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Invariant Totally Geodesic Unit Vector Fields on Three-Dimensional Lie Groups

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We give a complete list of left-invariant unit vector fields on three-dimensional Lie groups equipped with a left-invariant metric that generate a totally geodesic submanifold in the unit tangent bundle of a group equipped with the Sasaki metric. As a result we obtain that each three-dimensional Lie group admits totally geodesic unit vector field under some conditions on structural constants. From a geometrical viewpoint, the field is either parallel or a characteristic vector field of a natural almost contact structure on the group.

Key words: Sasaki metric, totally geodesic unit vector field, almost contact structure, Sasakian structure.

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Introduction

The problem on description of all totally geodesic submanifolds in tangent and unit tangent bundle of space forms was formulated by A. Borisenko in [2, Probl. 1]. In general setting the problem is unsolved up to now. More progress is achieved for a special class of submanifolds in the unit tangent bundle formed by unit vector fields on the base manifold. We begin with a definition.

Let (M^n, g) be a Riemannian manifold and (T_1M^n, g_s) its unit tangent bundle with Sasaki metric. Consider a unit vector field ξ as a mapping

$$\xi : M^n \to T_1M^n.$$

Definition 1. *A unit vector field ξ on the Riemannian manifold M^n is called totally geodesic if the image of the (local) imbedding $\xi : M^n \to T_1M^n$ is a totally geodesic submanifold in the unit tangent bundle T_1M^n with the Sasaki metric.*

In the two-dimensional case the problem is solved [13]. In the case of higher dimensions only partial results are known. A. Borisenko conjectured that the Hopf unit vector field on each odd-dimensional sphere is totally geodesic. The conjecture was approved in a more general case. *If M^{2m+1} is a Sasakian manifold and ξ is a characteristic vector field of the Sasakian structure, then $\xi(M^{2m+1})$ is totally geodesic in T_1M^{2m+1}* [12].

Note that the Hopf vector field belongs to the class of left-invariant unit vector fields on S^3 as a Lie group with the left-invariant Riemannian metric. In this paper, we give a full description of three-dimensional Lie groups with the left-invariant metric which admit a totally geodesic left-invariant unit vector field and the fields themselves. As a consequence, we show that, in nontrivial cases, for each totally geodesic left-invariant unit vector field ξ the structure $\big(\phi = -\nabla\xi,\ \xi,\ \eta = g(\xi,\cdot)\big)$ is an almost contact structure on the corresponding Lie group and ξ is a characteristic vector field of this structure. If ξ is a Killing unit vector field, then the structure is Sasakian.

It is worthwhile to note that in a similar way one can define a *locally minimal* unit vector field as a field of *zero mean curvature*. A number of examples of locally minimal unit vector fields were found recently [5, 6]. In particular, K. Tsukada and L. Vanhecke [9] described all minimal left-invariant unit vector fields on three-dimensional Lie groups with the left-invariant metric. While the totally geodesic unit vector fields form a subclass in a class of minimal unit vector fields, no method to distinguish minimal and totally geodesic fields was proposed in [9].

The paper is organized as follows. In Section 1, we give some preliminaries and formulate the results. In Section 2, we consider the unimodular Lie groups. We prove that if a totally geodesic unit vector field exists on a given group, then it is an eigenvector of the Ricci tensor which corresponds to the Ricci principal curvature $\rho = 2$. Moreover, we give a complete list of totally geodesic unit vector fields on a corresponding Lie group as well as the conditions on the structure constants of the group. In a series of Props. 2.2–2.6, we give a description of totally geodesic unit vector fields in unimodular case from the contact geometry viewpoint. In Section 3 we consider the nonunimodular case. We give an explicit expression for the totally geodesic unit vector field as well as the conditions on the structure constants of the corresponding group. Finally, Prop. 3.1 gives a geometrical characterization of the totally geodesic unit vector field and clarifies a structure of the corresponding nonunimodular Lie group.

1. Preliminaries and Results

Let (M, g) be an n-dimensional Riemannian manifold with metric g and TM be its tangent bundle. Denote by $\pi : TM \to M$ the bundle projection. Denote by Q a point on TM. Then $Q = (q, \xi)$, where $q \in M$ and $\xi \in T_qM$. Let

$\tilde X, \tilde Y \in T_Q TM$. A natural (Sasaki) Riemannian metric $\tilde g$ on the tangent bundle is defined by the following scalar product

$$\tilde g(\tilde X, \tilde Y)\big|_Q = \tilde g(\pi_* \tilde X, \pi_* \tilde Y)\big|_q + \tilde g(K\tilde X, K\tilde Y)\big|_q,$$

where π_* and K are the differentials of the bundle projection and the connection map [3] respectively. A unit tangent bundle T_1M is a subbundle in TM and a hypersurface in $(TM, \tilde g)$ with a pull-back metric.

Suppose that $u := (u^1, \dots, u^n)$ are local coordinates on M. Denote by $(u, \xi) := (u^1, \dots, u^n; \xi^1, \dots, \xi^n)$ the natural local coordinates in the tangent bundle TM. If $\xi(u)$ is a unit vector field on M, then it defines a mapping $\xi : M \to T_1M$, given by $\xi(u) = (u, \xi(u))$. The image $\xi(M)$ is a submanifold in T_1M with a pull-back metric.

Denote by ∇ the Levi–Civita connection on M. Introduce a pointwise linear operator $A_\xi : T_qM^n \to \xi_q^\perp$ by

$$A_\xi X = -\nabla_X \xi.$$

From the definition of the connection map it follows that the pull-back metric on $\xi(M)$ is defined by

$$\tilde g(\xi_* X, \xi_* Y\rangle)\big|_{(u,\xi(u))} = g(X,Y)\big|_q + g(A_\xi X, A_\xi Y)\big|_q.$$

From intrinsic viewpoint, this metric can be considered as a metric on M additively deformed by the field ξ.

When $\xi^\perp$ is an integrable distribution, the unit vector field ξ is called *holonomic*, otherwise it is called *nonholonomic*. In holonomic case the operator A_ξ is symmetric (w.r. to metric g) and is known as Weingarten or the *shape operator* for each hypersurface of the foliation. In general (nonholonomic) case, A_ξ is not symmetric but formally satisfies the Codazzi equation. Namely, a covariant derivative of A_ξ is defined by

$$(\nabla_X A_\xi)Y = -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi.$$

Then for the curvature operator of M we have

$$R(X,Y)\xi = (\nabla_Y A_\xi)X - (\nabla_X A_\xi)Y,$$

which gives a Codazzi-type equation. From this viewpoint, it is natural to call the operator A_ξ a *nonholonomic shape* operator.

Introduce a symmetric tensor field

$$Hess_\xi(X,Y) = \frac{1}{2}\big[(\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y\big], \tag{1}$$

which is a *symmetric part* of the covariant derivative of A_ξ. The trace $-\sum_{i=1}^n Hess_\xi(e_i, e_i) := \Delta\xi$, where $e_1, \dots, e_n$ is an orthonormal frame, known as the *rough Laplacian* [1] of the field ξ. Therefore, one can treat the tensor field (1) as a *rough Hessian* of the field.

A unit vector field is called *harmonic*, if it is a critical point of the energy functional of mapping $\xi : M^n \to T_1M^n$. This definition presumes the variation within the class of unit vector fields. A unit vector field is harmonic if and only if $\Delta\xi = -|\nabla\xi|^2\xi$ (see [10]). There exist harmonic unit vector fields that fail to be critical within a wider class of all mappings $f : M^n \to T_1M^n$ [4]. Introduce a tensor field

$$Hm_\xi(X, Y) = \frac{1}{2}\big[R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X\big].$$

A *harmonic* unit vector field ξ defines a *harmonic mapping* $\xi : M^n \to T_1M^n$ if and only if $\sum_{i=1}^n Hm_\xi(e_i, e_i) = 0$ (see [4]). The following lemma [14] gives the condition on ξ to be totally geodesic in terms of $Hess_\xi$ and Hm_ξ.

Lemma 1.1. *A unit vector field ξ on a given Riemannian manifold M^n is totally geodesic if and only if*

$$Hess_\xi(X, Y) + A_\xi Hm_\xi(X, Y) - g(A_\xi X, A_\xi Y)\,\xi = 0$$

for all vector fields X, Y on M^n.

For the sake of brevity, denote

$$TG_\xi(X, Y) := Hess_\xi(X, Y) + A_\xi Hm_\xi(X, Y) - g(A_\xi X, A_\xi Y)\,\xi. \tag{2}$$

The treatment of three-dimensional Lie groups with the left-invariant metrics is based on J. Milnor's description of three-dimensional Lie groups via the structure constants [8] and splits into two natural cases.

The unimodular case. In this case, there is an orthonormal frame e_1, e_2, e_3 of its Lie algebra such that the bracket operations are defined by

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3. \tag{3}$$

The constants $\lambda_1, \lambda_2, \lambda_3$ completely determine a topological structure of the corresponding Lie group as in the following table:

Signs of $\lambda_1, \lambda_2, \lambda_3$	Associated Lie group
$+,+,+$	$SU(2)$ or $SO(3)$
$+,+,-$	$SL(2,\mathbb{R})$ or $O(1,2)$
$+,+,0$	$E(2)$
$+,-,0$	$E(1,1)$
$+,0,0$	Nil^3 (Heisenberg group)
$0,0,0$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$

.

The main result for this case is the following.

Theorem 1.1. *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\, i = 1, 2, 3\}$ be an orthonormal basis for the Lie algebra satisfying* (3)*. Moreover, assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then the left-invariant totally geodesic unit vector fields on G are given as follows:*

G	*Conditions on* $\lambda_1, \lambda_2, \lambda_3$	ξ
SU(2)	$\lambda_1 = \lambda_2 = \lambda_3 = 2$	*arbitrary left-invariant*
	$\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$	$\pm e_3$
	$\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$	$\cos t\, e_1 + \sin t\, e_2$
	$\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$	$\pm e_1$
	$\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$	$\cos t\, e_2 + \sin t\, e_3$
	$\lambda_1 > \lambda_2 > \lambda_3 > 0, \quad \lambda_m^2 - (\lambda_i - \lambda_k)^2 = 4$	$\pm e_m$ *(i,k,m=1,2,3)*
SL(2,R)	$\lambda_3^2 - (\lambda_1 - \lambda_2)^2 = 4$	$\pm e_3$
	$\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$	$\pm e_1$
E(2)	$\lambda_1 = \lambda_2 > 0, \quad \lambda_3 = 0$	$\pm e_3$, $\cos t\, e_1 + \sin t\, e_2$
	$\lambda_1^2 - \lambda_2^2 = 4,\ \lambda_1 > \lambda_2 > 0, \quad \lambda_3 = 0$	$\pm e_1$
E(1,1)	$\lambda_1^2 - \lambda_2^2 = -4,\ \lambda_1 > 0, \lambda_2 < 0, \quad \lambda_3 = 0$	$\pm e_2$
	$\lambda_1^2 - \lambda_2^2 = 4,\ \lambda_1 > 0, \lambda_2 < 0, \quad \lambda_3 = 0$	$\pm e_1$
Heisenberg group	$\lambda_1 = 2, \quad \lambda_2 = 0, \lambda_3 = 0$	$\pm e_1$
$R \oplus R \oplus R$	$\lambda_1 = \lambda_2 = \lambda_3 = 0$	*arbitrary left-invariant*

The case of nonunimodular groups. Let e_1 be a unit vector orthogonal to the unimodular kernel U and choose an orthonormal basis $\{e_2, e_3\}$ of U which diagonalizes the symmetric part of $ad_{e_1}\big|_U$. Then the bracket operation can be expressed as

$$[e_1, e_2] = \alpha\, e_2 + \beta\, e_3, \quad [e_1, e_3] = -\beta\, e_2 + \delta\, e_3, \quad [e_2, e_3] = 0. \tag{4}$$

If necessary, by changing e_1 to $-e_1$, we can assume $\alpha + \delta > 0$ and by possibly alternating e_2 and e_3, we may also suppose $\alpha \geq \delta$ [9].

The main result in this case is the following one.

Theorem 1.2. *Let G be a nonunimodular Lie group with the basis satisfying (4) of its Lie algebra. In assumption $\alpha+\delta>0$ and $\alpha\geq\delta$, the left-invariant totally geodesic unit vector fields on G are given as follows:*

Conditions on α,β,δ	*Left-invariant totally geodesic unit vector field*	*Geometrical structure of G*
$\beta=\delta=0$	$\pm e_3$	$L^2(-\alpha)\times E^1$
$\beta=\pm1,\quad \alpha\delta=-1$	$\pm\left(\beta\dfrac{1}{\sqrt{1+\alpha^2}}\,e_2+\dfrac{\alpha}{\sqrt{1+\alpha^2}}\,e_3\right)$	*Sasakian manifold*

.

2. Unimodular Case

Choose the orthonormal frame as in (3). Define connection numbers by

$$\mu_i=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3)-\lambda_i.$$

Then the Levi–Civita covariant derivatives can be expressed via the cross-products as follows $\nabla_{e_i}e_k=\mu_i\,e_i\times e_k$. For any left-invariant unit vector field $\xi=x_1e_1+x_2e_2+x_3e_3$ we have

$$\nabla_{e_i}\xi=\mu_i\,e_i\times\xi.$$

Denote $N_i=e_i\times\xi$. Then

$$\nabla_{e_i}\xi=\mu_i\,e_i\times\xi=\mu_i\,N_i.$$

As a consequence, the matrix of the Weingarten operator takes the form

$$A_\xi=\begin{pmatrix}0 & -\mu_2x_3 & \mu_3x_2\\ \mu_1x_3 & 0 & -\mu_3x_1\\ -\mu_1x_2 & \mu_2x_1 & 0\end{pmatrix}. \tag{5}$$

The following technical lemma can be checked by direct computation.

Lemma 2.1. *Let G be a three-dimensional unimodular Lie group with the left-invariant metric g and let $\{e_i,\ i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying (3). Then for any left-invariant unit vector field $\xi=x_1e_1+x_2e_2+x_3e_3$ we have*

$$\begin{aligned}
&A_\xi e_i=-\mu_i\,e_i\times\xi=-\mu_i\,N_i,\\
&(\nabla_{e_i}A_\xi)e_i=\mu_i^2(\xi-x_ie_i),\\
&(\nabla_{e_i}A_\xi)e_k=\varepsilon_{ikm}\mu_i\mu_mN_m-\mu_i\mu_kx_ie_k,\quad i\neq k,\\
&R(e_i,e_k)\xi=-\varepsilon_{ikm}\sigma_{ik}N_m,
\end{aligned}$$

where $\sigma_{ik}=\sigma_{ki}=\mu_i\mu_m+\mu_k\mu_m-\mu_i\mu_k$ and $\varepsilon_{ikm}=g(e_i\times e_k,e_m)$.

Remark that the chosen frame diagonalizes the Ricci tensor [8]. Moreover, $2\mu_i\mu_k = \rho_m$, where ρ_m is the principal Ricci curvature and $i \neq k \neq m$. It also worthwhile to mention that $\sigma_{ik} = \frac{1}{2}(\rho_k + \rho_i - \rho_m)$ is a sectional curvature of the left-invariant metric in the direction of $e_i \wedge e_k$.

The following Lemma is also a result of direct computations.

Lemma 2.2. *Let G be a three-dimensional unimodular Lie group with the left-invariant metric g and let $\{e_i,\ i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3)*. Then the left-invariant unit vector field $\xi = x_1e_1 + x_2e_2 + x_3e_3$ is totally geodesic if and only if for any $i \neq k \neq m$*

$$\begin{aligned}
TG(e_i,e_i) \quad &= x_i\mu_i\Big\{x_m(\sigma_{ik}\mu_k - \mu_i)N_k - x_k(\sigma_{im}\mu_m - \mu_i)N_m\Big\} = 0,\\
2TG(e_i,e_k) \quad &= \varepsilon_{ikm}\Big\{-x_ix_m\mu_i(\sigma_{ik}\mu_i - \mu_k)N_i + x_kx_m\mu_k(\sigma_{ik}\mu_k - \mu_i)N_k\\
&+\Big(\mu_i\mu_m(1-\sigma_{km}) - \mu_k\mu_m(1-\sigma_{im}) + \mu_i(\sigma_{km}\mu_m - \mu_k)x_i^2\\
&-\mu_k(\sigma_{im}\mu_m - \mu_i)x_k^2\Big)N_m\Big\} = 0,
\end{aligned}$$

where $\sigma_{ik} = \sigma_{ki} = \mu_i\mu_m + \mu_k\mu_m - \mu_i\mu_k$ and $\varepsilon_{ikm} = g(e_i \times e_k, e_m)$.

Now we can prove the main Lemma.

Lemma 2.3. *Let G be a three-dimensional unimodular Lie group with the left-invariant metric and let $\{e_i,\ i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3)*. Denote by ρ_1,ρ_2,ρ_3 the principal Ricci curvatures of the given group. Then the set of left-invariant totally geodesic unit vector fields can be described as follows:*

ρ_1	ρ_2	ρ_3	μ_1	μ_2	μ_3	ξ
0	0	0	0	0	0	*arbitrary left-invariant*
0	0	0	$\neq 0$	0	0	$\pm e_1,\ \cos t\, e_2 + \sin t\, e_3$
0	0	0	0	$\neq 0$	0	$\pm e_2,\ \cos t\, e_1 + \sin t\, e_3$
0	0	0	0	0	$\neq 0$	$\pm e_3,\ \cos t\, e_1 + \sin t\, e_2$
2						$\pm e_1$
	2					$\pm e_2$
		2				$\pm e_3$
2	2					$\cos t\, e_1 + \sin t\, e_2$
2		2				$\cos t\, e_1 + \sin t\, e_3$
	2	2				$\cos t\, e_2 + \sin t\, e_3$
2	2	2				*arbitrary left-invariant*

.

P r o o f. Rewrite the result of Lem. 2.2 for various combinations of indices to get

$$(1,1)\quad x_1\mu_1\Big\{x_3(\sigma_{12}\mu_2-\mu_1)N_2-x_2(\sigma_{13}\mu_3-\mu_1)N_3\Big\}=0,$$

$$(2,2)\quad x_2\mu_2\Big\{x_3(\sigma_{21}\mu_1-\mu_2)N_1-x_1(\sigma_{23}\mu_3-\mu_2)N_3\Big\}=0,$$

$$(3,3)\quad x_3\mu_3\Big\{x_2(\sigma_{31}\mu_1-\mu_3)N_1-x_1(\sigma_{32}\mu_2-\mu_3)N_2\Big\}=0,$$

$$\begin{aligned}
(1,2)\quad & -x_1x_3\mu_1(\sigma_{12}\mu_1-\mu_2)N_1+x_2x_3\mu_2(\sigma_{12}\mu_2-\mu_1)N_2+\Big(\mu_1\mu_3(1-\sigma_{23})\\
& -\mu_2\mu_3(1-\sigma_{13})+\mu_1(\sigma_{23}\mu_3-\mu_2)x_1^2-\mu_2(\sigma_{13}\mu_3-\mu_1)x_2^2\Big)N_3=0,\\
(2,3)\quad & -x_2x_1\mu_2(\sigma_{23}\mu_2-\mu_3)N_2+x_3x_1\mu_3(\sigma_{23}\mu_3-\mu_2)N_3+\Big(\mu_2\mu_1(1-\sigma_{31})\\
& -\mu_3\mu_1(1-\sigma_{21})+\mu_2(\sigma_{31}\mu_1-\mu_3)x_2^2-\mu_3(\sigma_{21}\mu_1-\mu_2)x_3^2\Big)N_1=0,\\
(3,1)\quad & -x_3x_2\mu_3(\sigma_{13}\mu_3-\mu_1)N_3+x_1x_2\mu_1(\sigma_{13}\mu_1-\mu_3)N_1+\Big(\mu_3\mu_2(1-\sigma_{12})\\
& -\mu_1\mu_2(1-\sigma_{32})+\mu_3(\sigma_{12}\mu_2-\mu_1)x_3^2-\mu_1(\sigma_{32}\mu_2-\mu_3)x_1^2\Big)N_2=0.
\end{aligned}$$

The vectors N_1, N_2 and N_3 are linearly dependent:

$$x_1N_1+x_2N_2+x_3N_3=0,$$

but linearly independent in the pairs for general (not specific) fields ξ.

The case $x_1\neq 0, x_2\neq 0, x_3\neq 0$.

The subcase 1: $\mu_1=0, \mu_2=0, \mu_3=0$. All equations are fulfilled evidently. Therefore, the *arbitrary left-invariant vector field is totally geodesic in this case*, and we get the first row in the table.

The subcase 2: $\mu_1=0$, $\mu_2\neq 0$ or $\mu_3\neq 0$. Then from (2,2) and (3,3) we see, that $\mu_2=0, \mu_3=0$, which gives a contradiction. In a similar way we exclude the cases when $\mu_i=0$, but $\mu_k^2+\mu_m^2\neq 0$ for arbitrary triple of different indices (i,k,m).

The subcase 3: $\mu_1\neq 0, \mu_2\neq 0, \mu_3\neq 0$. Since N_1, N_2 and N_3 are linearly independent in pairs, from (1,1), (2,2) and (3,3) we conclude:

$$\begin{cases}\sigma_{12}\mu_2-\mu_1=0,\\ \sigma_{12}\mu_1-\mu_2=0,\end{cases}\quad \begin{cases}\sigma_{13}\mu_3-\mu_1=0,\\ \sigma_{13}\mu_1-\mu_3=0,\end{cases}\quad \begin{cases}\sigma_{23}\mu_2-\mu_3=0,\\ \sigma_{23}\mu_3-\mu_2=0.\end{cases}\tag{6}$$

As a consequence, we get

$$(\sigma_{12}-1)(\mu_1+\mu_2)=0,\quad (\sigma_{13}-1)(\mu_1+\mu_3)=0,\quad (\sigma_{23}-1)(\mu_2+\mu_3)=0.$$

Taking into account (6), the rest of the equations yields

$$\begin{cases} \mu_1\mu_3(1-\sigma_{23}) - \mu_2\mu_3(1-\sigma_{13}) = 0, \\ \mu_1\mu_2(1-\sigma_{13}) - \mu_1\mu_3(1-\sigma_{12}) = 0, \\ \mu_2\mu_3(1-\sigma_{12}) - \mu_1\mu_2(1-\sigma_{23}) = 0. \end{cases}$$

Since $\mu_i \neq 0$, $i = 1,2,3$, we conclude $\sigma_{ik} = 1$, $i,k = 1,2,3$, and therefore $\rho_i = 2$, $i = 1,2,3$. This is the case of the last row in the table.

The case $x_1 \neq 0$, $x_2 \neq 0$, $x_3 = 0$. In this case $x_1N_1 + x_2N_2 = 0$, but N_1, N_3 and N_2, N_3 are *linearly independent* in pairs. Rewrite the system for this case as follows:

$$(1,1) \quad \mu_1(\sigma_{13}\mu_3 - \mu_1) = 0, \quad (2,2) \quad \mu_2(\sigma_{23}\mu_3 - \mu_2) = 0, \quad (3,3) \quad \equiv 0,$$

$$\begin{aligned} (1,2) \quad & \mu_1\mu_3(1-\sigma_{23}) - \mu_2\mu_3(1-\sigma_{13}) + \mu_1(\sigma_{23}\mu_3 - \mu_2)x_1^2 \\ & \qquad - \mu_2(\sigma_{13}\mu_3 - \mu_1)x_2^2 = 0, \\ (2,3) \quad & x_1^2\mu_2(\sigma_{23}\mu_2 - \mu_3) + \mu_1\mu_2(1-\sigma_{31} - \mu_1\mu_3(1-\sigma_{21}) \\ & \qquad + \mu_2(\sigma_{13}\mu_1 - \mu_3)x_2^2 = 0, \\ (3,1) \quad & -x_2^2\mu_1(\sigma_{13}\mu_1 - \mu_3 + \mu_2\mu_3(1-\sigma_{12}) - \mu_1\mu_2(1-\sigma_{32}) \\ & \qquad - \mu_1(\sigma_{23}\mu_2 - \mu_3)x_1^2 = 0. \end{aligned}$$

Set $\mu_1 = \mu_2 = 0$. Then the system is fulfilled for the arbitrary μ_3. The case $\mu_3 = 0$ has already been considered. The case $\mu_3 \neq 0$ gives the $\cos t\, e_1 + \sin t\, e_2$ in the **4-th row** of the table.
Set $\mu_1 = 0, \mu_2 \neq 0$. Then $\sigma_{12} = \mu_2\mu_3, \sigma_{13} = \mu_2\mu_3, \sigma_{23} = -\mu_2\mu_3$. The equation (2,2) yields $-\mu_2^2(\mu_3^2 + 1) = 0$, which gives a contradiction.
Set $\mu_1 \neq 0, \mu_2 = 0$. Then $\sigma_{12} = \mu_1\mu_3, \sigma_{13} = -\mu_1\mu_3, \sigma_{23} = \mu_1\mu_3$. The equation (1,1) yields $-\mu_1^2(\mu_3^2 + 1) = 0$, which gives a contradiction.
Set $\mu_1 \neq 0, \mu_2 \neq 0$. Then $\mu_1 = \sigma_{13}\mu_3, \mu_2 = \sigma_{23}\mu_3$ and the substitution into (1,2) yields $\mu_3^3(\mu_2 - \mu_1) = 0$. The case $\mu_3 = 0$ contradicts $\mu_1 \neq 0, \mu_2 \neq 0$, as one can see from (1,1) and (2,2). Thus, set $\mu_1 = \mu_2 = \mu \neq 0$. Then $\sigma_{13} = \sigma_{23} = \mu^2$ and from (1,1) and (2,2) we conclude

$$\mu\mu_3 - 1 = 0. \tag{7}$$

In this case we have

$$\sigma_{12} = 2 - \mu^2, \quad \sigma_{13} = \mu^2, \quad \sigma_{23} = \mu^2. \tag{8}$$

If we plug (7) and (8) into the system, then we get an identity. Since $\mu\mu_3 = 1$ means that $\rho_1 = \rho_2 = 2$, we get the **8-th row** of the table.

The case $x_1 \neq 0, x_2 = 0, x_3 \neq 0$ after similar computations results $\cos t\, e_1 + \sin t\, e_3$ in the **3-rd** and in the **9-th rows** of the table.

The case $x_1 = 0, x_2 \neq 0, x_3 \neq 0$ results $\cos t\, e_2 + \sin t\, e_3$ in the **2-nd** and in the **10-th** rows of the table.

The case $x_1 = 1,\ x_2 = 0, x_3 = 0$. In this case $N_1 = 0$ and the equations (1,1), (2,2), (3,3) and (2,3) are fulfilled regardless the geometry of the group. The equations (1,2) and (1,3) take the forms

$$(1,2) \quad \mu_1\mu_3(1-\sigma_{23}) - \mu_2\mu_3(1-\sigma_{13}) + \mu_1(\sigma_{23}\mu_3 - \mu_2) = 0,$$
$$(1,3) \quad \mu_2\mu_3(1-\sigma_{12}) - \mu_1\mu_2(1-\sigma_{23}) - \mu_1(\sigma_{23}\mu_2 - \mu_3) = 0.$$

After simplifications, we get

$$(1,2) \quad \sigma_{13}(\mu_2\mu_3 - 1) = 0, \qquad (1,3) \quad \sigma_{12}(\mu_2\mu_3 - 1) = 0.$$

The case $\mu_2\mu_3 = 1$ means $\rho_1 = 2$, and we have the **5-th row** of the table. Consider the case $\sigma_{12} = 0, \sigma_{13} = 0$ which is equivalent to $\mu_2\mu_3 = 0$ and $\mu_1(\mu_2 - \mu_3) = 0$. We have four possible solutions:

$$(i)\ \mu_1 = 0, \mu_2 = 0, \mu_3 = 0; \quad (ii)\ \mu_1 = 0, \mu_2 = 0, \mu_3 \neq 0;$$
$$(iii)\ \mu_1 = 0, \mu_2 \neq 0, \mu_3 = 0; \quad (iv)\ \mu_1 \neq 0, \mu_2 = 0, \mu_3 = 0.$$

The case (i) is already included into the 1-st row of the table; the case (ii) is already included into $\cos t\, e_1 + \sin t\, e_2$ case in the 4-st one of the table; the case (iii) is already included into $\cos t\, e_1 + \sin t\, e_3$ case in the 3-rd row of the table. The case (iv) is a new one and yields e_1 in the **2-nd** row of the table.

The case $x_1 = 0, x_2 = 1, x_3 = 0$ yields e_2 in the **3-rd** and the **6-th** rows of the table. **The case** $x_1 = 0, x_2 = 0, x_3 = 1$ yields e_3 in the **4-th** and the **7-th** rows of the table. ■

If we specify the result of Lem. 2.3 to each unimodular group, then we get the result of Th. 1.1.

2.1. Geometrical Characterization of Totally Geodesic Unit Vector Fields on Unimodular Groups

Let M be an odd-dimensional smooth manifold. Denote by ϕ, ξ, η a $(1,1)$-tensor field, a vector field and a 1-form on M respectively. A triple (ϕ, ξ, η) is called an *almost contact structure* on M if

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{9}$$

for any vector field X on M. The manifold M with an almost contact structure is called an *almost contact* manifold. If M is endowed with the Riemannian metric $g(\cdot,\cdot)$ such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(\xi, X) \tag{10}$$

for all vector fields X and Y on M, then a quadruple (ϕ, ξ, η, g) is called an *almost contact* ***metric*** *structure* and the manifold is called an *almost contact* ***metric*** *manifold.* The first of the conditions above is called a *compatibility condition* for ϕ and g. If the 2-form $d\eta$, given by $d\eta(X,Y) = \frac{1}{2}\big(X\eta(Y) - Y\eta(X) - \eta([X,Y])\big)$, satisfies

$$d\eta(X,Y) = g(X, \phi Y)\,,$$

then the structure (ϕ, ξ, η, g) is called **a contact metric structure** and the manifold with a contact metric structure is called a **contact metric manifold**. A contact metric manifold is called *K-contact* if ξ is a Killing vector field. The Nijenhuis torsion of tensor field T of type $(1,1)$ is given by

$$[T,T](X,Y) = T^2[X,Y] + [TX,TY] - T[TX,Y] - T[X,TY]$$

and defines a $(1,2)$-tensor field on M. An almost contact structure (ϕ, ξ, η) is called *normal* if

$$[\phi,\phi](X,Y) + 2d\eta(X,Y)\,\xi = 0. \tag{11}$$

Finally, a contact metric structure (ϕ, ξ, η, g) is called *Sasakian*, if it is normal. A manifold with the Sasakian structure is called a *Sasakian manifold.* In the Sasakian manifold necessarily $\phi = A_\xi$ and $\eta = g(\xi, \cdot)$. The unit vector field ξ is called a *characteristic vector field* of the Sasakian structure and is a Killing one. This vector field is always totally geodesic [12].

In the three-dimensional case we have a stronger result.

Theorem 2.1. [12]. *Let ξ be a unit Killing vector field on a three-dimensional Riemannian manifold M^3. If $\xi(M^3)$ is totally geodesic in T_1M^3, then either*

$$\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\,\cdot\,)\big)$$

is a Sasakian structure on M^3, or $M^3 = M^2 \times E^1$ metrically and ξ is a unit vector field of the Euclidean factor.

Now we can give a geometrical description of totally geodesic unit vector fields.

Proposition 2.1. *Let ξ be a left-invariant totally geodesic unit vector field on $SU(2)$ with the left-invariant metric g and let $\{e_i,\ i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3)*. Assume in addition that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\,\cdot\,)\big) \tag{12}$$

is an almost contact structure on $SU(2)$. Moreover,

- *if $\lambda_1 = \lambda_2 = \lambda_3 = 2$ or $\lambda_1 = \lambda_2 > \lambda_3 = 2$ or $\lambda_1 = 2 > \lambda_2 = \lambda_3$, then the structure is Sasakian;*

- *if* $\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$ *or* $\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$, *then the structure is neither normal nor metric;*

- *if* $\lambda_1 > \lambda_2 > \lambda_3$, *then the structure is normal only for*

$$\xi = e_1, \quad \lambda_1 = \lambda_2 + \frac{1}{\lambda_2}, \quad \lambda_3 = \frac{1}{\lambda_2}, \quad \lambda_2 > 1.$$

P r o o f. Consider the cases from Th. 1.1.

• In the case of $\lambda_1 = \lambda_2 = \lambda_3 = 2$ we have $\mu_1 = \mu_2 = \mu_3 = 1$ and hence

$$A_\xi = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

Therefore, the field ξ is the Killing one. By Theorem 2.1, the structure (12) is Sasakian.

For $\lambda_1 = \lambda_2 = \lambda > \lambda_3 = 2$ we have $\mu_1 = 1, \mu_2 = 1, \mu_3 = \lambda - 1$ and $\xi = \pm e_3$. For $\xi = +\, e_3$ we find

$$A_\xi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so ξ is again a Killing unit vector field and the structure (12) is Sasakian.

For $\lambda_1 = 2 > \lambda_2 = \lambda_3 = \lambda > 0$, we have $\mu_1 = -1 + \lambda, \mu_2 = 1, \mu_3 = 1$ and $\xi = \pm e_1$. For $\xi = +\, e_1$ we find

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and see that again ξ is the Killing unit vector field. Therefore, the structure (12) is Sasakian.

• Consider the case $\lambda_1 = \lambda_2 = \lambda > 2 > \lambda_3 = \lambda - \sqrt{\lambda^2 - 4}$ and $\xi = x_1 e_1 + x_2 e_2$. We have

$$\mu_1 = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}), \quad \mu_2 = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}), \quad \mu_3 = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}).$$

For brevity, put $\theta = \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$ and $\bar\theta = \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})$. Then

$$\mu_1 = \theta, \quad \mu_2 = \theta, \quad \mu_3 = \bar\theta, \quad \theta\bar\theta = 1, \quad \theta \neq 1, \bar\theta \neq 1,$$

and for this case we have

$$\phi = A_\xi = \begin{pmatrix} 0 & 0 & \bar\theta x_2 \\ 0 & 0 & -\bar\theta x_1 \\ -\theta x_2 & \theta x_1 & 0 \end{pmatrix}.$$

Since $\theta \neq \bar{\theta}$, the field ξ is never a Killing one but it is geodesic, since $A_\xi \xi = 0$. The structure (12) is an almost contact one on $SU(2)$. Indeed,

$$\phi^2 = A_\xi^2 = \begin{pmatrix} -x_2^2 & x_1x_2 & 0 \\ x_1x_2 & -x_1^2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \phi^2 Z = -Z + g(\xi, Z)\,\xi.$$

This structure is not metric. For the compatibility condition (10), we have

$$\begin{aligned} g(\phi Z, \phi W) &= \theta^2(z_1w_1 + z_2w_2) + \bar{\theta}^2 x_3w_3 - g(\xi, Z)\, g(\xi, W) \\ &\neq g(Z, W) \ - g(\xi, Z)\, g(\xi, W)\,. \end{aligned}$$

This structure is not normal. To prove this, check the normality condition (9). We have

$$[\phi, \phi](e_1, e_2) = \theta(\theta^2 - 1)e_3 \neq 2d\eta(e_1, e_2)\xi.$$

In a similar way we can analyze the case $\lambda_1 = \lambda + \sqrt{\lambda^2 - 4} > \lambda = \lambda_2 = \lambda_3 > 2$ with the same result.

- Consider the case $\lambda_1 > \lambda_2 > \lambda_3$, $\xi = \pm e_i$. We have

$$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3), \quad \mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3).$$

Set $\xi = e_1$. The condition $\lambda_1^2 - (\lambda_2 - \lambda_3)^2 = 4$ means that $\mu_2\mu_3 = 1$. The matrix A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu_3 \\ 0 & \mu_2 & 0 \end{pmatrix} = \phi.$$

Since $\mu_2 \neq \mu_3$, the field ξ is not a Killing one, but it is geodesic. The structure (12) is almost contact. Indeed,

$$\phi^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu_2\mu_3 & 0 \\ 0 & & -\mu_3\mu_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and hence

$$\phi^2 Z = -Z + g(\xi, Z)\,\xi.$$

The structure is *normal* if and only if

$$\lambda_1 = \lambda_2 + \frac{1}{\lambda_2}, \quad \lambda_3 = \frac{1}{\lambda_2}, \quad \lambda_2 > 1. \tag{13}$$

Indeed, note that $\phi e_1 = 0$, $\phi e_2 = \mu_2 e_3$, $\phi e_3 = -\mu_3 e_2$. Now put $Z = e_1, W = e_2$. Then we have

$$[\phi, \phi][e_1, e_2] = (-\lambda_3 + \mu_2^2\lambda_2)\, e_3, \qquad d\eta(e_1, e_2) = 0.$$

Therefore, the first necessary condition of normality is $\lambda_3 = \mu_2^2\lambda_2$. Since $\mu_2\mu_3 = 1$, we can rewrite this condition as

$$\lambda_3\mu_3 = \lambda_2\mu_2. \tag{14}$$

Put $Z = e_1, W = e_3$. Then we have

$$[\phi,\phi][e_1,e_3] = (\lambda_2 - \mu_2^2\lambda_3)\, e_2, \qquad d\eta(e_1,e_3) = 0.$$

The second necessary condition of normality is $\lambda_2 = \mu_3^2\lambda_3$, which is equivalent to (14).

Finally, put $Z = e_2, W = e_3$. Then we have

$$[\phi,\phi][e_2,e_3] = \lambda_1\, e_1, \qquad d\eta(e_2,e_3) = -\frac{1}{2}e_1$$

and (11) is fulfilled. The equation (14) can be simplified to

$$(\lambda_3 - \lambda_2)(\lambda_1 - (\lambda_2 + \lambda_3)) = 0.$$

Since $\lambda_2 \neq \lambda_3$, we get $\lambda_1 = \lambda_2 + \lambda_3$. Then $\mu_1 = 0$, $\mu_2 = \lambda_3$, $\mu_3 = \lambda_2$ and, from the condition $\mu_2\mu_3 = 1$, we find $\lambda_2\lambda_3 = 1$. Since $\lambda_1 > \lambda_2 > \lambda_3$, we get (13).

The structure *is not metric*, since

$$g(\phi Z, \phi W) \;= \mu_3^2 z_3 w_3 + \mu_2^2 z_2 w_2 \neq g(Z,W) \;- g(\xi, Z)\, g(\xi, W) \;= z_2 w_2 + z_3 w_3.$$

Making similar computations for $\xi = e_2$, we get the normality condition of the form $\lambda_2 = \lambda_1 + \lambda_3$ which contradicts the condition $\lambda_1 > \lambda_2 > \lambda_3$. The structure is not metric. Finally, for $\xi = e_3$, we get the normality condition of the form $\lambda_3 = \lambda_1 + \lambda_2$ which contradicts again the condition $\lambda_1 > \lambda_2 > \lambda_3$ and the structure is not metric again. ■

In a similar way we prove the following propositions.

Proposition 2.2. *Let ξ be a left-invariant totally geodesic unit vector field on $SL(2,R)$ with the left-invariant metric g and let $\{e_i,\, i = 1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3)*. Assume in addition that $\lambda_1 \geq \lambda_2 > 0$, $\lambda_3 < 0$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\,\cdot\,)\big)$$

is the almost contact structure on $SL(2,R)$, where $g(\cdot\,,\,\cdot)$ is the scalar product with respect to g. Moreover, if

- $\lambda_1 = \lambda_2, \quad \lambda_3 = -2$, *then the structure is Sasakian;*
- $\lambda_3 = -\sqrt{4 + (\lambda_1 - \lambda_2)^2} < -2$ *or* $\lambda_1 = \sqrt{4 + (\lambda_2 - \lambda_3)^2}$ *, then the structure is neither normal nor metric.*

Proposition 2.3. *Let ξ be a left-invariant totally geodesic unit vector field on $E(2)$ with the left-invariant metric g and let $\{e_i,\, i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3). *Assume in addition that $\lambda_1 \ge \lambda_2 > 0$, $\lambda_3 = 0$.*

If $\lambda_1 = \lambda_2 = \lambda > 0$, then the group is flat. Moreover,

- *if $\xi = e_3$, then ξ is a parallel vector field on $E(2)$;*
- *if $\xi = x_1 e_1 + x_2 e_2$, then ξ moves along e_3 with the constant angle speed λ.*

If $\lambda_1 > \lambda_2 > 0$, then $\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\cdot\,)\big)$ is an almost contact structure on $E(2)$. This structure is neither metric nor normal.

Proposition 2.4. *Let ξ be a left-invariant totally geodesic unit vector field on $E(1,1)$ with the left-invariant metric and let $\{e_i,\, i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3). *Assume in addition that $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 = 0$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\cdot\,)\big)$$

is an almost contact structure on $E(1,1)$. This structure is neither metric nor normal.

Proposition 2.5. *Let ξ be a left-invariant totally geodesic unit vector field on the Heisenberg group with the left-invariant metric and let $\{e_i,\, i=1,2,3\}$ be an orthonormal basis for the Lie algebra satisfying* (3). *Moreover, assume that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0$. Then*

$$\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi,\cdot\,)\big)$$

is a Sasakian structure.

3. Nonunimodular Case

Choose the orthonormal frame e_1, e_2, e_3 as in (4). Then the Levi–Civita connection is given by the following table:

∇	e_1	e_2	e_3
e_1	0	$\beta\, e_3$	$-\beta\, e_2$
e_2	$-\alpha\, e_2$	$\alpha\, e_1$	0
e_3	$-\delta\, e_3$	0	$\delta\, e_1$

.

For any left-invariant unit vector field $\xi = x_1e_1 + x_2e_2 + x_3e_3$ we have

$$\nabla_{e_1}\xi = \beta\ e_1 \times \xi, \quad \nabla_{e_2}\xi = -\alpha\ e_3 \times \xi, \quad \nabla_{e_3}\xi = \delta\ e_2 \times \xi.$$

Denote

$$\begin{aligned} N_1 &= e_1 \times \xi = -x_3\, e_2 + x_2\, e_3, \\ N_2 &= e_3 \times \xi = -x_2\, e_1 + x_1\, e_2, \\ N_3 &= e_2 \times \xi = x_3\, e_1 - x_1\, e_3. \end{aligned} \tag{15}$$

Then the matrix of A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & -\alpha\, x_2 & -\delta\, x_3 \\ \beta\, x_3 & \alpha\, x_1 & 0 \\ -\beta\, x_2 & 0 & \delta\, x_1 \end{pmatrix}. \tag{16}$$

A direct computation gives the following result.

Lemma 3.1. *The derivatives $(\nabla_{e_i}A_\xi)e_k$ of the Weingarten operator A_ξ for the left-invariant unit vector field are as in the following table:*

	e_1	e_2	e_3
e_1	$-\beta^2(x_1e_1 - \xi)$	$\beta\,\delta\, N_3 + \beta\,\alpha\, x_1e_3$	$\beta\,\alpha\, N_2 - \beta\,\delta\, x_1e_2$
e_2	$\alpha^2N_2 + \beta\,\alpha\, x_3e_1$	$\beta\,\alpha\, N_1 - \alpha^2(x_3e_3 - \xi)$	$\alpha\,\delta\, x_3e_2$
e_3	$-\delta^2N_3 - \beta\,\delta\, x_2e_1$	$\alpha\,\delta\, x_2e_3$	$\beta\,\delta\, N_1 - \delta^2(x_2e_2 - \xi)$

By a straightforward application of the Codazzi equation and Lem. 3.1, we can easily prove the following.

Lemma 3.2. *The curvature operator of the nonunimodular group with respect to the chosen frame takes the form*

$$\begin{aligned} R(e_1, e_2)\xi &= \alpha^2N_2 + \beta\,(\alpha - \delta)N_3, \\ R(e_1, e_3)\xi &= -\delta^2N_3 - \beta\,(\alpha - \delta)\,N_2, \\ R(e_2, e_3)\xi &= \alpha\,\delta\,N_1. \end{aligned}$$

The following Lemma gives the components of (2).

Lemma 3.3. *Let G be a nonunimodular Lie group with the basis satisfying (4). Then the left-invariant unit vector field $\xi = x_1e_1 + x_2e_2 + x_3e_3$ is totally geodesic if and only if it satisfies the following equations:*

$$\begin{aligned} (1,1)\quad \beta\, x_1\Big\{&\big[\beta\,[1 + \alpha\,(\alpha - \delta)]x_2 + \alpha^3x_3\big]N_2 \\ &- \big[\beta\,[1 - \delta\,(\alpha - \delta)]x_3 - \delta^3x_2\big]N_3\Big\} = 0, \end{aligned}$$

$$(2,2)\quad \alpha\Big\{\Big[\beta\,[1+\alpha^2(1-x_3^2)]-[\alpha+\beta^2(\alpha-\delta)]x_2x_3\Big]N_1 + \alpha\Big[1+\delta^2\Big]x_1x_3N_3\Big\}=0,$$

$$(3,3)\quad \delta\Big\{\Big[\beta\,[1+\delta^2(1-x_2^2)]+[\delta-\beta^2(\alpha-\delta)]x_2x_3\Big]N_1 - \delta\Big[1+\alpha^2\Big]x_1x_2N_2\Big\}=0,$$

$$(1,2)\quad \beta\,x_1\Big[[\alpha+\beta^2(\alpha-\delta)]x_2+\beta\,\alpha^2x_3\Big]N_1 + \alpha\Big[\alpha\,[1+\alpha^2(1-x_3^2)]-\beta\,[1+\alpha\,(\alpha-\delta)]x_2x_3\Big]N_2$$
$$+\Big[\alpha\,\delta\,[\beta\,\delta\,(1-x_1^2)-\delta^2x_2x_3+\beta\,(\alpha-\delta)(1-x_3^2)]+\beta\,\alpha\,(x_3^2-x_1^2)+\beta\,\delta\Big]N_3=0,$$

$$(1,3)\quad \beta\,x_1\Big[[\delta-\beta^2(\alpha-\delta)]x_3-\beta\,\delta^2x_2\Big]N_1$$
$$-\Big[\alpha\,\delta\,[\alpha\,\beta\,(1-x_1^2)+\alpha^2x_2x_3-\beta\,(\alpha-\delta)(1-x_2^2)]+\beta\,\alpha+\beta\,\delta\,(x_2^2-x_1^2)\Big]N_2$$
$$+\delta\Big[\beta\,[-1+\delta\,(\alpha-\delta)]x_2x_3-\delta\,[1+\delta^2(1-x_2^2)]\Big]N_3=0,$$

$$(2,3)\quad \Big[\beta\,[\alpha\,\delta\,(\alpha+\delta)x_2x_3-\beta\,(\alpha-\delta)(\alpha\,(1-x_3^2)+\delta\,(1-x_2^2))]$$
$$+\alpha\,\delta\,(x_2^2-x_3^2)\Big]N_1+\alpha\,\delta\Big[1+\alpha^2\Big]x_1x_3N_2-\alpha\,\delta\Big[1+\delta^2\Big]x_1x_2N_3=0.$$

The proof consists of rather long computations of the corresponding components $TG_\xi(e_i,e_k)$ for various combinations of (i,k) similar to those in the unimodular case.

P r o o f of the Theorem 1.2. Set $\xi=x_1e_1+x_2e_2+x_3e_3$ and suppose $x_1\neq 0$. From (15) it follows that $N_2\neq 0$, $N_3\neq 0$ and they are always linearly independent. Moreover, the vectors N_1 and N_3 are linearly dependent if and only if $x_3=0$. If $x_3\neq 0$, then the equation $(2,2)$ implies $x_3=0$ and we come to a contradiction.

Put $x_3=0$. If $x_2\neq 0$, then N_1 and N_2 are linearly independent and $(3,3)$ implies $\delta=0$. In this case we can rewrite $(1,1)$ as $\beta^2x_1x_2(1+\alpha^2)N_2=0$ and we have $\beta=0$. In this case the equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2=0$ and we have a contradiction.

Put $x_3=x_2=0$. In this case $\xi=e_1$, $N_1=0$, $N_2=e_2$ and $N_3=-e_3$. The equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2=0$. This gives a contradiction.

Suppose $\xi=x_2e_2+x_3e_3$. Since $x_1=0$, we have $N_1\neq 0$ and N_1 is linearly independent with either N_2 or N_3.

Suppose $\beta=0$. Then $(2,2)$ implies $-\alpha^2x_2x_3=0$ and we have the following cases:

- Case $x_3 = 0$. Then $N_1 = \pm e_3$, $N_2 = \mp e_1$, $N_3 = 0$ and the equation $(1,2)$ takes the form $\alpha^2(1+\alpha^2)N_2 = 0$, which is a contradiction.
- Case $x_2 = 0$. Then $N_1 = \mp e_2$, $N_2 = 0$, $N_3 = \pm e_1$. The equation $(1,3)$ then takes the form $-\delta^2(1+\delta^2)N_3 = 0$ and we should set $\delta = 0$. It is easy to check that if $\beta = \delta = 0$, then all equations are fulfilled. Moreover, the field $\xi = \pm e_3$ becomes a parallel vector field, since $A_\xi \equiv 0$.

Suppose $\beta \neq 0$, $\delta = 0$. Then $(1,3)$ implies $\beta\,\alpha\,N_2 = 0$ and we have $x_2 = 0$. In this case $x_3^2 = 1$ and $(2,2)$ yields $\alpha\,\beta\,N_1 = 0$. This gives a contradiction.

Suppose $\beta \neq 0, \delta \neq 0$. In this case we apply a different method based on the explicit expression for the second fundamental form of $\xi(M^n) \subset T_1M^n$ [11].

Lemma 3.4. *Let ξ be a unit vector field on a Riemannian manifold M^{n+1}. The components of second fundamental form of $\xi(M) \subset T_1M^{n+1}$ are given by*

$$\begin{aligned}\tilde\Omega_{\sigma|ij} &= \tfrac12\Lambda_{\sigma ij}\Big\{-\big\langle(\nabla_{e_i}A_\xi)e_j + (\nabla_{e_j}A_\xi)e_i, f_\sigma\big\rangle \\ &\quad +\lambda_\sigma\left[\lambda_j\big\langle R(e_\sigma,e_i)\xi, f_j\big\rangle + \lambda_i\big\langle R(e_\sigma,e_j)\xi, f_i\big\rangle\right]\Big\},\end{aligned}$$

where $\Lambda_{\sigma ij} = [(1+\lambda_\sigma^2)(1+\lambda_i^2)(1+\lambda_j^2)]^{-1/2}$, $\lambda_0 = 0, \lambda_1, \dots, \lambda_n$ are the singular values of the matrix A_ξ and $e_0, e_1, \dots, e_n$; $f_1, \dots, f_n$ are the orthonormal frames of singular vectors $(i,j = 0,1,\dots,n;\ \sigma = 1,\dots,n)$.

Since $x_1 = 0$, the matrix (16) takes the form

$$A_\xi = \begin{pmatrix} 0 & -\alpha\,x_2 & -\delta\,x_3 \\ \beta\,x_3 & 0 & 0 \\ -\beta\,x_2 & 0 & 0 \end{pmatrix}.$$

Denote by $\tilde e_0$, $\tilde e_1$, $\tilde e_2$; $\tilde f_1$, $\tilde f_2$ the orthonormal singular frames of A_ξ. The matrix $A_\xi^tA_\xi$ takes the form

$$A_\xi^tA_\xi = \begin{pmatrix} \beta^2 & 0 & 0 \\ 0 & \alpha^2x_2^2 & \alpha\,\delta\,x_2x_3 \\ 0 & \alpha\,\delta\,x_2x_3 & \delta^2x_3^2 \end{pmatrix}. \tag{17}$$

The eigenvalues are $\left[0, \beta^2, \alpha^2x_2^2 + \delta^2x_3^2\right]$. Denote $m = \sqrt{\alpha^2x_2^2 + \delta^2x_3^2}$. Then the singular values are

$$\lambda_0 = 0,\ \lambda_1 = |\beta|,\ \lambda_2 = m.$$

The singular frame $\tilde e_0, \tilde e_1, \tilde e_2$ consists of the eigenvectors of the matrix (17), namely,

$$\tilde e_0 = \frac{1}{m}\big(-\delta\, x_3\, e_2 + \alpha\, x_2\, e_3\big), \quad \tilde e_1 = e_1, \quad \tilde e_2 = \frac{1}{m}\big(\alpha\, x_2\, e_2 + \delta\, x_3\, e_3\big).$$

To find $\tilde f_1$ and $\tilde f_2$, compute

$$A_\xi\, \tilde e_1 = \beta\,\big(x_3\, e_2 - x_2\, e_3\big), \quad A_\xi\, \tilde e_2 = -m\, e_1.$$

Denote $\varepsilon = sign(\beta)$. Then

$$\tilde f_1 = \varepsilon\big(x_3\, e_2 - x_2\, e_3\big), \quad \tilde f_2 = -e_1.$$

Now we have

$$\tilde\Omega_{\sigma|00} = -\frac{1}{\sqrt{1+\lambda_\sigma^2}} g((\nabla_{\tilde e_0} A_\xi)\, \tilde e_0,\, \tilde f_\sigma)\,.$$

If ξ is totally geodesic, then ξ satisfies

$$0 = (\nabla_{\tilde e_0} A_\xi)\, \tilde e_0 = \nabla_{\tilde e_0}(A_\xi\, \tilde e_0) - A_\xi \nabla_{\tilde e_0}\tilde e_0 = A_\xi A_{\tilde e_0}\, \tilde e_0.$$

Since (16) is applicable to any left-invariant unit vector field, we easily calculate

$$A_{\tilde e_0}\, \tilde e_0 = -\frac{1}{m^2}\alpha\,\delta\,(\delta\, x_3^2 + \alpha\, x_2^2)\ \tilde e_1.$$

Therefore,

$$A_\xi A_{\tilde e_0}\, \tilde e_0 = -\varepsilon\beta\,\alpha\,\delta\,(\delta\, x_3^2 + \alpha\, x_2^2)\, \tilde f_1.$$

Since $\beta \neq 0, \alpha \neq 0$ and $\delta \neq 0$, we have

$$\begin{cases} \alpha\, x_2^2 + \delta\, x_3^2 = 0, \\ x_2^2 + x_3^2 = 1. \end{cases}$$

Solving the system, we get

$$x_2^2 = \frac{-\delta}{\alpha - \delta}, \quad x_3^2 = \frac{\alpha}{\alpha - \delta}.$$

Remind that $\alpha + \delta > 0, \quad \alpha \geq \delta$ by the choice of the frame. Therefore, the solution exists if $\delta < 0$ and, as a consequence, $\alpha > 0$. Thus,

$$\xi = \pm\sqrt{\frac{-\delta}{\alpha - \delta}}\ e_2 \pm \sqrt{\frac{\alpha}{\alpha - \delta}}\ e_3.$$

Denote $\theta = \pm 1$. Without loss of generality, we can put

$$\xi = \theta\sqrt{\frac{-\delta}{\alpha - \delta}}\ e_2 + \sqrt{\frac{\alpha}{\alpha - \delta}}\ e_3.$$

As a consequence, we get $m = \sqrt{-\alpha\,\delta}$. Moreover,

$$\frac{\alpha}{m}x_2 = \theta\,\frac{\alpha}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{-\delta}{\alpha-\delta}} = \theta\,x_3,$$
$$\frac{\delta}{m}x_3 = \frac{\delta}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{\alpha}{\alpha-\delta}} = \frac{-\sqrt{(-\delta\,)^2}}{\sqrt{-\alpha\,\delta}}\sqrt{\frac{\alpha}{\alpha-\delta}} = -\theta\,x_2$$

and we have

$$\tilde e_0 = \tfrac{1}{m}\big(-\delta\,x_3\,e_2 + \alpha\,x_2\,e_3\big) = \theta\,\xi, \quad \tilde e_1 = e_1 = -\,\tilde f_2,$$
$$\tilde e_2 = \tfrac{1}{m}\big(\alpha\,x_2\,e_2 + \delta\,x_3\,e_3\big) = \theta(x_3\,e_2 - x_2\,e_3) = \theta\varepsilon\;\tilde f_1.$$

With respect to this frame the matrix A_ξ takes the form

$$A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -m \\ 0 & \theta\beta & 0 \end{pmatrix}.$$

A simple calculation yields

∇	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-\theta m\,\tilde e_2$	$\theta m\,\tilde e_1$
$\tilde e_1$	$-\beta\,\tilde e_2$	0	$\beta\,\tilde e_0$
$\tilde e_2$	$\theta m\,\tilde e_1$	$-\theta m\,\tilde e_0 - (\alpha+\delta\,)\,\tilde e_2$	$(\alpha+\delta\,)\,\tilde e_1$

. (18)

With respect to a new frame, the derivatives $(\nabla_{\tilde e_i} A_\xi)\,\tilde e_k$ form the following table:

	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-m(\theta m - \beta\,)\;\tilde e_1$	$m(\theta m - \beta\,)\;\tilde e_2$
$\tilde e_1$	$-m\beta\;\tilde e_1$	$\theta\beta^2\;\tilde e_0$	0
$\tilde e_2$	$-m\beta\;\tilde e_2$	$-(\alpha+\delta\,)(m-\theta\beta\,)\;\tilde e_1$	$\theta m^2\,\tilde e_0 + (\alpha+\delta\,)(m-\theta\beta\,)\;\tilde e_2$

.

Finally, the necessary components of the curvature operator can be found from the latter table and take the form

$$R(\tilde e_0, \tilde e_1)\xi = m(\theta m - 2\beta\,)\;\tilde e_1, \quad R(\tilde e_0, \tilde e_2)\xi = -\theta m^2\;\tilde e_2,$$
$$R(\tilde e_1, \tilde e_2)\xi = -(\alpha+\delta\,)(m-\theta\beta\,)\;\tilde e_1. \tag{19}$$

Now, we can compute all the entries of the matrices $\tilde\Omega_\sigma$. As a result, we have

$$\tilde\Omega_1 = \begin{pmatrix} 0 & 0 & \dfrac{1}{2}\dfrac{\varepsilon\theta m(m-2\theta\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} \\ 0 & 0 & 0 \\ \dfrac{1}{2}\dfrac{\varepsilon\theta m(m-2\theta\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} & 0 & \dfrac{\varepsilon\theta(\alpha+\delta)(\theta m-\beta)(m\beta-\theta)}{\sqrt{(1+\beta^2)(1+m^2)}} \end{pmatrix},$$

$$\tilde\Omega_2 = \begin{pmatrix} 0 & \dfrac{m^2(m\beta-\theta)}{2\sqrt{(1+\beta^2)(1+m^2)}} & 0 \\ \dfrac{m^2(m\beta-\theta)}{2\sqrt{(1+\beta^2)(1+m^2)}} & 0 & \dfrac{(\alpha+\delta)(\theta\beta-m)}{2\sqrt{(1+\beta^2)}} \\ 0 & \dfrac{(\alpha+\delta)(\theta\beta-m)}{2\sqrt{(1+\beta^2)}} & 0 \end{pmatrix}.$$

Thus, for a totally geodesic field ξ, we have a unique possible solution $\beta=\theta m$, $m\beta=\theta$. It follows that $-\alpha\delta=m^2=1$, $\beta=\theta(=\pm1)$. As a consequence,

$$\pm\xi = \beta\frac{1}{\sqrt{1+\alpha^2}}\,e_2 + \frac{\alpha}{\sqrt{1+\alpha^2}}\,e_3$$

is the corresponding totally geodesic unit vector field.

Now we give a geometrical description of totally geodesic unit vector field and the group.

Proposition 3.1. *Let G be a nonunimodular three-dimensional Lie group with the left-invariant metric. Suppose that G admits a left-invariant totally geodesic unit vector field ξ. Then either*

- *$G=L^2(-\alpha^2)\times E^1$, where $L^2(-\alpha^2)$ is the Lobachevski plane of curvature $-\alpha^2$, and ξ is a parallel unit vector field on G tangent to the Euclidean factor, or*
- *G admits a Sasakian structure; moreover, G admits two hyperfoliations $\mathcal{L}_1,\mathcal{L}_2$ such that:*

  (i) the foliations $\mathcal{L}_1$ and $\mathcal{L}_2$ are intrinsically flat, mutually orthogonal and have constant extrinsic curvature;

  (ii) one of them, say $\mathcal{L}_2$, is minimal;

  (iii) the integral trajectories of the field ξ are $\mathcal{L}_1\cap\mathcal{L}_2$.

P r o o f. Suppose ξ is as in the hypothesis. Consider the case $\beta = \delta = 0$ and $\xi = e_3$ of Th. 1.2. The bracket operations take the form

$$[e_1, e_2] = \alpha\, e_2, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0,$$

and we conclude that the group admits three integrable distributions, namely, $e_1 \wedge e_2$, $e_1 \wedge e_3$ and $e_2 \wedge e_3$. The table of the Levi–Civita connection takes the form

∇	e_1	e_2	e_3
e_1	0	0	0
e_2	$-\alpha\, e_2$	$-\alpha\, e_1$	0
e_3	0	0	0

.

The only nonzero component of the curvature tensor of the group is of the form $R(e_1, e_2)e_2 = -\alpha^2 e_1$. Thus, $G = L^2(-\alpha) \times R^1$ and the field $\xi = e_3$ is a parallel unit vector field on G tangent to the Euclidean factor.

Consider the second case of Th. 1.2. If $\beta = \theta$, $m = \sqrt{-\alpha\,\delta} = 1$, then with respect to the singular frame, the matrix A_ξ takes the form $A_\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and hence, $\xi = \theta\, \tilde e_0$ is a Killing unit vector field. Therefore, by Th. 2.1, the structure $\big(\phi = A_\xi,\ \xi,\ \eta = g(\xi, \cdot)\big)$ is Sasakian.

We can also say more about this Sasakian structure. The table (18) in the case under consideration takes the form

∇	$\tilde e_0$	$\tilde e_1$	$\tilde e_2$
$\tilde e_0$	0	$-\theta\, \tilde e_2$	$\theta\, \tilde e_1$
$\tilde e_1$	$-\theta\, \tilde e_2$	0	$\theta\, \tilde e_0$
$\tilde e_2$	$\theta\, \tilde e_1$	$-\theta\, \tilde e_0 - (\alpha + \delta)\, \tilde e_2$	$(\alpha + \delta)\, \tilde e_1$

, (20)

and hence, for the brackets we have

$$[\,\tilde e_0,\, \tilde e_1] = 0, \quad [\,\tilde e_0,\, \tilde e_2] = 0, \quad [\,\tilde e_1,\, \tilde e_2] = 2\theta\, \tilde e_0 + (\alpha + \delta)\ \tilde e_2. \tag{21}$$

From (21) we see that the distributions $\tilde e_0 \wedge \tilde e_2$ and $\tilde e_0 \wedge \tilde e_1$ are integrable. Denote by $\mathcal{L}_1$ and $\mathcal{L}_2$ the corresponding foliations generated by these distributions. Then the integral trajectories of the field ξ are exactly $\mathcal{L}_1 \cap \mathcal{L}_2$.

Denote by $\Omega^{(1)}$ and $\Omega^{(2)}$ the second fundamental forms of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Since $\tilde{e}_1$ and $\tilde{e}_2$ are unit normal vector fields for the corresponding foliations, from (20) we can easily find

$$\Omega^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & \alpha+\delta \end{pmatrix}, \qquad \Omega^{(2)} = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

and see that $\mathcal{L}_2$ is a minimal foliation.

Putting $\xi = \theta\,\tilde{e}_0$, we find from (19) the corresponding curvature components:

$$R(\tilde{e}_0, \tilde{e}_2)\,\tilde{e}_0 = -\,\tilde{e}_2, \quad R(\tilde{e}_0, \tilde{e}_1)\,\tilde{e}_0 = -\,\tilde{e}_1.$$

Denote by $K^{(i)}_{int}$ and $K^{(i)}_{ext}$ the intrinsic and extrinsic curvatures of the corresponding foliations $(i = 1, 2)$. Then $K^{(i)}_{ext} = \left\langle R(\tilde{e}_0, \tilde{e}_i)\,\tilde{e}_i, \tilde{e}_0\right\rangle = 1$. The Gauss equation implies

$$K^{(i)}_{int} = K^{(i)}_{ext} + \det\Omega^{(i)} = 0.$$

Therefore, both foliations are *intrinsically flat* and have a *constant extrinsic curvature* $K^{(i)}_{ext} = 1$. ■

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