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Dominated Convergence and Egorov Theorems for Filter Convergence

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We study the filters, such that for convergence with respect to this filters the Lebesgue dominated convergence theorem and the Egorov theorem on almost uniform convergence are valid (the Lebesgue filters and the Egorov filters, respectively). Some characterizations of the Egorov and the Lebesgue filters are given. It is shown that the class of Egorov filters is a proper subset of the class of Lebesgue filters, in particular, statistical convergence filter is the Lebesgue but not the Egorov filter. It is also shown that there are no free Lebesgue ultrafilters. Significant attention is paid to the filters generated by a matrix summability method.

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1. Introduction

The aim of the paper is to study the classical Lebesgue integration theory results — the Lebesgue dominated convergence theorem and the Egorov theorem in a general setting when the ordinary convergence of sequences is replaced by a filter convergence. We show that for some filters this theorems are valid and for some are not, and moreover that the set of filters on $\mathbb{N}$ for which the dominated

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convergence theorem takes place is strictly wider than the corresponding set of filters for the Egorov theorem.

Recall that a *filter* $\mathcal{F}$ on $\mathbb{N}$ is a nonempty collection of subsets of $\mathbb{N}$ satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$ if $B \supset A$ then $B \in \mathcal{F}$.

A sequence $a_n \in \mathbb{R}$ is said to be *$\mathcal{F}$-convergent* to a (and we write $a = \mathcal{F} - \lim a_n$ or $a_n \to_{\mathcal{F}} a$) if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |a_n - a| < \varepsilon\}$ belongs to $\mathcal{F}$.

In particular if one takes as $\mathcal{F}$ the filter of those sets whose complement is finite (*the* F*réchet filter*) then $\mathcal{F}$-convergence coincides with the ordinary one.

The natural ordering on the set of filters on $\mathbb{N}$ is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. If G is a centered collection of subsets (i.e., all finite intersections of the elements of G are nonempty), then there is a filter containing all the elements of G. The smallest filter, containing all the elements of G is called *the filter generated by G*.

Let $\mathcal{F}$ be a filter. A collection of subsets $G \subset \mathcal{F}$ is called *the base of $\mathcal{F}$* if for every $A \in \mathcal{F}$ there is a $B \in G$ such that $B \subset A$.

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it dominates the Fréchet filter. Below when we say "filter" we mean a free filter on $\mathbb{N}$. In particular every ordinary convergent sequence will be automatically $\mathcal{F}$-convergent.

A maximal in the natural ordering filter is called an *ultrafilter*. The Zorn lemma implies that every filter is dominated by an ultrafilter. A filter $\mathcal{F}$ on $\mathbb{N}$ is an ultrafilter if and only if for every $A \subset \mathbb{N}$ either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{F}$. More about filters, ultrafilters and their applications one can find in every advanced General Topology textbook, for example in [11].

All over the paper (Ω, Σ, μ) stands for a finite measure space, and when we say "a function on Ω" we mean a real-valued and measurable function. For a $\Delta \subset \Omega$ denote Σ_Δ the collection of intersections of Δ with elements of Σ. Below we several times make use of the following simple remark:

Theorem 1.1. *The outer measure μ^* generated by μ is countably additive on Σ_Δ even in the case of $\Delta \notin \Sigma$.*

Definition 1.2. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a Egorov filter (or is said to have the Egorov property) if for every measure space (Ω, Σ, μ), for every point-wise $\mathcal{F}$-convergent to 0 sequence of functions f_n on Ω and for every $\varepsilon > 0$ there is a subset $B \in \Sigma$ with $\mu(B) < \varepsilon$ such that* $\sup_{t \in \Omega \setminus B} |f_n(t)| \to_{\mathcal{F}} 0$.

Definition 1.3. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a Lebesgue filter (or is said to have the Lebesgue property) if the following statement takes place: for every measure space (Ω, Σ, μ), for every point-wise $\mathcal{F}$-convergent to 0 sequence of functions f_n on Ω if $|f_n|$ are dominated by a fixed integrable function $g \in L_1(\Omega, \Sigma, \mu)$ then* $\int_\Omega f_n d\mu \to_{\mathcal{F}} 0$.

Below we give some characterizations of Egorov and Lebesgue filters; we show that the Egorov property implies the Lebesgue one and that there are no ultrafilters with the Lebesgue property. Then we pass to a wide class of the Lebesgue filters — to filters generated by a summability method. Under a natural restriction on the summability method these filters are the examples of filters without the Egorov property.

2. Characterizations of Egorov and Lebesgue Filters

Let us start with some observations to simplify the conditions of Egorov and Lebesgue theorems. Namely it is sufficient to consider f_n in Defs.1.2 and 1.3 being the functions that take only values 0 and 1, or in other words, it is enough to consider f_n of the form $f_n = \chi_{A_n}$, $A_n \in \Sigma$. More precisely:

Theorem 2.1. *$\mathcal{F}$ is a Egorov filter if and only if for every measure space (Ω, Σ, μ), for every sequence $A_n \in \Sigma$ such that $f_n = \chi_{A_n}$ point-wise $\mathcal{F}$-converge to 0 and for every $\varepsilon > 0$ there is a subset $B \in \Sigma$ with $\mu(B) < \varepsilon$ and there is an $I \in \mathcal{F}$ such that $A_n \subset B$ for all $n \in I$.*

P r o o f. The "only if" part of the theorem is just a particular case of Def. 1.2 when all the f_n take only values 0 and 1. So let us prove the "if" part.

Step 1. The functions f_n in Def. 1.2 can be taken positive (the corresponding statements for f_n and for $|f_n|$ are equivalent) and moreover taking values from $[0, 1]$, since we may replace f_n by $\min\{f_n, 1\}$.

Step 2. Each of f_n can be uniformly approximated by a simple function $g_n : \Omega \to [0, 1]$ taking only irrational values; $|f_n - g_n| < 1/n$. If the statement of Def. 1.2 holds true for these g_n, then it holds true for f_n as well. So we may suppose that $f_n : \Omega \to [0, 1]$ are simple functions taking only irrational values.

Step 3. Now for every irrational $x \in [0, 1]$ let us write down its binary expansion $x = \sum_{m=1}^{\infty} 2^{-m} a_m(x)$; $a_m(x) \in \{0, 1\}$. The functions $a_m(x)$ are continuous on the set of irrationals, so their compositions with f_n are measurable functions and we have expansions

$$f_n(t) = \sum_{m=1}^{\infty} 2^{-m} a_m(f_n(t)).$$

Denote $A_{m,n} = \operatorname{supp}(a_m \circ f_n)$, then $a_m \circ f_n = \chi_{A_{m,n}}$. According to our assumption $f_n(t) \to_{\mathcal{F}} 0$ for all $t \in \Omega$, hence for a fixed $m \in \mathbb{N}$ $a_m(f_n(t)) \to_{\mathcal{F}} 0$ as $n \to \infty$. Applying the conditions of the theorem for $\varepsilon/2^m$, we get for every $m \in \mathbb{N}$ a subset $B_m \in \Sigma$ with $\mu(B_m) < \varepsilon/2^m$ and an $I_m \in \mathcal{F}$ such that $A_{m,n} \subset B_m$ for all $n \in I_m$. Put $B = \bigcup_m B_m$. Then $\mu(B) < \varepsilon$. Moreover, for all $n \in \bigcap_{k=1}^m I_k$

$$\sup_{t\in\Omega\setminus B} f_n(t) = \sup_{t\in\Omega\setminus B}\sum_{k=1}^{\infty} 2^{-k}a_k(f_n(t)) \le \sup_{t\in\Omega\setminus(\bigcup_{k=1}^m B_k)}\sum_{k=1}^{\infty} 2^{-k}a_k(f_n(t))$$
$$= \sup_{t\in\Omega\setminus(\bigcup_{k=1}^m B_k)}\sum_{k=m+1}^{\infty} 2^{-k}a_k(f_n(t)) \le \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m}.$$

This means that $\sup_{t\in\Omega\setminus B} f_n(t) \to_{\mathcal{F}} 0$, which completes the proof. ■

Theorem 2.2. *$\mathcal{F}$ is a Lebesgue filter if and only if for every measure space (Ω,Σ,μ), for every sequence $A_n \in \Sigma$ if $f_n = \chi_{A_n}$ point-wise $\mathcal{F}$-converge to 0, then $\mu(A_n) \to_{\mathcal{F}} 0$.*

P r o o f. Like the previous theorem the "only if" part is evident. The proof of the "if" part needs the same three steps as above.

Step 1. The functions f_n in the Def. 1.3 can be taken positive (it is sufficient to prove the statement for positive functions $f_n^+ = \max\{f_n, 0\}$ and $f_n^- = f_n^+ - f_n$ and to use the formula $f_n = f_n^+ - f_n^-$). Moreover, passing from the measure $d\mu$ and functions f_n to $(g+1)d\mu$ and $f_n/(g+1)$ respectively, we can reduce the task to the case of functions taking values from $[0,1]$.

The Steps 2 and 3 can be taken almost word-to-word from the proof of Th. 2.2 with the only difference that in the Step 3 we do not need B_n and B; $I_m \in \mathcal{F}$ must be selected in such a way that $\mu(A_{m,n}) < \varepsilon$ for $n \in I_m$ and in the final part we estimate $\int_\Omega f_n d\mu$ for $n \in \bigcap_{k=1}^m I_k$:

$$\int_\Omega f_n d\mu = \sum_{k=1}^{\infty} 2^{-k}\mu(A_{k,n}) \le \sum_{k=1}^{m} 2^{-k}\varepsilon + \sum_{k=m+1}^{\infty} 2^{-k}\mu(\Omega) \le \varepsilon + 2^{-m}\mu(\Omega).$$

■

Corollary 2.3. *The Egorov property of a filter implies the Lebesgue property.*

A set of naturals is called *stationary* with respect to a filter $\mathcal{F}$ (or just $\mathcal{F}$-stationary) if it has nonempty intersection with each member of the filter. Denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. For a $J \in \mathcal{F}^*$ we call the collection of sets $\{I \cap J : I \in \mathcal{F}\}$ *the trace of $\mathcal{F}$ on J* (which is evidently a filter on J), and by $\mathcal{F}(J)$ we denote the filter on $\mathbb{N}$ generated by the trace of $\mathcal{F}$ on J. Clearly $\mathcal{F}(J)$ dominates $\mathcal{F}$. Any subset of naturals is either a member of $\mathcal{F}$ or the complement of a member of $\mathcal{F}$ or the set and its complement are both $\mathcal{F}$-stationary sets. $\mathcal{F}^*$ is precisely the union of all ultrafilters dominating $\mathcal{F}$. $\mathcal{F}^*$ is a filter base if and only if it is equal to $\mathcal{F}$ and $\mathcal{F}$ is an ultrafilter.

Proposition 2.4. *Let $\mathbb{N} = J_0 \sqcup J_1$ be any disjoint partition of naturals into $\mathcal{F}$-stationary sets. Then $\mathcal{F}$ does not have the Egorov property if and only if either $\mathcal{F}(J_0)$ or $\mathcal{F}(J_1)$ is not a Egorov filter.*

P r o o f. First, we show that if $\mathcal{F}(J_0)$ and $\mathcal{F}(J_1)$ are both Egorov filters, then $\mathcal{F}$ is a Egorov filter. Notice that χ_{A_n} point-wise $\mathcal{F}$-converges if and only if it simultaneously converges point-wise in $\mathcal{F}(J_0)$ and $\mathcal{F}(J_1)$ senses. For a given measure μ and $\varepsilon > 0$ let us choose B_0 and B_1 with the measures smaller than $\varepsilon/2$ and I_0, I_1 from $\mathcal{F}(J_0)$ and $\mathcal{F}(J_1)$, respectively, as in Th. 2.1. Then $B = B_0 \sqcup B_1$, $I = I_0 \sqcup I_1 \in \mathcal{F}$ are fit to satisfy the conditions of Th. 2.1.

Let now, for instance, $\mathcal{F}(J_0)$ be not a Egorov filter. Then for some measure μ and measurable A_n with $\mathcal{F}(J_0) - \lim \chi_{A_n} = 0$ there is $\varepsilon > 0$ such that for each $I \cap J_0 \in \mathcal{F}(J_0)$, $I \in \mathcal{F}$, we have $\mu(\bigcup_{n \in I \cap J_0} A_n) \geq \varepsilon$. Now to get an example of $\hat{A}_n$ with $\chi_{\hat{A}_n}$ point-wise $\mathcal{F}$-converging to 0 and with $\mu(\bigcup_{n \in I} \hat{A}_n) \geq \varepsilon$ we can take $\hat{A}_n = A_n$ when $n \in J_0$ and to set the rest of $\hat{A}_n$ to be empty. ∎

Definition 2.5. *We say that a filter $\mathcal{F}$ is nowhere Egorov (nowhere Lebesgue) if and only if its trace on each $\mathcal{F}$-stationary set generates a non-Egorov (non-Lebesgue) filter.*

In the sense of this definition the preceding proposition says that each Egorov filter must be an everywhere Egorov filter. But it is easy to see that a filter without the Egorov property is not necessarily a nowhere Egorov filter.

Denote by $\widetilde{\mathbb{N}}$ the set of all free ultrafilters $\mathcal{U}$ on $\mathbb{N}$, equipped with the topology defined by means of its base $\{\tilde{A} : A \subset \mathbb{N}\}$, where $\tilde{A} = \{\mathcal{U} \in \widetilde{\mathbb{N}} : A \in \mathcal{U}\}$. Remark, that in this topology the basic open sets $\tilde{A}$ are at the same time closed. $\widetilde{\mathbb{N}}$ can be identified with $\beta\mathbb{N}\backslash\mathbb{N}$ where $\beta\mathbb{N}$ denotes the Stone–Čech compactification of $\mathbb{N}$.

By the definition, the *support set of a filter* $\mathcal{F}$ is the set $K_{\mathcal{F}} = \bigcap\{\tilde{A} : A \in \mathcal{F}\}$. In other words, $K_{\mathcal{F}}$ is the set of all ultrafilters dominating $\mathcal{F}$. More about the support sets in connection with different types of convergence see in [1, 4, 10].

Proposition 2.6. *If $\mathcal{F}$ is a nowhere Egorov filter, then $K_{\mathcal{F}}$ is nowhere dense in $\widetilde{\mathbb{N}}$.*

P r o o f. Observe that $K_{\mathcal{F}}$ is closed in $\widetilde{\mathbb{N}}$. So it is nowhere dense in $\widetilde{\mathbb{N}}$ if and only if for any infinite $A \subset \mathbb{N}$ there is a $\mathcal{U} \in \tilde{A}$ such that $\mathcal{U} \notin K_{\mathcal{F}}$. That means that for each infinite $A \subset \mathbb{N}$ there is such a $\mathcal{U}$ containing A that there is $I \in \mathcal{U}$ (can be reckon as subset of A) which does not belong to any ultrafilter from $K_{\mathcal{F}}$. Or in terms of $\mathcal{F}$-stationary sets: each stationary set has an infinite nonstationary subset. In other words, we have that $K_{\mathcal{F}}$ is nowhere dense in $\widetilde{\mathbb{N}}$ provided the trace of $\mathcal{F}$ on any $D \in \mathcal{F}^*$ is not the Fréchet filter on D (or as we further say *there is no Fréchet stationary set with respect to* $\mathcal{F}$). Since the Fréchet filter has the Egorov property the claim is proved. ∎

In the next section we show that for the filter generated by a summability matrix the inverse implication is true as well.

The rest of this section is devoted to one more reformulation of the Egorov property (Th. 2.8) and the Lebesgue property (Th. 2.14) in a way that reduces the number of parameters in the definitions to minimum.

Lemma 2.7. *$\mathcal{F}$ does not have the Egorov property if and only if there exists a measure space (Ω, Σ, μ) such that:*
(1) $\Omega \subset \mathcal{F}$;
(2) $A_n = \{\omega \in \Omega : n \notin \omega\} \in \Sigma$;
(3) $\mu \not\equiv 0$ and for every $I \in \mathcal{F}$ the set $\bigcup_{n\in I} A_n$ has full measure.

P r o o f. As follows from Th. 2.1, $\mathcal{F}$ is not a Egorov filter if and only if there are $(\Omega_1, \Sigma_1, \nu)$, $C_n \in \Sigma_1$ and $\varepsilon > 0$ such that for each $t \in \Omega_1$ $\{n : t \notin C_n\} \in \mathcal{F}$, and for all B with $\nu(B) < \varepsilon$ and for every $I \in \mathcal{F}$ there is $n \in I$ such that C_n does not lie in B. In other words, $\nu(\bigcup_{n\in I} C_n) > \varepsilon$ for every $I \in \mathcal{F}$.

Let $I \in \mathcal{F}$, consider $B_I = \bigcup_{n\in I} C_n$. The family $\{B_I\}_{I\in\mathcal{F}}$ has the following property:

$$B_{I_1} \cap B_{I_2} \supset B_{I_1 \cap I_2}. \tag{2.1}$$

Denoting

$$\alpha = \inf_{I\in\mathcal{F}} \nu(B_I) > 0,$$

we choose $I_n \in \mathcal{F}$ such that $\nu(B_{I_n}) < \alpha + 1/n$. Without loss of generality, we can assume that $I_1 \supset I_2 \supset I_3 \supset \ldots$.

Introducing the notation $\widehat{\Omega} = \bigcap_{n=1}^{\infty} B_{I_n}$, observe that $\nu(B_{I_n}) \to \nu(\widehat{\Omega})$ when $n \to \infty$, thus $\nu(\widehat{\Omega}) = \alpha$. Due to (2.1) we have that $\nu(\widehat{\Omega} \cap B_I) = \alpha$ for every $I \in \mathcal{F}$.

From now on we deal with $\widehat{\Omega}$ instead of Ω_1, $\widehat{A}_n = C_n \cap \widehat{\Omega}$ instead of C_n, and $\widehat{B}_I = B_I \cap \widehat{\Omega} = \bigcup_{n\in I} \widehat{A}_n$ instead of B_I. Note that the condition on C_n still holds for $\widehat{A}_n$: $\{n : t \notin \widehat{A}_n\} \in \mathcal{F}$. And for any $I \in \mathcal{F}$, as we have already mentioned,

$$\nu(\widehat{B}_I) = \alpha. \tag{2.2}$$

Consider the natural map $G : \widehat{\Omega} \to \mathcal{F}$, $G(t) = \{n : t \notin \widehat{A}_n\}$. Define the measure space (Ω, Σ, μ) we need as the image of $(\widehat{\Omega}, \Sigma_1, \nu)$ under G, i.e., put $\Omega = G(\widehat{\Omega})$, let Σ be the collection of those $D \subset \Omega$ that $G^{-1}(D) \in \Sigma_1$ and put $\mu(D) = \nu(G^{-1}(D))$. Define $A_n = G(\widehat{A}_n)$. Observing that $t \in \widehat{A}_n$ if and only if $n \notin G(t)$, we obtain that $G^{-1}(A_n) = \widehat{A}_n$, which means that $A_n \in \Sigma$. To complete the proof remark that $\mu(\Omega) = \alpha$ and for every $I \in \mathcal{F}$

$$\mu\left(\bigcup_{n\in I} A_n\right) = \nu\left(G^{-1}\left(\bigcup_{n\in I} A_n\right)\right) = \nu(\widehat{B}_I) = \alpha.$$

■

Let us equip $2^{\mathbb{N}}$ (the collection of all subsets of $\mathbb{N}$) with the standard product topology and denote by $\mathcal{B}$ the Borel σ-algebra on $2^{\mathbb{N}}$.

Theorem 2.8. *$\mathcal{F}$ is not a Egorov filter if and only if there exists a Borel measure ν on $2^{\mathbb{N}}$ such that:*

(1) $\nu^*(\mathcal{F}) > 0$; and

(2) for every $I \in \mathcal{F}$ $C_I = \{\omega \subset \mathbb{N} : \omega \supset I\}$ is a ν-null set.

P r o o f. First we establish that if such a measure exists, then $\mathcal{F}$ is not a Egorov filter. Consider $\Omega = \mathcal{F}$, $\Sigma = \mathcal{B}_{\mathcal{F}}$, and equip $\mathcal{B}_{\mathcal{F}}$ with the measure $\mu = \nu^*$ (Th. 1.1). Now we are in the conditions of the preceding criterion: the third condition follows from the observation that $C_I = \{\omega \in \mathcal{F} : \omega \supset I\}$ is the complement to $\bigcup_{n\in I} A_n$ from Lem. 2.7 and the second one from observation that A_n are closed in $\mathcal{F}$.

Now suppose that $\mathcal{F}$ possesses the Egorov property and so there is a measure μ satisfying conditions (1), (2), (3) of Lem. 2.7. Note that in induced topology on Ω the natural neighborhoods base of $J \in \Omega$ is formed by

$$\begin{aligned} U_n(J) &= \big\{\omega \in \Omega :\ \omega \cap \{1,2,\dots,n\} = J \cap \{1,2,\dots,n\}\big\} \\ &= \left(\bigcap_{i\in\{1,2,\dots,n\}\cap J} \Omega\backslash A_i\right) \backslash \left(\bigcup_{i\in\{1,2,\dots,n\}\backslash J} \Omega\backslash A_i\right). \end{aligned}$$

Thus $U_n(J) \in \Sigma$ and hence Σ contains the σ-algebra of Borel sets on Ω. To complete the proof put $\nu(A) = \mu(A \cap \Omega)$ for all $A \in \mathcal{B}$. ■

Corollary 2.9. *Every filter $\mathcal{F}$ with a countable base possesses the Egorov property.*

P r o o f. Let $\mathcal{G} = \{I_n\}_{n=1}^{\infty}$ be a base of $\mathcal{F}$. Suppose that condition (2) of the theorem holds. Then

$$0 = \nu\Big(\bigcup_{n=1}^{\infty} C_{I_n}\Big) = \nu(\mathcal{F}),$$

and hence condition (1) does not hold. This establishes the claim. ■

Now we can apply Th. 2.8 to show that all ultrafilters do not have the Egorov property (remind that we consider only free filters and ultrafilters).

Corollary 2.10. *Free ultrafilters do not have the Egorov property.*

P r o o f. Consider the standard product measure ν on $2^{\mathbb{N}}$. If $\mathcal{U}$ is an ultrafilter, then $2^{\mathbb{N}} = \mathcal{U} \sqcup \{\mathbb{N}\backslash u : u \in \mathcal{U}\}$. But, as $u \mapsto \mathbb{N}\backslash u$ is a preserving measure bijection of $2^{\mathbb{N}}$, we have that $\nu^*(\mathcal{U}) = \nu^*(\{\mathbb{N}\backslash u : u \in \mathcal{U}\})$ and both must be at least $1/2$ (to be precise $\nu^*(\mathcal{U}) = 1$, see [6, Lem. 464Ca]). Since $u \in \mathcal{U}$ is infinite all the $C_u = \{\omega \subset \mathbb{N} : \omega \supset u\}$ are ν-null sets. ■

As we see, a wide class of filters that have no countable base are not the Egorov filters, among them, as we show in the next section, there is the filter of statistical convergence. Before proceeding further it is useful to have an example of the Egorov filter that has no countable base.

Example 2.11. *A filter with the Egorov property that has no countable base.*

Let $\{n_{i,j}\}_{i,j=1}^{\infty}$ be an enumeration of $\mathbb{N}$. For each sequence of naturals $\{k_j\}$ define an element of the filter base $\mathcal{G}$ in the following way: $J_{(k_j)} = \{n_{i,j} : i \geq k_j,\ \ j = 1, 2, \dots\}$. It is easy to see that the filter Φ generated by $\mathcal{G}$ has no countable base. A real-valued sequence $\{x_n\}$ Φ-converge to 0 if and only if for each $j \in \mathbb{N}$ $x_{n_{i,j}} \to 0$ as $i \to \infty$.

Let $\{f_n\}$ be a point-wise Φ-convergent to 0 sequence of functions on Ω. For any $\varepsilon > 0$ put $\varepsilon_k = \varepsilon/2^k$, $k \in \mathbb{N}$. Applying the ordinary Egorov theorem for $f_{n_{i,k}}$ for each k, we get B_k with $\mu(B_k) < \varepsilon_k$ such that the sequence $f_{n_{i,k}}$ uniformly converges to 0 on $\Omega \backslash B_k$ as $i \to \infty$. Put $B := \bigcup_{k=1}^{\infty} B_k$. Then $\mu(B) < \varepsilon$ and f_n uniformly Φ-converge to 0 on $\Omega \backslash B$.

Let us proceed now with the Lebesgue property. Recall some facts.

Fact 1. Any μ-almost everywhere (a.e.) convergent sequence of functions on Ω is convergent in measure μ.

Fact 2. If for a given sequence of measurable functions $\{f_n\}$ there are positive scalars a_n, ε_n such that $\lim_n a_n = 0$, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $\mu(\{t : |f_n| > a_n\}) < \varepsilon_n$ for all $n \in \mathbb{N}$ then f_n converge a.e. to 0.

Theorem 2.12. *For a fixed filter $\mathcal{F}$ on $\mathbb{N}$ the following three properties of a sequence f_n on (Ω, Σ, μ) are equivalent:*

(1) f_n is $\mathcal{F}$-convergent to 0 in measure;

(2) every $J \in \mathcal{F}^*$ contains an infinite subset M such that f_n converge a.e. to 0 along M;

(3) for every $J \in \mathcal{F}^*$ there is an infinite subset $M \subset J$ such that f_n converge in measure to 0 along M.

P r o o f. (2.12)$\Rightarrow$ (2.12). Let $J \in \mathcal{F}^*$ and let a_n,ε_n be as in Fact 2. From $\mathcal{F}$-convergence in measure follows that for every $n \in \mathbb{N}$ there is $I_n \in \mathcal{F}$ such that for all $i \in I_n$ $\mu(\{t : |f_i| > a_n\} < \varepsilon_n\}$. Let us select an increasing sequence m_n such that $m_n \in I_n \cap J$. Then $g_n := f_{m_n}$ satisfies the conditions of Fact 2 and thus f_i converge a.e. to 0 along $M := \{m_n\}$.

The implication (2.12)$\Rightarrow$ (2.12) evidently follows from Fact 1, so let us prove that (2.12)$\Rightarrow$ (2.12). Suppose f_n do not $\mathcal{F}$-converge in measure. Then there are positive scalars a,ε such that in each $I \in \mathcal{F}$ there is an j such that $\mu(\{t : |f_j| > a) > \varepsilon$. Consequently $J = \{j \in \mathbb{N} : \mu(\{t : |f_j| > a) > \varepsilon\}$ is stationary and contains no subset along which f_n converge to 0 in measure. ■

And once more the "Lebesgue" analogues for the properties of the Egorov (non-Egorov) filters.

Lemma 2.13. *$\mathcal{F}$ does not have the Lebesgue property if and only if there exists a measure space (Ω, Σ, μ) such that:*

(1) $\Omega \subset \mathcal{F}$;

(2) $A_n = \{\omega \in \Omega : n \notin \omega\} \in \Sigma$;

(3) there is a $J \in \mathcal{F}^*$ *such that*

$$\inf_{n \in J} \mu(A_n) > 0,$$

or alternatively, for every infinite subset $M \subset J$

$$\mu(\{\omega \in \Omega : |M \backslash \omega| = \infty\}) > 0.$$

P r o o f. Theorem 2.2 says that $\mathcal{F}$ does not have the Lebesgue property when there are $(\Omega_1, \Sigma_1, \nu)$, $C_n \in \Sigma_1$ such that for each $t \in \Omega_1$ $\{n : t \notin C_n\} \in \mathcal{F}$ and the sequence $\nu(C_n)$ does not $\mathcal{F}$-converge to 0, or in other words, χ_{C_n} do not $\mathcal{F}$-converge to 0 in measure. Due to the item (2.12) of the previous theorem, there is a $J \in \mathcal{F}^*$ such that the sequence $\{\nu(C_n)\}_{n \in J}$ does not have converging to 0 subsequences, or in other words $\inf_{n \in J} \nu(C_n) > 0$. Alternatively, due to the item (2.12) of the same theorem, there is a $J \in \mathcal{F}^*$ (in fact J can be left the same) such that χ_{C_n} do not converge a.e. along any subsequence of J. This means that

$$\nu\Big(\bigcap_{n=1}^{\infty} \bigcup_{m > n, m \in M} C_m\Big) > 0 \tag{2.3}$$

for every infinite subset $M \subset J$.

Now, in the same way as we did in Lemma 2.7 we apply map $G : \Omega_1 \to \mathcal{F}$, $G(t) = \{n : t \notin C_n\}$ to the original measure space in order to get the measure space (Ω, Σ, μ) we need. Then $A_n = \{\omega \in \Omega : \ n \notin \omega\}$ equals $G(C_n)$ and $\{\omega \in \Omega : |M \backslash \omega| = \infty\} = \bigcap_{n=1}^{\infty} \bigcup_{m > n,\ m \in M} A_m = G\left(\bigcap_{n=1}^{\infty} \bigcup_{m > n,\ m \in M} C_m\right)$ which owing to (2.3) completes the proof. ■

Theorem 2.14. *$\mathcal{F}$ is not a Lebesgue filter if and only if there exists a Borel measure ν on $2^{\mathbb{N}}$ such that there is a $J \in \mathcal{F}^*$ such that*

$$\inf_{n\in J} \nu^*(\{I \in \mathcal{F} : n \notin I\}) > 0,$$

or alternatively, for each infinite subset $M \subset J$

$$\nu^*(\{I \in \mathcal{F} : |M \backslash I| = \infty\}) > 0.$$

P r o o f. The argumentation is the same as in Th. 2.8. Let such a measure exists. Considering $\Omega = \mathcal{F}$, $\Sigma = \mathcal{B}_{\mathcal{F}}$ and equipping $\mathcal{B}_{\mathcal{F}}$ with the measure $\mu = \nu^*$ we find ourselves in the conditions of preceding criterion.

The converse results from Lem. 2.13 by putting $\nu(A) = \mu(A \cap \Omega)$ for all Borel subsets of $2^{\mathbb{N}}$. ■

Corollary 2.15. *Let ν be the usual product measure on $2^{\mathbb{N}}$. If $\mathcal{F}$ has the Lebesgue property then $\nu^*(\mathcal{F}) = 0$.*

P r o o f. Suppose the contrary, let $\nu^*(\mathcal{F}) = a > 0$. Let us show that, for instance, an alternative condition of the non-Lebesgue property is valid with $\mathbb{N}$ as a stationary set from this condition (which leads to a contradiction). Denoting $A_m = \{I \in \mathcal{F} : m \notin I\}$, for any infinite $M \subset \mathbb{N}$, we have $\mathcal{F} = (\bigcup_{m\in M} A_m) \sqcup (\{I \in \mathcal{F} : M \subset I\})$. Since M is infinite $\nu^*(\{I \in \mathcal{F} : M \subset I\}) \le \nu(\{\omega \subset \mathbb{N} : M \subset \omega\}) = 0$. Thus for any infinite M $\nu^*(\bigcup_{m\in M} A_m)) \ge \nu^*(\mathcal{F}) = a$ and hence for $M_n = M \cap \{n, n+1, n+2, \dots\}$ as well. Thus applying Th. 1.1, we obtain:

$$\nu^*(\{I \in \mathcal{F} : |M \backslash I| = \infty\}) = \nu^*(\bigcap_{n=1}^{\infty} \bigcup_{m\in M_n} A_m) \ge a > 0.$$

■

Corollary 2.16. *An ultrafilter does not have the Lebesgue property.*

For a given non-Lebesgue filter $\mathcal{F}$ Th. 2.14 suggests to consider filter $\mathcal{F}(J)$, where J is the stationary set from the criterion. It is evident that any filter dominating $\mathcal{F}(J)$ satisfies the condition in its turn. Consequently we have the following result.

Corollary 2.17. *If $\mathcal{F}_0$ is not a Lebesgue filter, then there is a $J \in \mathcal{F}^*$ such that each $\mathcal{F} \succ \mathcal{F}_0(J)$ does not have the Lebesgue property.*

Now we are going to consider the filters generated by summability matrices. We show that all of them possess the Lebesgue property, characterize the nowhere Egorov ones and give a sufficient condition for a filter to be non-Egorov.

3. Filters Generated by Summability Matrices

In this section we study the matrix generalization of statistical convergence with respect to the Egorov and the Lebesgue properties. Though matrix summability methods and statistical convergence were introduced separately and, until recently, followed independent lines of development, they are closely related. The definition of statistical convergence was introduced by H. Fast [5] with the natural density of a set in $\mathbb{N}$ being used. A real valued sequence x_k is *statistically convergent to* x if for every $\varepsilon > 0$ the set $\{k : |x_k - x| > \varepsilon\}$ has natural density 0 where the natural density of a subset $A \subset \mathbb{N}$ is defined to be $\delta(A) := \lim_n n^{-1}|\{k \le n : k \in A\}|$. Statistical convergence is a generalization of the usual notion of convergence, and in its turn has been extended in a variety of ways. A number of authors replaced the natural density with the one generated by a matrix summability method ([4–9]), or more generally considered statistical convergence determined by a finitely additive set function satisfying some elementary properties ([2]). An overview of the theory of statistical convergence the reader can find in one of the most recent papers [3]. In this section we use an extension of Fast's definition of statistical convergence where the natural density is replaced by a matrix generated as presented in [4].

An $\mathbb{N} \times \mathbb{N}$ matrix $\varphi = (\varphi_{i,j})$ is said to be a *summability matrix* if:

(1) $\varphi_{i,j} \ge 0$ for all i and j;

(2) $\sum_{j=1}^{\infty} \varphi_{n,j} \le 1$ for every $n \in \mathbb{N}$;

(3) $\limsup_{n\to\infty} \sum_{j=1}^{\infty} \varphi_{n,j} > 0$;

(4) $\lim_{n\to\infty} \varphi_{n,j} = 0$ for every $n \in \mathbb{N}$.

Usually in literature the following *regularity condition* is also demanded from a summability matrix: $\lim_{n\to\infty} \sum_{j=1}^{\infty} \varphi_{n,j} = 1$, but for our purposes it is more convenient to consider nonregular matrices as well.

For a summability matrix φ and $I \subset \mathbb{N}$ let

$$d_\varphi(I) = \lim_{i\to\infty} \sum_{j=1}^{\infty} \varphi_{i,j} \chi_I(j),$$

when this limit exists. Because $d_\varphi(I)$ does not exist for some subsets of $\mathbb{N}$, it is sometimes convenient to use the upper density $\overline{d}_\varphi(I) := \limsup_{i\to\infty} \sum_{j=1}^{\infty} \varphi_{i,j} \chi_I(j)$. We say that a set $I \subset N$ is *φ-null* if $d_\varphi(I) = 0$, and *φ-nonthin* if $\overline{d}_\varphi(I) > 0$. Having introduced matrix generated density d_φ, a sequence x_k is said to be *φ-statistically convergent to* x provided for every $\varepsilon > 0$, $d_\varphi(\{k : |x_k - x| > \varepsilon\}) = 0$.

For a summability matrix φ denote $\mathcal{F}_\varphi = \{I \subset \mathbb{N} : d_\varphi(\mathbb{N} \setminus I) = 0\}$ and note that $\mathcal{F}_\varphi$ is a filter. As it is easy to see, $\mathcal{F}_\varphi$-convergence and φ-statistical convergence coincide, and a set J is $\mathcal{F}_\varphi$-stationary when it is φ-nonthin.

If $\varphi = C$ is a Cèsaro matrix (i.e., $\varphi_{i,j} = 1/i$ for $j \le i$ and $\varphi_{i,j} = 0$ otherwise), then $\mathcal{F}_C$ concurs with the usual filter of statistical convergence and d_C is the usual natural density function. Note that the filter Φ from Ex. 2.11 can be also generated by a summability matrix. To define such a summability matrix put $\varphi_{i,n_{l,k}} = 2^{-k}$, where $\{n_{l,k}\}$ are from the definition of Φ. Define all the rest of $\varphi_{i,j}$ as zeros.

Each summability matrix φ is equivalent to a summability triangular matrix T, i.e., for any $I \subset \mathbb{N}$ $d_\varphi(I) = d_T(I)$. To establish this we note that since series $\sum_{j=1}^{\infty} \varphi_{i,j}$ converge for each i, then there are N_i such that $\sum_{j=N_i}^{\infty} \varphi_{i,j} \le 2^{-i}$. Thus for any $I \subset \mathbb{N}$

$$d_\varphi(I) = \lim_{i\to\infty} \sum_{j=1}^{N_i} \varphi_{i,j}\chi_I(j).$$

Consequently, if we erase all the elements $\varphi_{i,j}$ with $j > N_i$ (writing zeros instead of them) we pass to an equivalent matrix. Adding to this new matrix first $N_1 - 1$ zero rows and for every i repeating the i-th row of $\{\varphi_{i,j}\}$ $N_{i+1} - N_i - 1$ times, we reduce our matrix to an equivalent triangular matrix. So for the remainder of the note all the summability matrices are triangular.

Recall that for a summability matrix φ, a scalar valued sequence x_k is said to be strongly φ-summable if there is a scalar x such that $\lim_i \sum_j \varphi_{i,j}|x - x_i| = 0$. It is known that a bounded sequence is φ-statistically convergent if and only if it is strongly φ-summable [2, Th. 8]. Let us apply this fact.

Proposition 3.1. *If $\mathcal{F}_\varphi$ is a filter generated by a summability matrix φ, then $\mathcal{F}_\varphi$ is a Lebesgue filter.*

P r o o f. To establish this let us use the very first reformulation of the Lebesgue property (Th. 2.2). Let a measure space (Ω, Σ, μ) and $A_n \in \Sigma$ such that χ_{A_n} point-wise $\mathcal{F}_\varphi$-converge to 0 be given. In terms of strong φ-summability this means that $S_i = \sum_j \varphi_{i,j}\chi_{A_j}$ point-wise converge to 0. Note that $S_i \le 1$ for all i and are integrable. We can apply classical dominated convergence theorem to get

$$0 = \lim_i \int_\Omega S_i \, d\mu = \lim_i \sum_{j=1}^{i} \varphi_{i,j}\mu(A_j),$$

so, once more using the connection of strong φ-summability with φ-statistical convergence, $\mu(A_n) \to_{\mathcal{F}_\varphi} 0$. ∎

Corollary 3.2. *$\nu^*(\mathcal{F}_\varphi) = 0$, where ν is the usual product measure on $2^{\mathbb{N}}$ and φ is an arbitrary summability matrix.*

Thus for a filter generated by a summability matrix it is the Egorov property to be studied. Before we proceed, let us introduce some more notations: a *lacunary sequence* is an increasing sequence of integers $\{n_i\}$ such that $n_0 = 0$ and $n_i - n_{i-1} \to \infty$ as $i \to \infty$. We set $(n_{i-1}, n_i] = \{n : n_{i-1} < n \leq n_i\}$. For two sequences $\{n_i^1\}$ and $\{n_i^2\}$ we write $\{n_i^1\} \succ \{n_i^2\}$ if every $n_i^1 \geq n_i^2$.

From now on λ is the Lebesgue measure on [0,1] and $\mathcal{F}$ is a filter on $\mathbb{N}$.

Definition 3.3. *A lacunary sequence $\{n_i\}$ is called $\mathcal{F}$-special if $\mathbb{N}\backslash\{a_i\} \in \mathcal{F}$ for every $\{a_i\} \subset \mathbb{N}$ such that $a_i \in (n_{i-1}, n_i]$ for all i.*

Lemma 3.4. *If there is an $\mathcal{F}$-special lacunary sequence $\{n_i\}$ and there are $\alpha_n \geq 0,\ \alpha > 0$ such that:*

$$\sum_{k \in (n_{i-1}, n_i]} \alpha_k \leq 1 \tag{3.1}$$

and for every $I \in \mathcal{F}$

$$\sup_{i \in \mathbb{N}} \sum_{k \in (n_{i-1}, n_i]} \alpha_k \chi_I(k) \geq \alpha, \tag{3.2}$$

then $\mathcal{F}$ is not a Egorov filter.

P r o o f. Using the condition (3.1) one can easily construct such a sequence A_k of the Lebesgue measurable subsets of [0,1] that

(1) A_j for $j \in (n_{i-1}, n_i]$ are disjoint;

(2) $\lambda(A_j) = \alpha_j$ and $\bigsqcup_{j \in (n_{i-1}, n_i]} A_j \subset [0, 1]$.

The condition () guaranties that for each $t \in [0, 1]$ there is no more than one $a_i \in (n_{i-1}, n_i]$ such that $t \in A_{a_i}$. Because of this the definition of $\mathcal{F}$-special lacunary sequence ensures that χ_{A_k} point-wise $\mathcal{F}$-converge to 0. As it was observed in the proof of Lem. 2.7, Th. 2.1 guarantees that the filter does not have the Egorov property when there is such an $\varepsilon > 0$ that $\lambda(\bigcup_{n \in I} A_n) > \varepsilon$ for every $I \in \mathcal{F}$. The obvious inequality

$$\lambda(\bigcup_{n \in I} A_n) \geq \sup_{i \in \mathbb{N}} \sum_{k \in (n_{i-1}, n_i]} \lambda(A_k)\chi_I(k) = \sup_{i \in \mathbb{N}} \sum_{k \in (n_{i-1}, n_i]} \alpha_k \chi_I(k) \geq \alpha$$

establishes the claim. ∎

Note that for a summability matrix φ Def. 3.3 of $\mathcal{F}$-special sequence $\{n_i\}$ can be rewritten as follows: for any $\{a_i\} \subset \mathbb{N}$ with $a_i \in (n_{i-1}, n_i]$, $i = 1, 2, \ldots$

$$\lim_{i\to\infty} \sum_{j=1}^{i} \varphi_{i,j} \chi_{\{a_k\}}(j) = 0. \tag{3.3}$$

Lemma 3.5. *For a summability matrix φ and the corresponding filter $\mathcal{F} = \mathcal{F}_\varphi$ the following assertions are equivalent:*

(1) There is an $\mathcal{F}$-special lacunary sequence $\{n_i\}$.

(2) The matrix φ satisfies

$$\lim_{i\to\infty} \max_{j\in\mathbb{N}} \varphi_{i,j} = 0. \tag{3.4}$$

Moreover, under the second condition $\{n_i\}$ can be selected in such a way that every lacunary sequence $\{m_i\} \succ \{n_i\}$ is also $\mathcal{F}$-special.

P r o o f. (1) $\Rightarrow$ (2). If $\{n_i\}$ is $\mathcal{F}$-special but (2) is not true, then there is a $\varepsilon > 0$ and there is an increasing sequence of naturals i_k with the corresponding $b_k \le i_k$, such that $\varphi_{i_k,b_k} \ge \varepsilon$. For every k let $m(k)$ be the index for which $b_k \in (n_{m(k)-1}, n_{m(k)}]$. Since each column of φ tends to zero, only finite number of $m(k)$ for different k can coincide. So passing if necessary to a subsequence of indices k we may assume that $m(1) < m(2) < \ldots$. Now selecting an arbitrary sequence $a_i \in (n_{i-1}, n_i]$ of naturals in such a way that $a_{m(k)} = b_k$, we get

$$\sum_{j=1}^{i_k} \varphi_{i_k,j} \chi_{\{a_k\}}(j) \ge \varphi_{i_k,b_k} \ge \varepsilon.$$

This contradicts the property (3.3) of $\mathcal{F}$-special sequence.

(2) $\Rightarrow$ (1). If (2) holds, then there are n_k such that for all $i \ge n_{k-1}$ $\max_{j\in\mathbb{N}} \varphi_{i,j} < (k)^{-2}$. Now for any $\{a_k\} \succ \{n_k\}$ and any $i \in (n_{k-1}, n_k]$

$$\sum_{j=1}^{i} \varphi_{i,j} \chi_{\{a_k\}}(j) = \sum_{j=1}^{n_k} \varphi_{i,j} \chi_{\{a_k\}}(j) = \sum_{a_m:\, a_m \le n_k} \varphi_{i,a_m} \le k \max_{j\in\mathbb{N}} \varphi_{i,j} < k^{-1},$$

and thus converge to 0 when $i \to \infty$. So condition (3.3) holds and lemma is proved. ■

Theorem 3.6. *Under the condition (3.4) $\mathcal{F} = \mathcal{F}_\varphi$ is not a Egorov filter. Moreover, for every sequence $\{m_i\}$ there are an $\mathcal{F}$-special lacunary sequence $\{n_i\} \succ \{m_i\}$ and corresponding α_k and α such that the conditions of Lem. 3.4 are fulfilled.*

P r o o f. Fix a sequence $\{m_i\}$. Due to Lem. 3.5 there is an $\mathcal{F}$-special lacunary sequence $\{n_i^0\} \succ \{m_i\}$, such that all lacunary sequences $\{n_i\} \succ \{n_i^0\}$ are $\mathcal{F}$-special.

By the definition of summability matrix there are $\{i_m\}$ and $\alpha > 0$ such that for all m

$$\sum_{k=1}^{i_m} \varphi_{i_m,k} \geq 3\alpha.$$

Since the columns of φ converge to 0 we can choose a lacunary sequence $\{n_i\} \succ \{n_i^0\}$ such that $\{n_i\}_{i=1}^{\infty} \subset \{i_m\}$, and for all $i \in \mathbb{N}$

$$\sum_{k \in (n_{i-1}, n_i]} \varphi_{n_i,k} > 2\alpha.$$

For every $k \in (n_{i-1}, n_i]$ put $\alpha_k = \varphi_{n_i,k}$. Let us prove that this sequence of α_k satisfies the conditions of Lem. 3.4 for $\{n_i\}$.

Let $I \in \mathcal{F}_\varphi$. Then $\mathbb{N} \backslash I$ is a φ-null set and so for i sufficiently large $\sum_{k \in (n_{i-1}, n_i]} \varphi_{n_i,k} \chi_{\mathbb{N} \backslash I}(k) < \alpha$, and thus for such i

$$\sum_{k \in (n_{i-1}, n_i]} \alpha_k \chi_I(k) = \sum_{k \in (n_{i-1}, n_i]} \varphi_{n_i,k} \chi_I(k) > \alpha$$

and we are in the conditions of Lem. 3.4. ■

Theorem 3.7. *Let $\mathcal{F}_\varphi$ be a filter generated by a summability matrix φ. The following assertions are equivalent:*

(1) $\mathcal{F}_\varphi$ is nowhere Egorov;

(2) $K_{\mathcal{F}_\varphi}$ is nowhere dense in $\widetilde{\mathbb{N}}$;

(3) there is no Fréchet stationary set with respect to $\mathcal{F}_\varphi$;

(4) the condition (3.4) holds true.

P r o o f. (1) $\Rightarrow$ (2). As we have already shown (Prop. 2.6) it is true for an arbitrary nowhere Egorov filter.

(2) $\Rightarrow$ (3). Once again see the proof of Prop. 2.6.

(3) $\Rightarrow$ (4). This result for regular summability matrices is shown in [10]. For the general case it is true as well. Namely, if (4) does not hold, then there is an $\varepsilon > 0$ and there are increasing sequences of naturals $\{i_m\}$ and $\{j_m\}$ such that $\varphi_{i_m,j_m} \geq \varepsilon$ for all m. Under these conditions $J = \{j_m\}$ is an $\mathcal{F}_\varphi$-stationary set, such that the trace of $\mathcal{F}_\varphi$ on J coincides with the Fréchet filter on J.

(4) $\Rightarrow$ (1). Remark that for a $J \in \mathcal{F}_\varphi^*$ the filter $\mathcal{F}_\varphi(J)$ is generated by the following summability matrix ψ: $\psi_{i,j} = \varphi_{i,j} \chi_J(j)$. If the condition (3.4) holds,

then $\lim_{i\to\infty} \max_{j\in\mathbb{N}} \psi_{i,j} = \lim_{i\to\infty} \max_{j\in J} \varphi_{i,j} = 0$. So Theorem 3.6 ensures us that $\mathcal{F}_\varphi(J)$ does not have the Egorov property, and hence $\mathcal{F}_\varphi$ is nowhere Egorov. ■

It is easy to show that there are non-Egorov filters that are not nowhere Egorov ones. As a consequence we get an example of the Egorov filter dominating a non-Egorov filter. Moreover:

Example 3.8. *There are some filters $\mathcal{F}_4 \succ \mathcal{F}_3 \succ \mathcal{F}_2 \succ \mathcal{F}_1$, such that $\mathcal{F}_1$, $\mathcal{F}_3$ are the Egorov filters and $\mathcal{F}_2$, $\mathcal{F}_4$ are not.*

For $\mathcal{F}_1$ we take a Frèchet filter on $\mathbb{N}$. Then let $\mathbb{N} = \{n_i\} \sqcup \{m_i\}$ be a disjoint partition of naturals. For $\mathcal{F}_2$ we take the filter whose restriction on $\{n_i\}$ is the image $\widehat{\mathcal{F}}$ of the statistical convergence filter under map $i \to n_i$ and the restriction of $\mathcal{F}_2$ on $\{m_i\}$ is $\mathcal{F}(\{m_i\})$ — the Frèchet filter on $\{m_i\}$, i.e., $\omega \in \mathcal{F}_2$ if and only if $\omega \cap \{n_i\} \in \widehat{\mathcal{F}}$ and $\omega \cap \{m_i\} \in \mathcal{F}(\{m_i\})$. $\mathcal{F}_2$ dominates $\mathcal{F}_1$, but $\mathcal{F}_2$ does not have the Egorov property because the trace of $\mathcal{F}_2$ on $\{n_i\}$ is a non-Egorov filter $\widehat{\mathcal{F}}$. Then take $\mathcal{F}(\{m_i\})$ for the base of $\mathcal{F}_3$. Then $\mathcal{F}_3$ is a Egorov filter that dominates $\mathcal{F}_2$. And finally, for $\mathcal{F}_4$ one can take, say, an ultrafilter dominating $\mathcal{F}_3$.

The next theorem shows that for the filters generated by summability matrices there is a connection between domination and the Egorov property.

Theorem 3.9. *Let φ^1 and φ^2 be summability matrices, $\mathcal{F}_{\varphi^2} \succ \mathcal{F}_{\varphi^1}$ such that $\lim_{i\to\infty} \max_j \varphi^1_{i,j} = 0$. Then any filter $\mathcal{F}$ such that $\mathcal{F}_{\varphi^2} \succ \mathcal{F} \succ \mathcal{F}_{\varphi^1}$ is not a Egorov filter.*

P r o o f. Since $\mathcal{F}_{\varphi^2} \succ \mathcal{F}_{\varphi^1}$ it follows that $K_{\mathcal{F}_{\varphi^2}} \subset K_{\mathcal{F}_{\varphi^1}}$. By Theorem 3.7 $K_{\mathcal{F}_{\varphi^1}}$ and hence $K_{\mathcal{F}_{\varphi^2}}$ are nowhere dense in $\widetilde{\mathbb{N}}$, thus $\lim_{i\to\infty} \max_{j\in\mathbb{N}} \varphi^2_{i,j} = 0$ too. Applying Lemma 3.5 (the "moreover" part), we can find a lacunary sequence $\{m_i\}$ such that every lacunary sequence $\{n_i\} \succ \{m_i\}$ is at the same time $\mathcal{F}_{\varphi^1}$ and $\mathcal{F}_{\varphi^2}$-special. By Theorem 3.6 there are $\{n_i\} \succ \{m_i\}$ and corresponding α_k and α such that the conditions of Lem. 3.4 are fulfilled for the filter $\mathcal{F}_{\varphi^2}$. Now the inequality $\mathcal{F} \succ \mathcal{F}_{\varphi^1}$ ensures that the $\mathcal{F}_{\varphi^1}$-special sequence $\{n_i\}$ is at the same time $\mathcal{F}$-special; and due to the inequality $\mathcal{F}_{\varphi^2} \succ \mathcal{F}$, the conditions of Lem. 3.4 are fulfilled for the filter $\mathcal{F}$ and for the sequences n_i, α_k and α. ■

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