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On Stability of Polynomially Bounded Operators

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We prove that if T is a polynomially bounded operator and the peripheral spectrum of T has zero measure, then $T^n x \to 0$ for all x in X if and only if T^* has no nontrivial invariant subspace on which it is invertible and doubly power bounded.

Key words: polynomially bounded operator, Banach space, invariant subspace.
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1. Introduction

Let X be a Banach space. A linear bounded operator T on X is called *polynomially bounded* if there exists a constant M such that

$$\|p(T)\| \le M \sup_{|z|\le 1} \|p(z)\|, \tag{1}$$

for every polynomial p.

It is a well known theorem of Sz. Nagy and C. Foias [8] that if X is a Hilbert space and T is a completely nonunitary contraction on X with spectrum $\sigma(T)$ such that $m(\sigma(T) \cap \Gamma) = 0$, where Γ denotes the unit circle and m is the Lebesgue measure on Γ, then $\|T^n x\| \to 0$ as $n \to \infty$, for all x in X. According to von Neumann's inequality (see e.g. [8]), every contraction operator T satisfies (1) with $M = 1$, hence every contraction is a power bounded operator. However, G. Pisier [9] has shown that not every polynomially bounded operator on a Hilbert space is similar to a contraction. The proof of the above result of Sz. Nagy and C. Foias uses the theory of unitary dilations of contractions and, therefore, cannot be extended to polynomially bounded operators on a Hilbert space.

In this note, we extend the Nagy–Foias theorem to polynomially bounded operators on Banach spaces.

Throughout the paper, D is the open unit dist, Γ is the unit circle and $A(D)$ is the disk algebra of functions analytic in D and continuous in $\overline{D}$.

2. The Limit Isometry

Let T be a power bounded operator on a Banach space X, i.e., T satisfies the condition $\sup_{n\geq 0}\|T^n\| < \infty$. By introducing the equivalent norm $|||x||| = \sup_{n\geq 0}\|T^n x\|$, we can always assume, without loss of generality, that T is a contraction. This implies that $\lim_{n\to\infty}\|T^n x\|$ exists for all x in X.

The following construction associates with T another Banach space E, a natural homomorphism Q from X to E and an isometry V on E such that $QT = VQ$ and $\sigma(V) \subset \sigma(T)$. This construction has proved useful in many investigations on the asymptotic behavior of semigroups of operators (see [2, 7, 10–13]).

Lemma 1. *Let T be a power bounded on a Banach space X. There exists a Banach space E, a bounded linear map Q of X into E with dense range, and an isometric operator V on E, with the following properties:*

1) $Qx = 0$ if and only if $\inf_{n\geq 0}\|T^n x\| = 0$;

2) $QT = VQ$ $(s \in S)$;

3) $\sigma(V) \subset \sigma(T), P\sigma(V^*) \subset P\sigma(T^*)$.

The operator V in Lem. 1 is called *the limit isometry* of T. Recall the construction of E, Q and V. First, a seminorm on X is defined by

$$l(x) = \lim_{n\to\infty}\|T^n x\|,\ x \in X.$$

Let $L = ker(l) = \{x \in X : l(x) = 0\}$. Consider the quotient space $\widehat{X} = X/L$, the canonical homomorphism $Q : X \to \widehat{X}$, $Qx = \hat{x}$, and define a norm in $\widehat{X}$ by

$$\hat{l}(\hat{x}) = l(x),\ x \in X.$$

The operators T generate the corresponding operator $\widehat{T}$ on $\widehat{X}$ in the natural way, namely

$$\widehat{T}\hat{x} := \widehat{Tx},\ x \in X.$$

Clearly, $\widehat{T}$ is an isometric operator on the normed space $\widehat{X}$, since

$$\hat{l}(\widehat{T}\hat{x}) = \lim_{n\to\infty}\|T^n(Tx)\| = \hat{l}(\hat{x}),\ x \in X.$$

We denote by E the completion of $\widehat{X}$ in the norm $\hat{l}$, and by V the continuous extension of $\widehat{T}$ from $\widehat{X}$ to E. All properties 1)–3) can be verified directly.

An operator T is called *stable*, if the discrete semigroup $\{T^n\}_{n\geq 0}$ is stable, i.e., $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$. Note that in the above construction the subspace E is nonzero if and only if T is nonstable. On the other hand, if $\inf_{n\geq 0} \|T^n x\| > 0$ for all $x \in X,\ x \neq 0$, then T is said to be of *class* C_1. From $\sigma(V) \subset \sigma(T)$ it follows that if $\sigma(T)$ does not contain the unit circle, then $\sigma(V)$ also does not contain the unit circle, so that V is an invertible isometry.

3. Stability of $\{T^n\}$

An important property of polynomially bounded invertible isometries is that they possess a functional calculus for continuous functions on their spectra.

Lemma 2. *Let V be a polynomially bounded invertible isometry on a Banach space E. Then the algebra $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$.*

P r o o f. It was shown in [6] that there is a homomorphism $\varphi : C(\Gamma) \to L(E)$ such that $\|\varphi\| \leq M$, i.e., there is a functional calculus on $C(\Gamma)$ which satisfies: $\|f(T)\| \leq M\|f\|_\infty$. Moreover, $f(T)$ is completely determined by its values on $\sigma(V)$, and the spectral mapping theorem holds: $\sigma(f(V)) = f(\sigma(V))$. Therefore, the functional calculus can be defined for $C(\sigma(V))$, and we have

$$\sup_{\lambda\in\sigma(V)} |f(\lambda)| \leq \|f(V)\| \leq M \sup_{\lambda\in\sigma(V)} |f(\lambda)|,$$

i.e., the homomorphism is in fact an isomorphism.

Now let T be a polynomially bounded operator on a Banach space X. Assume that T is not stable, i.e., there exists $x \in X$ such that $\|T^n x\|$ does not converge to 0. Then the Banach space E, defined in Lemma 1, is nonzero, and we can speak about the limit isometry V. Assume that V is invertible (which holds, e.g., if T has a dense range or $\sigma(T)$ does not contain the whole unit circle).

Lemma 3. *Let T be polynomially bounded, nonstable, and let E and V be as in Lemma 1 such that V is an invertible isometry. Then there exists a family of measures μ_{z,z^*}, where $z \in E, z^* \in E^*$, such that*

$$\langle f(V)z, z^*\rangle = \int_{\sigma(V)} f(\lambda) d\mu_{z,z^*}(\lambda) \tag{2}$$

for every function f in $C(\sigma(V))$.

P r o o f. Since T also is polynomially bounded, it follows easily that V also is polynomially bounded. In fact, we have

$$\begin{aligned} \hat{l}(p(\widehat{T})\hat{x}) &= \lim_{n\to\infty} \|T^n p(T)x\| \leq \|p(T)\| \lim_{n\to\infty} \|T^n x\| \\ &= \|p(T)\|\hat{l}(\hat{x}) \leq M \sup_{|z|\leq 1} |p(z)|\hat{l}(\hat{x}), \end{aligned}$$

which implies $\|p(\widehat{T})\| \leq M \sup_{|z|\leq 1} |p(z)|$, hence $\|p(V)\| \leq M \sup_{|z|\leq 1} |p(z)|$, i.e., V is polynomially bounded. Lemma 2 implies that $\mathcal{A}(V)$ is isomorphic to $C(\sigma(V))$. Therefore, for each $z \in E, z^* \in E^*$, the mapping $f \mapsto \langle f(V)z, z^*\rangle$ is a continuous linear functional on $C(\sigma(V))$. Hence, by Riesz's theorem, for every $z \in E, z \in E^*$, there exists a measure μ_{z,z^*} on $\sigma(V)$ such that (2) holds.

Note that, in general, V does not have a spectral measure, i.e., it is not a spectral operator in the sense of N. Dunford [4]. But formula (2), which resembles the functional calculus for spectral operators of scalar type and holds in our case only for continuous functions f on the spectrum of V, will be one of the main ingredients in the proof of Lemma 5 below.

Lemma 4. *Suppose that T is a polynomially bounded operator on a Banach space X. Then for every function $f \in A(D)$ one can define a bounded linear operator $f(T)$ on X such that:*

1) If $f = 1$, then $f(T) = I$;

2) If $f(\lambda) = \lambda$, then $f(T) = T$;

3) The mapping $f \mapsto f(T)$ is an algebra homomorphism from $A(D)$ into $L(X)$ satisfying $\|f(T)\| \leq M\|f\|_\infty$.

The proof of Lemma 4 is straightforward. In fact, we first define $f(T)$ for polynomials f in the standard way. Then, using von Neumann's inequality, we can extend this definition to the functions of the class $A(D)$ using approximations.

In the sequel, an invertible operator S on X is called *doubly power bounded* provided that both S and S^{-1} are power bounded, i.e., if $\sup_{n\in\mathbf{Z}} \|S^n\| < \infty$. It is easy to see that if S is doubly power bounded, then S is an (invertible) isometry in the equivalent norm $|||x||| = \sup_{n\in\mathbf{Z}} \|S^n x\|$, $x \in X$.

Lemma 5. *Assume that:*

1) T is polynomially bounded operator on a Banach space X.

2) There does not exist an invariant subspace K with respect to T^* such that $T^*|K$ is invertible and doubly power bounded.

Then the measures μ_{z,z^*} are absolutely continuous with respect to the Lebesgue measure.

P r o o f. Assuming the contrary, i.e., there exist $z \in E$, $z^* \in E^*$ such that μ_{z,z^*} is not absolutely continuous with respect to the Lebesgue measure m. This implies that there exists a compact set K with $m(K) = 0$ and $\mu_{z,z^*}(K) \neq 0$.

By Fatou's theorem (see e.g. [6, p. 80]), there exists a function $h \in A(D)$ such that

$$h(\lambda) = 1, \text{ if } \lambda \in K \text{ and } |h(\lambda)| < 1 \text{ if } \lambda \in \overline{D} \setminus K < 1. \tag{3}$$

Let $\tilde{h}(\lambda) := \overline{h(\bar{\lambda})}$. Then $\tilde{h} \in A(D), \|\tilde{h}\| = 1$. Since V^* also is a polynomially bounded invertible isometry, $\tilde{h}^n(V^*)$ is defined and satisfies

$$\sup_{n \geq 0} \|\tilde{h}^n(V^*)\| \leq M < \infty. \tag{4}$$

Fix a nonzero functional z^* in E^*. By (4) and the weak* compactness of the unit ball in E^*, there exists a subsequence n_k such that $\tilde{h}^{n_k}(V^*)z^* \to z_0^*$ in the (E^*, E)-topology. Define two functionals x^* and x_0^* in E^* by

$$x^*(x) = z^*(\hat{x}), \quad x_0^*(x) = z_0^*(\hat{x}), \; x \in X. \tag{5}$$

Then, for every vector x in X, (5) implies that

$$(\tilde{h}^{n_k}(T^*)x^*)(x) = x^*(h^{n_k}(T)x) = z^*((\widehat{h^{n_k}(T)}x))$$
$$= z^*(h^{n_k}(V)\hat{x}) = (\tilde{h}^{n_k}(V^*)z^*)(\hat{x}).$$

Therefore,

$$\lim_{k\to\infty}(h^{n_k}(T^*)x^*)(x) = \lim_{k\to\infty}(\tilde{h}^{n_k}(V^*)z^*)(\hat{x})$$
$$= z_0^*(\hat{x}) = x_0^*(x),$$

i.e., $\tilde{h}^{n_k}(T^*)x^*$ converges to x_0^* in the (X, X^*)-topology. Now we have, by adopting (4)–(6) and the Dominated Convergence Theorem,

$$x_0^*(y) = \lim_{k\to\infty}(\tilde{h}^{n_k}(T^*)x^*)(y)$$
$$= \lim_{k\to\infty} x_0^*(h^{n_k}(T)y) = \lim_{k\to\infty} z^*(h^{n_k}(V)\hat{y})$$
$$= \lim_{k\to\infty} \int_{-\pi}^{\pi} h^{n_k}(e^{i\lambda}) d\mu_{\hat{y},z^*}(\lambda) = \mu_{\hat{y},z^*}(K).$$

Since $\mu_{z,z^*}(K) \neq 0$, and $\widehat{X}$ is dense in E, there exists $\hat{y}$ such that $\mu_{\hat{y},z^*}(K) \neq 0$, so that $x_0^* \neq 0$.

By Rudin–Carleson's theorem (see e.g. [6, p. 80]), there exists a function $\phi \in A(D)$ such that

$$\phi(e^{i\lambda}) = e^{-i\lambda} \text{ for } \lambda \in K \text{ and } \|\phi\|_\infty = 1. \tag{6}$$

We show that

$$T^*\phi(T^*)x_0^* = x_0^*. \tag{7}$$

Indeed, we have, in view of (4)–(6),

$$([I - T^*\phi(T^*)]x_0^*)(y) = x_0^*([I - T\phi(T)]y)$$
$$= z_0^*([I - V\phi(V)]\hat{y}) = \lim_{k\to\infty} \tilde{[}h^{n_k}(V^*)z^*]([I - V\phi(V)]\hat{y})$$
$$= \lim_{k\to\infty} z^*(h^{n_k}(V)[I - V\phi(V)]\hat{y})$$
$$= \lim_{k\to\infty} \int_{-\pi}^{\pi} h^{n_k}(e^{i\lambda})(1 - e^{i\lambda}\phi(e^{i\lambda}))d\mu_{\hat{y},z^*}(\lambda)$$
$$= \int_K (1 - e^{i\lambda}\phi(e^{i\lambda}))d\mu_{\hat{y},z^*}(\lambda) = 0, \quad \text{for all } y \in X,$$

which implies that (7) holds.

Now let $W := \phi(T^*)$. Then (6) and (7) imply that $\sup_{n\geq 0} \|W^n\| \leq M$, and $(WT^*)^n(T^*)^k x_0^* = (T^*)^k x_0^*$, $k = 0, 1, 2, \ldots$. Let $K := \overline{span}\{(T^*)^k x_0^* : k \geq 0\}$. Then K is invariant subspace for T^*, $T^*|K$ is invertible (with the inverse equal W) and $\sup_{n\in\mathbf{Z}} \|(T^*|K)^n\| \leq M$, which is a contradiction.

R e m a r k. Lemma 5 has been proved in [10, Prop. 2.1], for contractions on Hilbert space, and here we generalized this proof.

From Lemma 5 we obtain the following result which is a generalization of the Nagy–Foias theorem.

Theorem 1. *Let T be a polynomially bounded operator on a Banach space X such that $\sigma(T) \cap \Gamma$ has measure 0. Then the following are equivalent:*

(i) $T^n x \to 0$ for every $x \in X$;

(ii) T^* does not have an invariant subspace $K \neq \{0\}$ on which $T^*|K$ is invertible and doubly power bounded.

P r o o f. Since $\sigma(T) \cap \Gamma$ has measure zero, it follows that $\sigma(V) \cap \Gamma$ also has measure zero, hence V is an invertible isometry.

Suppose that (ii) holds, we show that (i) holds. Assuming the contrary, we have $E \neq \{0\}$. By Lemma 5, the measures $m_{z,z^*}, z \in E, z^* \in E^*$, are absolutely continuous with respect to the Lebesgue measure. From $m(\sigma(V)) = 0$ it follows that $m_{z,z^*}(\sigma(V)) = 0$, i.e., all the measures m_{z,z^*} are zero, which is an absurd.

Now suppose that (i) holds but (ii) does not hold. Thus, there is a nonzero subspace K of X^* which is invariant under T^* and such that $T^*|K$ is invertible and $\sup_{n\in\mathbf{Z}} \|(T^*|K)^n\| < \infty$. Let $S = T^*|K$. Fix an element x^* in K, $x^* \neq 0$. Then $\{S^{-n}x^* : n \geq 0\}$ are uniformly bounded, hence $x^*(x) = (S^n(S^{-n})x^*)(x) = [(T^*)^n S^{-n} x^*](x) = [S^{-n}x^*](T^n x) \to 0$, for all $x \in X$, which is a contradiction.

Note that Theorem 1 can be regarded as an analogue of the stability results in [1, 7, 10] (see also [11–13])where the condition that $m(\sigma(T) \cap \Gamma) = 0$ is replaced by countability of $\sigma(T) \cap \Gamma$, and condition

$$T^* \text{ does not have an invariant subspace } K \neq \{0\}$$
$$\text{such that } T^*|K \text{ is invertible and doubly power bounded}$$

is replaced by

$$T^* \text{ does not have eigenvalues on the unit circle.}$$

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