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Subharmonic Almost Periodic Functions of Slow Growth

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We obtain a complete description of the Riesz measures of almost periodic subharmonic functions with at most of linear growth on $\mathbb{C}$. As a consequence we get a complete description of zero sets for the class of entire functions of exponential type with almost periodic modula.

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Bohr's Theorem (see [2], or [10, Ch. 6, § 1]) implies that any almost periodic function* on real axis $\mathbb{R}$ with the bounded spectrum is just the restriction to $\mathbb{R}$ of an entire almost periodic function f of exponential type. Moreover, f has no zeros outside some strip $|\mathrm{Im} z| \le H$ if and only if supremum and infimum of the spectrum f also belong to the spectrum. In [9] (see also [10, Appendix VI]) M.G. Krein and B.Ya. Levin obtained a complete description of zeros of functions from the last class. Namely, a set $\{a_k\}_{k\in\mathbb{Z}}$ in a horizontal strip of the finite width is just a zero set of an entire almost periodic function f of exponential type if and only if the set is almost periodic and has the representation

$$a_k = dk + \psi(k),\ k \in \mathbb{Z}, \tag{1}$$

where d is a constant, the function $\psi(k)$ is bounded, and the values

$$S_n = \lim_{r\to\infty} \sum_{|k|<r} [\psi(k+n) - \psi(k)] \frac{k}{k^2+1} \tag{2}$$

are bounded uniformly in $n \in \mathbb{Z}$.

*Explicit definitions of almost periodicity for the functions, measures, and discrete sets see in [10, Ch. 6 and Appendix VI], [11], or Sect. 3 of the present paper.

It can be proved that almost periodicity of $\{a_k\}$ yields representation (1) and a finite limit in (2) for every fixed $n \in \mathbb{Z}$. Also, one can obtain a complete description of zero sets for the class of entire functions of exponential type with almost periodic modulus and zeros in a horizontal strip of finite width: we should only replace the S_n by $\mathrm{Re} S_n$.

Observe that every entire function of exponential type bounded on $\mathbb{R}$ has the form

$$f(z) = Ce^{i\nu z} \lim_{r\to\infty} \prod_{|a_k|<r} \left(1 - \frac{z}{a_k}\right), \ \nu \in \mathbb{R} \tag{3}$$

([10, Ch. 5]; for simplicity we suppose $0 \notin \{a_k\}$); so we have an explicit representation for the functions from the classes mentioned above.

Note that in [5] one of the authors of the present paper obtained a complete description of zero sets for holomorphic almost periodic functions on the strip and on the plane without any growth conditions. An implicit representation for a special case of almost periodic holomorphic functions was obtained earlier in [3]. Besides, it was proved in [3] that zero sets of holomorphic functions with the almost periodic modulus on the strip (or on the plane) are just almost periodic discrete sets. This result is the consequence of a more general one: every almost periodic measure on a strip is just the Riesz measure of some subharmonic almost periodic function on the strip.

In Sect. 3 of our paper we obtain a complete description of the Riesz measures for almost periodic subharmonic functions of normal type with respect to order 1 (note that it is the smallest growth for the bounded on $\mathbb{R}$ subharmonic function). In particular, we consider the case of periodic subharmonic functions. As a consequence, we get a complete description of zero sets for the class of entire functions of exponential type with the almost periodic modulus without any additional requirements on distributions of zeros. Note that representation (1) with a bounded function $\psi(k)$ is incorrect here, therefore the methods used in paper [9] do not work in our case. The integral representation from [3] creates an almost periodic subharmonic function with a given almost periodic Riesz measure, but does not allow to control the growth of the function, therefore it is not fit for our problem as well.

We make use of a subharmonic analogue of the representation

$$\log|f(z)| = \int_0^\infty \frac{n(0,t) - n(z,t)}{t} dt - \nu y + \log|C|, \tag{4}$$

for functions of the form (3) (see [6] or review [7, p. 45]); here $n(c,t)$ is a number of zeros in the disc $\{z : \ |z-c| \le t\}$. We obtain this analogue in Sect. 2 of our paper. Also, we get a complete description of the Riesz measures for the bounded on $\mathbb{R}$ subharmonic functions with at most of linear growth on $\mathbb{C}$.

Here we base on a subharmonic analogue of (3) as well. Of course, this analogue can be obtained by repeating all the steps of the proof (3) for entire functions in [10], nevertheless we prefer to give a short proof in Sect. 1, using Azarin's theory of limit sets for subharmonic functions [1]. The idea of the proof belongs to Prof. A.F. Grishin, and the Authors are very grateful to him.

1. In this section we prove the following theorem:

Theorem 1. *Let $v(z)$ be a subharmonic function on $\mathbb{C}$ such that*

$$v^{+}(z) = O(|z|) \text{ as } z \to \infty \tag{5}$$

and

$$\sup_{x\in\mathbb{R}} v(x) < \infty. \tag{6}$$

Then

$$v(z) = \lim_{R\to\infty} \int\limits_{|w|<R} (\log|z-w| - \log^{+}|w|)d\mu(w) + A_1 y + A_2. \tag{7}$$

Here $z = x+iy$, $\mu = \frac{1}{2\pi}\triangle v$ is the Riesz measure of v, $A_1, A_2 \in \mathbb{R}$, and the limit exists uniformly on compact subsets in $\mathbb{C}$.

Note that the condition (5) means that v is at most of normal type with respect to the order 1.

Our proof of Th. 1 is based on Azarin's theory of limit sets [1]. Thus, if a subharmonic function v satisfies (5), then:

a) The family $v_t(z) = t^{-1}v(tz)$, $t > 1$ is a relatively compact set in the space of distributions $\mathcal{D}'(\mathbb{C})$; in other words, for every sequence of functions from this family there is a subsequence converging to a subharmonic function.

Note that the convergence in $\mathcal{D}'(\mathbb{C})$ is weak on all functions from the class of infinitely smooth compactly supported functions $\mathcal{D}(\mathbb{C})$; moreover, the class of subharmonic functions is closed with respect to this convergence.

b) If

$$v_\infty = \lim_{t\to\infty} v_t(z), \tag{8}$$

then the Riesz measure μ_∞ of the function v_∞ satisfies the equality

$$\mu_\infty = \lim_{t\to\infty} \mu_t, \tag{9}$$

where $\mu_t(E) = t^{-1}\mu(tE)$ for the Borel subsets of $\mathbb{C}$; limits (9) exist in the sense of weak convergence on the continuous compactly supported functions on $\mathbb{C}$. Moreover, in this case there exists

$$\lim_{R\to\infty} \int\limits_{1\le|w|<R} \frac{d\mu(w)}{w} \neq \infty. \tag{10}$$

If a subharmonic function satisfies (5) and (8), then it is called *a completely regular growth (with respect to the order 1).*

In what follows we need a simple criterion of the compactness for the family of subharmonic functions (see[1]).

Lemma A. *A family $\{u_\alpha\}$ of subharmonic functions on $\mathbb{C}$ is a relatively compact set in the space of distributions $\mathcal{D}'(\mathbb{C})$ if and only if*

a) $\sup_\alpha \sup_{z\in K} u_\alpha(z) < \infty$ *for all compacta* $K \subset \mathbb{C}$,

b) $\inf_\alpha \sup_{z\in K_0} u_\alpha(z) > -\infty$ *for some compact set* $K_0 \subset \mathbb{C}$.

Also, we need the following variant of the Fragmen–Lindelof theorem.

Theorem FL. *If a function v is subharmonic in the neighborhood of closure of the upper half-plain $\mathbb{C}^+ = \{z = x + iy :\ y > 0\}$ and satisfies conditions (5), (6), then for all $z \in \mathbb{C}^+$*

$$v(z) \le \sup_{x\in\mathbb{R}} v(x) + \sigma^+ y,$$

with $\sigma^+ = \limsup\limits_{y\to+\infty} y^{-1} v(iy)$.

The proof of this statement is the same as for holomorphic on $\mathbb{C}^+$ and continuous on $\overline{\mathbb{C}^+}$ functions (see, for example, [8, p. 28]).

First, let us prove a subharmonic analogue of the Cartwright theorem (a holomorphic case see, for example, in [10, Ch. V]).

Theorem 2. *Assume that a subharmonic function v on $\mathbb{C}$ satisfies (5) and (6). By definition, put*

$$\sigma_\pm = \limsup_{y\to\pm\infty} \frac{v(iy)}{|y|}. \tag{11}$$

Then $v(z)$ is a completely regular growth; moreover, the function v_∞ from (8) has the form

$$v_\infty(z) = \begin{cases} \sigma^+ y, & y \ge 0, \\ \sigma^- |y|, & y < 0, \end{cases} \tag{12}$$

R e m a r k. From Theorem FL it follows that if a subharmonic function v on $\mathbb{C}$ satisfies the conditions of Th. 2 with $\sigma^+ \le 0$ and $\sigma^- \le 0$, then v is a constant.

The proof is based on the following lemma

Lemma 1. *Let $u < 0$ be a subharmonic function on $\mathbb{C}^+$. Then for every $R < \infty$ and $r \in (0, R/2)$*

$$\frac{u(ir)}{r} \le \frac{C}{R^3} \int\limits_{|z-iR|<R} u(z)\, dm_2(z), \tag{13}$$

where C is an absolute constant and m_2 is a plain Lebesque measure.

P r o o f. Since Poisson formula for the disc $B(iR/2,R/2)=\{z:\ |z-iR/2|<R/2\}$, we have

$$u(ir)\le\frac{1}{2\pi}\int\limits_0^{2\pi}u(iR/2+e^{i\theta}R/2)\frac{(R/2)^2-(R/2-r)^2}{(R/2)^2+(R/2-r)^2-R(R/2-r)\cos(\pi/2+\theta)}d\theta.$$

Using the inequality $u<0$, replace the interval of integration by $[\pi/4,3\pi/4]$. We obtain

$$u(ir)\le\frac{r}{(R-r)}\frac{1}{4}\sup_{\theta\in[\pi/4,3\pi/4]}u\left(\frac{iR}{2}+\frac{R}{2}e^{i\theta}\right). \qquad (14)$$

Since for all $\theta\in[\pi/4,3\pi/4]$

$$B(iR,R/2)\subset B(iR/2+e^{i\theta}R/2,(1+\frac{\sqrt{2}}{2})R/2)\subset\mathbb{C}^+,$$

we have

$$u(\frac{iR}{2}+\frac{R}{2}e^{i\theta})\le\frac{4}{\pi R^2(1+\frac{\sqrt{2}}{2})^2}\int\limits_{|z-iR/2-e^{i\theta}R/2|\le(1+\frac{\sqrt{2}}{2})R/2}u(z)dm_2(z)$$

$$\le\frac{8}{\pi R^2(3+2\sqrt{2})}\int\limits_{|z-iR|<R/2}u(z)dm_2(z).$$

Then replace the average over the disc $B(iR,R/2)$ by the average over the disc $B(iR,R)$. So, the assertion of the lemma follows from (14).

P r o o f o f T h e o r e m 2. Without loss of generality, it can be assumed that $\sup\limits_{x\in\mathbb{R}}v(x)=0$. Put $u(z)=v(z)-\sigma_+y$. From Theorem FL it follows that $u(z)<0$ on $\mathbb{C}^+$, then

$$\limsup_{y\to+\infty}\frac{u(iy)}{y}=0. \qquad (15)$$

Fix $z_0\in\mathbb{C}^+$. Let φ, $0\le\varphi\le 1$, be an infinite differentiable and compactly supported function on $\mathbb{C}^+$, depending only on $|z-z_0|$. Apply Lemma 1 for the function $u_t(z)=u(tz)t^{-1}$. If $\mathrm{supp}\varphi\subset B(iR,R)$, then we get

$$\frac{u(itr)}{tr}\le\frac{C}{R^3}\int\limits_{|z-iR|<R}u_t(z)dm_2(z)\le\frac{C}{R^3}\int\limits_{\mathbb{C}^+}u_t(z)\varphi(z)dm_2(z).$$

Since (15), we see that for any $\varepsilon>0$ and $t>t(\varepsilon)$ there is $r\in(0,R/2)$ such that $u(itr)\ge-\varepsilon tr$. We obtain

$$\int\limits_{\mathbb{C}^+}u_t(z)\varphi(z)dm_2(z)\ge-\frac{\varepsilon R^3}{C}.$$

Hence,

$$\int_{\mathbb{C}^+} u_t(z)\varphi(z)dm_2(z) \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, $v_t(z) = u_t(z) + \sigma_+ y \to \sigma_+ y$ in the space $\mathcal{D}'(\mathbb{C}^+)$. Similarly, $v_t(z) \to -\sigma_- y$ in the space $\mathcal{D}'(\mathbb{C}^-)$, where $\mathbb{C}^- = \{z = x + iy : \ y < 0\}$. Every limit function for $v_t(z)$ is always subharmonic, therefore we get (12). So limit (8) exists. Theorem 2 is proved.

Consequence. *The Riesz measure of the limit function* $v_\infty(z)$ *equals*

$$\frac{\sigma_+ + \sigma_-}{2\pi} m_1(x),$$

where m_1 *is the Lebesgue measure on* $\mathbb{R}$.

P r o o f o f T h e o r e m 1. From the Jensen–Privalov formula for the subharmonic function, we get the estimate for $r \geq 1$

$$\mu(B(0,r)) \leq C_1 r, \tag{16}$$

where constant C_1 depends only on v. Using the Brelot–Hadamard theorem for the subharmonic function (see, for example, [12]), we obtain that there is a harmonic polynomial $H(z)$ of degree 1 such that

$$v(z) = \int_{|w|<1} \log|z-w|d\mu(w) + \int_{|w|\geq 1} \log\left(\left|1 - \frac{z}{w}\right| + \mathrm{Re}\frac{z}{w}\right) d\mu(w) + H(z). \tag{17}$$

Denote by $v^0(z)$ the first integral in (17). Since (10), we get

$$v(z) = v^0(z) + \lim_{R\to\infty} \int_{1\leq|w|<R} \log\left|1 - \frac{z}{w}\right| d\mu(w) + A_0 x + A_1 y + A_2. \tag{18}$$

The application of Th. 2 yields that the function v is a completely regular growth, hence the measures μ_t converge weakly to the measure

$$\mu_\infty = \frac{\sigma_+ + \sigma_-}{2\pi} m_1. \tag{19}$$

Let μ' be a restriction of the measure μ to $\mathbb{C} \setminus B(0,1)$. Obviously, the measures μ'_t converge weakly to the measure μ_∞ as well. Therefore,

$$v_t(z) = v_t^0(z) + \lim_{R\to\infty} \int_{|w|<R} \log\left|1 - \frac{z}{w}\right| d\mu'_t(w) + A_0 x + A_1 y + \frac{A_2}{t}. \tag{20}$$

Pass to a limit in (20) as $t \to \infty$ in the space $\mathcal{D}'(\mathbb{C})$. First, by Th. 2, the functions $v_t(z)$ converge to the function $v_\infty(z)$ from (12). Since $v^0(z) = O(\log|z|)$ as $|z| \to \infty$, we see that $v_t^0(z) \to 0$. By (16), we obtain $\mu_t'(B(0,R)) \le C_1 R$ for all $t \ge 1$ and $R > 0$. Therefore, we get uniformly in $t \ge 1$,

$$\int\limits_{|w|\ge R} \frac{d\mu_t'(w)}{|w|^2} = 2\int\limits_R^\infty \frac{\mu_t(B(0,s))}{s^3} ds - \frac{{\mu_t}'(R)}{R^2} \to 0, \tag{21}$$

as $R \to \infty$. Also, by (10), uniformly in $t \ge 1$, $R' \ge R$

$$\int\limits_{R\le|w|\le R'} \frac{d\mu_t'(w)}{w} = \int\limits_{Rt\le|w|\le R't} \frac{d\mu(w)}{w} \to 0, \tag{22}$$

as $R \to \infty$. For $|z| < C$ and a sufficiently large $|w|$

$$\left|\log\left|1 - \frac{z}{w}\right| + \mathrm{Re}\frac{z}{w}\right| \le \frac{|z|^2}{|w|^2}.$$

Therefore, taking into account (21) and (22), we obtain for all $\varphi \in \mathcal{D}(\mathbb{C})$ uniformly in $t \ge 1$

$$\int\limits_{\mathbb{C}} \left(\lim_{R\to\infty} \int\limits_{|w|\le R} \log\left|1 - \frac{z}{w}\right| d{\mu_t}'(w) \right) \varphi(z) dm_2(z)$$

$$= \lim_{R\to\infty} \int\limits_{|w|\le R} \left(\int\limits_{\mathbb{C}} \log\left|1 - \frac{z}{w}\right| \varphi(z) dm_2(z) \right) d{\mu_t}'(w). \tag{23}$$

Note that the measure μ_∞ does not charge any circle $|w| = R$, therefore the restrictions of the measures ${\mu_t}'(w)$ to any disc $B(0,R)$ converge weakly to the restriction of the measure $\mu_\infty(w)$. The function $\int \log|z - w|\varphi(z)dm_2(z)$ is continuous in the variable w, so we have

$$\lim_{t\to\infty} \int\limits_{|w|\le R} \int \log|z-w|\varphi(z)dm_2(z)d\mu_t(w) = \int\limits_{|w|\le R} \int \log|z-w|\varphi(z)dm_2(z)d\mu_\infty(w). \tag{24}$$

By the same reason for each $\delta > 0$

$$\lim_{t\to\infty} \int\limits_{\delta\le|w|\le R} \log|w|\mu_t(w) = \int\limits_{\delta\le|w|\le R} \log|w|\mu_\infty(w). \tag{25}$$

Furthermore,

$$\int\limits_{|w|\le\delta} \log|w|{\mu'}_t(w) = \log\delta\,\frac{\mu'(B(0,\delta t))}{t} - \int\limits_0^{\delta} \frac{\mu'(B(0,st))}{st}ds. \qquad (26)$$

Since $\mu'(B(0,r)) \le C_1 r$ for all $r>0$, we see that (26) tends to zero as $\delta\to 0$ uniformly in $t\ge 1$. Combining (23)–(25), and (26), we get the equality

$$v_\infty(z) = \lim_{R\to\infty} \int\limits_{|w|\le R} \log\left|1-\frac{z}{w}\right| d\mu_\infty(w) + A_0 x + A_1 y.$$

Take $y=0$. Since $v_\infty(x)=0$ and

$$\lim_{R\to\infty} \int\limits_{|u|\le R} \log\left|1-\frac{x}{u}\right| du = 0$$

for all $x\in\mathbb{R}$, we obtain $A_0=0$. Now the assertion of Th. 1 follows from (18).

2. Here we get a complete description of the Riesz measures for subharmonic functions with at most of linear growth on $\mathbb{C}$ (i.e., not exceeding $C(|z|+1)$ with $C<\infty$) and with some additional conditions (bounded on $\mathbb{R}$ or with the compact family of translations along $\mathbb{R}$). Holomorphic analogues of the corresponding theorems were obtained earlier by one of the Authors in [6].

First, prove some lemmas.

Lemma 2. *If a measure μ on $\mathbb{C}$ satisfies the condition*

$$\mu(B(0,R+1)) - \mu(B(0,R)) = \overline{o}(R) \text{ as } R\to\infty, \qquad (27)$$

and the limit

$$\lim_{R\to\infty} \int\limits_{|w|<R} (\log^+|z-w| - \log^+|w|)d\mu(w)$$

exists at some point $z\in\mathbb{C}$, then the limit equals

$$\int\limits_1^\infty \frac{\mu(B(0,t)) - \mu(B(z,t))}{t}dt.$$

P r o o f. For all $z\in\mathbb{C}$ and $R\in(|z|+1,\infty)$ we have

$$\int\limits_{|w|<R} \log^+|z-w|d\mu(w) - \int\limits_{|w|<R} \log^+|w|d\mu(w) = (\log R)\mu(B(z,R))$$

$$-\int_1^R \frac{\mu(B(z,t))}{t}dt - (\log R)\mu(B(0,R)) + \int_1^R \frac{\mu(B(0,t))}{t}dt$$

$$+ \int\limits_{|w|<R,\ |w-z|\geq R} \log^+|z-w|d\mu(w) - \int\limits_{|w|\geq R,\ |w-z|<R} \log^+|z-w|d\mu(w)$$

$$= \int_1^R \frac{\mu(B(0,t)) - \mu(B(z,t))}{t}dt$$

$$+ \int\limits_{|w|<R,\ |w-z|\geq R} \log\left|\frac{z-w}{R}\right| d\mu(w) - \int\limits_{|w|\geq R,\ |w-z|<R} \log\left|\frac{z-w}{R}\right| d\mu(w). \tag{28}$$

If $|w| < R$ and $|z-w| \geq R$ or $|w| \geq R$ and $|z-w| < R$, then we have

$$1 - \frac{|z|}{R} \leq \left|\frac{z-w}{R}\right| \leq 1 + \frac{|z|}{R}.$$

Therefore the integrand functions of the last two integrals in (28) are $O(1/R)$ as $R \to \infty$. The domains of integrations are the subsets of the ring $R - |z| \leq |w| \leq R + |z|$, hence, by (27), these integrals tend to 0 as $R \to \infty$. Lemma 2 is proved.

Lemma 3. *Assume that a measure μ satisfies (10), (16), and (27), and*

$$V(z) = \lim_{R\to\infty} \int\limits_{|w|<R} (\log|z-w| - \log^+|w|)d\mu(w). \tag{29}$$

Then V is a subharmonic function with the Riesz measure μ and

$$V(z) = \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(z,t))}{t}dt + \int\limits_{|w-z|<1} \log|z-w|d\mu(w) \tag{30}$$

for all $z \in \mathbb{C}$. Furthermore, the function

$$\tilde{V}(z) = \frac{1}{2\pi}\int_0^{2\pi} V(z+e^{i\theta})d\theta \tag{31}$$

satisfies the equality

$$\tilde{V}(z) = \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(z,t))}{t}dt. \tag{32}$$

P r o o f. It follows from (16) that the integral in (17) is a subharmonic function on $\mathbb{C}$ with the Riesz measure μ; besides, it satisfies (5) (see, for example, [12, Ch. 1]). If, in addition, μ satisfies (10), then the limit in (29) exists uniformly on bounded sets, and the function V coincides with the integral in (17) up to a linear term; so it has the same properties as well.

Using the equality $\int_0^{2\pi} \log|a+e^{i\theta}|d\theta = 2\pi\log^+|a|$, we get

$$\tilde{V}(z) = \lim_{R\to\infty} \int\limits_{|w|<R} (\log^+|z-w| - \log^+|w|)d\mu(w).$$

and $V(z) = \tilde{V}(z) + \int_{|w-z|<1} \log|z-w|d\mu(w)$. Then Lemma 2 implies (32) and (30). Lemma 3 is proved.

Lemma 4. *Assume that a subharmonic on $\mathbb{C}$ function v satisfies (5). Then the family of translations $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset in $\mathcal{D}'(\mathbb{C})$ if and only if the function*

$$\tilde{v}(z) = \frac{1}{2\pi}\int_0^{2\pi} v(z+e^{i\theta})d\theta \tag{33}$$

is bounded on $\mathbb{R}$; this function is bounded simultaneously with the function

$$\hat{v}(z) = \frac{1}{\pi}\int_{|w|<1} v(z+w)dm_2(w). \tag{34}$$

P r o o f. If the family $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset, then it is uniformly bounded from above on compacta in $\mathbb{C}$ and the function v is uniformly bounded from above on every strip $|y| < H$. Then by Lemma A there is a compact subset K_0 of $\mathbb{C}$ such that

$$\sup_{K_0} v(z+h) \geq C_2, \quad \forall h \in \mathbb{R}.$$

Take $d > \sup_{K_0}|z|$. Then for any $h \in \mathbb{R}$ there is a point $z(h)$, $|z(h)| < d$, such that

$$\frac{1}{(d+1)^2\pi} \int\limits_{|w-z(h)|\leq d+1} v(w+h)dm_2(w) \geq v(z(h)+h) > C_2 - 1.$$

Further, we have

$$\int\limits_{|w-z(h)|\leq d+1} v(w+h)dm_2(w) \leq \int\limits_{|w|\leq 1} v(w+h)dm_2(w) + (d^2+2d)\pi \sup_{|y|<d+1} v(z).$$

Therefore, $\inf_{h\in\mathbb{R}} \hat{v}(h) > -\infty$. Since $\tilde{v}(h) \geq \hat{v}(h)$, we see that the functions $\hat{v}$ and $\tilde{v}$ are bounded uniformly from below on $\mathbb{R}$. It is clear that these functions are bounded uniformly from above on $\mathbb{R}$ as well.

On the other hand, if $\tilde{v}(z)$ is bounded from below on $\mathbb{R}$, then $\inf_{h\in\mathbb{R}} \sup_{|w|=1} v(h+w) > -\infty$; if $\tilde{v}(h)$ is bounded from above on $\mathbb{R}$, then $v(h)$ is bounded from above on $\mathbb{R}$ as well. Since Theorem FL, we see that $v(z)$ is bounded from above on every strip $|y| < H$. It follows from Lemma A that $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact set. Hence $\hat{v}(x)$ is bounded on $\mathbb{R}$.

Now we can prove the theorems mentioned above.

Theorem 3. *For a measure μ on $\mathbb{C}$ to be the Riesz measure for some subharmonic function satisfying conditions (5) and (6), it is necessary and sufficient that the conditions (10), (16), (27), and*

$$\sup_{x\in\mathbb{R}} \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(x,t))}{t} dt < \infty. \tag{35}$$

were fulfilled.

P r o o f. If a subharmonic function v satisfies (5) and (6), then, by Th. 2, its Riesz measure μ satisfies (10) and the measures μ_t converge to the measure $\mu_\infty = (\sigma_+ + \sigma_-)(2\pi)^{-1} m_1$. The last measure does not charge any circle $|w| = R$, therefore (9) implies that $\mu(B(0,R)) = CR + \overline{o}(R)$ as $R \to \infty$. Hence we get (16) and (27). By Theorem 1, $v(z) = V(z) + A_1 y + A_2$. So the function V is also bounded from above on $\mathbb{R}$. Since Theorem FL, we see that the same is true for the function $\tilde{V}$ from (31). Now Lemma 3 implies (35).

Conversely, if a measure μ satisfies (10), (16), and (27), then, by Lemma 3, the subharmonic function V has the Riesz measure μ and satisfies (5) and (6). Theorem 3 is proved.

Theorem 4. *For a measure μ on $\mathbb{C}$ to be the Riesz measure for some subharmonic function v with the property (5) such that the family of translations $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset in $\mathcal{D}'(\mathbb{C})$, it is necessary and sufficient that the conditions (10), (16), (27), and*

$$\sup_{x\in\mathbb{R}} \left| \int_1^\infty \frac{\mu(B(0,t)) - \mu(B(x,t))}{t} dt \right| < \infty \tag{36}$$

were fulfilled.

P r o o f. If a function v satisfies (5), and the function $\tilde{v}$ from (33) is bounded on $\mathbb{R}$, then v is bounded from above on $\mathbb{R}$. Theorem 3 implies conditions (10), (16), and (27) for the Riesz measure μ of the function v. By Theorem 1, $\tilde{v}$ equals $\tilde{V}$ from (31) on $\mathbb{R}$ up to a constant term, therefore Lemmas 3 and 4 imply (36).

Conversely, if a measure μ satisfies (10), (16), (27), and (36), then V from (29) satisfies (5), and the function $\tilde{V}$ is bounded on $\mathbb{R}$.

Hence the assertion of Th. 4 follows from Lemma 4.

3. A continuous function $F(z)$ on a closed strip $\{z = x+iy :\ x \in \mathbb{R},\ |y| \le H\}$ with $H \ge 0$ is *almost periodic* if the family of translations $\{F(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact set with respect to the topology of uniform convergence on the strip; a function is almost periodic on an open strip (in particular, on $\mathbb{C}$), if it is almost periodic on every closed substrip of a finite width.

A measure (maybe complex) μ on $\mathbb{C}$ is called *almost periodic* if for any test-function $\varphi \in \mathcal{D}(\mathbb{C})$ the convolution $\int \varphi(w+t)d\mu(w)$ is an almost periodic function in $t \in \mathbb{R}$ ([11]).

The following statement is valid:

Theorem R (Theorem 1.8 [11]). *For a measure μ to be almost periodic it is necessary and sufficient to fulfil the following condition: for any sequence $\{h_n\} \subset \mathbb{R}$ there should exist a subsequence $\{h'_n\}$ such that the convolutions $\int \varphi(w+x+h'_n)d\mu(w)$ would converge uniformly with respect to $x \in \mathbb{R}$ and $\varphi \in L$, where L is a compact subset in $\mathcal{D}(\mathbb{C})$. Moreover, for a measure to be almost periodic it is sufficient to check this condition only for all single-point sets $L \subset \mathcal{D}(\mathbb{C})$.*

If we take $L = \{\varphi(z+iy)\}_{|y|\le H}$ for some $\varphi \in \mathcal{D}(\mathbb{C})$, we obtain that the convolutions $\int \varphi(w+z+h'_n)d\mu(w)$ actually converge uniformly on any strip $|y| \le H$, hence the function $\int \varphi(w+z)d\mu(w)$ is almost periodic on $\mathbb{C}$.

Further, a subharmonic function v on $\mathbb{C}$ is called *almost periodic*, if the measure $v(z)dm_2(z)$ is almost periodic (see [3]; an equivalent definition see in [4]).

It follows from the definition that the Riesz measure of an almost periodic subharmonic function is also almost periodic. Conversely, every almost periodic measure is the Riesz measure of some almost periodic subharmonic function (see [3], where the case of the strip is studied as well). Note that the family of translations $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset of $\mathcal{D}'(\mathbb{C})$ for every almost periodic subharmonic function v on $\mathbb{C}$. Note also that any almost periodic subharmonic function is bounded from above on every horizontal strip of a finite width (see [3]).

Here we obtain the following result:

Theorem 5. *A necessary and sufficient condition for a measure μ on $\mathbb{C}$ to be the Riesz measure of some almost periodic subharmonic function at most of linear growth is that the measure should be almost periodic and satisfy (16), (27), (10), and (36).*

The proof of the theorem is based on the following lemmas.

Lemma 5. *Suppose subharmonic functions $v_1(z)$ and $v_2(z)$ on $\mathbb{C}$ with the common Riesz measure satisfy (5) and (6); then $v_1(z) = v_2(z)+p_1+p_2y$. Further, if*

$$\sup_{\mathbb{R}} v_1(x) = \sup_{\mathbb{R}} v_2(x) \quad \textit{and} \quad \sigma_+(v_1) = \sigma_+(v_2),$$

where σ_+ is defined in (11), then $v_1 \equiv v_2$.

P r o o f. The first part follows from Th. 1, the second one is evident.

Lemma 6. *Assume that a subharmonic on $\mathbb{C}$ function v satisfies (5) and a family $\{v(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset of $\mathcal{D}'(\mathbb{C})$; if $v(z+h_n) \to v^*(z)$ in the space $\mathcal{D}'(\mathbb{C})$, then the family $\{v^*(z+h)\}_{h\in\mathbb{R}}$ is a relatively compact subset as well and*

$$\sup_{x\in\mathbb{R}} v^*(x) \le \sup_{x\in\mathbb{R}} v(x), \tag{37}$$

$$\inf_{t\in\mathbb{R}} \int\limits_{|z-t|<1} v^*(z)dm_2(z) \ge \inf_{t\in\mathbb{R}} \int\limits_{|z-t|<1} v(z)dm_2(z), \tag{38}$$

$$\sigma_+(v^*) \le \sigma_+(v), \qquad \sigma_-(v^*) \le \sigma_-(v). \tag{39}$$

P r o o f. Put $M = \sup_{x\in\mathbb{R}} v(x)$. By Theorem FL, we get for any $\varepsilon > 0$

$$v(z) \le M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < 2\varepsilon.$$

Let φ be a function from $\mathcal{D}(\mathbb{C})$ such that φ depends only on $|z|$, $\varphi \ge 0$, $\varphi(z) = 0$ for $|z| \ge \varepsilon$, $\int \varphi(z)dm_2(z) = 1$. Then

$$(v * \varphi)(z) \le M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < \varepsilon.$$

Therefore,

$$(v^* * \varphi)(z) \le M + 2\varepsilon \max\{\sigma_+(v), \sigma_-(v)\}, \quad |y| < \varepsilon.$$

Note that v^* is a subharmonic function, hence $(v^* * \varphi)(z) \ge v^*(z)$. Since ε is arbitrary, we obtain (37). By the same argument, for all $y \in \mathbb{R}$

$$\sup_{x\in\mathbb{R}} v^*(x+iy) \le \sup_{x\in\mathbb{R}} v(x+iy).$$

Therefore we obtain (5) and (39).

Further, the functions $v(z+h_n)$ are integrable on every disc and uniformly bounded from above, therefore we can replace the convergence of measures $v(z+$

$h_n)dm_2(z)$ in the sense of distributions by the weak convergence of measures. Since the limit measure $v^*(z)dm_2(z)$ does not charge any circle, we have

$$\lim_{n\to\infty} \int\limits_{|z-t|<1} v(z+h_n)dm_2(z) = \int\limits_{|z-t|<1} v^*(z)dm_2(z)$$

for each $t \in \mathbb{C}$. Hence we get (38). Taking into account Lemma 4, we see that the family $\{v^*(z+h)\}_{h\in\mathbb{R}}$ is a compact subset of $\mathcal{D}'(\mathbb{C})$. The lemma is proved.

Lemma 7. *Under the conditions of the previous lemma, suppose that the Riesz measure μ of the function $v(z)$ is almost periodic. Then inequalities (37)–(39) turn into equalities, the Riesz measure μ^* of the function $v^*(z)$ becomes almost periodic, and there is a subsequence $\{h_{n'}\}$ such that for every $\varphi \in \mathcal{D}(\mathbb{C})$*

$$\lim_{n'\to\infty} \sup_{t\in\mathbb{R}} \left| \int \varphi(w-t-h_{n'})d\mu(w) - \int \varphi(w-t)d\mu^*(w) \right| = 0. \tag{40}$$

P r o o f. For all $\varphi \in \mathcal{D}(\mathbb{C})$ we have

$$\lim_{n\to\infty} \int \varphi(z-h_n)v(z)dm_2(z) = \lim_{n\to\infty} \int \varphi(z)v(z+h_n)dm_2(z) = \int \varphi(z)v^*(z)dm_2(z).$$

Since $\mu = (2\pi)^{-1}\triangle v$, we obtain

$$\lim_{n\to\infty} \int \varphi(z-h_n)d\mu(z) = \int \varphi(z)d\mu^*(z). \tag{41}$$

From Theorem R it follows that there is a subsequence $\{h_{n'}\}$ such that for any $\varphi \in \mathcal{D}(\mathbb{C})$ the almost periodic functions $\int \varphi(z-t-h_{n'})d\mu(z)$ converge to an almost periodic function uniformly in $t \in \mathbb{R}$. If we replace z by $z-t$ in (41), then we get (40). Consequently, the function $\int \varphi(w-t)d\mu^*(w)$ is almost periodic in $t \in \mathbb{R}$, and μ^* is an almost periodic measure.

Passing to a subsequence again if necessary, we may assume that the functions $v^*(z-h_n)$ converge in the space $\mathcal{D}'(\mathbb{C})$ to some subharmonic function $v^{**}(z)$ with the Riesz measure μ^{**}. Therefore,

$$\lim_{n'\to\infty} \int \varphi(z+h_{n'})d\mu^*(z) = \int \varphi(z)d\mu^{**}(z).$$

On the other hand, it follows from (40) that

$$\lim_{n\to\infty} \left| \int \varphi(w)d\mu(w) - \int \varphi(w+h_n)d\mu^*(w) \right| = 0.$$

Here φ is an arbitrary function from $\mathcal{D}(\mathbb{C})$, hence, $\mu^{**} = \mu$. By Lemma 5, we get $v^{**}(z) = v(z) + D_1 + D_2 y$.

Since (37) is valid for the pairs v, v^* and v^*, v^{**}, we get $D_1 \le 0$. Then (38) for the pairs v, v^* and v^*, v^{**} implies $D_1 \ge 0$, and we obtain $D_1 = 0$ and the equality in (37) and (38). By the same way, we obtain the equalities in (39). Lemma 7 is proved.

P r o o f o f T h e o r e m 5. The necessity follows immediately from Th. 4. Let us prove a sufficiency. Suppose μ satisfies the conditions of Th. 5. Let V be the function from (29), and let $\{h_n\} \subset \mathbb{R}$ be an arbitrary sequence. It follows from Th. 4 that the family $\{V(z + h_n)\}$ is a relatively compact subset of $\mathcal{D}'(\mathbb{C})$. Therefore we can assume without loss of generality that $V(z + h_n) \to V^*(z)$ in $\mathcal{D}'(\mathbb{C})$. To prove the Th. 5, we need to check that

$$\int \varphi(z - t - h_n)V(z)dm_2(z) \to \int \varphi(z - t)V^*(z)dm_2(z)$$

uniformly in $t \in \mathbb{R}$ for any $\varphi \in \mathcal{D}(\mathbb{C})$.

Assume the contrary. Then there is $\varphi_0 \in \mathcal{D}(\mathbb{C})$, $\varepsilon_0 > 0$, and $\{t_n\} \subset \mathbb{R}$ such that

$$\left| \int \varphi_0(w)V(w + h_n + t_n)dm_2(w) - \int \varphi_0(w)V^*(w + t_n)dm_2(w) \right| \ge \varepsilon_0. \qquad (42)$$

(If necessary we can replace the sequence $\{h_n\}$ by a subsequence.)

We may assume also that $V(z + h_n + t_n) \to V^{**}(z)$, $V^*(z + t_n) \to V^{***}(z)$ in $\mathcal{D}'(\mathbb{C})$ as $n \to \infty$. By μ^*, μ^{**}, μ^{***} denote the Riesz measures of the functions V^*, V^{**}, V^{***}, respectively. Then we have

$$\lim_{n\to\infty} \int \varphi(w - h_n - t_n)d\mu(w) = \int \varphi(w)d\mu^{**}(w), \qquad (43)$$

$$\lim_{n\to\infty} \int \varphi(w - t_n)d\mu^*(w) = \int \varphi(w)d\mu^{***}(w) \qquad (44)$$

for any $\varphi \in \mathcal{D}(\mathbb{C})$.

On the other hand, the measure μ satisfies (40). Hence the integrals in the left-hand sides of (43) and (44) have the same limit, and $\mu^{**} = \mu^{***}$.

By Lemma 7 we obtain

$$\sup_{\mathbb{R}} V^{**}(x) = \sup_{\mathbb{R}} V(x) = \sup_{\mathbb{R}} V^*(x) = \sup_{\mathbb{R}} V^{***}(x)$$

and

$$\sigma_+(V^{**}) = \sigma_+(V) = \sigma_+(V^*) = \sigma_+(V^{***}).$$

Using Lemma 5, we get $V^{**} \equiv V^{***}$. This contradicts (42). Theorem 5 is proved.

Now let d be a divisor in $\mathbb{C}$, i.e., a sequence $\{a_k\} \subset \mathbb{C}$ without finite limit points such that any value may appear with a finite multiplicity. A divisor is called *almost periodic* if the discrete measure supported at the points a_k with the mass at every point being equal the multiplicity of the point in the sequence is almost periodic (see [12], [3]; in [3] there is an equivalent geometric definition). Moreover, almost periodic divisors are just the divisors of entire functions* with almost periodic modulus in every substrip $\{z = x + iy : |y| < H\}$. ([3]).

Theorem 6. *For a divisor $\{a_k\}$ to be the divisor of an entire function of exponential type with the almost periodic modulus, it is necessary and sufficient to fulfil the following conditions:*

a) the divisor should be almost periodic,

b) there should be a finite limit

$$\lim_{R\to\infty} \sum_{|a_k|<R} \frac{1}{a_k},$$

c) $n(0,t) = O(t)$,

d) $n(0,t+1) - n(0,t) = \overline{o}(t)$,

e)

$$\sup_{x\in\mathbb{R}} \left| \int_1^\infty \frac{n(0,t) - n(x,t)}{t} dt \right| < \infty;$$

here $n(c,t) = \mathrm{card}\{k : |a_k - c| \le t\}$.

P r o o f. By Theorem 6, conditions a)–e) mean just the existence of an almost periodic subharmonic function v at most of linear growth with the Riesz measure supported at the points $\{a_k\}$ with the mass being equal the multiplicity of the point in the sequence. Then $v(z) = \log |f(z)|$ for an entire function f of exponential type such that the divisor of f is $\{a_k\}$. Since v is almost periodic, we get that $|f|$ is almost periodic (see [3]). The theorem is proved.

Now we consider a periodic case.

Theorem 7. *A necessary and sufficient condition for a measure μ on $\mathbb{C}$ to be the Riesz measure of some periodic subharmonic function with period 1 at most of linear growth is that the measure should be stable with respect to the translation on 1 and*

$$\mu\{z = x + iy : 0 \le x < 1, \ y \in \mathbb{R}\} < \infty. \tag{45}$$

*A divisor $\{a_k\}$ is the divisor of a holomorphic function f when zeros of f coincide with the values $\{a_k\}$ and the multiplicity of every zero equals the multiplicity of corresponding a_k.

P r o o f. Let v be a subharmonic function such that $v(z+1) = v(z)$. It is clear that its Riesz measure is stable with respect to the translation on 1. By Theorem 1, it follows that μ satisfies (16). Using the equality

$$\mu\{z = x+iy:\ 0 \le x < 1,\ |y| < n\} = \frac{1}{2n}\mu\{z = x+iy:\ -n \le x < n,\ |y| < n\}, \tag{46}$$

we get (45).

Conversely, let μ be stable with respect to the translation on 1 and satisfy (45). Using (46), we obtain (16). Then for any $r > 0$

$$\mu\{z:\ r \le |z| < r+1\} \le \sum_{n\in\mathbb{Z},\, |n|\le r+1} \mu\{z:\ x\in[0,1),\ r \le |z+n| < r+1\}. \tag{47}$$

Fix $\delta \in (0,1)$. For $|n| < r(1-\delta) - 1$ we have

$$\{z:\ x\in[0,1),\ r \le |z+n| < r+1\} \subset \{z:\ x\in[0,1),\ |y| > r\delta\}$$

Hence (47) is majorized by

$$2r(1-\delta)\mu\{z:\ x\in[0,1),\ |y| > r\delta\} + (2\delta r + 5)\mu\{z:\ x\in[0,1),\ y\in\mathbb{R}\}. \tag{48}$$

It follows from (45) that for any $\varepsilon > 0$ there exist $\delta > 0$ and $r_0 < \infty$ such that for $r \ge r_0$ (48) is less than $r\varepsilon$. This yields (27).

Further, take $R > r > 1$. We have

$$\left| \int_{r<|z|<R} \frac{d\mu(z)}{z} - \int_{r<|[x]+iy|<R} \frac{d\mu(z)}{z} \right|$$

$$\le \frac{1}{r-1}\mu\{z:\ r-1 < |z| < r+1\} + \frac{1}{R-1}\mu\{z:\ R-1 < |z| < R+1\}; \tag{49}$$

where $[x]$ is the integral part of real x. By (27), the right-hand side of (49) tends to zero as $r\to\infty$. Then we obtain

$$\int_{r<|[x]+iy|<R} \frac{d\mu(z)}{z} = \sum_{n\in\mathbb{Z}:\ |n|\le R}\ \int_{x\in[0,1),\ r<|n+iy|<R} \frac{d\mu(z)}{z+n}$$

$$= \int_{x\in[0,1),\ y\in\mathbb{R}}\ \sum_{n\in\mathbb{Z}:\ r<|n+iy|<R} \frac{\overline{z}+n}{|z+n|^2}\, d\mu(z). \tag{50}$$

Now for any $x\in[0,1)$, $y\in\mathbb{R}$ we have

$$\sum_{n\in\mathbb{Z}:\ |n+iy|>r} \frac{|\overline{z}|}{|z+n|^2} \le \sum_{n\in\mathbb{N}\bigcup\{0\}:\ n^2>r^2-y^2} \frac{1+|y|}{n^2+y^2} + \sum_{n\in\mathbb{N}:\ n^2>r^2-y^2} \frac{1+|y|}{(n-1)^2+y^2}$$

$$\leq 2 \int\limits_{\sqrt{\max\{0,r^2-y^2\}}}^{\infty} \frac{1+|y|}{t^2+y^2} dt + \frac{2(1+|y|)}{y^2} \chi_r(y); \tag{51}$$

here $\chi_r(y)$ is a characteristic function of the interval $(\sqrt{r^2-1}, \infty)$. Besides,

$$-\sum_{n\in\mathbb{Z}:\, r<|n+iy|<R} \frac{n}{|z+n|^2} = \sum_{n\in\mathbb{N}:\, r<|n+iy|<R} n\left(\frac{1}{|z-n|^2} - \frac{1}{|z+n|^2}\right)$$

$$= \sum_{n\in\mathbb{N}:\, r^2<n^2+y^2<R^2} \frac{4n^2 x}{((n-x)^2+y^2)((n+x)^2+y^2)}. \tag{52}$$

It is easy to see that the right-hand side of (52) is also majorized by (51). Both terms monotonically decrease to 0 as $r \to \infty$, hence (50) tends to 0 as $r \to \infty$ uniformly in R. It follows from (49) and (50) that (10) is valid. Finally, by (31) and (32), the integral

$$\int\limits_1^\infty \frac{\mu(B(0,t)) - \mu(R(z,t))}{t} dt$$

is bounded for $z = x \in [0,1]$. Since μ is stable with respect to the translation on 1, we get (36). Now the assertion of Th. 7 follows from Th. 5.

Consequence. *The necessary and sufficient conditions for a divisor $\{a_k\}$ to be the divisor of an entire periodic (with period 1) function of exponential type bounded on real axis are that the divisor should be periodic with period 1 and its restriction to the strip $\{z:\ 0 \leq x < 1,\ y \in \mathbb{R}\}$ should be finite.*

In this case the corresponding function is a finite product of the elementary functions $\sqrt{1-\cos 2\pi(z-\gamma_k)}$ with $\mathrm{Re}\gamma_k \in [0,1)$; it is unique up to a multiplier $Ce^{i\nu z}$, $\nu \in \mathbb{R}$.

References

[1] *V.S. Azarin*, Asymptotic Behavior of Subharmonic Functions of Finite Order. — *Math. USSR Sb.* **36** (1979), 135–154. (Russian)

[2] *H. Bohr*, Zur Theorie der Fastperiodischen Funktionen, III Teil; Dirichletentwicklung Analytischer Funktionen. — *Acta Math.* **47** (1926), 237–281.

[3] *S.Yu. Favorov, A.Yu. Rashkovskii, and L.I. Ronkin*, Almost Periodic Divisors in a Strip. — *J. Anal. Math.* **74** (1998), 325–345.

[4] *S.Yu. Favorov and A.V. Rakhnin*, Subharmonic Almost Periodic Functions. — *J. Math. Phys., Anal., Geom.* **1** (2005), 209–224.

[5] *S.Yu. Favorov*, Zeros of Holomorphic Almost Periodic Functions. — *J. d'Analyse Math.* **84** (20015), 51–66.

[6] *S.Yu. Favorov*, Zero Sets of Entire Functions of Exponential Type with Some Conditions on Real Axis. — *Algebra and Analiz.* (To appear). (Russian)

[7] *B.N. Khabibullin*, Completeness of Exponential systems and Uniquness Sets. — Bushkir State Univ. Press, Ufa, 2006. (Russian)

[8] *P. Koosis*, The Logarithmic Integral. **1**. Cambridge Univ. Press, Cambridge, 1988.

[9] *M.G. Krein and B.Ja. Levin*, On Almost Periodic Functions of Exponential Type. — *Dokl. AN USSR* **64** (1949), No. 3, 285–287. (Russian)

[10] *B.Ya. Levin*, Distribution of Zeros of Entire Functions. **5**. AMS, Providence, RI, 1964.

[11] *L.I. Ronkin*, Almost Periodic Distributions and Divisors in Tube Domains. — *Zap. Nauchn. Sem. POMI* **247** (1997), 210–236. (Russian)

[12] *L.I. Ronkin*, Introduction to the Theory of Entire Functions of Several Variables. Transl. Math. Monogr. **44**. AMS, Providence, RI, 1974.

[13] *H. Tornehave*, Systems of Zeros of Holomorphic Almost Periodic Functions. Kobenhavns Univ. Mat. Inst. Preprint (1988), No. 30, 52 p.