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On a Convergence of Formal Power Series Under a Special Condition on the Gelfond–Leont'ev Derivatives

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For a formal power series the conditions on the Gelfond-Leont'ev derivatives are found, under which the series represents a function, analytic in the disk $\{z : |z| < R\}$, $0 < R \le +\infty$.

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1. Introduction

Let $(f_k)_{k=0}^{\infty}$ be an arbitrary sequence of complex numbers. For $0 < R \le \infty$ by $A(R)$ we denote the class of analytic functions

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \tag{1}$$

in the disk $\{z : |z| < R\}$. The denotement $f \in A(0)$ means further that either $f \in A(R)$ for some $R > 0$ or the series (1) converges only at the point $z = 0$, i.e., $A(0)$ is a class of formal power series. Clearly, $A(R_2) \subset A(R_1)$ for all $0 \le R_1 \le R_2 \le \infty$. We say that $f \in A^+(R)$ if $f \in A(R)$ and $f_k > 0$ for all $k \ge 0$.

For $f \in A(0)$ and $l(z) = \sum\limits_{k=0}^{\infty} l_k z^k \in A^+(0)$ the formal power series

$$D_l^n f(z) = \sum_{k=0}^{\infty} \frac{l_k}{l_{k+n}} f_{k+n} z^k \tag{2}$$

is called [1–2] the Gelfond–Leont'ev derivative of the order n. If $l(z) = e^z$, that is $l_k = 1/k!$, then $D_l^n f(z) = f^{(n)}(z)$ is a usual derivative of the order n. We can assume that $l_0 = 1$.

As in [2], let Λ be a class of all positive sequences $\lambda = (\lambda_k)$ with $\lambda_1 \geq 1$, and let $\Lambda^* = \{\lambda \in \Lambda : \ln \lambda_k \leq ak$ for every $k \in \mathbb{N}$ and some $a \in [0, +\infty)\}$. We say that $f \in A_\lambda(0)$ if $f \in A(0)$ and $|f_k| \leq \lambda_k |f_1|$ for all $k \geq 1$. Finally, let N be a class of increasing sequences (n_p) of nonnegative integers, $n_0 = 0$.

Studying of conditions on the Gelfond–Leont'ev derivatives, under which series (1) represents an entire function, was started in [2]. In particular, the following theorems are proved.

Theorem A. *Let $(n_p) \in N$. In order that for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(\infty)$ the condition $(\forall p \in \mathbb{Z}_+)\{D_l^{n_p} f \in A_\lambda(0)\}$ implies $f \in A(\infty)$, it is necessary and sufficient that $\varlimsup\limits_{p\to+\infty} (n_{p+1} - n_p) < \infty$.*

Theorem B. *Let $(n_p) \in N$, $l \in A^+(\infty)$ and the sequence $(l_{k-1}l_{k+1}/l_k^2)$ be nondecreasing. In order that for every $\lambda \in \Lambda^*$ and $f \in A(0)$ the condition $(\forall p \in \mathbb{Z}_+)\{D_l^{n_p} f \in A_\lambda(0)\}$ implies $f \in A(\infty)$, it is necessary and sufficient that*

$$\lim_{p\to+\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{l_{n_p+1}} - \sum_{j=1}^{p} \ln \frac{1}{l_{n_j - n_{j-1}+1}} \right\} = +\infty. \tag{3}$$

A problem on finding conditions on $l \in A^+(0)$, $\lambda \in \Lambda$ and $(n_p) \in N$, under which the condition $(\forall p \in \mathbb{Z}_+)\{D_l^{n_p} f \in A_\lambda(0)\}$ implies $f \in A(R)$, $R > 0$, is natural. In [3] the following analog of Th. A is proved.

Theorem C. *Let $(n_p) \in N$ and let $R[f]$ and $R[l]$ be the radii of developments into power series of f and l. The condition $\varlimsup\limits_{p\to\infty} (n_{p+1} - n_p) < +\infty$ is necessary and sufficient in order that for every $\lambda \in \Lambda$, $f \in A(0)$ and $l \in A^+(0)$ the condition $(\forall p \in \mathbb{Z}_+)\{D_l^{n_p} f \in A_\lambda(0)\}$ implies the inequality $R[f] \geq PR[l]$ with some constant $P > 0$.*

The main result of this paper is the following analog of Th. B.

Theorem 1. *Let $(n_p) \in N$. In order that for every $f \in A(0)$, $l \in A^+(0)$ and $\lambda \in \Lambda$ such that the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing and $\lambda_{k-1}\lambda_{k+1}/\lambda_k^2 \geq 1$, $k \geq 2$, the condition $(\forall p \in \mathbb{Z}_+)\{D_l^{n_p} f \in A_\lambda(0)\}$ implies $f \in A(R)$, it is necessary and sufficient that*

$$\varliminf_{p\to+\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j - n_{j-1}+1}}{l_{n_j - n_{j-1}+1}} \right\} \geq \ln R. \tag{4}$$

None of the conditions on $\lambda \in \Lambda$ and $l \in A^+(0)$ in Th. 1 can be dropped in general.

2. Proof of Theorem 1

In [2] the following lemma is proved.

Lemma 1. *If* $\lambda \in \Lambda$, $(n_p) \in N$, $f \in A(0)$, $l \in A^+(0)$ *and* $D_l^{n_p} f \in A_\lambda(0)$ *for all* $p \in \mathbb{Z}_+$ *then*

$$|f_{n_p+k}| \le |f_1| l_1^p l_{n_p+k} \frac{\lambda_k}{l_k} \prod_{j=1}^{p} \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \tag{5}$$

for all $p \in \mathbb{Z}_+$ *and* $k = 2, \dots, n_{p+1} - n_p + 1$.

First we prove the following theorem using Lem. 1.

Theorem 2. *Let* $(n_p) \in N$ *and the sequence* $\lambda \in \Lambda$ *and the function* $l \in A^+(0)$ *be such that for all* $p \in \mathbb{Z}_+$ *and* $k = 2, \dots, n_{p+1} - n_p$

$$\ln \frac{l_{n_p+k-1} l_{n_p+k+1}}{l_{n_p+k}^2} - \ln \frac{l_{k-1} l_{k+1}}{l_k^2} + \ln \frac{\lambda_{k-1}\lambda_{k+1}}{\lambda_k^2} \ge 0. \tag{6}$$

If $D_l^{n_p} f \in A_\lambda(0)$ *for all* $p \in \mathbb{Z}_+$ *then the estimate*

$$\ln R[f] \ge \varliminf_{p\to+\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} \tag{7}$$

is true and sharp.

P r o o f. From (5) for $p \to \infty$ we have

$$\frac{\ln |f_{n_p+k}|}{n_p+k}$$

$$\le \frac{1}{n_p+k} \left\{ \ln l_{n_p+k} - \ln l_k + \ln \lambda_k + p \ln l_1 + \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} + o(1). \tag{8}$$

We put

$$A_p = p \ln l_1 + \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}$$

and

$$\gamma_k = \gamma_{k,p} = \frac{1}{n_p+k} \{ \ln l_{n_p+k} - \ln l_k + \ln \lambda_k + A_p \}, \quad k = 1, 2, \dots, n_{p+1} - n_p + 1.$$

Then

$$\gamma_k - \gamma_{k-1} = \frac{\delta_k}{(n_p+k)(n_p+k-1)}, \quad k = 2, \dots, n_{p+1} - n_p + 1, \tag{9}$$

where

$$\delta_k = (n_p + k - 1)(\ln l_{n_p+k} - \ln l_k + \ln \lambda_k) \\ - (n_p + k)(\ln l_{n_p+k-1} - \ln l_{k-1} + \ln \lambda_{k-1}) - A_p.$$

In view of (6)

$$\delta_{k+1} - \delta_k = (n_p + k)\left(\ln \frac{l_{n_p+k-1} l_{n_p+k+1}}{l^2_{n_p+k}} - \ln \frac{l_{k-1} l_{k+1}}{l^2_k} + \ln \frac{\lambda_{k-1}\lambda_{k+1}}{\lambda^2_k}\right) \geq 0,$$

$$k = 2, \ldots, n_{p+1} - n_p,$$

i.e., $\delta_2 \leq \cdots \leq \delta_{n_{p+1}-n_p+1}$. If all $\delta_k \geq 0$, then in view of (9) $\gamma_k \geq \gamma_{k-1}$ for all $k = 2, \ldots, n_{p+1} - n_p + 1$ and $\max\{\gamma_k : 2 \leq k \leq n_{p+1} - n_p + 1\} = \gamma_{n_{p+1}-n_p+1}$. If all $\delta_k \leq 0$, then $\gamma_k \leq \gamma_{k-1}$ for all $k = 2, \ldots, n_{p+1} - n_p + 1$ and $\max\{\gamma_k : 2 \leq k \leq n_{p+1} - n_p + 1\} = \gamma_1$. Finally, if $\delta_2 \leq \cdots \leq \delta_{k_0-1} < 0 \leq \delta_{k_0} \leq \ldots \delta_{n_{p+1}-n_p+1}$ for some k_0, $2 \leq k_0 \leq n_{p+1} - n_p + 1$, then $\gamma_{k_0-1} < \gamma_{k_0-2} < \cdots < \gamma_1$ and $\gamma_{k_0-1} \leq \gamma_{k_0} \leq \cdots < \gamma_{n_{p+1}-n_p+1}$. Thus,

$$\max\{\gamma_k : 1 \leq k \leq n_{p+1} - n_p + 1\} = \max\{\gamma_1, \gamma_{n_{p+1}-n_p+1}\}.$$

Since

$$\gamma_1 = \frac{1}{n_p + 1}\{\ln l_{n_p+1} - \ln l_1 + \ln \lambda_1 + A_p\},$$

and

$$\gamma_{n_{p+1}-n_p+1} = \frac{1}{n_{p+1} + 1}\{\ln l_{n_{p+1}+1} - \ln l_{n_{p+1}-n_p+1} + \ln \lambda_{n_{p+1}-n_p+1} + A_p\}$$

$$= \frac{1}{n_{p+1} + 1}\{\ln l_{n_{p+1}+1} - \ln l_1 + A_{p+1}\},$$

from (8) for $1 \leq k \leq n_{p+1} - n_p + 1$ we have

$$\frac{\ln |f_{n_p+k}|}{n_p + k} \leq \max\left\{\frac{\ln l_{n_p+1} + A_p}{n_p + 1}, \frac{\ln l_{n_{p+1}+1} + A_{p+1}}{n_{p+1} + 1}\right\} + o(1), \quad p \to \infty,$$

i.e., for $p \to \infty$

$$\frac{1}{n_p + k} \ln \frac{1}{|f_{n_p+k}|}$$

$$\geq \min\left\{\frac{1}{n_p + 1}\left(\frac{1}{\ln l_{n_p+1}} - A_p\right), \frac{1}{n_{p+1} + 1}\left(\ln \frac{1}{l_{n_{p+1}+1}} - A_{p+1}\right)\right\} + o(1).$$

Hence it follows

$$\ln R[f] \geq \underline{\lim}_{p\to\infty} \frac{1}{n_p + 1}\left(\frac{1}{\ln l_{n_p+1}} - A_p\right),$$

that is in view of the definition of A_p the estimate (7) is proved.

For the proof of its sharpness we consider a power series

$$f(z) = \sum_{k=0}^{\infty} f_{n_k+1} z^{n_k+1}. \tag{10}$$

Since for the series (10)

$$D_l^{n_p} f(z) = \sum_{k=p}^{\infty} \frac{l_{n_k-n_p+1}}{l_{n_k+1}} f_{n_k+1} z^{n_k-n_p+1},$$

then $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$ if and only if for all $p \in \mathbb{Z}_+$ and $k > p$

$$\frac{l_{n_k-n_p+1}}{l_{n_k+1}} |f_{n_k+1}| \le \lambda_{n_k-n_p+1} \frac{l_1}{l_{n_p+1}} |f_{n_p+1}|. \tag{11}$$

It is easy to see that if $f_1 > 0$ and

$$f_{n_k+1} = f_1 l_1^{k-1} l_{n_k+1} \prod_{j=1}^{k} \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}, \quad k \ge 1, \tag{12}$$

then (11) holds if and only if for all $p \in \mathbb{Z}_+$ и $k > p$

$$\prod_{j=p+1}^{k} \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \le l_1^{p+1-k} \frac{\lambda_{n_k-n_p+1}}{l_{n_k-n_p+1}}. \tag{13}$$

We suppose that $l_1 \ge 1$, and $\lambda_k/l_k = \exp\{(k-1)\varphi(k-1)\}$, $k \ge 2$, where φ is positive, continuous and nondecreasing function on $[0, +\infty)$. Then

$$\prod_{j=p+1}^{k} \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \le \prod_{j=p+1}^{k} e^{(n_j-n_{j-1})\varphi(n_j-n_{j-1})} \le \prod_{j=p+1}^{k} e^{(n_j-n_{j-1})\varphi(n_k-n_p)}$$

$$= e^{(n_k-n_p)\varphi(n_k-n_p)} = \frac{\lambda_{n_k-n_p+1}}{l_{n_k-n_p+1}} \le l_1^{p+1-k} \frac{\lambda_{n_k-n_p+1}}{l_{n_k-n_p+1}},$$

i.e., (13) holds and, thus, $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$. Since for the series (10) with the coefficients (12) the equality

$$\ln R[f] = \varliminf_{p\to+\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} \tag{14}$$

is true, then we need to show that there exist sequences (l_k) and (λ_k) such that $\lambda_k/l_k = \exp\{(k-1)\varphi(k-1)\}$, $k \ge 2$, and the condition (6) holds.

Since for $\lambda_k/l_k = \exp\{(k-1)\varphi(k-1)\}$ the condition (6) takes the form

$$\ln \frac{l_{n_p+k-1} l_{n_p+k+1}}{l^2_{n_p+k}} + (k-2)\varphi(k-2) + k\varphi(k) - 2(k-1)\varphi(k-1) \ge 0,$$

it is sufficient to choose a sequence (l_k) such that $l_{k-1}l_{k+1} \ge l_k^2$, $k \ge 2$, and a function φ such that the function $x\varphi(x)$ is convex. The proof of Th. 2 is complete.

P r o o f of Theorem 1. At first we remark that if $\lambda \in \Lambda$, $l \in A^+(0)$, the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing and $\lambda_{k-1}\lambda_{k+1}/\lambda_k^2 \ge 1$, $k \ge 2$, then the condition (6) of Th. 2 holds. Therefore, if (4) holds, then (7) implies the inequality $R[f] \ge R$, i.e. $f \in A(R)$. The sufficiency of (4) is proved.

On the other hand, from the proof of Th. 2 it follows that there exist $f \in A(0)$, $\lambda \in \Lambda$, $l \in A^+(0)$ (for example, $l_k = 1$ and $\lambda_k = \exp\{(k-1)\varphi(k-1)\}$, $k \ge 2$) such that the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing, $\lambda_{k-1}\lambda_{k+1}/\lambda_k^2 \ge 1$ for $k \ge 2$ and $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$ and the equality (14) holds. Therefore, if the condition (4) does not hold, then for the series (10) with the coefficients (12) we have

$$\varliminf_{p\to+\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} < \ln R,$$

i.e., $f \notin A(R)$. Theorem 1 is proved.

3. Essentiality of the Conditions in Theorems 1–2

We suppose that $n_p = 2^p$ for $p \ge 1$ (thus, $n_{p+1} - n_p = n_p$ for $p \ge 2$) and consider a power series

$$f(z) = \sum_{k=0}^{\infty} \left(f_{n_k} z^{n_k} + f_{n_k+1} z^{n_k+1} \right), \tag{15}$$

where $f_0 = 0$, $f_1 = 1$, $f_{n_1} = \lambda_{n_1}$,

$$f_{n_k} = l_{n_k} \mu_{n_{k-1}} \prod_{j=0}^{k-2} \mu_{n_j+1},\ k \ge 2, \quad f_{n_k+1} = l_{n_k+1} \prod_{j=0}^{k-1} \mu_{n_j+1},\ k \ge 1, \tag{16}$$

and (μ_n) is an arbitrary sequence of positive numbers. Since for the series (15)

$$D_l^{n_p} f(z) = \sum_{k=p}^{\infty} \left(\frac{l_{n_k-n_p}}{l_{n_k}} f_{n_k} z^{n_k-n_p} + \frac{l_{n_k-n_p+1}}{l_{n_k+1}} f_{n_k+1} z^{n_k-n_p+1} \right),$$

then $D_l^{n_p} f \in A_\lambda(0)$ if and only if for all $k \geq p+1$

$$\frac{l_{n_k-n_p+1}}{l_{n_k+1}} f_{n_k+1} \leq \lambda_{n_k-n_p+1} \frac{l_1}{l_{n_p+1}} f_{n_p+1}, \quad \frac{l_{n_k-n_p}}{l_{n_k}} f_{n_k} \leq \lambda_{n_k-n_p} \frac{l_1}{l_{n_p+1}} f_{n_p+1}.$$

If $l_1 = 1$ then hence it follows that $D_l^{n_p} f \in A_\lambda(0)$ for all $p \geq 0$ if and only if for all $p \geq 1$

$$\mu_{n_p} \leq \frac{\lambda_{n_{p+1}-n_p}}{l_{n_{p+1}-n_p}} = \frac{\lambda_{n_p}}{l_{n_p}} \tag{17}$$

and for all $p \geq 0$

$$\prod_{j=p}^{k-1} \mu_{n_j+1} \leq \frac{\lambda_{n_k-n_p+1}}{l_{n_k-n_p+1}}, \ k \geq p+1, \quad \mu_{n_{k-1}} \prod_{j=p}^{k-2} \mu_{n_j+1} \leq \frac{\lambda_{n_k-n_p}}{l_{n_k-n_p}}, \ k \geq p+2. \tag{18}$$

Choosing properly the sequences (l_k), (λ_k) and (μ_k), we can show that the conditions in Ths. 1 and 2 are essential.

For example, if $l_k = \lambda_k$ and $\mu_k = 1$ for all $k \geq 1$, then the inequalities (17) and (18) are obvious and $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$.

Besides, if $l_{2j} = e^{-2ja}$, $l_{2j+1} = e^{-(2j+1)b}$ and $b > a$, then the condition (6) does not hold,

$$\varliminf_{p \to +\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} = b$$

and

$$\ln R[f] = \varliminf_{p \to +\infty} \frac{1}{n_p} \ln \frac{1}{l_{n_p}} = a,$$

i.e., the inequality (7) does not hold and, thus, *the condition (6) in Th. 2 can not be dropped in general.*

Now we show that *the condition $\lambda_{k-1}\lambda_{k+1}/\lambda_k^2 \geq 1$, $k \geq 2$, in Th. 1 can not be dropped in general.* For this purpose we put $l_k = 1$ and $\mu_k = \lambda_k$ for $k \geq 1$, and we choose the sequence (λ_k) such that $\lambda_{2j+1} = 1$, $\lambda_{2(j+1)} \geq \lambda_{2j}$ for all $j \geq 1$ and $\ln \lambda_{n_k} = n_k$, $k \geq 1$. Due to the choice $l \in A^+(0)$, the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing and it is easy to verify the fulfillment of conditions (17) and (18), i.e., $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$. Besides,

$$\varliminf_{p \to +\infty} \frac{1}{n_p+1} \left\{ \ln \frac{1}{l_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}} \right\} = 0,$$

and

$$\ln R[f] = \varliminf_{p \to +\infty} \frac{1}{n_p} \ln \frac{1}{f_{n_p}} = \varliminf_{p \to +\infty} \frac{1}{n_p} \ln \frac{1}{\lambda_{n_{p-1}}} = -\frac{1}{2} < 0,$$

i.e., the condition (4) holds with $R = 1$, but $f \notin A(R)$.

Finally, we show that *the condition of nondecreasing for the sequence* $(l_{k-1}l_{k+1}/l_k^2)$ *in Th. 1 can not be dropped in general.* We choose $\lambda_k = e^{k^2}$, $l_{2k} = e^{-(2k)^2}$, $l_{2k+1} = e^{-12(2k)^2}$ and $\mu_k = 1/l_k$. Then $\lambda_{k-1}\lambda_{k+1}/\lambda_k^2 \geq 1$, $k \geq 2$, and the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is not nondecreasing. The inequality (17) is obvious and for $k \geq p+1$

$$\sum_{j=p}^{k-1} \ln \mu_{n_j+1} = -\sum_{j=p+1}^{k} \ln l_{n_j-n_{j-1}+1} = 12\sum_{j=p+1}^{k}(n_j - n_{j-1})^2 \leq 12(n_k - n_p)^2$$

$$= -\ln l_{n_k-n_p+1} < \ln \frac{\lambda_{n_k-n_p+1}}{l_{n_k-n_p+1}},$$

that is the first inequality in (18) holds. Further, for $k \geq p+2$ we have

$$\ln \mu_{n_{k-1}} + \sum_{j=p}^{k-2} \ln \mu_{n_j+1} = -\ln l_{n_{k-1}} - \sum_{j=p}^{k-2} \ln l_{n_j+1} = n_{k-1}^2 + 12\sum_{j=p}^{k-2} n_j^2$$

$$= 4^{k-1} + 12\sum_{j=p}^{k-2} 4^j = 4^{k-1} + 4^k - 4^{p+1} < 2(2^k - 2^p)^2 = 2(n_k - n_p)^2 = \ln \frac{\lambda_{n_k-n_p}}{l_{n_k-n_p}},$$

that is the second inequality in (18) holds and, thus, $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$. Besides,

$$\varliminf_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln \frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p} \ln \frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}\right\}$$

$$= \varliminf_{p\to+\infty} \frac{1}{n_p}\left\{12n_p^2 - \sum_{j=1}^{p}((n_j - n_{j-1}+1)^2 + 12(n_j - n_{j-1})^2)\right\}$$

$$= \varliminf_{p\to+\infty} \frac{1}{n_p}\left\{12n_p^2 - 13\sum_{j=1}^{p}(n_j - n_{j-1})^2 - 2\sum_{j=1}^{p}(n_j - n_{j-1}) - \sum_{j=1}^{p} 1\right\}$$

$$= \varliminf_{p\to+\infty} \frac{1}{2^p}\left\{12\ 4^p - \frac{13}{3}(4^p - 1) - 2^{p+1} - p\right\} = +\infty$$

and

$$\ln R[f] = \varliminf_{p\to+\infty} \frac{1}{n_p} \ln \frac{1}{f_{n_p}} = \varliminf_{p\to+\infty} \frac{1}{n_p}\left\{\ln \frac{1}{l_{n_p}} - \ln \frac{1}{l_{n_{p-1}}} - \sum_{j=0}^{p-2} \ln \frac{1}{l_{n_j+1}}\right\}$$

$$\varliminf_{p\to+\infty}\frac{1}{n_p}\left\{n_p^2-n_{p-1}^2-12\sum_{j=0}^{p-2}n_j^2\right\}=\varliminf_{p\to+\infty}\frac{1}{2^p}\left\{4^p-4^{p-1}-12\sum_{j=0}^{p-2}4^j\right\}$$

$$=\varliminf_{p\to+\infty}\frac{1}{2^p}(-4^{p-1}+4)=-\infty,$$

that is the condition (4) holds with $R=+\infty$, but $f\notin A(\infty)$.

4. Supplements and Remarks

Here we consider the case when the sequence $\lambda\in\Lambda$ satisfies a condition of the form $\lambda\in\Lambda^*$.

Proposition 1. *Let $(n_p)\in N$, the function $l\in A^+(0)$ be such that the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing and $\ln\lambda_k\le a(k-1)$ for all $k\ge 1$ and some $a\in(0,+\infty)$. If $D_l^{n_p}f\in A_\lambda(0)$ for all $p\in\mathbb{Z}_+$, then the estimate*

$$\ln R[f]\ge\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}}-p\ln l_1-\sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\}-a \qquad (19)$$

is true and sharp.

Indeed, from the conditions $\ln\lambda_k\le a(k-1)$ for all $k\ge 1$ and $D_l^{n_p}f\in A_\lambda(0)$ for all $p\in\mathbb{Z}_+$ it follows that $D_l^{n_p}f\in A_{\lambda^*}(0)$ for all $p\in\mathbb{Z}_+$, where $\ln\lambda_k^*=a(k-1)$. It is clear that $\lambda_{k-1}^*\lambda_{k+1}^*=(\lambda_k^*)^2$ and, since the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing, the condition (6) of Th. 2 holds. Therefore, from (7) we obtain

$$\ln R[f]$$

$$\ge\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}}-p\ln l_1-\sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}-\sum_{j=1}^{p}\ln\lambda^*_{n_j-n_{j-1}+1}\right\}$$

$$\ge\varliminf_{p\to+\infty}\frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}}-p\ln l_1-\sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}-a\sum_{j=1}^{p}(n_j-n_{j-1})\right\},$$

whence the inequality (19) follows.

For the proof of sharpness of the inequality (19) it is sufficient to consider the series (10) with the coefficients (12) and choose $\lambda_k=l_k=e^{a(k-1)}$. Then the inequality (13) holds (thus, $D_l^{n_p}f\in A_\lambda(0)$ for all $p\in\mathbb{Z}_+$) and

$$\ln R[f]=\varliminf_{p\to+\infty}\frac{1}{n_p+1}\ln\frac{1}{f_{n_p+1}}=\varliminf_{p\to+\infty}\frac{1}{n_p+1}\ln\frac{1}{l_{n_p+1}}=-a$$

$$= \varliminf_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\} - a.$$

Proposition 1 is proved.

We remark that the condition $\ln \lambda_k \le a(k-1)$ in Prop. 1 can not be replaced in general by the condition $\ln \lambda_k \le ak$ and moreover by the condition $\varlimsup\limits_{k\to\infty} (\ln \lambda_n)/n = a$. Indeed, let $n_p = p + [\sqrt{p}]$ for all $p \ge 0$, $\lambda_k = e^{ak}$, and $l_k = e^{bk}$ for all $k \ge 2$, $b > a$, and $l_1 = 1$. It is easy to verify that for such λ_k and l_k the inequality (13) holds. Therefore, for the function (10) with the coefficients (12) we have $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$. Besides,

$$\ln R[f] = \varliminf_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{\lambda_{n_j-n_{j-1}+1}}{l_{n_j-n_{j-1}+1}}\right\}$$

$$= \varliminf_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\} - \lim_{p\to\infty}\frac{a(n_p+p)}{n_p+1}$$

$$= \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\} - 2a,$$

that is the inequality (19) does not hold.

We remark that from the proof of Prop. 1 it follows that if the sequence $(l_{k-1}l_{k+1}/l_k^2)$ is nondecreasing, $\lambda_k = 1$ for all $k \ge 1$ and $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$, then

$$\ln R[f] \ge \varliminf_{p\to+\infty} \frac{1}{n_p+1}\left\{\ln\frac{1}{l_{n_p+1}} - p\ln l_1 - \sum_{j=1}^{p}\ln\frac{1}{l_{n_j-n_{j-1}+1}}\right\}, \qquad (20)$$

and moreover the condition $\lambda_k = 1$ can not be replaced in general by the condition $\ln \lambda_k = o(k)$, $k \to \infty$. However the following proposition is true.

Proposition 2. *Let $(n_p) \in N$, $\ln \lambda_k = o(k)$ as $k \to \infty$, $l \in A^+(0)$ and the sequence $(\mu_{k-1}\mu_{k+1}/\mu_k^2)$ is nondecreasing, where $\mu_k = l_k/\lambda_k$. If $D_l^{n_p} f \in A_\lambda(0)$ for all $p \in \mathbb{Z}_+$ then the estimate (20) is true and sharp.*

Indeed, from the inequality (5) we have

$$|f_{n_p+k}| \le |f_1| l_1^p \lambda_{n_p+k} \frac{\mu_{n_p+k}}{\mu_k} \prod_{j=1}^{p} \frac{1}{\mu_{n_j-n_{j-1}+1}}$$

for all $p \in \mathbb{Z}_+$ и $k = 2, \dots, n_{p+1} - n_p + 1$, whence in view of the condition $\ln \lambda_k = o(k),\ k \to \infty$, we have

$$\frac{\ln |f_{n_p+k}|}{n_p + k}$$

$$\le \frac{1}{n_p + k} \left\{ \ln \mu_{n_p+k} - \ln \mu_k + p \ln l_1 + \sum_{j=1}^{p} \ln \frac{1}{\mu_{n_j - n_{j-1}+1}} \right\} + o(1), \quad p \to \infty.$$

Since the sequence $(\mu_{k-1}\mu_{k+1}/\mu_k^2)$ is nondecreasing, hence as in the proof of Th. 2 we obtain for all $p \in \mathbb{Z}_+$ and $k = 2, \dots, n_{p+1} - n_p + 1$

$$\frac{1}{n_p + k} \ln \frac{1}{|f_{n_p+k}|} \ge \min \left\{ \frac{1}{n_p + 1} \left(\ln \frac{1}{\mu_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{1}{\mu_{n_j - n_{j-1}+1}} \right), \right.$$

$$\left. \frac{1}{n_{p+1} + 1} \left(\ln \frac{1}{\mu_{n_{p+1}+1}} - (p+1) \ln l_1 - \sum_{j=1}^{p+1} \ln \frac{1}{\mu_{n_j - n_{j-1}+1}} \right) \right\} + o(1), \quad p \to \infty,$$

that is

$$\ln R[f] \ge \varliminf_{p \to +\infty} \frac{1}{n_p + 1} \left\{ \ln \frac{1}{\mu_{n_p+1}} - p \ln l_1 - \sum_{j=1}^{p} \ln \frac{1}{\mu_{n_j - n_{j-1}+1}} \right\}.$$

Since $\mu_k = l_k/\lambda_k$ and $\ln \lambda_k = o(k), k \to \infty$, hence we obtain the inequality (20). For the proof of its sharpness it is sufficient to consider the series (10) with the coefficients (12), where $\lambda_1 = 1$, $\lambda_k = k - 1$ and $l_k = (k-1)e^{k-1}$ for $k \ge 2$. Proposition 2 is proved.

From the proof of Prop. 2 one can see that in Th. A nondecreasing of sequence $(l_{k-1}l_{k+1}/l_k^2)$ can be replaced by the following condition: *there exists a positive sequence* (ν_k) *such that* $\ln \nu_k = O(k)$, $k \to \infty$, *and* $(\mu_{k-1}\mu_{k+1}/\mu_k^2)$ *does not decrease, where* $\mu_k = l_k \nu_k$.

Finally, the following proposition supplements Th. A.

Proposition 3. *For all* $\lambda \in \Lambda$ *and* $l \in A^+(0)$ *there exists* $f \in A(0)$ *such that* $D_l^n f \in A_\lambda(0)$ *for all* $n \ge 0$ *and* $R[f] = +\infty$.

Indeed, there exists an increasing to $+\infty$ function φ such that

$$\max \left\{ -\frac{2}{k-1} \ln \frac{1}{l_{k-1}}, -\frac{1}{k} \ln \frac{\lambda_1 \lambda_k}{l_k} \right\} \le \varphi(k), \quad k \ge 1.$$

We put $f_k = l_k \exp\{-(k+1)\varphi(k+1)\}$, $k \geq 1$. Then

$$\frac{1}{k}\ln\frac{1}{f_k} \geq \frac{1}{k}\ln\frac{1}{l_k} + \varphi(k+1) \to +\infty, \quad k \to \infty,$$

and for all $n \geq 0$ and $k \geq 1$

$$\frac{f_{k+n}}{l_{k+n}} = e^{-(k+n+1)\varphi(k+n+1)} \leq e^{-k\varphi(k)} e^{-(n+1)\varphi(n+1)} \leq \frac{l_1\lambda_k}{l_k}\frac{f_{n+1}}{l_{n+1}},$$

that is $R[f] = +\infty$ and $D_l^n f \in A_\lambda(0)$ for all $n \geq 0$. Proposition 3 is proved.

We remark that in view of Th. A one can not replace $R[f] = +\infty$ by $R[f] = R \in (0, +\infty)$ in the last proposition.

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