{q}_k \le 2C \sum_{j=n}^{\infty} j q_j$$

(the latter inequality easily follows from the definition of $\hat{q}_k$). ■

R e m a r k. The constants C that enter (3.23), (3.26) and (3.28) are inefficient. This circumstance makes no problem when studying the limit set for the discrete spectrum. In contrast to this case, the efficient constants are called for when dealing with the domains which contain the whole discrete spectrum. Such constants will be obtained in the next section under additional assumptions of "non asymptotic flavor".

Throughout the rest of the section we assume that condition (3.27) is satisfied. It is clear that equation (3.8) can be rewritten for the functions $\tilde{V}_n(z)$, defined in (3.18), as

$$\begin{aligned} \tilde{V}_n(z) &= (\lambda - \tilde{B}_n) z \tilde{V}_{n+1}(z) - \tilde{C}_n z^2 \tilde{V}_{n+2}(z) \\ &= (z^2 + 1 - \tilde{B}_n z)\tilde{V}_{n+1}(z) - \tilde{C}_n z^2 \tilde{V}_{n+2}(z)\,. \end{aligned} \tag{3.38}$$

Let us now expand $\tilde{V}_n(z)$ in the Taylor series taking into account definition (3.18) and (3.26)

$$\tilde{V}_n(z) = I + \sum_{j=1}^{\infty} K(n,j) z^j\,. \tag{3.39}$$

Here $\|K(n,j)\|_{n,j=1}^{\infty}$ is the operator which transforms the Jost solutions of (2.1) for $D = D_0$ to that of (2.1) for D. If we plug (3.39) into (3.38) and match the coefficients for the same powers z^j we have

$$j=1: \quad K(n,1) = K(n+1,1) - \tilde{B}_n \,,$$
$$j=2: \quad K(n,2) = I + K(n+1,2) - \tilde{B}_n K(n+1,1) - \tilde{C}_n \,,$$
$$j \geq 2: \quad K(n,j+1)$$
$$= K(n+1,j-1) + K(n+1,j+1) - \tilde{B}_n K(n+1,j) - \tilde{C}_n K(n+2,j-1).$$

Summing up each of these expressions for $k = n, n+1, \ldots$, it is not hard to verify that

$$K(n,1) = -\sum_{k=n+1}^{\infty} \tilde{B}_{k-1} \,, \tag{3.40}$$

$$K(n,2) = -\sum_{k=n+1}^{\infty} \left\{ \tilde{B}_{k-1} K(k,1) + (\tilde{C}_{k-1} - I) \right\} , \tag{3.41}$$

$$K(n,j+1) = K(n+1,j-1) - \sum_{k=n+1}^{\infty} \left\{ \tilde{B}_{k-1} K(k,j) + \left(\tilde{C}_{k-1} - I\right) K(k+1,j-1) \right\} . \tag{3.42}$$

In the last step we used $K(n,j) \to 0$ for $n \to \infty$ and any fixed j, which follows from the Cauchy inequality and (3.28)

$$\|K(n,j)\| \leq \max \|\tilde{V}_n(z) - I\| \leq C \sum_{k=n}^{\infty} k q_k \,.$$

From (3.40)–(3.42), using the induction on j, we obtain

$$\|K(n,j)\| \leq \kappa(n,j) \kappa\left(n + \left[\frac{j}{2}\right]\right), \quad n \in \mathbb{Z}_+, \tag{3.43}$$

where $\kappa(n)$ and $\kappa(n,m)$ are defined by

$$\kappa(n) := \sum_{j=n}^{\infty} g_j, \quad \kappa(n,m) := \prod_{j=n+1}^{n+m-1} (1 + \kappa(j)) = \prod_{j=1}^{m-1} (1 + \kappa(n+j)) \,, \tag{3.44}$$

and

$$g_j = \|\tilde{B}_j\| + \|I - \tilde{C}_j\|. \tag{3.45}$$

In fact, for $j = 1$ we have $\kappa(n,1) = 1$, $\left[\frac{1}{2}\right] = 0$ and $\|K(n,1)\| \le \kappa(n+1) \le \kappa(n)$. Further, for $j = 2$,

$$\begin{aligned}
\|K(n,2)\| &\le \sum_{k=n+1}^{\infty} \left\{\|\tilde{B}_{k-1}\|\|K(k,1)\| + \|\tilde{C}_{k-1} - I\|\right\} \\
&\le \sum_{k=n+1}^{\infty} \left\{(\|K(k,1)\| + 1)\left(\|\tilde{B}_{k-1}\| + \|\tilde{C}_{k-1} - I\|\right)\right\| \\
&\le \sum_{k=n+1}^{\infty} g_k\,(1+\kappa(k)) \le (1+\kappa(n+1))\,\kappa(n+1) = \kappa(n,2)\kappa(n+1).
\end{aligned}$$

When we pass from the even $j = 2l$ to the odd $2l+1$, we have by (3.42) and by the inductive hypothesis:

$$\|K(n,2l+1)\| \le \kappa(n+1,2l-1))\kappa(n+l)$$
$$+ \sum_{k=n+1}^{\infty} \left\{\|\tilde{B}_{k-1}\|\|K(k,2l)\| + \|\tilde{C}_{k-1} - I\|\|K(k+1,2l-1)\|\right\}.$$

But, according to the inductive hypothesis,

$$K(k,2l)\| \le \kappa(k,2l)\kappa(k+l)\,;$$
$$\|K(k+1,2l-1)\| \le \kappa(k+1,2l-1)\kappa(k+l) \le \kappa(k,2l)\kappa(k+l).$$

Using these inequalities, we obtain

$$\begin{aligned}
\|K(n,2l+1)\| &\le \kappa(n+1,2l-1)\kappa(n+l) + \sum_{k=n+1}^{\infty} d_k\kappa(k,2l)\kappa(k+l) \\
&\le \kappa(n+1,2l-1)\kappa(n+l) + \kappa(n+l)\kappa(n+1,2l)\kappa(n+1) \\
&= \kappa(n+l)\left\{\kappa(n+1,2l-1) + \kappa(n+1,2l)\kappa(n+1)\right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\kappa(n+1,2l-1) + \kappa(n+1,2l)\kappa(n+1) \\
&= \prod_{j=n+2}^{j=n+2l-1} (1+\kappa(j)) + \prod_{j=n+2}^{j=n+2l} (1+\kappa(j))\,\kappa(n+1) \\
&= \prod_{j=n+2}^{j=n+2l-1} (1+\kappa(j))\left\{1 + (1+\kappa(n+2l))\,\kappa(n+1)\right\} \\
&\le \prod_{j=n+2}^{j=n+2l-1} (1+\kappa(j))\left\{1 + \kappa(n+2l) + (1+\kappa(n+2l))\,\kappa(n+1)\right\} \\
&= \prod_{j=n+2}^{j=n+2l-1} (1+\kappa(j))\left\{(1+\kappa(n+2l))\,(1+\kappa(n+2l))\right\} \\
&= \kappa(n,2l+1),
\end{aligned}$$

from which we have

$$\|K(n,2l+1)\| \le \kappa(n+2l)\kappa(n+l),$$

as needed. Analogous calculations help us to pass from the odd $j=2l+1$ to the even $j+1=2l+2$.

It is easy to see that $\kappa(n)$ and $\kappa(n,m)$ in (3.44) can be replaced by

$$\tilde{\kappa}(n) := C\sum_{j=n}^{\infty} q_j, \quad \tilde{\kappa}(n,m) := \prod_{j=n+1}^{n+m-1}(1+\tilde{\kappa}(j)) = \prod_{j=1}^{m-1}(1+\tilde{\kappa}(n+j)). \tag{3.46}$$

Further, it is evident that $\{\kappa(n)\} \in \ell^1$ and the sequences $\tilde{\kappa}(\cdot)$ and $\tilde{\kappa}(\cdot,m)$ decrease monotonically. Hence

$$\|K(n,m)\| \le \prod_{j=1}^{\infty}(1+\tilde{\kappa}(j)) \sum_{k=n+\left[\frac{m}{2}\right]}^{\infty} q_k. \tag{3.47}$$

Taking the latter expression with $n=0$, we come to the following.

Theorem 3.4. *Under hypothesis* (3.27) *the Taylor coefficients of the matrix-valued function*

$$\Delta(z) := V_0(z) = \sum_{n=0}^{\infty} \delta(n) z^n \tag{3.48}$$

admit the bound

$$\|\delta(n)\| \le C \prod_{j=1}^{\infty}(1+\tilde{\kappa}(j)) \sum_{k=\left[\frac{n}{2}\right]}^{\infty} q_k, \tag{3.49}$$

where $[x]$ *is an integer part of* x. *In particular,* Δ *belongs to the space* W_+ *of absolutely convergent Taylor matrix-valued series.*

Let us now denote by Δ_{ij} the entries of the matrix Δ:

$$\Delta := \|\Delta_{ij}\|_{i,j=1}^{p}.$$

Corollary 3.5. *Let for the banded matrix* D *the numbers*

$$M_{n+1} := \sum_{k=0}^{\infty}(k+1)^{n+1} q_k < \infty. \tag{3.50}$$

Then the n*-th derivative* $\Delta^{(n)}(z) = V_0^{(n)}(z)$ *belongs to* W_+ *and*

$$\max_{z\in\overline{\mathbb{D}}} \left|\Delta_{ij}^{(n)}(z)\right| \le C(D)\,\frac{4^n}{n+1}\, M_{n+1}, \quad i,j=1,2\dots,p. \tag{3.51}$$

P r o o f. The statement is a simple consequence of (3.50), the series expansion

$$\Delta^{(n)}(z) = \sum_{j=0}^{\infty}(j+1)\dots(j+n)\delta(j+n)z^j,$$

bounds (3.49) and the obvious inequality $|\Delta^{(n)}_{ij}(z)| \le \|\Delta^{(n)}\|$. ∎

4. The Limit Set and Location of the Discrete Spectrum

Consider the matrix-valued Jost solutions $V_k(z)$, which exist $\forall z \in \mathbb{C}\backslash\{0\}$. The p scalar solutions $\{v^{(l)}_j\}_{j\ge 1-p}$, $l = 1,2,\dots,p$ of equation (2.4), constructed from $V_k(z)$ by formulae (3.7 and (3.3), are linearly independent and belong to ℓ^2 due to asymptotic formula (3.26). According to Prop. 2.2, the number $\eta = \zeta + \zeta^{-1}$ is an eigenvalue for the operator D if and only if the determinant of the matrix-valued function $\Delta(z) = V_0(z)$ vanishes at the point ζ. Thus, the study of the discrete spectrum of the operator D is reduced to the study of zeros of the function

$$\gamma(z) := \det\Delta(z). \tag{4.1}$$

The main topic considered in this section is the limit set E_D of the discrete spectrum of the operator D. Remind that $E_D \subset [-2,2]$.

Let $D \in \mathcal{P}_p(\beta)$ (see Def. 1.1). Since $\|L_k\|$ and $\|L_k^{-1}\|$ are uniformly bounded, it is clear from 3.24 that h_n, defined in (3.21), satisfies the same inequality:

$$h_n \le C_1 \exp(-C_2\,(n+1)^{\beta}) \tag{4.2}$$

(with the same exponent β, but other constants $C_1, C_2 > 0$).

We are in a position now to prove the first result announced in the introduction.

Theorem 4.1. *Let $D \in \mathcal{P}(\beta)$ where $0 < \beta < \frac{1}{2}$. Then E_D is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies*

$$\dim E_D \le \tau(E_D) \le \frac{1-2\beta}{1-\beta}\,, \tag{4.3}$$

where $\dim E_D$ is the Hausdorff dimension of E_D. Moreover, if $J \in \mathcal{P}(\frac{1}{2})$ then $E_D = \emptyset$, i.e., the discrete spectrum is finite.

P r o o f. Denote by $\mathcal{A}$ the set of all functions, analytic inside $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Recall that the set E on the unit circle $\mathbb{T}$ is called a zero set for a class $\mathcal{X} \subset \mathcal{A}$ of functions, if there exists a nontrivial function $f \in \mathcal{X}$, which vanishes on E. We want to show that the function γ belongs to a certain class

$\mathcal{X}$ (the Gevré class, see below) with the known properties of its zero sets. Note that, since $\mathcal{X} \subset \mathcal{A}$, then, according to the Fatou theorem, the zero set has the Lebesgue measure zero.

We begin with certain bounds for the derivatives of the function γ, which can be obtained from Th. 3.4 and Cor. 3.5. For this we write

$$\gamma(z) := \det \Delta(z) = \sum_{\pi} \operatorname{sign} \pi \, \Delta_{1,\pi(1)} \Delta_{2,\pi(2)} \dots \Delta_{p,\pi(p)} \,,$$

where π are the permutations of the set $\{1, 2, \dots, p\}$. For the n-th derivative of γ

$$\gamma^{(n)}(z) = \sum_{\pi} \operatorname{sign} \pi \sum_{\substack{\sum_1^p k_j = n \\ k_j \ge 0}} \begin{pmatrix} n \\ k_1, k_2, \dots, k_p \end{pmatrix} \Delta_{1,\pi(1)}^{(k_1)} \Delta_{2,\pi(2)}^{(k_2)} \dots \Delta_{p,\pi(p)}^{(k_p)} \qquad (4.4)$$

holds, where the multinomial coefficients $\begin{pmatrix} n \\ k_1, k_2, \dots, k_p \end{pmatrix}$ are defined from the identity

$$(x_1 + x_2 + \dots + x_p)^p = \sum_{\substack{\sum_1^p k_j = n \\ k_j \ge 0}} \begin{pmatrix} n \\ k_1, k_2, \dots, k_p \end{pmatrix} x_1^{k_1} x_2^{k_2} \dots x_p^{k_p}. \qquad (4.5)$$

(4.5) with $x_1 = x_2 = \dots = x_p = 1$ gives

$$\sum_{\substack{\sum_1^p k_j = n \\ k_j \ge 0}} \begin{pmatrix} n \\ k_1, k_2, \dots, k_p \end{pmatrix} = p^n \,.$$

From (4.4) and (3.51) we immediately derive

$$\max_{z \in \overline{\mathbb{D}}} \left| \gamma^{(n)}(z) \right| \le C^p p! \, 4^n \sum_{\substack{\sum_1^p k_j = n \\ k_j \ge 0}} \begin{pmatrix} n \\ k_1, k_2, \dots, k_p \end{pmatrix} M_{k_1+1} M_{k_2+1} \dots M_{k_p+1}, \qquad (4.6)$$

where the numbers M_r are defined in (3.50). To estimate the product $M_{k_1+1} \dots M_{k_p+1}$ note first, that by (4.2)

$$M_r \le C \sum_{k=0}^{\infty} (k+1)^r \exp\left(-\frac{C}{2}(k+1)^{\beta}\right) \exp\left(-\frac{C}{2}(k+1)^{\beta}\right).$$

An elementary analysis of the function $u(x) = x^r \exp(-\frac{C}{2} x^{\beta})$ gives

$$\max_{x \ge 0} u(x) = u(x_0) = \left(\frac{2r}{C\beta}\right)^{r/\beta} e^{-r/\beta} = \left(\frac{2}{C\beta e}\right)^{r/\beta} r^{r/\beta}, \quad x_0 = \left(\frac{2r}{C\beta}\right)^{1/\beta},$$

so that

$$M_r \le B\left(\frac{2}{C\beta e}\right)^{r/\beta} r^{r/\beta}, \qquad B = C\sum_{k=0}^{\infty} \exp\left\{-\frac{C}{2}(k+1)^{\beta}\right\}.$$

Hence,

$$M_{k_1+1}\dots M_{k_p+1} \le B^p\left(\frac{2}{C\beta e}\right)^{\frac{n+p}{\beta}} (k_1+1)^{\frac{k_1+1}{\beta}} \dots (k_p+1)^{\frac{k_p+1}{\beta}}.$$

The inequalities

$$\left(\frac{n+1}{n}\right)^{n/\beta} < e^{1/\beta}, \quad (n+1)^{1/\beta} < e^{(n+1)/\beta}, \quad (n+1)^{(n+1)/\beta} < e^{1/\beta} n^{n/\beta} e^{(n+1)/\beta}$$

lead to the bound

$$M_{k_1+1}\dots M_{k_p+1} \le B_2\left(\frac{2}{C\beta}\right)^{n/\beta} k_1^{k_1/\beta}\dots k_p^{k_p/\beta}.$$

Substituting the latter into (4.6) gives

$$\max_{z\in\overline{\mathbb{D}}} |\gamma^{(n)}(z)| \le C4^n p^n \left(\frac{2}{C\beta}\right)^{n/\beta} n^{k_1/\beta}\dots n^{k_p/\beta} \le C\tilde{C}^n n^{n/\beta}, \quad n \ge 0. \tag{4.7}$$

In other words, the function γ for $D \in \mathcal{P}(\beta)$ belongs to the *Gevré class* $\mathcal{G}_\beta$.

The rest of the proof goes along the same line of reasoning as in [2]. Suppose first that $0 < \beta < 1/2$. The celebrated Carleson's theorem [7] gives a complete description of zero sets for $\mathcal{G}_\beta$. It claims that

$$\sum_{j=1}^{\infty} |l_j|^{\frac{1-2\beta}{1-\beta}} < \infty,$$

where $\{l_j\}$ are adjacent arcs of the zeros set of γ. Hence the right inequality in (4.3) is obtained. The left inequality is a general fact of the fractal dimension theory.

When $\beta = 1/2$ the Gevré class $\mathcal{G}_{1/2}$ is known to be quasi-analytic, i.e., it doesn't contain nontrivial function f, such that $f^{(n)}(\zeta_0) = 0$ for all $n \ge 0$ and some $\zeta_0 \in \mathbb{T}$. So each function $f \in \mathcal{G}_{1/2}$ may have only finite number of zeros inside the unit disk, which, due to the relation between the discrete spectrum of D and zeros of γ, proves the second statement of the theorem. ∎

It turns out that the exponent $1/2$ in Th. 1 is sharp in the following sense.

Theorem 4.2. *For arbitrary $\varepsilon > 0$ and different points $-2 < \nu_i < 2$, $i = 1, \dots, p$, there exists an operator $D \in \mathcal{P}_p(\frac{1}{2} - \varepsilon)$ such that its discrete spectrum $\sigma_{\rm d}(D)$ is infinite and, moreover, $E_D = \{\nu_1, \dots, \nu_p\}$.*

P r o o f. The proof of the theorem is based on the similar result, proved in [2] for the Jacobi matrices, that is, for $p = 1$. If $p > 1$ than given $\nu_1, \dots, \nu_p$ we construct the complex Jacobi matrices

$$J(i) = \begin{pmatrix} b_0(i) & a_0(i) & & & \\ a_0(i) & b_1(i) & a_1(i) & & \\ & a_1(i) & b_2(i) & a_2(i) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \begin{array}{c} a_j(i) > 0,\ b_j(i) \in \mathbb{R}, \\ j \geq 1,\ i = 1, 2, \dots, p, \end{array} \tag{4.8}$$

which belong to the class $\mathcal{P}_1(\frac{1}{2} - \varepsilon)$ and have the only point of accumulation of the discrete spectrum $E_{J(i)} = \nu_i$.

Consider now the p-banded matrix $D = \|d_{ij}\|_{i,j=1}^{\infty}$ with

$$d_{pn+i,\,pn+i} = b_n(i), \quad d_{pn+i,\,p(n+1)+i} = d_{p(n+1)+i,\,pn+i} = a_n(i),$$

$$d_{pn+i,\,pn+i+j} = d_{pn+i+j,\,pn+i} = 0, \qquad j \neq 0, p, \quad i = 1, \dots, p.$$

Since $J(i) \in \mathcal{P}_1(\frac{1}{2} - \varepsilon)$, it is easily seen that $D \in \mathcal{P}_p(\frac{1}{2} - \varepsilon)$.

The space ℓ^2, where the operator D acts, can be decomposed as follows:

$$\ell^2 = \bigoplus_{i=1}^{p} L_i$$

with $L_i = \mathrm{Lin}\{\hat{e}_i, \hat{e}_{p+i}, \hat{e}_{2p+i}, \dots, \hat{e}_{np+i}, \dots\}$ and $\{\hat{e}_k\}_{k \in \mathbb{N}}$ the standard basis vectors in ℓ^2. It is shown directly that the subspaces L_i are invariant for D.

The restriction $D_i = D\,|L_i$ of D on the subspace L_j has the matrix representation $J(i)$. Since $D = \bigoplus_{i=1}^{p} D_i$ and every operator D_i has a discrete spectrum with the only accumulation point ν_i, the discrete spectrum D is the union of the discrete spectra of the operators $J(i)$ and it has p accumulation points $\nu_1, \nu_2, \dots, \nu_p$, as needed. ■

The second issue we address in this section concerns the domains which contain the whole discrete spectrum, and conditions for the lack of the discrete spectrum.

We begin with (3.33) and (3.37) for $n = 0$

$$\|\Delta(z) - I\| \leq \frac{2\sigma_0(0)}{|z - z^{-1}|} \exp\left\{\frac{2\sigma_0(0)}{|z - z^{-1}|}\right\}, \quad \|\Delta(z) - I\| \leq \sigma_1(0) \exp\{\sigma_1(0)\}, \tag{4.9}$$

with $\Delta = \tilde{V}_0$, $\sigma_0(0) = \sum_{k=1}^{\infty} h_k$, $\sigma_1(0) = \sum_{k=1}^{\infty} k h_k$, h_k defined in (3.21), which hold under assumptions (3.25) and (3.27), respectively. To work with (4.9) we have to find the *efficient* constant C which enters (3.23). To do this let us go back to (3.10) and assume that

$$q := \sup_{n\geq 1} q_n = \sup_{n\geq 1}\Big(|d_{n,n-p}-1| + |d_{n,n+p}-1| + \sum_{r=-p+1}^{p-1} |d_{n,n+r}|\Big) < 1, \qquad (4.10)$$

which now implies

$$\sup_{k\geq 1} \|A_k - I\| \leq q, \qquad \sup_{k\geq 1} \|A_k\| \leq 1+q < 2. \qquad (4.11)$$

It is not hard to show now that $\sup_{k\geq 1} \|A_k^{-1}\| \leq (1-q)^{-1}$,

$$\sup_{k\geq 1} \|L_k\| \leq \exp\{\sum_{j=1}^{\infty} \|A_j - I\|\}, \qquad \sup_{k\geq 1} \|L_k^{-1}\| \leq \exp\Big\{\frac{1}{1-q}\sum_{j=1}^{\infty} \|A_j - I\|\Big\}$$

and

$$\sup_{k\geq 1} \|L_k\|\|L_k^{-1}\| \leq \exp\Big\{\frac{2-q}{1-q}\sum_{j=1}^{\infty} \|A_j - I\|\Big\} \leq \exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}, \quad Q_0 := \sum_{k=1}^{\infty} q_k\,.$$

Hence, instead of (3.23), we have by (3.22)

$$h_k \leq 2\exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}(\hat{q}_{k-1} + \hat{q}_k)$$

and so

$$\sigma_0(0) \leq 4\exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}\,Q_0, \quad \sigma_1(0) \leq 4\exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}\,Q_1, \quad Q_1 := \sum_{k=1}^{\infty} kq_k. \qquad (4.12)$$

Let now t be a unique root of the equation

$$te^t = 1, \quad t \approx 0.567. \qquad (4.13)$$

Theorem 4.3. *Assume that $Q_0 < \infty$ and $q < 1$. Then the domain*

$$G(D) = \{z + z^{-1} : \quad z \in \Omega\}$$

with

$$\Omega := \{z \in \mathbb{D} : |z - z^{-1}| > \frac{8Q_0}{t}\,\exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}$$

is free from the discrete spectrum $\sigma_{\rm d}(D)$. Moreover, assume that $Q_1 < \infty$. Then D has no discrete spectrum as long as

$$\exp\Big\{\frac{2-q}{1-q}\,Q_0\Big\}\,Q_1 < \frac{4}{t}.$$

P r o o f. It is clear from the first inequality in (4.9) that $\|\Delta(z) - I\| < 1$ (and the more so, $\det \Delta(z) \neq 0$) whenever $|z - z^{-1}| > 2\sigma_0(0)t^{-1}$. The rest follows immediately from the first bound in (4.12) and the relation between eigenvalues of D and zeros of Δ. The second statement is proved in exactly the same way by using the second inequality in (4.9) and the second bound in (4.12). ■

R e m a r k. Suppose that

$$c = \frac{8Q_0}{t} \exp\left\{\frac{2-q}{1-q} Q_0\right\} < 2.$$

Then $\sigma_{\rm d}(D)$ is contained in the union of two symmetric rectangles

$$\sigma_{\rm d}(D) \subset \left\{\lambda :\ \sqrt{4-c^2} < |\mathrm{Re}\,\lambda| < \sqrt{4+c^2},\quad |\mathrm{Im}\,\lambda| < \frac{c^2}{4}\right\}.$$

5. The Discrete Spectrum of Doubly-Infinite Banded Matrices

A doubly-infinite complex matrix $D = \|d_{ij}\|_{i,j=-\infty}^{\infty}$ is called the *banded matrix of order* p if

$$d_{ij} = 0, \quad |i-j| > p, \qquad d_{ij} \neq 0, \quad |i-j| = p, \qquad d_{ij} \in \mathbb{C}. \tag{5.1}$$

There is a nice "doubling method" that relates a doubly-infinite p-banded matrix to a certain semi-infinite banded matrix of order $2p$. Indeed, let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard basis in $\ell^2(\mathbb{Z})$. Consider a transformation $U : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{N})$, defined by

$$Ue_k = \hat{e}_{-2k}, \quad k < 0, \qquad Ue_k = \hat{e}_{2k+1}, \quad k \geq 0,$$

where $\{\hat{e}_k\}_{k\in\mathbb{N}}$ is the standard basis in $\ell^2(\mathbb{N})$. The transformation U is clearly isometric and it is easy to check directly that the matrix

$$\hat{D} := UDU^{-1} \tag{5.2}$$

is the semi-infinite band matrix of order $2p$ unitarily equivalent to D. For instance, in the case of Jacobi matrices ($p = 1$)

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_{-1} & c_0 & & & \\ & & a_0 & b_0 & c_1 & & \\ & & & a_1 & b_1 & c_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \hat{J} = \begin{pmatrix} \hat{B}_0 & \hat{C}_1 & & & \\ \hat{A}_1 & \hat{B}_1 & \hat{C}_2 & & \\ & \hat{A}_2 & \hat{B}_2 & \hat{C}_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

with

$$\hat{B}_0 = \begin{pmatrix} b_0 & a_0 \\ c_0 & b_{-1} \end{pmatrix}, \quad \hat{B}_k = \begin{pmatrix} b_k & 0 \\ 0 & b_{-k-1} \end{pmatrix},$$
$$\hat{A}_k = \begin{pmatrix} a_k & 0 \\ 0 & c_{-k} \end{pmatrix}, \quad \hat{C}_k = \begin{pmatrix} c_k & 0 \\ 0 & a_{-k} \end{pmatrix}, \quad k \in \mathbb{N}.$$

Similarly to the semi-infinite case (1.4), we put

$$q_n := |d_{n,n-p} - 1| + \sum_{j=-p+1}^{p-1} |d_{n,n+j}| + |d_{n,n+p} - 1|, \quad m \in \mathbb{Z}. \tag{5.3}$$

We say that the matrix D belongs to the class $\mathcal{P}_p(\beta)$, $0 < \beta < 1$, if

$$q_n \le C_1 \exp(-C_2 |n|^\beta), \qquad C_1, C_2 > 0, \qquad n \in \mathbb{Z}. \tag{5.4}$$

It is clear from the construction that the semi-infinite matrix $\hat{D}$ (5.2) belongs to $\mathcal{P}_{2p}(\beta)$ whenever the doubly-infinite matrix D belongs to $\mathcal{P}_p(\beta)$. So the results of the previous section can be easily derived for doubly-infinite banded matrices as well. For instance, the following statement holds.

Theorem 5.1. *Let $D \in \mathcal{P}_p(\beta)$ where $0 < \beta < \frac{1}{2}$. Then E_D is a closed point set of the Lebesgue measure zero and its convergence exponent satisfies*

$$\dim E_D \le \tau(E_D) \le \frac{1-2\beta}{1-\beta}, \tag{5.5}$$

where $\dim E_D$ is the Hausdorff dimension of E_D. Moreover, if $D \in \mathcal{P}_p(\frac{1}{2})$ then $E_D = \emptyset$, i.e., the discrete spectrum is finite.

6. The Discrete Spectrum of Asymptotically Periodic Jacobi Matrices

The goal of this section is to apply the results obtained above to the study of the spectrum of doubly-infinite asymptotically periodic complex Jacobi matrices.

Definition 6.1. *A doubly-infinite complex Jacobi matrix*

$$J^0 = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & a^0_{-1} & b^0_{-1} & c^0_0 & & \\ & & a^0_0 & b^0_0 & c^0_1 & \\ & & & a^0_1 & b^0_1 & c^0_2 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad b^0_0 = (J^0 e_0, e_0), \tag{6.1}$$

$a^0_n c^0_n \ne 0$, $a^0_n, b^0_n, c^0_n \in \mathbb{C}$, *is called p-periodic if*

$$a^0_{n+p} = a^0_n, \quad b^0_{n+p} = b^0_n, \quad c^0_{n+p} = c^0_n, \quad n \in \mathbb{Z}.$$

The following result due to P.B. Naiman [8, 9] is the key ingredient in our argument.

Theorem. *Each p-periodic Jacobi matrix J^0 satisfies the algebraic equation of the p-th degree*

$$P(J^0) = E_p,$$

where $E_p = \{e_{jk}\}$ is a p-banded matrix with the entries

$$e_{ij} = \begin{cases} \alpha := \prod_{k=1}^{p} a_k^0, & i - j = p, \\ \delta := \prod_{k=1}^{p} c_k^0, & i - j = -p, \\ 0, & |i - j| \neq p. \end{cases} \tag{6.2}$$

Next, let

$$\Gamma := P^{(-1)}(\Pi) = \{z \in \mathbb{C} : P(z) \in \Pi\}$$

be the preimage of the ellipse in the complex plane

$$\Pi := \{z \in \mathbb{C} : z = \alpha e^{ipt} + \delta e^{-ipt}, \quad 0 \le t \le \frac{2\pi}{p}\}.$$

Then the spectrum $\sigma(J^0) = \Gamma$.

The polynomial P is known as the *Burchnall–Chaundy polynomial for J^0* and given explicitly in [9, p. 141]:

$$P(\lambda) = \det\bigl(\lambda - J(1,p)\bigr) - a_1 c_1 \det\bigl(\lambda - J(2, p-1)\bigr),$$

$$J(m,n) := \begin{pmatrix} b_m^0 & c_{m+1}^0 & & \\ a_{m+1}^0 & b_{m+1}^0 & \ddots & \\ & \ddots & \ddots & c_n^0 \\ & & a_n^0 & b_n^0 \end{pmatrix}.$$

There is also another way to exhibit the spectrum of any p-periodic Jacobi matrix J^0. Consider the $p \times p$-matrix

$$\Lambda_p(t) := \begin{pmatrix} b_1^0 & c_2^0 & & a_1^0 e^{ipt} \\ a_2^0 & b_2^0 & \ddots & \\ & \ddots & \ddots & c_p^0 \\ c_1^0 e^{-ipt} & & a_p^0 & b_p^0 \end{pmatrix}.$$

with the eigenvalues $\{\lambda_j(t)\}_{j=1}^p$, and put

$$\operatorname{range}\lambda_j := \{w \in \mathbb{C} : w = \lambda_j(t), \quad 0 \le t \le \frac{2\pi}{p}\}.$$

Then

$$\sigma(J^0) = \bigcup_{j=1}^{p} \operatorname{range}\lambda_j.$$

Indeed, it is easy to compute

$$\det\left(\Lambda(t) - \lambda\right) = (-1)^p \left(P(\lambda) - \alpha e^{ipt} - \delta e^{-ipt}\right).$$

For further information about the spectra of the periodic Jacobi matrices with algebro-geometric potential see [10].

For the rest of the paper we restrict our consideration by the so called *quasi-symmetric* matrices with

$$\prod_{k=1}^{p} a_k^0 = \prod_{k=1}^{p} c_k^0, \qquad \alpha = \delta. \tag{6.3}$$

The latter certainly holds for the symmetric periodic matrices ($a_n^0 = c_n^0$). In this case we have $Q(J^0) = D_0$ with D_0 defined in (1.3) for the polynomial $Q = \alpha^{-1}P$, and

$$\sigma(J^0) = Q^{(-1)}\big([-2, 2]\big). \tag{6.4}$$

It is not hard to make sure that $\sigma(J^0)$ is a collection of finitely many algebraic arcs with no closed loops, that is, the complement $\mathbb{C}\backslash\sigma(J^0)$ is a connected set. Indeed, if any of these arcs formed a closed loop $\Gamma_1 \subset \Gamma$ with the interior domain G_1, then $\Im Q$ would vanish on Γ_1 and so $\Im Q \equiv 0$ in G_1, which is impossible, unless Q is a real constant.

Let now J be a complex asymptotically p-periodic doubly-infinite Jacobi matrix with the quasi-symmetric p-periodic background J^0 (6.1)

$$J = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ & a_{-1} & b_{-1} & c_0 & & & \\ & & a_0 & b_0 & c_1 & & \\ & & & a_1 & b_1 & c_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_n c_n \ne 0, \tag{6.5}$$

and

$$\omega_n := |a_n - a_n^0| + |b_n - b_n^0| + |c_{n+1} - c_{n+1}^0| \to 0, \quad k \to \pm\infty.$$

In other words, $J-J^0$ is a compact operator in $\ell^2(\mathbb{Z})$. Since $\mathbb{C}\backslash\sigma(J^0)$ is connected, a version of the Weyl theorem holds (Cf. [5], Lem. I.5.2)

$$\sigma(J)=\sigma(J^0)\cup\sigma_{\mathrm{d}}(J),$$

where the discrete spectrum $\sigma_{\mathrm{d}}(J)$ is an at most denumerable set of eigenvalues of finite algebraic multiplicity off $\sigma(J^0)$, which can accumulate only to $\sigma(J^0)$.

We proceed with two simple propositions.

Proposition 6.2. *Let D_{m_1} and D_{m_2} be two banded matrices of orders m_1 and m_2, respectively. Then*

1. *$D=D_{m_1}D_{m_2}$ is the banded matrix of order $m=m_1+m_2$.*
2. *If $m_1<m_2$ then $D_{m_1}+D_{m_2}$ is the banded matrix of order m_2.*

In particular, if T is a polynomial of degree p and D is a banded matrix of order m, then $T(D)$ is the banded matrix of order pm.

P r o o f. (2) is obvious. To prove (1) let us show that the elements of the extreme diagonals of D do not vanish. Indeed, let $D_{m_l}=\{d_{ij}^{(l)}\}$, $l=1,2$, and $D=\{d_{ij}\}$. Then

$$d_{k,\ k+m}=\sum_{j=1}^{\infty}d_{k\,j}^{(1)}d_{j,\,k+m}^{(2)}=d_{k\,k+m_1}^{(1)}d_{k+m_1,\,k+m}^{(2)}\neq 0,\quad k\in\mathbb{N}.$$

Similarly, $d_{k,k-m}=d_{k,k-m_1}^{(1)}d_{k-m_1,k-m}^{(2)}\neq 0$. The rest is plain. ∎

Proposition 6.3. *Let J_1 and J_2 be two bounded (with bounded entries) complex Jacobi matrices. Put*

$$\omega_k=\sum_j|(J_1-J_2)_{kj}|=\sum_j|\langle(J_1-J_2)e_j,e_k\rangle|\,.$$

Then for any polynomial T of degree p we have

$$\sum_j|(T(J_1)-T(J_2))_{kj}|=\sum_j|\langle(T(J_1)-T(J_2))e_j,e_k\rangle|\le C\sum_{s=k-p}^{k+p}\omega_s,$$

where a positive constant C depends on J_1, J_2 and T.

P r o o f. For $T(z) = z^n$ the statement follows immediately from the banded structure of the powers J_l^m, $(J_l^*)^s$, $l = 1, 2$ and the equality

$$J_1^n - J_2^n = \sum_{j=0}^{n-1} J_1^{n-j-1}(J_1 - J_2)J_2^j.$$

The rest is straightforward. ■

Definition 6.4. *We say that the matrix* J (6.5) *belongs to the class* $\mathcal{P}(\beta, J^0)$, *if*

$$\omega_n \le C_1 \exp(-C_2 |n|^\beta), \qquad 0 < \beta < 1, \quad C_1, C_2 > 0; \ n \in \mathbb{Z}. \tag{6.6}$$

Our main result claims that $\sigma_{\mathrm{d}}(J)$ is a finite set as long as $J \in \mathcal{P}(1/2, J^0)$.

Theorem 6.5. *Let J be an asymptotically p-periodic doubly-infinite Jacobi matrix* (6.5) *with the quasisymmetric background* J^0 (6.1), (6.3). *If* $J \in \mathcal{P}(1/2, J^0)$, *then $\sigma(J)$ is the union of p algebraic arcs and a finite number of eigenvalues of the finite algebraic multiplicities off these arcs.*

P r o o f. Let $T = Q = \alpha^{-1}P$ be the Burchnall–Chaundy polynomial for J^0. By Prop. 6.2 $Q(J)$ is the p-banded matrix. Since $J \in \mathcal{P}(1/2, J^0)$ the matrix $Q(J)$ is close to $Q(J^0) = D_0$ with D_0 defined in (1.3) in the sense of Prop. 6.3

$$\sum_j |(Q(J) - D_0)_{kj}| \le C_1 e^{-C_2|k|^{1/2}}$$

and so $Q(J) \in \mathcal{P}_p(1/2)$. According to Th. 5.1 $\sigma(Q(J)) = [-2, 2] \cup E$, where the set E of the eigenvalues off $[-2, 2]$ is now finite.

On the other hand, as we know, $\sigma(J) = \Gamma \cup F$, where the set F of the eigenvalues off $\Gamma = \sigma(J^0)$ is at most denumerable, and $\Gamma = Q^{(-1)}([-2, 2])$. Therefore, by the Spectral Mapping Theorem $Q(F) = E$ and so F is a finite set, as claimed. ■

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References

[1] *I. Egorova and L. Golinskii*, On Location of Discrete Spectrum for Complex Jacobi Matrices. — *Proc. AMS* **133** (2005), 12, 3635–3641.

[2] *I. Egorova and L. Golinskii*, On limit sets for the Discrete Spectrum of Complex Jacobi Matrices. — *Mat. Sb.* **196** (2005), 6, 43–70. (Russian)

[3] *B.S. Pavlov*, On Nonselfadjoint Schrödinger Operator I. — *Probl. Math. Phys.*, LGU **1** (1966), 102–132. (Russian)

[4] *B.S. Pavlov*, On Nonselfadjoint Schrödinger Operator II. — *Probl. Math. Phys.*, LGU **1** (1966), 102–132. (Russian)

[5] *I. Gohberg and M. Krein*, Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space. Nauka, Moscow, 1965. (Russian)

[6] *O. Perron*, Über Summengleichungen und Poincarésche Differenzengleichingen. — *Math. Annal.* **84** (1921), 1–15. (German)

[7] *L. Carleson*, Sets of Uniqueness for Functions Analytic in the Unit Disc. — *Acta Math.* **87** (1952), 325–345.

[8] *P.B. Naiman*, On the Theory of Periodic and Limit-Periodic Jacobi Operators. — *Sov. Math. Dokl.* **3** (1962), 4, 383–385. (Russian)

[9] *P.B. Naiman*, To the Spectral Theory of the Non-Symmetric Periodic Jacobi Matrices. — *Notes Depart. Math. and Mech. Kharkov State Univ. and Kharkov Math. Soc.* **XXX** (1964), 4, 138–151. (Russian)

[10] *V. Batchenko and F. Gesztesy*, On the Spectrum of Jacobi Operators with Quasi-periodic Algebro-Geometric Coefficients. — *Int. Math. Res. Papers* **10** (2005), 511–563.