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Uniform Approximation of $sgn(x)$ by Rational Functions with Prescribed Poles

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For $a \in (0,1)$ let $L_m^k(a)$ be an error of the best approximation of the function sgn (x) on two symmetric intervals $[-1,-a]\cup[a,1]$ by rational functions with the only possible poles of degree $2k-1$ at the origin and of $2m-1$ at infinity. Then the following limit exists

$$\lim_{m\to\infty} L_m^k(a)\left(\frac{1+a}{1-a}\right)^{m-\frac{1}{2}}(2m-1)^{k+\frac{1}{2}} = \frac{2}{\pi}\left(\frac{1-a^2}{2a}\right)^{k+\frac{1}{2}}\Gamma\left(k+\frac{1}{2}\right). \tag{0.1}$$

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Dedicated to the memory of B.Ya. Levin

1. Introduction

This is the second step (for the first one see [5]) on the way to understand better the difficulties that up to now do not allow to find the Bernstein constant. Recall that Sergey Natanovich Bernstein found [3, 4] that for the error $E_n(p)$ of the best uniform approximation of $|x|^p$, p being not an even integer, on $[-1, 1]$ by polynomials of degree n the following limit exists:

$$\lim_{m\to\infty} n^p E_n(p) = \mu(p) > 0.$$

For $p = 1$ this result was obtained by Bernstein in 1914, and he posed the question, whether one could express $\mu(1)$ in terms of the known transcendental functions. This question is still open.

Actually, we solve here a problem on asymptotics of the best approximation of $\operatorname{sgn}(x)$ on the union of two intervals $[-1, -a]\cup[a, 1]$ by rational functions. In 1877, E.I. Zolotarev [6, 2] found an explicit expression, in terms of elliptic functions, of the rational function of the given degree which is uniformly closest to $\operatorname{sgn}(x)$ on this set. This result was a subject of a number of generalizations, and it has applications in electric engineering. In Zolotarev's case position of the poles of the rational function is free, the natural question is to find the best approximation when the poles and their multiplicities are fixed. In [5] A. Eremenko and the second co-author of the current paper solved the polynomial case. Here we allow the rational function to have one more pole in $(-a, a)$, more precisely, admitted are two poles — one at infinity and one in the origin.

Thus the problem is:

Problem 1.1. *For $k, m \in \mathbb{N}$, find the best approximation of the function* $\operatorname{sgn}(x)$*, $|x| \in [a, 1]$, by functions of the form*

$$f(x) = \frac{a_{-(2k-1)}}{x^{2k-1}} + ... + a_{2m-1}x^{2m-1}$$

and the approximation error $L^k_m(a)$.

One can be interested in many different asymptotics for $L^k_m(a)$ when m or k, or both of them go to infinity in a certain prescribed way. In this paper we concentrate on the case when k is fixed and $m \to \infty$. Note, however, that due to the evident symmetry $L^k_m(a) = L^m_k(a)$ and a bit less evident (6.2) we have simultaneously asymptotic for $k \to \infty$, m is fixed and $k \to \infty$, $m \to \infty$ so that $k = m$.

As it appears the tricks which are used in [5] to find precise asymptotic work in this general case (so we have a method in hands):

1. For each certain k and m we reveal the structure of the extremal function by representing it with the help of an explicitly given conformal mapping.

2. The system of conformal mappings (k is fixed, m is a parameter) converges (in the Caratheodory sense) after appropriate renormalization. The limit map does not depend on a, thus we obtain asymptotics for $L_m^k(a)$ in terms of a-depending parameters, that we use for renormalization, (an explicit formula) and a k-depending constant, say Y_k, which is a certain characteristic of this final conformal map (kind of capacity).

Of course, it is very tempting to guess Y_k directly from the given explicitly conformal map. It might be that we have here special functions that are given in such a form that we are unable to recognize them. In any case, we would consider this way of finding Y_k as a very interesting open problem. However we are able to find Y_k using the third step below our strategy. Problem 1.1 in an evident way is equivalent to

Problem 1.2. *For $p = 2k-1$ and $n = 2(k+m-1)$, find the best weighted polynomial approximation and the minimal deviation*

$$E_n^*(p,a) = \inf_{\{P:\deg P\le n\}} \sup_{|x|\in[a,1]} \left|\frac{|x|^p - P(x)}{x^p}\right|. \tag{1.2}$$

Thus we have $E_n^*(p,a) = L_m^k(a)$. Note that Bernstein himself solved the unweighted problem.

Problem 1.3. *For a fixed non even p, find asymptotics for the minimal deviation*

$$E_n(p,a) = \inf_{\{P:\deg P\le n\}} \sup_{|x|\in[a,1]} \left||x|^p - P(x)\right|, \tag{1.3}$$

when n goes to infinity through the even integers.

3. Due to the evident relation

$$\lim_{a\to 1}\lim_{n\to\infty}\frac{E_n^*(p,a)}{E_n(p,a)} = 1,$$

we can recalculate the constant in Probl. 1.3 to the constant related to Probl. 1.2 and thus to get explicitly $e^{Y_k} = \frac{\Gamma\left(k+\frac{1}{2}\right)}{\pi}$.

This interplay between Problems 1.2 and 1.3 indicates that most likely one can find our asymptotic formula (0.1) by using original Bernstein's method, though up to the last step our consideration is very direct and simple. However we can go in the opposite direction. In particular, in this work we show that the extremal polynomials of Probl. 1.3, at least for $p = 1$, also have special representations in terms of conformal mappings. The boundaries of the corresponding domains are not so explicit as in Probl. 1.1, they are described in terms of certain functional equations with an unknown function being involved, its Hilbert transform and independent variable (7.2). Precise constants that characterize these equations (counterparts of the constants Y_k), related to the conformal mappings and their asymptotics leave enough space for the hope that for $a = 0$ one also would be able to characterize very similar equations in terms of classical constants.

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2. Special Functions

In this section we introduce certain special conformal mappings that we need in what follows. They are marked by a natural parameter k, but in this section k can be just real, $k > 1/2$.

For the given k, consider the domain

$$\Pi_k = \mathbb{C}_+ \setminus \{w : \mathrm{Re} w = -\log t,\ |\mathrm{Im} w - k\pi| \le \arccos t,\ t \in (0, 1]\}. \tag{2.1}$$

Define the conformal map

$$H_k : \mathbb{C}_+ \to \Pi_k,$$

normalized by $H_k(0) = \infty_1$, $H_k(\infty) = \infty_2$ (on the boundary we have two infinite points denoted ∞_1, ∞_2 respectively), and moreover

$$H_k(\zeta) = \zeta + \dots, \quad \zeta \to \infty$$

(that is the leading coefficient is fixed). By D_k we denote the positive number such that $H_k(-D_k) = 0$.

Note that for H_k we have the following integral representation

$$H_k(\zeta) = \zeta + D_k + \int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k}\right) \rho_k(t) dt, \tag{2.2}$$

where $\rho_k(t) = \frac{1}{\pi} \mathrm{Im} H_k(t)$. Evidently $\rho_k(t) \to k + \frac{1}{2}$, $t \to +\infty$.

Lemma 2.1. *The function H_k possesses the asymptotic*

$$\lim_{\zeta \to -\infty} \left\{ H_k(\zeta) - \zeta + \left(k + \frac{1}{2}\right) \log(-\zeta) \right\} = Y_k, \tag{2.3}$$

where

$$Y_k := D_k + \left(k + \frac{1}{2}\right) \log D_k - \int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2}\right)}{t + D_k}\, dt. \tag{2.4}$$

P r o o f. Since

$$\int_0^\infty \left(\frac{1}{t - \zeta} - \frac{1}{t + D_k}\right) \left(\rho_k(t) - \left(k + \frac{1}{2}\right)\right) dt \to - \int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2}\right)}{t + D_k} dt \tag{2.5}$$

and

$$\left(k + \frac{1}{2}\right) \int_0^\infty \left(\frac{1}{t - \zeta} - \frac{1}{t + D_k}\right) dt = - \left(k + \frac{1}{2}\right) (\log(-\zeta) - \log D_k), \tag{2.6}$$

we get (2.3). ∎

Finally, note that Y_k, as it was defined here, has sense for all real $k > \frac{1}{2}$. As it is shown in Sect. 5, for the integer k we have

$$Y_k = \log \Gamma \left(k + \frac{1}{2}\right) - \log \pi.$$

We do not know wether these values coincide for non integers k.

3. Extremal Problem

Problems 1.1 and 1.2 are related in a trivial way. Recall, for $p = 2k - 1$ and $n = 2(k + m - 1)$, we have

$$E_n^*(p, a) = L_m^k(a) = \inf_{\{P : \deg P \leq 2(m+k-1)\}} \sup_{|x| \in [a,1]} \left| \frac{|x|^{2k-1} - P(x)}{x^{2k-1}} \right|, \tag{3.1}$$

where $a \in (0,1)$, $k, m \in \mathbb{N}$. Evidently, $L_m^k(a)$ can be rewritten in the terms of the best approximation of the function $\operatorname{sgn}(x)$ by functions of the form

$$f(x) = \frac{a_{-(2k-1)}}{x^{2k-1}} + ... + a_{2m-1} x^{2m-1}.$$

Also, it is trivial that in the first case the extremal polynomial is even and the extremal function $f = f(x; k, m; a)$ is odd.

For a parameter $B > 0$, and $k, m \in \mathbb{N}$, we denote by $\Omega_m^k(B)$ a subdomain of the half-strip

$$\{w = u + iv : v > 0,\ 0 < u < (k + m)\pi\}$$

that we obtain by deleting the subregion

$$\{w = u + iv : |u - \pi k| \leq \arccos\left(\frac{\cosh B}{\cosh v}\right),\ v \geq B\}. \tag{3.2}$$

Let $\phi(z) = \phi(z; k, m; B)$ be a conformal map of the first quadrant onto $\Omega_m^k(B)$ such that $\phi(0) = \infty_1$, $\phi(1) = (k+m)\pi$, $\phi(\infty) = \infty_2$. Let $a = \phi^{-1}(0)$. Then a is a continuous strictly increasing function of B, moreover $\lim_{B\to 0} a(B) = 0$ and $\lim_{B\to\infty} a(B) = 1$. Thus we may consider the inverse function $B(a) = B_m^k(a)$, $a \in (0,1)$.

Theorem 3.1. *The error of the best approximation is*

$$L_m^k(a) = \frac{1}{\cosh B_m^k(a)} \tag{3.3}$$

and the extremal function is of the form

$$f(x; k, m; a) = 1 - (-1)^k L_m^k(a) \cos\phi(x; k, m; B(a)), \quad x > 0.$$

P r o o f. Basically the proof is the same as in [5]. A comparably important difference is as follows. We have to note and prove that on the imaginary axis the extremal function has precisely one zero (there are no critical points and the behavior at $i0$ and at $i\infty$ is evident). At this point $\phi = k\pi + iB$ and we have (3.3). ∎

4. Asymptotics

Theorem 4.1. *The following limit exists*

$$\lim_{m\to\infty}\left\{B_m^k(a) - \left(m - \frac{1}{2}\right)\log\frac{1+a}{1-a} - \left(k + \frac{1}{2}\right)\log(2m-1)\right\} = \left(k + \frac{1}{2}\right)\log\frac{a}{1-a^2} - Y_k. \tag{4.1}$$

P r o o f. As in [5], we use the symmetry principle and make convenient changes of variables to have a conformal map $\Phi_m(Z) = \Phi(Z; k, m; B)$ of the upper plane in the region

$$i(\Omega_m^k(B) \cup \overline{\Omega_m^k(B)}) \cup (0, i\pi(m+k)).$$

This conformal map has the following boundary correspondence

$$\Phi_m : (-C_m, -A_m, 0, A_m, C_m) \to (-\infty_2, -\infty_1, 0, \infty_1, \infty_2),$$

here $A_m = aC_m$ and C_m will be chosen later.

For Φ_m we have the following integral representation

$$\Phi_m(Z) = \left(m - \frac{1}{2}\right) \log \frac{1 + \frac{Z}{C_m}}{1 - \frac{Z}{C_m}} + \int_{A_m}^{\infty} \left[\frac{1}{X - Z} - \frac{1}{X + Z}\right] v_m(X)\, dX,$$

where

$$v_m(X) = \begin{cases} \frac{1}{\pi}\mathrm{Im}\Phi_m(X), & A_m \le X \le C_m \\ k + \frac{1}{2}, & X > C_m. \end{cases} \tag{4.2}$$

Put now

$$H_m^k(\zeta) = \Phi_m(Z) - B_m, \quad Z = A_m + \zeta,$$

then

$$H_m^k(\zeta) = \left(m - \frac{1}{2}\right) \log \frac{1 + a + \frac{\zeta}{C_m}}{1 - a - \frac{\zeta}{C_m}} + \int_0^{\infty} \left[\frac{1}{t - \zeta} - \frac{1}{t + 2A_m + \zeta}\right] \hat{v}_m(t)\, dt - B_m,$$

where $\hat{v}_m(t) = v_m(t + A_m)$. Let us rewrite H_m^k in the form that is close to the integral representation of H_k:

$$\begin{aligned} H_m^k(\zeta) = &\left(m - \frac{1}{2}\right) \log \frac{1 + \frac{\zeta}{C_m(1+a)}}{1 - \frac{\zeta}{C_m(1-a)}} + D_k + \int_0^{\infty} \left[\frac{1}{t - \zeta} - \frac{1}{t + D_k}\right] \hat{v}_m(t)\, dt \\ &+ \left(m - \frac{1}{2}\right) \log \frac{1 + a}{1 - a} - D_k + \int_0^{\infty} \left[\frac{1}{t + D_k} - \frac{1}{t + 2A_m + \zeta}\right] \hat{v}_m(t)\, dt - B_m. \end{aligned} \tag{4.3}$$

Now, we put

$$C_m = \frac{2m - 1}{1 - a^2}.$$

In this case the first line in (4.3) converges to $H_k(\zeta)$. Since

$$\lim_{m \to \infty} \int_0^{\infty} \left[\frac{1}{t + D_k} - \frac{1}{t + A_m + \zeta}\right] \left(\hat{v}_m(t) - \left(k + \frac{1}{2}\right)\right) dt = \int_0^{\infty} \frac{\rho_k(t) - \left(k + \frac{1}{2}\right)}{t + D_k}\, dt \tag{4.4}$$

and

$$\int_0^{\infty} \left[\frac{1}{t + D_k} - \frac{1}{t + 2A_m + \zeta}\right] dt = \log \frac{2A_m}{D_k} + \log\left(1 + \frac{\zeta}{2A_m}\right), \tag{4.5}$$

we have from the second line in (4.3) that

$$\lim_{m\to\infty}\left\{B_m-\left(m-\frac{1}{2}\right)\log\frac{1+a}{1-a}-\left(k+\frac{1}{2}\right)\log 2A_m\right\}$$
$$=-D_k-\left(k+\frac{1}{2}\right)\log D_k+\int_0^\infty\frac{\rho_k(t)-\left(k+\frac{1}{2}\right)}{t+D_k}dt=-Y_k. \tag{4.6}$$

Thus we get (4.1). In order to prove (0.1) we have to find the constant $2e^{Y_k}$. ■

5. The Constant

From the point of view of the best weighted polynomial approximation of the function $|x|^p$ (see Sect. 3) our current result has the form

$$\lim_{m\to\infty}\left(\frac{1+a}{1-a}\right)^{\frac{n}{2}+1}n^{\frac{p}{2}+1}E_n^*(p,a)=\left(\frac{(1+a)^2}{2a}\right)^{\frac{p}{2}+1}c(p). \tag{5.1}$$

On the other hand for the uniform approximation of $|x|^p$ (see details in Appendix 1)

$$\lim_{m\to\infty}\left(\frac{1+a}{1-a}\right)^{\frac{n}{2}+1}n^{\frac{p}{2}+1}E_n(p,a)=2^{\frac{p}{2}+1}a^{\frac{p}{2}-1}\frac{(1+a)^2}{2\left|\Gamma\left(-\frac{p}{2}\right)\right|}. \tag{5.2}$$

Since

$$\lim_{a\to1}\lim_{n\to\infty}\frac{E_n^*(p,a)}{E_n(p,a)}=1,$$

we obtain

$$c(p)\left|\Gamma\left(-\frac{p}{2}\right)\right|=2.$$

Using $\left|\Gamma\left(-\frac{p}{2}\right)\right|\Gamma\left(\frac{p}{2}+1\right)=\pi$, we have

$$c(p)=\frac{2}{\pi}\Gamma\left(\frac{p}{2}+1\right).$$

This finishes the proof of (0.1).

6. Case $m=k$, $m\to\infty$

It is quite evident that the final configuration of the conformal mapping in this case should be just a symmetrization of the map that we had in the case $k=0$, $m\to\infty$. However it is even much simpler to make this reduction by a suitable change of variables. First, we put $a=\alpha^2$, then $x\in[a,1]$ means $y=\frac{x}{\alpha}\in[\alpha,\alpha^{-1}]$

and we have one more symmetry $y \mapsto 1/y$. Therefore the extremal function is symmetric and possesses the representation

$$\tilde{f}(y; m, m) := f(x; m, m; a) = P_{2m-1}\left(\frac{y + y^{-1}}{\alpha + \alpha^{-1}}\right), \tag{6.1}$$

where $P_{2m-1}(t)$ is the best polynomial approximation of sgn (t) on $\left[-1, -\frac{2\alpha}{1+\alpha^2}\right] \cup \left[\frac{2\alpha}{1+\alpha^2}, 1\right]$. Thus we have

$$L_m^m(a) = L_m^0\left(\frac{2\sqrt{a}}{1+a}\right), \tag{6.2}$$

and

$$\lim_{m\to\infty} L_m^m(a)\left(\frac{1+\sqrt{a}}{1-\sqrt{a}}\right)^{2m-1}(2m-1)^{\frac{1}{2}} = \frac{1-a}{\sqrt{\pi\sqrt{a}(1+a)}}. \tag{6.3}$$

7. Unweighted Extremal Polynomial via Conformal Mapping

Let $P_m(z, a)$ be the best uniform (unweighted) approximation of $|x|$ by polynomials of degree not more than $2m$ on two intervals $[-1, -a] \cup [a, 1]$ and $L = L_m(a)$ be an approximation error.

In this section we prove

Theorem 7.1. *There is a curve $\gamma = \gamma_m(a)$ inside the half–strip*

$$\{w = u + iv : u \in (0, (m+1)\pi),\ v > 0\} \tag{7.1}$$

such that the extremal polynomial possesses the representation

$$P_m(z, a) = z + L\cos\phi_m(z, a),$$

where $\phi_m(z, a)$ is the conformal map of the first quadrant onto the region in the half-strip (7.1) *bounded on the left by $\gamma_m(a)$, which is normalized by $\phi_m(a, a) = 0$, $\phi_m(1, a) = (m+1)\pi$ and $\phi_m(\infty, a) = \infty$. Moreover, the curve γ is an image of the imaginary half–axis under this conformal map that satisfies the following functional equation*

$$\gamma_m(a) = \{u + iv = \phi_m(iy, a) :\ L\sin u(y)\sinh v(y) = y,\ y > 0\}. \tag{7.2}$$

P r o o f. First we clarify the shape of the extremal polynomial. In particular, we prove that $P_m(0, a) > L$. On the way we show the fact that is probably interesting on its own: $P'_m(x, a)$ looks much similar to the polynomial of the best approximation of sgn (x) on two symmetric intervals [5], with the only difference that the deviations in area should be equal, instead of the maximum modulus. However it can be shown that $P'_m(x,a)$ is not the best L^1 approximation of sgn (x).

Due to the symmetry of $P_m(x,a)$, we can use the Chebyshev theorem with respect to the best approximation of $\sqrt{x}$ on $[a^2,1]$ by polynomials of degree m. It gives us that $P_m(z,a)$ has $m+2$ points $\{x_j\}$ on the interval $[a,1]$ where $P_m(x_j,a) = x_j \pm L$ (the right half of the Chebyshev set in this case). Moreover, $x_0 = a$ and $x_{m+1} = 1$. At all other points, in addition, we have $P'_m(x_j,a) = 1$, $1 \le j \le m$. Between each two of them we have a point y_j, where $P''_m(y_j) = 0$. Therefore we obtain $2(m-1)$ zeros of the second derivative in $(-1,-a) \cup (a,1)$ and this is precisely its degree. Thus there is *no other critical points* of $P'_m(z,a)$, in particular, in $(-a,a)$ and on the imaginary axis.

From the first consequence, we conclude that on $(-a,a)$ the $P'_m(z,a)$ increases. That is on (a,x_1) the graph of $P_m(z,a)$ is under the line $x \pm L$, depending on the value $P_m(a,a)$, that, recall, should be $a+L$ or $a-L$. Therefore, it is under the line $x+L$ and $P_m(a,a)-a = L$, $P_m(x_1,a)-x_1 = -L$. Continuing in this way we get values of $P_m(x_j,a)$ at all other points x_j by alternance principle. Note that as a byproduct we get

$$\int_{x_{i-1}}^{x_i} |P'_m(x,a)-1|\,dx = 2L$$

for all $1 \le i \le m+1$.

From the second consequence we have that $\mathrm{Im}P'_m(iy) \ge 0$ on the imaginary axis, that is $P_m(iy,a)$, being real, decreases with y, starting from $P_m(0,a) > L$ to $-\infty$. From this remark and the argument principle we deduce that the equation

$$P_m(z,a) - z = tL \tag{7.3}$$

has no solution in the open first quarter for all $t \in (-1,1)$.

Indeed, since $P_m(z,a)-z$ alternates between $\pm L$ in the interval [a,1], (7.3) has $m+1$ solutions, which we denote by $x_j(t)$. Consider now the contour that runs on the positive real axis to $x_j(t)-\epsilon$, then it goes around $x_j(t)$ on the half-circle of the radius ϵ clockwise. After the last of x_j's we continue to go along the contour to the big positive R. The next piece of the contour is a quarter-circle up to imaginary axis. Finally, from iR we go back to the origin. On each half-circle of the radius ϵ the argument of the function changes by $-\pi$. On the quarter-circle it changes by about $\deg P_m(z,a) \times \frac{\pi}{2} = m\pi$. On the imaginary axis we have $\mathrm{Re}(P_m(iy,a)-iy) = P_m(iy,a)$ and $\mathrm{Im}(P_m(iy,a)-iy) = -y$. Since $P_m(iy,a)$ decreases much faster than $-y$ (degree of P_m is at least two), the change of the argument on the last piece of the contour is about π. Thus the whole change is $-(m+1)\pi + m\pi + \pi = 0$. Since the function has no poles, it has no zeros in the region.

Thus $\arccos \frac{P_m(z,a)-z}{L}$ is well defined in the quarter-plane. We finish the proof by inspection of the boundary correspondence. ■

Note two facts: the curve (7.2) has the asymptote $u \to \pi$, $v \to +\infty$ ($y \to +\infty$) and we have uniqueness of the solution of the functional equation (7.2) due to uniqueness of the extremal polynomial.

8. Appendix 1

As it is said in [1], problem 42:

$$E_l\left[\frac{1}{(b+x)^s}\right] \sim \frac{l^{s-1}}{|\Gamma(s)|}\frac{(b-\sqrt{b^2-1})^l}{(b^2-1)^{\frac{s+1}{2}}}, \quad b>1,\ s\neq 0, \tag{8.1}$$

where $E_l[f(x)]$ is an error of approximation of the function $f(x)$ on the interval $[-1,1]$ by polynomials of degree not more than l.

We change the variable

$$y = \frac{b+x}{b+1}$$

and put $a^2 = \frac{b-1}{b+1}$. Then we have

$$\inf_{P:\deg P\le l}\ \max_{y\in[a^2,1]} |y^{-s} - P(y)| = (1+b)^s E_l\left[\frac{1}{(b+x)^s}\right].$$

That is

$$E_{2l}(-2s,a) = (1+b)^s E_l\left[\frac{1}{(b+x)^s}\right]. \tag{8.2}$$

Note that

$$b = \frac{1+a^2}{1-a^2}, \quad b^2-1 = \frac{4a^2}{(1-a^2)^2},$$

and therefore

$$\sqrt{b^2-1} = \frac{2a}{1-a^2}, \quad b-\sqrt{b^2-1} = \frac{1-a}{1+a}.$$

Thus from (8.1) and (8.2) we get

$$\begin{aligned} E_{2l}(-2s,a) \sim &\left(\frac{2}{1-a^2}\right)^s \frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^l \left(\frac{1-a^2}{2a}\right)^{s+1} \\ =&a^{-s}\frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^l\left(\frac{1-a^2}{2a}\right) \\ =&a^{-s-1}\frac{l^{s-1}}{|\Gamma(s)|}\left(\frac{1-a}{1+a}\right)^{l+1}\frac{(1+a)^2}{2}. \end{aligned}$$

9. Appendix 2

Here we present a "solvable model" for the problem under consideration: we replace the comparably complicated configuration (3.2), that we remove from the strip, by just two slits. We used this model on the first step of rough understanding of the form of asymptotic and it might be useful for the reader, in particular, it contains the hint that in a nonmodel case the asymptotic of $L_m^{qm}(a)$ for $m \to \infty$ can also be found for an arbitrary $q \in \mathbb{N}$ fixed, see (9.12).

For $B > 0$, consider the conformal mapping $w = \phi(z)$ of the upper half-plane $\mathbb{C}_+$ on the strip

$$\Pi = \{w : 0 < \mathrm{Im} w < (k+m)\pi\} \tag{9.1}$$

with the cut

$$\gamma_B = \{w : \mathrm{Im} w = k\pi, |\mathrm{Re} w| \geq B\} \tag{9.2}$$

under the normalizations

$$\phi(0) = 0, \quad \phi(\pm 1) = \pm\infty_2, \tag{9.3}$$

where ∞_2 denotes the point on the boundary of the domain when we go to infinity on the level $k\pi < \mathrm{Im} w < (k+m)\pi$. By ∞_1 we denote the point on the boundary that corresponds to the level $0 < \mathrm{Im} w < k\pi$. Put $a = \phi^{(-1)}(+\infty_1)$ (therefore $-a = \phi^{(-1)}(-\infty_1)$).

Let us find a precise formula for this map as well as the relation between a and B. We have

$$\begin{aligned} \phi(z) =& k \int_a^\infty \left(\frac{1}{x-z} - \frac{1}{x+z} \right) dx + m \int_1^\infty \left(\frac{1}{x-z} - \frac{1}{x+z} \right) dx \\ &+ k \log \frac{x-z}{x+z} \Big|_a^\infty + m \log \frac{x-z}{x+z} \Big|_1^\infty \\ =& k \log \frac{a+z}{a-z} + m \log \frac{1+z}{1-z}. \end{aligned} \tag{9.4}$$

Further, for $a < x < 1$ we have

$$\mathrm{Re}\phi(x) = k \log \frac{x+a}{x-a} + m \log \frac{1+x}{1-x} \tag{9.5}$$

and B corresponds to the critical value of this function on the given interval. For the critical point c we have

$$(\mathrm{Re}\phi)'(c) = -\frac{2ka}{c^2 - a^2} + \frac{2m}{1-c^2} = 0. \tag{9.6}$$

Therefore

$$c = \sqrt{\frac{ma^2 + ka}{m + ka}} \tag{9.7}$$

and

$$B = k \log \frac{c+a}{c-a} + m \log \frac{1+c}{1-c}. \tag{9.8}$$

Let us mention that the relation between a and B is monotonic, and a runs from 0 to 1 as B runs from 0 to ∞.

As the next step, we calculate the asymptotic behavior of B for the fixed a as $m \to \infty$. First, we write the asymptotic for c

$$c = \sqrt{\frac{ma^2 + ka}{m + ka}} = a + \frac{k}{2m}(1 - a^2) + \dots . \tag{9.9}$$

Therefore

$$\begin{aligned} B =& k \log \left(2a + \frac{k}{2m}(1 - a^2) + \dots \right) - k \log \left(\frac{k}{2m}(1 - a^2) + \dots \right) \\ &+ m \log \frac{1 + a + \frac{k}{2m}(1 - a^2) + \dots}{1 - a - \frac{k}{2m}(1 - a^2) + \dots} \\ =& k \log \frac{2a}{1 - a^2} + k \log \frac{2m}{k} + \dots \\ &+ m \log \frac{1+a}{1-a} + m \log \frac{1 + \frac{k}{2m}(1 - a) + \dots}{1 - \frac{k}{2m}(1 + a) + \dots} \\ =& m \log \frac{1+a}{1-a} + k \log 2m + k \log \frac{2a}{1 - a^2} + k - k \log k + \dots . \end{aligned} \tag{9.10}$$

Actually, it was important for us to note that in the second (logarithmic) term in asymptotic we have the factor k.

To finish this section let us discuss asymptotic for the case

$$k = qm, \quad m \to \infty$$

for a fixed q. Note that now c is just a constant

$$c = \sqrt{\frac{a^2 + qa}{1 + qa}} \tag{9.11}$$

and we have

$$B = m \left(q \log \frac{c+a}{c-a} + \log \frac{1+c}{1-c} \right), \tag{9.12}$$

and $B = 2m \log \frac{1+\sqrt{a}}{1-\sqrt{a}}$ for $q = 1$.

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