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Klein–Gordon Equation as a Result of Wave Equation Averaging on the Riemannian Manifold of Complex Microstructure

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An asymptotic behavior of solution of the Cauchy problem for the wave equation is studied on the Riemannian manifold M^ε depending on a small parameter ε. It is supposed that a topological type of M^ε increases as $\varepsilon \to 0$. The averaged equation is derived, it describes the asymptotic behavior of the original Cauchy problem as $\varepsilon \to 0$.

Key words: Riemannian manifolds, wave equation, asymptotic behavior, homogenization.
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Introduction

We denote by M_3^ε the 3-dimensional Riemannian manifold depending on a small parameter ε and described in the following way. Let D_j^ε be a union of "holes" in $\mathbb{R}^3$ — balls of the radius ε^3 with the centers $x_j^\varepsilon \in \mathbb{R}^3$ and distributed periodically in $\mathbb{R}^3$ with the period ε. Let

$$\Omega^\varepsilon = \mathbb{R}^3 \backslash \bigcup_j D_j^\varepsilon.$$

We consider two copies Ω_1^ε and Ω_2^ε of the domain Ω^ε. Let Ω_1^ε and Ω_2^ε be the upper and the lower sheets respectively. The boundaries of these sheets consist of the spheres $\partial D_{1j}^\varepsilon$ and $\partial D_{2j}^\varepsilon$. We join $\partial D_{1j}^\varepsilon$ and $\partial D_{2j}^\varepsilon$ by means of 3-dimensional tubes ("wormholes") $G_j^\varepsilon = S_j^\varepsilon \times [0,1]$, where S_j^ε is a sphere in $\mathbb{R}^3$ of the radius ε^3.

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Then we obtain the 3-dimensional oriented manifold

$$M_3^\varepsilon = (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \bigcup \left(\bigcup_{j=1}^{N(\varepsilon)} G_j^\varepsilon \right).$$

We introduce differential structure on M_3^ε in the standard way (see, e.g., [1]). This manifold is illustrated in figure. The points of M_3^ε we denote by $\tilde{x}$

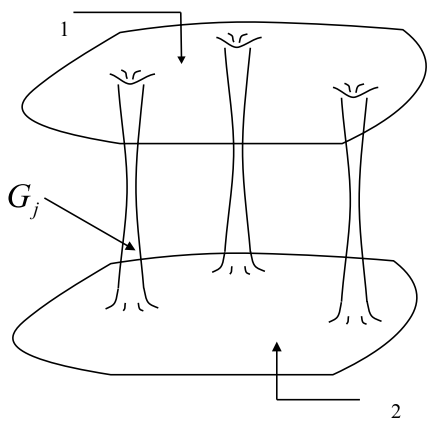

Figure. Manifold M_3^ε.

We define a Riemannian structure on M_3^ε by the metric tensor $g_{ik}^\varepsilon(\tilde{x})$ depending on ε.

By M_4^ε we denote the 4-dimensional manifold (space and time)

$$M_4^\varepsilon = M_3^\varepsilon \times \mathbb{R}$$

and introduce a pseudo Riemannian metric on M_4^ε by the formula

$$ds^2 = [c_\varepsilon(\tilde{x})]^2 dt^2 - \sum_{i,k=1}^{3} g_{ik}^\varepsilon dx_i dx_k,$$

where $c_\varepsilon(\tilde{x}) > 0$.

The Cauchy problem for the wave equation is considered on M_4^ε:

$$\Box^\varepsilon u^\varepsilon \equiv \frac{1}{c_\varepsilon(\tilde{x})^2} \frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{1}{c_\varepsilon(\tilde{x})\sqrt{G}^\varepsilon} \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i} \left(g_\varepsilon^{ik} c_\varepsilon(\tilde{x}) \sqrt{G}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_k} \right) = 0, \quad (0.1)$$

$$u^\varepsilon(\tilde{x}, 0) = f^\varepsilon, \quad (0.2)$$

$$u_t^\varepsilon(\tilde{x}, 0) = g^\varepsilon. \quad (0.3)$$

($G^\varepsilon = \det g^\varepsilon_{ik}$, g^{ik}_ε, $i,k = 1,2,3$, are the components of the tensor inverse to g^ε_{ik}.)

Suppose that the following conditions hold: on the upper sheet the metric coincides with the Euclidean metric (outside of some small neighborhoods of the "holes" D^ε_j), and $c^\varepsilon(\tilde{x}) = 1$; while on the lower sheet the metric increases or $c^\varepsilon \to 0$ (the proper time becomes slower) as $\varepsilon \to 0$ (in the latter case it is possible to choose the radiuses of "holes" more than ε^3).

Then we have the following result. The solution of the problem (0.1)–(0.3) converges on the upper sheet to the solution of the Cauchy problem for the Klein–Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u^\varepsilon + mu = 0, \quad \tilde{x} \in \mathbb{R}^3, t > 0,$$
$$u(x,0) = f(x),$$
$$u_t(x,0) = g(x),$$

where m definitely depends on the characteristics of "wormholes", metric and the function $c_\varepsilon(\tilde{x})$.

This fact admits the following interesting physics interpretation: as a result of connection with the lower sheet by means of "wormholes" G^ε_j a scalar massless particle gets a mass m as $\varepsilon \to 0$. In the paper this fact is proved in a more general statement.

1. The Increasing Metric Case

1.1. The Problem Setting and the Statement of Main Result

Let $\{D^\varepsilon_j, j = 1 \ldots N(\varepsilon)\}$ be a system of disjoint small domains in $\mathbb{R}^3$ with a smooth border. Suppose that this system depends on the parameter $\varepsilon > 0$ in such a way that the diameters of the sets D^ε_j tend to zero as $\varepsilon \to 0$, and their total number $N(\varepsilon)$ tends to infinity. Denote $\Omega^\varepsilon = \mathbb{R}^3 \setminus \bigcup_{j=1}^{N(\varepsilon)} D^\varepsilon_j$. We consider two copies Ω^ε_1 and Ω^ε_2 of the set Ω^ε. Let Ω^ε_1 and Ω^ε_2 be called the upper and the lower sheets, respectively.

Let G^ε_j be 3-dimensional manifolds with the boundaries consisting of two disjoined components Γ^ε_{1j} and Γ^ε_{2j} being diffeomorphic to ∂D^ε_j.

By means of these diffeomorphisms and taking account of orientation, we glue Γ^ε_{1j} to the copy of ∂D^ε_j on the upper sheet and Γ^ε_{2j} to the copy of ∂D^ε_j on the lower sheet.

As a result we obtain a differentiable manifold M^ε:

$$M^\varepsilon = (\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2) \bigcup \left(\bigcup_{j=1}^{N(\varepsilon)} G^\varepsilon_j \right).$$

We denote by $\tilde{x}$ the points of this manifold. If the point $\tilde{x} \in \Omega_k^\varepsilon$, then we assign a pair (x, k) to $\tilde{x}$, where $x \in \mathbb{R}^3$ is a coordinate.

Let $B(D_j^\varepsilon)$ be the smallest ball containing D_j^ε, with the center x_j^ε and the radius d_j^ε. We denote

$$\Omega_k'^\varepsilon = \left\{ \tilde{x} = (x, k) \in \Omega_k^\varepsilon : \quad x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{N(\varepsilon)} B(D_j^\varepsilon) \right\}.$$

On M^ε we introduce a metric $g_{ik}^\varepsilon(\tilde{x})$ that coincides with the Euclidean metric in $\Omega_1'^\varepsilon$ ($g_{ik}^\varepsilon = \delta_{ik}$) and increases in $\Omega_2'^\varepsilon$ as follows:

$$g_{ik}^\varepsilon = \frac{\delta_{ik}}{\mu_\varepsilon}, \quad \mu_\varepsilon > 0 \quad and \quad \mu_\varepsilon \to 0, \ \varepsilon \to 0. \tag{1.1}$$

Consider the following Cauchy problem on M^ε:

$$\Box^\varepsilon u^\varepsilon \equiv \frac{\partial^2 u^\varepsilon(\tilde{x}, t)}{\partial t^2} - \Delta^\varepsilon u^\varepsilon(\tilde{x}, t) = 0, \quad \tilde{x} \in M^\varepsilon, \ t > 0, \tag{1.2}$$

$$u^\varepsilon(\tilde{x}, 0) = f^\varepsilon(\tilde{x}), \tag{1.3}$$

$$u_t^\varepsilon(\tilde{x}, 0) = g^\varepsilon(\tilde{x}), \tag{1.4}$$

with Δ^ε being the Laplace–Beltrami operator on M^ε

$$\Delta^\varepsilon = \frac{1}{\sqrt{G}^\varepsilon} \sum_{i,k=1}^{3} \frac{\partial}{\partial x_i} \left(\sqrt{G}^\varepsilon g_\varepsilon^{ik} \frac{\partial}{\partial x_k} \right),$$

where $G^\varepsilon = \det g_{ik}^\varepsilon$, g_ε^{ik}, $i, k = 1, 2, 3$, are the components of the tensor inverse to g_{ik}^ε, and f^ε, g^ε are the smooth functions.

The purpose of this paper is to describe the asymptotic behavior of $u^\varepsilon(\tilde{x}, t)$ on the upper sheet as $\varepsilon \to 0$.

Introduce the notation: $r_j^\varepsilon = \operatorname{dist} \left(B(D_j^\varepsilon); \bigcup\limits_{i \neq j} B(D_i^\varepsilon) \right)$,

$$B_{kj}^\varepsilon = \left\{ \tilde{x} = (x, k) \in \Omega_k^\varepsilon : d_j^\varepsilon < |x - x_j^\varepsilon| < d_j^\varepsilon + \frac{r_j^\varepsilon}{2} \right\},$$

$$G_j'^\varepsilon = G_j^\varepsilon \bigcup_{k=1}^{2} \left\{ \tilde{x} = (x, k) \in \Omega_k^\varepsilon : x \in B(D_j^\varepsilon) \setminus D_j^\varepsilon \right\},$$

$$\tilde{G}_j^\varepsilon = G_j'^\varepsilon \bigcup_{k=1}^{2} B_{kj}^\varepsilon, \quad S_{kj}^\varepsilon = \partial \tilde{G}_j^\varepsilon \cap \Omega_k^\varepsilon.$$

We consider the following boundary-value problem in the domain $\tilde{G}_j^\varepsilon$:

$$\Delta^\varepsilon v = 0, \quad \tilde{x} \in \tilde{G}_j^\varepsilon, \tag{1.5}$$

$$v = 1, \tilde{x} \in S_{1j}^\varepsilon, \tag{1.6}$$

$$v = 0, \tilde{x} \in S_{2j}^\varepsilon. \tag{1.7}$$

Let $v_j^\varepsilon = v_j^\varepsilon(\tilde{x})$ be the solution of (1.5)–(1.7). We set

$$V_j^\varepsilon = \int\limits_{\tilde{G}_j^\varepsilon} \sum_{i,k=1}^{3} g_\varepsilon^{ik} \frac{\partial v_j^\varepsilon}{\partial x_i} \frac{\partial v_j^\varepsilon}{\partial x_k} d\tilde{x},$$

where $d\tilde{x} = \sqrt{G^\varepsilon} dx_1 dx_2 dx_3$ is the volume element on M^ε, and introduce the generalized function

$$V^\varepsilon(x) = \sum_{j=1}^{N(\varepsilon)} V_j^\varepsilon \delta(x - x_j^\varepsilon).$$

We introduce the following functional spaces:
$L^2(M^\varepsilon)$ is the Hilbert space of real valued functions on M^ε with the norm

$$\|u^\varepsilon\|_{0\varepsilon} = \left\{ \int\limits_{M^\varepsilon} (u^\varepsilon)^2 d\tilde{x} \right\}^{1/2},$$

$H^1(M^\varepsilon)$ is the Hilbert space of real valued functions on M^ε with the norm

$$\|u^\varepsilon\|_{1\varepsilon} = \left\{ \int\limits_{M^\varepsilon} \left(\sum_{i,k=1}^{3} g_\varepsilon^{ik} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_k} + (u^\varepsilon)^2 \right) d\tilde{x} \right\}^{1/2}.$$

We say that the function $f^\varepsilon \in L^2(M^\varepsilon)$ converges on the upper sheet to $f \in L^2(\mathbb{R}^3)$, if for any bounded domain $\Omega \subset \mathbb{R}^3$

$$\lim_{\varepsilon \to 0} \|Q^\varepsilon f^\varepsilon - f\|_{L^2(\Omega)} = 0, \tag{1.8}$$

where the operator $Q^\varepsilon : L^2(M^\varepsilon) \to L^2(\mathbb{R}^3)$ is defined by the formula

$$[Q^\varepsilon f^\varepsilon](x) = \begin{cases} f^\varepsilon(\tilde{x}), & \tilde{x} = (x, 1) \in \Omega_1'^\varepsilon, \\ 0, & x \in \bigcup\limits_{j=1}^{N(\varepsilon)} B(D_j^\varepsilon). \end{cases}$$

Similarly, we say that $u^\varepsilon(\tilde{x},t) \in L^2(M^\varepsilon \times [0,T])$ converges on the upper sheet to $u(x,t) \in L^2(\mathbb{R}^3 \times [0,T])$, if for any bounded domain $\Omega \subset \mathbb{R}^3$

$$\lim_{\varepsilon\to 0} \int\limits_0^T \|Q^\varepsilon u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^2(\Omega)} dt = 0. \tag{1.9}$$

Let us formulate the basic theorem

Theorem 1. *Suppose that the following conditions hold:*

(i) $\lim\limits_{\varepsilon\to 0} \max\limits_j d_j^\varepsilon = \lim\limits_{\varepsilon\to 0} \max\limits_j r_j^\varepsilon = 0$;

(ii) *for any domain* $\mathrm{G} \subset \mathbb{R}^3$

$$\overline{\lim_{\varepsilon\to 0}} \sum_{x_j^\varepsilon \in \mathrm{G}} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \le C_1 \mathrm{meas}\mathrm{G},$$

and $r_j^\varepsilon > C_0 d_j^\varepsilon$ *(here* $0 < C_0,\ C_1 < \infty$*);*

(iii) $\lim\limits_{\varepsilon\to 0} \max\limits_j \frac{(r_j^\varepsilon)^8}{\mu_\varepsilon^3} = 0$;

(iv) *there exists a limit (in* $D'(\mathbb{R}^3)$*)*

$$\lim_{\varepsilon\to 0} V^\varepsilon(x) = V(x),$$

where $V(x)$ *is a measurable bounded nonnegative function;*

(v) *for any domain* $\mathrm{G} \subset \mathbb{R}^3$

$$\sum_{x_j^\varepsilon \in \mathrm{G}} \mathrm{meas} G_j'^\varepsilon \to 0 \ (\varepsilon \to 0);$$

(vi) *norms* $\|f^\varepsilon\|_{1\varepsilon}$ *are uniformly bounded with respect to* ε*; when* $\varepsilon \to 0$*:* $f^\varepsilon(\tilde{x})$ *and* $g^\varepsilon(\tilde{x})$ *converge in the sense of (1.8) to the functions* $f \in H^1(\mathbb{R}^3)$ *and* $g \in L^2(\mathbb{R}^3)$*, respectively, and*

$$\int\limits_{\Omega_2'^\varepsilon \bigcup\limits_j G_j'^\varepsilon} \left(|f^\varepsilon|^2 + |g^\varepsilon|^2\right) d\tilde{x} \to 0,\ \varepsilon \to 0.$$

Then the solution of the problem (1.2)–(1.4) $u^\varepsilon(\tilde{x},t)$ *converges in the sense of (1.9) to the solution* $u(x,t)$ *of the following problem:*

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - V(x)u, \quad \tilde{x} \in \mathbb{R}^3, t > 0, \tag{1.10}$$

$$u(x,0) = f(x), \tag{1.11}$$

$$u_t(x,0) = g(x). \tag{1.12}$$

The proof of the theorem is based on a study of asymptotic behavior of the operator $-\Delta^\varepsilon$ resolvent as $\varepsilon \to 0$.

1.2. Asymptotic Behavior of the Solution of the Stationary Problem

We consider the following problem:

$$-\Delta^\varepsilon u^\varepsilon + \lambda u^\varepsilon = \mathcal{F}^\varepsilon, \quad \tilde{x} \in M^\varepsilon, \tag{1.13}$$

$$u^\varepsilon \in H^1(M^\varepsilon), \tag{1.14}$$

where $\lambda > 0, \mathcal{F}^\varepsilon \in L^2(M^\varepsilon)$.

As it is known, there exists a unique solution $u^\varepsilon(x,\lambda)$ of this problem. The following theorem describes the asymptotic behavior of $u^\varepsilon(x,\lambda)$ on the upper sheet.

Theorem 2. *Suppose that conditions* (i)–(v) *of the Th. 1 hold and suppose* (vi$'$) $\mathcal{F}^\varepsilon(x)$ *converges in the sense of (1.8) to the function* $\mathcal{F}(x) \in L^2(\mathbb{R}^3)$ *as* $\varepsilon \to 0$ *and*

$$\int\limits_{\Omega_2'^\varepsilon \bigcup\limits_j G_j'^\varepsilon} (\mathcal{F}^\varepsilon)^2 d\tilde{x} \to 0 (\varepsilon \to 0).$$

Then the solution of the problem (1.13)–(1.14) converges in the sense of (1.8) to the solution $u(x)$ of the following problem:

$$-\Delta u + \lambda u + V(x) u = \mathcal{F}, \quad x \in \mathbb{R}^3, \tag{1.15}$$

$$u^\varepsilon \in H^1(\mathbb{R}^3). \tag{1.16}$$

P r o o f. As we know, the solution $u^\varepsilon(x,\lambda)$ of the problem (1.13)–(1.14) minimizes the functional

$$J^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} \left\{ \sum_{i,k=1}^{3} g_\varepsilon^{ik} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_k} + \lambda (u^\varepsilon)^2 - 2\mathcal{F}^\varepsilon u^\varepsilon \right\} d\tilde{x} \tag{1.17}$$

in the class of functions $H^1(M^\varepsilon)$, while the solution $u(x,\lambda)$ of the problem (1.15)–(1.16) minimizes the functional

$$J[u] = \int\limits_{\mathbb{R}^3} \left\{ |\nabla u|^2 + \lambda u^2 + V(x) u - 2\mathcal{F} u \right\} dx \tag{1.18}$$

in the class $H^1(\mathbb{R}^3)$. The converse assertions are also true. Therefore it is sufficient to show that the solution of the problem of minimizing (1.17) converges to the solution of the problem of minimizing (1.18).

Consider an abstract scheme for solving the problem. Let H^ε be a Hilbert space depending on the parameter $\varepsilon > 0$, $(u^\varepsilon, v^\varepsilon)_\varepsilon, \|u^\varepsilon\|_\varepsilon$ be a scalar product and a norm in this space, F^ε be the continuous linear functionals in H^ε which are uniformly bounded with respect to ε. Let H be a Hilbert space with the scalar product (u, v) and norm $\|u\|$, F be a continuous linear functional in H.

Consider the following two problems of minimization:

$$\|u^\varepsilon\|_\varepsilon^2 + F^\varepsilon[u^\varepsilon] \to \inf, \quad u^\varepsilon \in H^\varepsilon, \tag{1.19}$$

$$\|u\|^2 + F[u] \to \inf, \quad u \in H. \tag{1.20}$$

We have the following theorem proved in [3].

Theorem 3. *Let M be a dense subset of H, and let $\Pi^\varepsilon : H^\varepsilon \to H$ and $P^\varepsilon : M \to H^\varepsilon$ be the operators satisfying the following conditions:*

(a) $\|\Pi^\varepsilon w^\varepsilon\| \le C\|w^\varepsilon\|, \forall w^\varepsilon \in H^\varepsilon$;

(b_1) $\Pi^\varepsilon P^\varepsilon w \to w$ weakly in H as $\varepsilon \to 0, \forall w \in M$;

(b_2) $\lim\limits_{\varepsilon\to 0} \|P^\varepsilon w\|_\varepsilon = \|w\|, \forall w \in M$;

(b_3) for any sequence $\gamma^\varepsilon \in H^\varepsilon$, such that $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly as $\varepsilon \to 0$, for any $w \in M$ one has

$$\overline{\lim_{\varepsilon\to 0}} |(P^\varepsilon w, \gamma^\varepsilon)_\varepsilon| \le C\|w\|\|\gamma\|;$$

(c) for any sequence $\gamma^\varepsilon \in H^\varepsilon$, such that $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} F^\varepsilon[\gamma^\varepsilon] = F[\gamma].$$

Then the solution u^ε of the minimization problem (1.19) converges to the solution of the minimization problem (1.20) in the following sense:

$$\Pi^\varepsilon u^\varepsilon \underset{\varepsilon\to 0}{\to} u \text{ weakly in } H.$$

Note, that Th. 3 holds true if conditions (b_3) and (c) hold only for such sequences γ^ε that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε (as in the proof of Th. 3, the conditions (b_3) and (c) are used only with these sequences).

Now we consider our abstract scheme. Let H^ε be the Hilbert space $H^1(M^\varepsilon)$ of the functions on M^ε with the scalar product

$$(u^\varepsilon, v^\varepsilon)_\varepsilon = \int\limits_{M^\varepsilon} \left\{ \sum_{i,k=1}^{3} g_\varepsilon^{ik} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v^\varepsilon}{\partial x_k} + \lambda u^\varepsilon v^\varepsilon \right\} d\tilde{x}, \tag{1.21}$$

and let F^ε be the linear functional on it defined by the formula

$$F^\varepsilon[u^\varepsilon] = \int\limits_{M^\varepsilon} -2\mathcal{F}^\varepsilon u^\varepsilon d\tilde{x}. \tag{1.22}$$

Let H be the Hilbert space $H^1(\mathbb{R}^3)$ with the scalar product

$$(u,v)=\int\limits_{\mathbb{R}^3}\left\{\sum_{i=1}^{3}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+\lambda uv+V(x)uv\right\}dx \tag{1.23}$$

and F be the linear functional on it defined by the formula

$$F[u]=\int\limits_{\mathbb{R}^3}-2\mathcal{F}u dx. \tag{1.24}$$

Since $|F^\varepsilon[u^\varepsilon]|\leq 2\|\mathcal{F}^\varepsilon\|_{0_\varepsilon}\|u^\varepsilon\|_{0_\varepsilon}\leq 2\|\mathcal{F}^\varepsilon\|_{0_\varepsilon}\|u^\varepsilon\|_\varepsilon$ and norms $\|\mathcal{F}^\varepsilon\|_{0_\varepsilon}$ are uniformly bounded with respect to ε, then the functionals F^ε are uniformly bounded with respect to ε.

Now we introduce the operators Π^ε and P^ε satisfying the conditions (a)–(c) of Th. 3. Let $u^\varepsilon\in H^1(M^\varepsilon)$, u'^ε be a contraction of u^ε on $\Omega_1'^\varepsilon$. Then u'^ε can be extended to $\bigcup\limits_{j=1}^{N(\varepsilon)}B(D_j^\varepsilon)$ so that the obtained function $\tilde{u}'^\varepsilon$ belongs to the space $H^1(\mathbb{R}^3)$ and satisfies the inequality

$$\|\tilde{u}'^\varepsilon\|\leq C\|u^\varepsilon\|_{\varepsilon,\Omega_1^\varepsilon}, \tag{1.25}$$

where C does not depend on ε [4].

Since this kind of extensions is not unique, we require the norms of the extended function in the space $H^1(\bigcup\limits_j B(D_j^\varepsilon))$ to be minimal. Then we obtain a unique extension $\tilde{u}^\varepsilon$. For this reason we set $\Pi^\varepsilon u^\varepsilon=\tilde{u}^\varepsilon$.

It follows from (1.25) that the condition (a) of Th. 3 holds.

We introduce an operator P^ε. Let $\varphi(r)$ be a twice continuously differentiable non-negative function on the half-line $[0,\infty)$, which is equal to 1 for $r\in[0,1/4]$ and to 0 for $r\geq 1/2$. We set

$$\varphi_j^\varepsilon=\varphi\left(\frac{|x-x_j^\varepsilon|-d_j^\varepsilon}{r_j^\varepsilon}\right),\quad \varphi_{j0}^\varepsilon=\varphi\left(\frac{|x-x_j^\varepsilon|-d_j^\varepsilon}{C_0 d_j^\varepsilon}\right).$$

Let $M=C_0^2(\mathbb{R}^3)$ be a dense subset of $H^1(\mathbb{R}^3)$ and let $w\in M$. Define the operator P^ε by the equalities

$$[P^\varepsilon w](\tilde{x})=\begin{cases} w(x)+\sum\limits_{j=0}^{N(\varepsilon)}(w_j^\varepsilon-w(x))\varphi_{j0}^\varepsilon+\sum\limits_{j=0}^{N(\varepsilon)}(v_j^\varepsilon(x)-1)\varphi_j^\varepsilon w_j^\varepsilon, & \tilde{x}\in\Omega_1'^\varepsilon,\\ v_j^\varepsilon(\tilde{x})w_j^\varepsilon, & \tilde{x}\in G_j'^\varepsilon,\\ \sum\limits_{j=0}^{N(\varepsilon)}v_j^\varepsilon(x)\varphi_j^\varepsilon w_j^\varepsilon, & \tilde{x}\in\Omega_2'^\varepsilon,\end{cases} \tag{1.26}$$

where $w_j^\varepsilon=w(x_j^\varepsilon)$.

To be sure that the conditions (b_1)–(b_3) hold we use the following estimates for the solution v_j^ε of the problem (1.5)–(1.7).

Lemma 1. *Let* $\tilde{x} = (x,k) \in B_{kj}^\varepsilon$ *and* $|x - x_j^\varepsilon| \geq d_j^\varepsilon(1 + C_0)$. *Then*

$$|D^\alpha(v_j^\varepsilon(\tilde{x}) - \delta_{1k})| \leq C\frac{d_j^\varepsilon}{|x - x_j^\varepsilon|^{1+|\alpha|}},$$

where $|\alpha| = 0, 1$, *and* C *does not depend on* ϵ.

Lemma 2. *The following estimates are valid:*

$$\int\limits_{B_{kj}^\varepsilon} |v_j^\varepsilon - \delta_{1k}|^2 d\tilde{x} \leq \begin{cases} C(d_j^\varepsilon)^3(1 + r_j^\varepsilon/d_j^\varepsilon), & k = 1, \\ C(d_j^\varepsilon)^3(1 + r_j^\varepsilon/d_j^\varepsilon)\mu_\varepsilon^{-3/2}, & k = 2. \end{cases}$$

The proofs of these lemmas are carried out in the same way as those of Lems. 1, 2 in [5], by using the inequality $0 \leq v_j^\varepsilon \leq 1$, which follows from the maximum principle.

We verify that the condition (b_2) holds. Let $w \in M$, $\mathrm{G} = \mathrm{supp}(w)$. Then

$$\|P^\varepsilon u^\varepsilon\|_\varepsilon^2 = \int\limits_{\Omega_1'^\varepsilon} \left\{\sum_{i=1}^3 \left(\frac{\partial w}{\partial x_i}\right)^2 + \lambda w^2\right\} dx + \sum_{x_j^\varepsilon \in \mathrm{G}} \left\{ \int\limits_{B_{1j}^\varepsilon} \sum_{i=1}^3 \left(\frac{\partial v_j^\varepsilon}{\partial x_i}\right)^2 w_j^2 dx \right.$$
$$\left. + \int\limits_{G_j'^\varepsilon} \sum_{i,k=1}^3 g_\varepsilon^{ik} \frac{\partial v_j^\varepsilon}{\partial x_i}\frac{\partial v_j^\varepsilon}{\partial x_k} w_j^2 d\tilde{x} + \int\limits_{B_{2j}^\varepsilon} \mu_\varepsilon \sum_{i=1}^3 \left(\frac{\partial v_j^\varepsilon}{\partial x_i}\right)^2 w_j^2 d\tilde{x} \right\}$$
$$+ \Delta_1(\varepsilon) + \Delta_2(\varepsilon) + \Delta_G(\varepsilon), \tag{1.27}$$

where $\Delta_1(\varepsilon), \Delta_2(\varepsilon), \Delta_G(\varepsilon)$ are the remaining integrals over $\Omega_1'^\varepsilon, \Omega_2'^\varepsilon, G_j'^\varepsilon$, estimated as follows:

$$|\Delta_1(\varepsilon)| \leq c_1(w) \sum_{x_j^\varepsilon \in \mathrm{G}} (d_j^\varepsilon)^3 + c_2(w) \left\{ \sum_{x_j^\varepsilon \in \mathrm{G}} \int\limits_{B_{1j}} \left(\sum_{i=1}^3 \left|\frac{\partial v_j}{\partial x_i}\right| + |v_j - 1| \right) dx \right.$$
$$\left. + \sum_{x_j^\varepsilon \in \mathrm{G}} \int\limits_{R_{1j}^\varepsilon} \left(\sum_{i=1}^3 \left(\frac{\partial v_j^\varepsilon}{\partial x_i}\right)^2 + (v_j^\varepsilon - 1)^2 \frac{1}{(r_j^\varepsilon)^2} \right) dx + \sum_{x_j^\varepsilon \in \mathrm{G}} \int\limits_{B_{1j}^\varepsilon} (v_j^\varepsilon - 1)^2 dx \right\},$$

$$|\Delta_G(\varepsilon)| \leq c_3(w) \sum_{x_j^\varepsilon \in \mathrm{G}} \int\limits_{G_j'^\varepsilon} (v_j^\varepsilon)^2,$$

$$|\Delta_2(\varepsilon)| \leq c_4(w)\left\{\sum_{x_j^\varepsilon\in \mathrm{G}}\int\limits_{R_{2j}^\varepsilon}\left[\sum_{i=1}^{3}\left(\frac{\partial v_j^\varepsilon}{\partial x_i}\right)^2+(v_j^\varepsilon)^2\frac{1}{(r_j^\varepsilon)^2}\right]\mu_\varepsilon d\tilde{x}+\sum_{x_j^\varepsilon\in \mathrm{G}}\int\limits_{B_{2j}^\varepsilon}(v_j^\varepsilon)^2 d\tilde{x}\right\},$$

where $R_{kj}^\varepsilon = \left\{\tilde{x}=(x,k)\in\Omega_k^{\prime\varepsilon} : d_j^\varepsilon+\frac{r_j^\varepsilon}{4}<|x-x_j^\varepsilon|<d_j^\varepsilon+\frac{r_j^\varepsilon}{2}\right\}$.

We estimate the integrals over R_{kj}^ε by means of Lem. 1 and the integrals over B_{kj}^ε by means of Lem. 2. Taking into account that in $\Omega_2^{\prime\varepsilon} : d\tilde{x}=\mu_\varepsilon^{-3/2}dx_1dx_2dx_3$, we have

$$\Delta_1(\varepsilon)\leq c(w)\left\{\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3}\max_j d_j^\varepsilon(r_j^\varepsilon)^3+\left(\sum_{x_j^\varepsilon\in \mathrm{G}}(r_j^\varepsilon)^3\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3}\right)^{1/2}\right.$$

$$\left.\times\left(\max r_j^\varepsilon+\max(r_j^\varepsilon)^2\right)+\left(\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{r_j^{\varepsilon 3}}\max_j(r_j^\varepsilon)^2\right)+\left(\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3}\max_j(r_j^\varepsilon)^4\right)\right\},$$

$$\Delta_2(\varepsilon)\leq c(w)\left\{\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{r_j^{\varepsilon 3}}\max_j\frac{(r_j^\varepsilon)^2}{\mu_\varepsilon^{1/2}}+\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3}\max_j\frac{(r_j^\varepsilon)^4}{\mu_\varepsilon^{3/2}}\right\},$$

$$\Delta_G(\varepsilon)\leq c(w)\sum_{x_j^\varepsilon\in \mathrm{G}}\mathrm{meas}G_j^{\prime\varepsilon}.$$

By conditions (i)–(iii) and (v) of the theorem

$$\lim_{\varepsilon\to 0}(\Delta_1(\varepsilon)+\Delta_G(\varepsilon)+\Delta_2(\varepsilon))=0. \tag{1.28}$$

It follows from (1.27), (1.28) and the condition (iv) that

$$\lim_{\varepsilon\to 0}\|P^\varepsilon w\|_\varepsilon=\|w\|.$$

Thus the condition (b_2) holds.

Now we verify the condition (b_3). Let $w\in M$, $\mathrm{G}=\mathrm{supp}(w)$, the sequence γ^ε is such that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε and such that $\Pi^\varepsilon\gamma^\varepsilon\to\gamma(\varepsilon\to 0)$ weakly in H. Denote by $(u,v)_1$ the following scalar product in $H^1(\mathbb{R}^3)$

$$(u,v)_1=\int\limits_{\mathbb{R}^3}\left[\sum_{i=1}^{3}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}+\lambda uv\right]dx.$$

Integrating by parts, we have

$$(P^\varepsilon w,\gamma^\varepsilon)_\varepsilon=(w,\Pi^\varepsilon\gamma^\varepsilon)_1+I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon, \tag{1.29}$$

where

$$I_1^\varepsilon = -\sum_{x_j^\varepsilon \in \mathrm{G}} w_j^\varepsilon \Bigg[\int\limits_{R_{1j}^\varepsilon} \left(2\sum_{i=1}^{3} \frac{\partial v_j^\varepsilon}{\partial x_i} \frac{\partial \varphi_j^\varepsilon}{\partial x_i} + (v_j^\varepsilon - 1)\Delta\varphi_j^\varepsilon \right) \gamma^\varepsilon dx$$

$$+ \int\limits_{R_{2j}^\varepsilon} \mu_\varepsilon \left(2\sum_{i=1}^{3} \frac{\partial v_j^\varepsilon}{\partial x_i} \frac{\partial \varphi_j^\varepsilon}{\partial x_i} + v_j^\varepsilon \Delta\varphi_j^\varepsilon \right) \gamma^\varepsilon d\tilde{x} \Bigg];$$

$$I_2^\varepsilon = \lambda \sum_{x_j^\varepsilon \in \mathrm{G}} w_j^\varepsilon \Bigg[\int\limits_{B_{1j}^\varepsilon} (v_j^\varepsilon - 1)\varphi_j^\varepsilon \gamma^\varepsilon dx + \int\limits_{B_{2j}^\varepsilon} v_j^\varepsilon \varphi_j^\varepsilon \gamma^\varepsilon d\tilde{x} + \int\limits_{G_j'^\varepsilon} v_j^\varepsilon \gamma^\varepsilon d\tilde{x} \Bigg];$$

$$|I_3^\varepsilon| \le C(w) \Bigg[\sum_{x_j^\varepsilon \in \mathrm{G}} (d_j^\varepsilon)^3 \Bigg]^{1/2} \|\Pi^\varepsilon \gamma^\varepsilon\|_1 \le C(w) \left(\max_j (d_j^\varepsilon) \max_j (r_j^\varepsilon)^3 \sum_{x_j^\varepsilon \in \mathrm{G}} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \right)^{1/2}.$$

Since the norms $\|\Pi^\varepsilon \gamma^\varepsilon\|_1$ are uniformly bounded with respect to ε, it follows from (i), (ii):

$$\lim_{\varepsilon \to 0} I_3^\varepsilon = 0. \tag{1.30}$$

Estimating the integrals over R_{kj}^ε by means of Lem. 1 and the integrals over B_{kj}^ε by means of Lem. 2, we have the following estimates:

$$|I_1^\varepsilon| \le C \Bigg[\sum_{x_j^\varepsilon \in \mathrm{G}} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} |w_j^\varepsilon|^2 \Bigg]^{1/2} \|\Pi^\varepsilon \gamma^\varepsilon\|_{0\mathrm{G}} + C(w) \Bigg[\sum_{x_j^\varepsilon \in \mathrm{G}} \mu_\varepsilon^{1/2} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \Bigg]^{1/2} \|\gamma^\varepsilon\|_{0\varepsilon};$$

$$|I_2^\varepsilon| \le C(w) \Bigg\{ \Bigg[\max_j (r_j^\varepsilon)^4 \sum_{x_j^\varepsilon \in \mathrm{G}} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \Bigg]^{1/2}$$

$$+ \Bigg[\max_j \frac{(r_j^\varepsilon)^4}{\mu_\varepsilon^{3/2}} \sum_{x_j^\varepsilon \in \mathrm{G}} \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \Bigg]^{1/2} + \Big[\mathrm{meas} \bigcup G_j'^\varepsilon \Big]^{1/2} \Bigg\} \|\gamma^\varepsilon\|_{0\varepsilon},$$

where $\|\cdot\|_{0\mathrm{G}}$ denotes the norm in $L^2(G^3)$.

Since $\Pi^\varepsilon \gamma^\varepsilon$ converges weakly in $H^1(\mathbb{R}^3)$ to γ, from the embedding theorem we have

$$\lim_{\varepsilon \to 0} \|\Pi^\varepsilon \gamma^\varepsilon\|_{0\mathrm{G}} = \|\gamma\|_{0\mathrm{G}} \le \|\gamma\|.$$

Since $\|\gamma^\varepsilon\|_{0\varepsilon} \le \|\gamma^\varepsilon\|_\varepsilon \le C$, by the conditions (i)–(iii), (v)

$$\overline{\lim_{\varepsilon\to 0}}|I_1| \le C\|w\|_0\|\gamma\|_0 \le C\|w\|\|\gamma\|, \tag{1.31}$$

$$\lim_{\varepsilon\to 0} I_2 = 0. \tag{1.32}$$

It follows from (1.29)–(1.32) that the condition (b_3) holds.

We verify the condition (b_1). Let $w \in M$. Since

$$\|\Pi^\varepsilon P^\varepsilon w\| \le c\|P^\varepsilon w\| \to c\|w\|,$$

we have

$$\|\Pi^\varepsilon P^\varepsilon w\| \le C(w) \quad \text{uniformly with respect to } \varepsilon(\varepsilon < \varepsilon_0).$$

Moreover, in the same way as in (b_2), it is easy to show that $\Pi^\varepsilon P^\varepsilon w \to w$ strongly in $L^2(\mathbb{R}^3)$; thus the condition (b_1) also holds .

And, finally, verify that condition (c) holds. Let sequence $\gamma^\varepsilon \in H^\varepsilon$ be such that the norms $\|\gamma^\varepsilon\|_\varepsilon$ are uniformly bounded with respect to ε and $\Pi^\varepsilon\gamma^\varepsilon \to \gamma$ weakly in H. Then

$$|F^\varepsilon[\gamma^\varepsilon] - F[\gamma]| \;\le\; 2\left|\int_{\mathbb{R}^3}(Q^\varepsilon\mathcal{F}^\varepsilon \cdot \Pi^\varepsilon\gamma^\varepsilon - \mathcal{F}\gamma)dx\right| + 2\left|\int_{\Omega_2'^\varepsilon \bigcup_j G_j'^\varepsilon} \mathcal{F}^\varepsilon\gamma^\varepsilon d\tilde{x}\right|. \tag{1.33}$$

It follows from (1.33) and the condition (vi$'$) that $|F^\varepsilon[\gamma^\varepsilon] - F[\gamma]| \to 0(\varepsilon \to 0)$; so the condition (c) holds.

Thus all the conditions of Th. 3 hold. Hence $\Pi^\varepsilon u^\varepsilon \to u$ weakly in H. Therefore by the embedding theorem $\Pi^\varepsilon u^\varepsilon \to u$ strongly in $L^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^3$. Moreover, for any bounded domain $\mathrm{G} \subset \mathbb{R}^3$

$$\sum_{x_j^\varepsilon\in \mathrm{G}} \mathrm{meas} B(D_j^\varepsilon) = C\sum_{x_j^\varepsilon\in \mathrm{G}}(d_j^\varepsilon)^3 \le C\sum_{x_j^\varepsilon\in \mathrm{G}}\frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3}\max_j(r_j^\varepsilon)^3\max_j d_j^\varepsilon \to 0, \varepsilon \to 0. \tag{1.34}$$

Then by (1.34) $u^\varepsilon \to u$ in the sense (1.9). The theorem is proved.

1.3. Asymptotic Behavior of the Solution of the Nonstationary Problem

We consider the following problem for the complex values of the parameter λ:

$$-\Delta^\varepsilon u^\varepsilon + \lambda^2 u^\varepsilon = \lambda f^\varepsilon + g^\varepsilon, \quad \tilde{x} \in M^\varepsilon, \tag{1.35}$$

$$u^\varepsilon \in H^1(M^\varepsilon). \tag{1.36}$$

The solution $u(\tilde{x},\lambda)$ of this problem is the Laplace transform of the solution of the problem (1.2)–(1.4). It is proved that for $\lambda^2 > 0$ (that is $\lambda \in \mathbb{R}\backslash\{0\}$) the solution of the problem (1.35)–(1.36) converges in the sense of (1.8) to the solution of the problem

$$-\Delta u + \lambda^2 u + V(x)u = \lambda f + g, \quad x \in \mathbb{R}^3, \tag{1.37}$$

$$u \in H^1(\mathbb{R}^3). \tag{1.38}$$

Theorem 2′. *Let $\Lambda = \{\lambda : \mathrm{Re}\lambda \geq \delta > 0\}$ and all conditions of Th. 2 hold. Then the solution of the problem (1.35)–(1.36) $u^\varepsilon(\cdot,\lambda)$ is a holomorphic function in Λ^*, the following estimate holds:*

$$\|Q^\varepsilon u^\varepsilon(\cdot,\lambda)\|_0 \leq C, \tag{1.39}$$

where C does not depend on λ and ε, and $u^\varepsilon(\cdot,\lambda)$ converges (uniformly on each bounded subset of Λ) in the sense of (1.8) to the solution $u(\cdot,\lambda)$ of the problem (1.37)–(1.38). Besides, $u(\cdot,\lambda)$ is holomorphic in Λ and satisfies the estimate (1.39).

P r o o f. Since $-\Delta^\varepsilon$ induces a nonnegative selfadjoint operator in $L^2(M^\varepsilon)$, then for all $\lambda \in \Lambda$

$$u^\varepsilon = \int\limits_0^\infty \frac{dE_t(\lambda f^\varepsilon + g^\varepsilon)}{t+\lambda^2},$$

where E_t is a resolution of identity of the operator $-\Delta^\varepsilon$.

Then

$$\|u^\varepsilon\|_{0_\varepsilon} \leq \frac{|\lambda|\cdot\|f^\varepsilon\|_{0_\varepsilon} + \|g^\varepsilon\|_{0_\varepsilon}}{\mathrm{dist}\,(\lambda^2,(-\infty,0])}\,, \tag{1.40}$$

$$\mathrm{dist}\left(\lambda^2,(-\infty,0]\right) = \begin{cases} |\mathrm{Im}\lambda^2|, & |\arg\lambda| > \pi/4 \\ |\lambda^2|, & |\arg\lambda| \leq \pi/4 \end{cases}$$

$$= \begin{cases} |2\mathrm{Im}\lambda\mathrm{Re}\lambda|, & |\arg\lambda| > \pi/4 \\ |\lambda^2|, & |\arg\lambda| \leq \pi/4 \end{cases} \geq \begin{cases} 2\delta|\lambda|\sin\pi/4, & |\arg\lambda| > \pi/4 \\ \delta|\lambda|, & |\arg\lambda| \leq \pi/4 \end{cases}. \tag{1.41}$$

It follows from (1.40),(1.41) and the condition (vi) of Th. 1 that (1.39) holds.

Moreover, $Q^\varepsilon u^\varepsilon(\cdot,\lambda)$ and $u(\cdot,\lambda)$ are holomorphic functions for $\mathrm{Re}\lambda \neq 0$ (as the resolvent is holomorphic outside of the operator spectrum).

Since $Q^\varepsilon u^\varepsilon(x,\lambda)$ converges to $u(x,\lambda)$ in $L^2(\mathbb{R}^3)$ for $\lambda \in \mathbb{R}\backslash\{0\}$, then by the Vitaly theorem, when $\lambda \in \Lambda$, then $Q^\varepsilon u^\varepsilon(x,\lambda)$ converges (uniformly on

* As a function of complex variable with values in $L^2(M^\varepsilon)$.

each bounded subset of Λ) in $L^2(\mathbb{R}^3)$ to the function $U(x,\lambda)$. Moreover, $U(x,\lambda)$ is holomorphic and satisfies the estimate (1.39). Since $U(x,\lambda)=u(x,\lambda)$ for $\lambda\in\mathbb{R}\backslash\{0\}$, then by the uniqueness theorem $U(x,\lambda)=u(x,\lambda)$ for $\lambda\in\Lambda$. The theorem is proved.

First, we prove that $Q^\varepsilon u^\epsilon(x,t)$ converges to $u(x,t)$ weakly in $L^2(\mathbb{R}^3\times[0,T])$. Let $\mathcal{D}$ be a set of functions of the form

$$g(x,t)=\varphi(x)\psi(t),$$

where $\varphi(t)\in L^2(\mathbb{R}^3)$, $\psi(x)\in C_0^2[0,T]$.

We note, that in view of properties of the Laplace transform, since the solution $u^\varepsilon(\tilde{x},\lambda)$ of the problem (1.13)–(1.14) is the Laplace transform for $u^\varepsilon(\tilde{x},t)$, then $\frac{u^\varepsilon(\tilde{x},\lambda)}{\lambda^2}$ is the Laplace transform for $\int\limits_0^t\int\limits_0^s u^\varepsilon(\tilde{x},\tau)d\tau ds$, and one has

$$\int\limits_0^t\int\limits_0^s u^\varepsilon(\tilde{x},\tau)d\tau ds=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{u^\varepsilon(\tilde{x},\lambda)}{\lambda^2}e^{\lambda t}d\lambda.$$

Integrating by parts we have

$$\begin{aligned}\int\limits_0^T\int\limits_{\mathbb{R}^3} Q^\varepsilon u^\varepsilon(x,t)\varphi(x)\psi(t)dxdt&=\int\limits_{\mathbb{R}^3}\varphi(x)\int\limits_0^T\left(\int\limits_0^t\int\limits_0^s Q^\varepsilon u^\varepsilon(x,\tau)d\tau ds\right)\psi'' dtdx\\&=\int\limits_{\mathbb{R}^3}\varphi(x)\int\limits_0^T\frac{1}{2\pi i}\left(\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{Q^\varepsilon u^\varepsilon(x,\lambda)}{\lambda^2}e^{\lambda t}d\lambda\right)\psi'' dtdx\\&=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{\lambda^2}\int\limits_{\mathbb{R}^3}\varphi(x)Q^\varepsilon u^\varepsilon(x,\lambda)dx\int\limits_0^T e^{\lambda t}\psi'' dt\right)d\lambda.\end{aligned}\tag{1.42}$$

Here the change of order of integration is valid since, in view of the following inequalities:

$$\int\limits_{\mathbb{R}^3}|Q^\varepsilon u^\varepsilon(x,\lambda)||\varphi(x)|dx\le\|Q^\varepsilon u^\varepsilon\|_0\|\varphi\|_0\le C(\varphi),$$

$$\int\limits_0^T|e^{\lambda t}||\psi''|dt\le C(\psi),\ \operatorname{Re}\lambda=\sigma,$$

the last repeated integral in (1.42) converges absolutely.

By Theorem 2,

$$\int_{\mathbb{R}^3} Q^\varepsilon u^\varepsilon(x,\lambda)\varphi(x)dx \to \int_{\mathbb{R}^3} u(x,\lambda)\varphi(x)dx.$$

Since the norms $\|Q^\varepsilon u^\varepsilon(\cdot,\lambda)\|_{0_\varepsilon}$ are uniformly bounded with respect to ε and the convergence is uniform on each bounded subset of Λ, we may pass to the limit under the integral sign in (1.42).

On the other hand, similarly, we have

$$\int_0^T\int_{\mathbb{R}^3} u(x,t)\varphi(x)\psi(t)dxdt = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{1}{\lambda^2}\int_{\mathbb{R}^3}\varphi(x)u(x,\lambda)dx\int_0^T e^{\lambda t}\psi'' dt\right)d\lambda.$$

We will prove later that the set $\{u^\varepsilon\}$ is uniformly bounded in $H^1(M^\varepsilon\times[0,T])$; hence, since $\mathcal{D}$ is a dense subset in $L^2(\mathbb{R}^3\times[0,T])$, then $Q^\varepsilon u^\varepsilon(x,t)$ converges to $u(x,t)$ weakly in $L^2(\mathbb{R}^3\times[0,T])$. In view of (1.34), $\Pi^\varepsilon u^\varepsilon(x,t)$ converges to $u(x,t)$ weakly in $L^2(\mathbb{R}^3\times[0,T])$, where the operator $\Pi^\varepsilon: H^1(M^\varepsilon)\to H^1(\mathbb{R}^3)$ is defined in the proof of Th. 2.

Prove that the set $\{\Pi^\varepsilon u^\varepsilon\}$ is uniformly bounded in $H^1(M^\varepsilon\times[0,T])$. In view of (1.2), it is easy to see that the following equation holds

$$\int_{M^\varepsilon}\left[\left(\frac{\partial u^\varepsilon(\tilde{x},t)}{\partial t}\right)^2+\sum_{i,k=1}^3 g_\varepsilon^{ik}\frac{\partial u^\varepsilon(\tilde{x},t)}{\partial x_i}\frac{\partial u^\varepsilon(\tilde{x},t)}{\partial x_k}\right]d\tilde{x}$$

$$=\int_{M^\varepsilon}\left[(g^\varepsilon(\tilde{x}))^2+\sum_{i,k=1}^3 g_\varepsilon^{ik}(\tilde{x})\frac{\partial f^\varepsilon(\tilde{x})}{\partial x_i}\frac{\partial f^\varepsilon(\tilde{x})}{\partial x_k}\right]d\tilde{x}. \tag{1.43}$$

In addition, $\forall t<T$

$$\int_{M^\varepsilon}(u^\varepsilon)^2 d\tilde{x} = \int_{M^\varepsilon}\left(\int_0^t\frac{\partial u^\varepsilon(\tilde{x},t)}{\partial t}dt+f^\varepsilon(\tilde{x})\right)^2 d\tilde{x}$$

$$\le 2\int_{M^\varepsilon}\left(T\int_0^T\left(\frac{\partial u^\varepsilon(\tilde{x},t)}{\partial t}\right)^2 dt+(f^\varepsilon(\tilde{x}))^2\right)d\tilde{x}. \tag{1.44}$$

By (1.43), (1.44) and the condition (vi) it follows that the set $\{u^\varepsilon\}$ is uniformly bounded in $H^1(M^\varepsilon\times[0,T])$, so in view of (1.25), the set $\{\Pi^\varepsilon u^\varepsilon\}$ is uniformly bounded in $H^1(\mathbb{R}^3\times[0,T])$. Then for any bounded domain $\Omega\subset\mathbb{R}^3$ the set $\{\Pi^\varepsilon u^\varepsilon\}$ is compact in $L^2(\Omega\times[0,T])$. Hence $\Pi^\varepsilon u^\varepsilon\to u$ strongly in $L^2(\Omega\times[0,T])$. Hence u^ε converges to u in the sense of (1.9). Theorem 1 is proved.

1.4. Example

We consider a concrete example of the Riemannian manifold M^ε. Let D_j^ε be a system of balls with the radius $d^\varepsilon = a\varepsilon^3$ and the centers at x_j^ε periodically distributed in $\Omega \subset \mathbb{R}^3$, that is

$$D_j^\varepsilon = \Big\{ x \in \mathbb{R}^3 : \ |x - x_j^\varepsilon| \le d^\varepsilon, \ x_j^\varepsilon = \varepsilon \sum_{i=1}^{3} e_i z_i^j \Big\},$$

where $r^\varepsilon = \varepsilon$ is a period, $\{e_i, i = 1, 2, 3\}$ is an orthonormal basis in $\mathbb{R}^3$, $z_i^j \in \mathbb{Z}$ and $D_j^\varepsilon \subset \Omega, j = 1 \ldots N(\varepsilon)$.

G_j^ε is the "pipe", that is $G_j^\varepsilon = \Big\{ (\varphi, \psi, z) : \ \varphi \in [0, 2\pi], \psi \in [-\pi/2, \pi/2],$ $z \in [0, 1] \Big\} = S^3 \times [0, 1]$.

On M^ε we introduce a metric $g_{ik}^\varepsilon(\tilde{x})$ that coincides with the Euclidean metric in $\Omega_1'^\varepsilon$, increases in $\Omega_2'^\varepsilon$ in the sense of (1.1), and in the "pipe" $G_j'^\varepsilon$ we define it by the following formula for the square of the element of length:

$$ds^2 = q_j^\varepsilon dz^2 + (d^\varepsilon)^2 (\cos^2 \psi d\varphi^2 + d\psi^2), \tag{1.45}$$

where $q_j^\varepsilon > 0$.

We introduce the spherical coordinates (r, φ, ψ) in B_{kj}^ε

$$\begin{cases} x_1 = r \sin\varphi \cos\psi, \\ x_2 = r \cos\varphi \cos\psi, \\ x_3 = r \sin\psi, \quad r \in [d^\varepsilon, r^\varepsilon/2], \psi \in [-\pi/2, \pi/2], \varphi \in [0, 2\pi]. \end{cases} \tag{1.46}$$

We will state the function v_j^ε, supposing that it depends neither on φ, nor on ψ. We denote

$$v_j^\varepsilon = \begin{cases} v_1, \text{ if } \tilde{x} \in B_{1j}^\varepsilon, \\ v_g, \text{ if } \tilde{x} \in G_j'^\varepsilon, \\ v_2, \text{ if } \tilde{x} \in B_{2j}^\varepsilon. \end{cases}$$

Then v_1, v_2, v_g satisfy the equations

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_1}{\partial r} \right) = 0, & d^\varepsilon < r < r^\varepsilon/2, \\ \frac{\partial^2 v_g}{\partial z^2} = 0, & 0 < z < 1, \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_2}{\partial r} \right) = 0, & d^\varepsilon < r < r^\varepsilon/2; \end{cases}$$

the boundary conditions

$$\begin{cases} v_1(r^\varepsilon/2) = 1, \\ v_2(r^\varepsilon/2) = 0; \end{cases}$$

and conditions on the boundaries of the upper and the lower sheets

$$\begin{cases} v_1(d^\varepsilon) = v_g(0), \\ v_2(d^\varepsilon) = v_g(1); \end{cases} \qquad \begin{cases} \frac{\partial v_1}{\partial r} = -\frac{1}{\sqrt{q_j^\varepsilon}}\frac{\partial v_g}{\partial z}, \\ \frac{1}{\sqrt{\mu}_\varepsilon}\frac{\partial v_2}{\partial r} = \frac{1}{\sqrt{q_j^\varepsilon}}\frac{\partial v_g}{\partial z}. \end{cases}$$

Hence we have

$$v_j^\varepsilon = \begin{cases} \frac{a_1}{r} + b_1, & \tilde{x} \in B_{1j}^\varepsilon, \\ Az + B, & \tilde{x} \in G_j^{'\varepsilon}, \\ \frac{a_2}{r} + b_2, & \tilde{x} \in B_{2j}^\varepsilon, \end{cases}$$

where

$$a_1 = -d^\varepsilon \left(\frac{\sqrt{q_j^\varepsilon}}{d^\varepsilon} + \left(1 + \sqrt{\mu}_\varepsilon\right)\left(1 - \frac{2d^\varepsilon}{r^\varepsilon}\right) \right)^{-1}, \quad A = a_1 \frac{\sqrt{q_j^\varepsilon}}{(d^\varepsilon)^2}, \quad a_2 = -a_1\sqrt{\mu}_\varepsilon\,.$$

If q_j^ε can be represented in the form $q_j^\varepsilon = q^\varepsilon l_j^2$, $l_j = l(x_j^\varepsilon)$, where $l(x)$ is a continuous function on Ω and there exists the limit $q = \lim\limits_{\varepsilon\to 0} \dfrac{\sqrt{q}^\varepsilon}{d^\varepsilon} < \infty$, then

$$V_j^\varepsilon = \int\limits_{\tilde{G}_j^\varepsilon} \sum_{i,k=1}^{3} g_\varepsilon^{ik} \frac{\partial v_j^\varepsilon}{\partial x_i}\frac{\partial v_j^\varepsilon}{\partial x_k} d\tilde{x} = \int\limits_{d^\varepsilon}^{r^\varepsilon/2} \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \left(\frac{a_1}{r^2}\right)^2 r^2 \cos\psi d\psi d\varphi dr$$

$$+ \int\limits_0^1 \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \frac{A^2}{\sqrt{q_j^\varepsilon}} (d_j^\varepsilon)^2 \cos\psi d\psi d\varphi dz + \int\limits_{d^\varepsilon}^{r^\varepsilon/2} \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} {\mu_\varepsilon}^{-1/2} \left(\frac{a_2}{r^2}\right)^2 r^2 \cos\psi d\psi d\varphi dr$$

$$= 4\pi(a_1^2 + {\mu_\varepsilon}^{-1/2} a_2^2)\left(\frac{1}{d^\varepsilon} - \frac{2}{r^\varepsilon}\right) + 4\pi \frac{A^2 (d^\varepsilon)^2}{\sqrt{q_j^\varepsilon}} = 4\pi d^\varepsilon \left(\frac{l_j \sqrt{g}^\varepsilon}{d^\varepsilon} + 1\right)^{-1} (1 + \bar{o}(1)).$$

Since x_j^ε are distributed periodically in Ω and $d^\epsilon = a(r^\epsilon)^3$, it follows that $V(x) = \dfrac{4a\pi}{ql(x)+1}\chi_\Omega(x)$ ($\chi_\Omega(x)$ is a characteristic function of Ω), and the averaged equation has the form:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{4a\pi\chi_\Omega}{ql(x)+1} u = 0.$$

2. Case of the Delay of Proper Time

We consider the manifold $M_4^\varepsilon = M_3^\varepsilon \times \mathbb{R}$, where M_3^ε is a particular case of the manifold M^ε considered in Sect. 1.4 (without assumption that x_j^ε are distributed periodically in Ω).

Introduce a metric on M_3^ε similarly to 1.4 and define a metric in M_4^ε by the following formula:

$$ds^2 = [c_\varepsilon(\tilde{x})]^2\, dt^2 - \sum_{i,k=1}^{3} g_{ik}^\varepsilon dx_i dx_k,$$

where

$$c_\varepsilon(\tilde{x}) = \begin{cases} 1, \tilde{x} \in \Omega_1^{\prime\varepsilon} \bigcup\limits_{j=1}^{N(\varepsilon)} G_j^{\prime\varepsilon}, \\ \tilde{c}_\varepsilon, \tilde{x} \in \Omega_2^{\prime\varepsilon}, \end{cases} \qquad \tilde{c}_\varepsilon \to 0,\ \varepsilon \to 0,$$

and $g^\varepsilon(\tilde{x}) = \{g_{ik}^\varepsilon, i,k = 1,2,3\}$ is a metric tensor on M_3^ε.

We consider the Cauchy problem on M_4^ε

$$\Box^\varepsilon u^\varepsilon \equiv \frac{1}{c_\varepsilon(\tilde{x})^2}\frac{\partial^2 u^\varepsilon}{\partial t^2} - \frac{1}{c_\varepsilon(\tilde{x})\sqrt{G}^\varepsilon}\sum_{i,k=1}^{3}\frac{\partial}{\partial x_i}\left(g_\varepsilon^{ik} c_\varepsilon(\tilde{x})\sqrt{G}^\varepsilon \frac{\partial u^\varepsilon}{\partial x_k}\right) = 0, \tag{2.1}$$

$$u^\varepsilon(\tilde{x}, 0) = f^\varepsilon, \tag{2.2}$$

$$u_t^\varepsilon(\tilde{x}, 0) = g^\varepsilon. \tag{2.3}$$

($G^\varepsilon = \det g_{ik}^\varepsilon$, g_ε^{ik}, $i,k = 1,2,3$ are the components of the tensor inverse to g_{ik}^ε.)

Denote

$$V_j^\varepsilon = \int\limits_{\tilde{G}_j^\varepsilon} \sum_{i,k=1}^{3} c_\varepsilon(\tilde{x}) g_\varepsilon^{ik} \frac{\partial v_j^\varepsilon}{\partial x_i}\frac{\partial v_j^\varepsilon}{\partial x_k} d\tilde{x}, \tag{2.4}$$

where v_j^ε is the solution of the problem

$$\sum_{i,k=1}^{3} \frac{1}{\sqrt{G}^\varepsilon}\frac{\partial}{\partial x_i}\left(g_\varepsilon^{ik}\sqrt{G}^\varepsilon c_\varepsilon(\tilde{x})\frac{\partial v}{\partial x_k}\right) = 0, \quad \tilde{x} \in \tilde{G}_j^\varepsilon, \tag{2.5}$$

$$v = 1, \tilde{x} \in S_{1j}^\varepsilon, \tag{2.6}$$

$$v = 0, \tilde{x} \in S_{2j}^\varepsilon. \tag{2.7}$$

It follows from the form of metric in $G_j^{\prime\varepsilon}$ (1.45), that

$$V_j^\varepsilon = 4\pi(a_1^2 + \tilde{c}_\varepsilon a_2^2)\left(\frac{1}{d_j^\varepsilon} - \frac{1}{\rho_j^\varepsilon}\right) + 4\pi\frac{A^2(d_j^\varepsilon)^2}{\sqrt{q_j^\varepsilon}},$$

where $\rho_j^\varepsilon = d_j^\varepsilon + r_j^\varepsilon/2$,

$$a_1 = -\tilde{c}_\varepsilon d_j^\varepsilon \left(\frac{\tilde{c}_\varepsilon \sqrt{q_j^\varepsilon}}{d_j^\varepsilon} + \Big(\tilde{c}_\varepsilon + 1\Big)\Big(1 - \frac{d_j^\varepsilon}{\rho_j^\varepsilon}\Big) \right)^{-1}, \quad A = a_1 \frac{\sqrt{q_j^\varepsilon}}{(d_j^\varepsilon)^2}, \quad a_2 = -a_1 \frac{1}{\tilde{c}_\varepsilon}.$$

Introduce the following generalized function

$$V^\varepsilon(x) = \sum_{j=1}^{N(\varepsilon)} V_j^\varepsilon \delta(x - x_j^\varepsilon) \tag{2.8}$$

and let the following limit exist in $D'(\mathbb{R}^3)$

$$\lim_{\varepsilon\to 0} V^\varepsilon(x) = V(x), \tag{2.9}$$

where $V(x)$ is a measurable bounded nonnegative function.

Theorem 4. *Suppose the conditions* (i),(iv) *of Th. 1 hold and:*
(ii$'$) *for any domain* $\mathrm{G} \subset \mathbb{R}^3$

$$\overline{\lim_{\varepsilon\to 0}} \sum_{x_j^\varepsilon \in \mathrm{G}} \tilde{c}_\varepsilon^2 \frac{(d_j^\varepsilon)^2}{(r_j^\varepsilon)^3} \le C_1 \mathrm{meas} \mathrm{G},$$

and $r_j^\varepsilon > C_0 d_j^\varepsilon$ *(here* $0 < C_0, C_1 < \infty$*);*

(iii$'$) $\displaystyle\lim_{\varepsilon\to 0} \max_j \frac{(r_j^\varepsilon)^4}{\tilde{c}_\varepsilon^3} = 0$;

(vi$'$) *the norms* $\|f^\varepsilon\|_{1\varepsilon}$ *are uniformly bounded with respect to* ε*; when* $\varepsilon \to 0$, $f^\varepsilon(\tilde{x})$ *and* $g^\varepsilon(\tilde{x})$ *converge in the sense of(1.8) to the functions* $f \in H^1(\mathbb{R}^3)$ *and* $g \in L^2(\mathbb{R}^3)$*, respectively, and*

$$\int_{\Omega_2'^\varepsilon \bigcup_j G_j'^\varepsilon} \frac{|f^\varepsilon|^2 + |g^\varepsilon|^2}{c_\varepsilon(\tilde{x})} d\tilde{x} \to 0, \ \varepsilon \to 0.$$

Then the solution of the problem (2.1)–(2.3) $u^\varepsilon(\tilde{x}, t)$ *converges in the sense of (1.9) to the solution* $u(x,t)$ *of the problem (1.10)–(1.12), where* $V(x)$ *is defined by (2.8)–(2.9).*

P r o o f. The theorem is proved in the same way as Th. 1. But here, for the solution v_j^ε of the problem (2.5)–(2.7), a stronger estimate than in Lem. 1 is used. Namely, as it appears from an explicit form of the function v_j^ε, the following inequality holds:

$$|D^\alpha(v_j^\varepsilon(\tilde{x}) - 1)| \le C \frac{\tilde{c}_\varepsilon d_j^\varepsilon}{|x - x_j^\varepsilon|^{1+|\alpha|}}, \quad \tilde{x} = (x, 1) \in B_{1j}^\varepsilon, \tag{2.10}$$

where $|\alpha| = 0, 1$, and C does not depend on ε.

As for the rest, the theorem is proved in the same way as Th. 1.

R e m a r k. This result is obtained for the discontinuous metric tensor $g^{\varepsilon}(\tilde{x})$. However, $g^{\varepsilon}(\tilde{x})$ can be approximated by the smooth tensor $g^{\varepsilon}_{\delta}(\tilde{x})$ that coincides with $g^{\varepsilon}(\tilde{x})$ outside of the small neighborhood of $\partial D^{\varepsilon}_{kj}$. We introduce the local coordinates (x_1, x_2, x_3) in the neighborhood of the point $\tilde{x} \in \partial D^{\varepsilon}_{kj}$, such that $r = d^{\varepsilon}_j + |x_1|(x_1 \leq 0)$, $z = x_1(x_1 \geq 0)$, $x_2 = \varphi$, $x_3 = \psi$ and set

$$ds^2 = q^{\varepsilon}_{j_\delta}(x_1)dx_1^2 + \left[(d^{\varepsilon}_j)^2(1 - \varphi^{\varepsilon}_{j_\delta}(x_1)) + (d^{\varepsilon}_j + |x_1|)^2\varphi^{\varepsilon}_{j_\delta}(x_1)\right](\cos^2 x_3 dx_2^2 + dx_3^2),$$

where $q^{\varepsilon}_{j_\delta}(x_1)$ is a smooth nonnegative function equal to 1 for $x_1 \leq 0$ and q^{ε}_j for $x_1 \geq \delta > 0$, $\varphi^{\varepsilon}_{j_\delta}(x_1) = (q^{\varepsilon}_{j_\delta}(x_1) - q^{\varepsilon}_j)/(1 - q^{\varepsilon}_j)$. Then $g^{\varepsilon}(\tilde{x}) = g^{\varepsilon}_{\delta}(\tilde{x})$ for $x_1 \notin (0, \delta)$.

In the same way we can approximate $c_{\varepsilon}(\tilde{x})$ by the function $c_{\varepsilon_\delta}(\tilde{x})$. Similarly, we introduce the local coordinates (x_1, x_2, x_3) in the neighborhood of the point $\tilde{x} \in \partial D^{\varepsilon}_{2j}$ and set

$$c_{\varepsilon_\delta}(\tilde{x}) = 1 - \varphi^{\varepsilon}_{j_\delta}(x_1) + \tilde{c}_{\varepsilon}\varphi^{\varepsilon}_{j_\delta}(x_1).$$

We suppose $\delta = \bar{o}(d^{\varepsilon}_j)$. Then the function $V(x)$, calculated by the formulae (2.4),(2.8),(2.9), but with a tensor $g^{\varepsilon}_{\delta}(\tilde{x})$ and a function $c_{\varepsilon_\delta}(\tilde{x})$, will be equal to the function $V(x)$ calculated with a tensor $g^{\varepsilon}(\tilde{x})$ and a function $c_{\varepsilon}(\tilde{x})$.

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