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# A Multidimensional Version of Levin's Secular Constant Theorem and its Applications

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We study holomorphic almost periodic functions on a tube domain with the spectrum in a cone. We extend to this case Levin's theorem on a connection between the Jessen function, secular constant, and the Phragmen–Lindelöf indicator. Then we obtain a multidimensional version of Picard's theorem on exceptional values for our class.

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An almost periodic function with the bounded from below spectrum has some specific properties. Namely, it extends to the upper half-plane as a holomorphic almost periodic function $f$ of exponential type (H. Bohr [2]), then $\log|f|$ and the mean value of $\log|f|$ over a horizontal line (the so-called the Jessen function) are of the same growth along the imaginary positive semi-axis (B. Jessen, H. Tornehave [7] and B.Ja. Levin [9]). The last result (together with the discovered by Ph. Hartman [6], and B. Jessen, H. Tornehave [7] connection between the Jessen function, mean motions of $\arg f(z)$, and a distribution of zeros for holomorphic almost periodic functions on a strip) shows the regularity of functions of this important class.

In the end of the last century, L.I. Ronkin created the theory of holomorphic almost periodic functions and mappings defined on the tube domains of multidimensional complex space [11, 12, 14]. The Jessen function of several variables, introduced by him, plays the main role in the value distribution theory for almost periodic holomorphic mappings.

Here we continue studying the class of almost periodic functions on a tube domain with the spectrum in a cone done in [4] and [5]. Namely, we find a connection between the asymptotic behavior of the Jessen function and the polar indicator.

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Then we introduce a multidimensional analogue of the secular constant and study its asymptotic behavior. Also, we obtain a multidimensional version of Picard's theorem on exceptional values for our class.

Let us give a more detailed description of the subject.

Suppose $f$ is a $2\pi$-periodic function with the convergent Fourier series $f(x) = \sum_{n\geq n_0} a_n e^{inx}$, $n_0 \leq 0$, $a_{n_0} \neq 0$. Then $f(z) = \sum_{n\geq n_0} a_n e^{inz}$, $z = x + iy$, is a natural extension of $f(x)$ to the upper half-plane $\mathbb{C}^+$. Clearly, $f(z)$ is a holomorphic function of exponential type $|n_0|$ without zeros in some half-plane $y > y_0$ and

$$\lim_{y\to+\infty} y^{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(x+iy)|dx = \lim_{y\to+\infty} y^{-1}\log|f(iy)| = -n_0.$$

In [2] and [7], these properties were generalized to almost periodic functions $f$ with bounded from below spectrum under the condition $\Lambda^0 = \inf \operatorname{sp} f \in \operatorname{sp} f$. One should only replace the mean value over the period by the *Jessen* function

$$J_f(y) = \lim_{S\to\infty}(2S)^{-1}\int_{-S}^{S}\log|f(x+iy)|dx \tag{1}$$

the number $n_0$ by $\Lambda^0$, and make use of the Phragmen–Lindelöf Principle (see a footnote in the proof of Th. 1).

Note that the limit in (1) exists for every holomorphic almost periodic function on a strip $\{z = x + iy : a < y < b\}$ and the function $J_f(y)$ is convex on $(a, b)$. Then for all $y \in (a, b)$, maybe except some countable set $E_f$, we have

$$J_f'(y) = -c_f(y), \tag{2}$$

where

$$c_f(y) = \lim_{\gamma-\beta\to\infty}\frac{\arg f(\gamma+iy) - \arg f(\beta+iy)}{\gamma-\beta}$$

is the *mean motion*, or *secular number*, of the function $f$; here $\arg f(x+iy)$ is a continuous branch of the argument of $f$ on the line $y = \text{const}$. By the way, equality (2) and the Argument principle imply that the number $N(-S, S, y_1, y_2)$ of zeros of the function $f$ in the rectangle $\{|x| < S, y_1 < y < y_2\}^*$ has a density

$$\lim_{S\to\infty}(2S)^{-1}N(-S, S, y_1, y_2) = J_f'(y_2) - J_f'(y_1) \tag{3}$$

for all $y_1, y_2 \notin E_f$. It can also be proved that $f$ has no zeros on a substrip $\{\alpha < y < \beta\}$ if and only if $J_f(y)$ is a linear function on the interval $(\alpha, \beta)$. In this case,

$$f(z) = e^{ic_f z + g(z)},$$

*Zeros should be counted with multiplicities.

where $g(z)$ is almost periodic on the strip $\{z = x + iy : x \in \mathbb{R},\ \alpha < y < \beta\}$.

Thus, an almost periodic function $f$ with the property $-\infty < \Lambda^0 = \inf \mathrm{sp} f \in \mathrm{sp} f$ is extended to $\mathbb{C}^+$ as a holomorphic almost periodic function. Then we get

$$-\Lambda^0 = \lim_{y\to+\infty} \frac{\log|f(iy)|}{y} = \lim_{y\to+\infty} \frac{J_f(y)}{y} = \lim_{y\to+\infty} J'_f(y) = -\lim_{y\to+\infty} c_f(y) \qquad (4)$$

(see, for example, [7, 10]).

In the case $\Lambda^0 \notin \mathrm{sp} f$, the function is also extended to $\mathbb{C}^+$ as a holomorphic almost periodic function; the equalities (4) are also valid, but the proof of the second equality is complicated, and this is the contents of Levin's Secular Constant Theorem [9, 10].

Note that there exists a natural connection between the distribution of zeros of an almost periodic holomorphic function on the upper half-plane and the configuration of its spectrum:

**Theorem B** ([1]). *Suppose that the spectrum* $\mathrm{sp} f$ *of an almost periodic function* $f$ *on* $\mathbb{C}^+$ *is bounded from below. Then:*

*1) if* $\Lambda^0 = \inf \mathrm{sp} f \ge 0$*, then* $f(z)$ *tends to a finite limit as* $y \to \infty$ *on* $\mathbb{C}^+$ *uniformly in* $x \in \mathbb{R}$*;*

*2) if* $\Lambda^0 = \inf \mathrm{sp} f < 0$ *and* $\Lambda^0 \in \mathrm{sp} f$*, then* $f(z) \to \infty$ *as* $y \to \infty$ *on* $\mathbb{C}^+$ *uniformly in* $x \in \mathbb{R}$*;*

*3) if* $\Lambda^0 = \inf \mathrm{sp} f < 0$ *and* $\Lambda^0 \notin \mathrm{sp} f$*, then the function* $f(z)$ *takes every complex value on the half-plane* $y > q \ge 0$ *for each* $q < \infty$*.*

To discuss the multidimensional case, we need the following definitions.

Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^p$, $z = x + iy \in \mathbb{C}^p$, $x \in \mathbb{R}^p$, $y \in \mathbb{R}^p$. By $\langle x, y\rangle$ or $\langle z, w\rangle$ denote the scalar product (or the Hermitian scalar product for $z, w \in \mathbb{C}^p$). By $|.|$ denote the Euclidean norm on $\mathbb{R}^p$ or $\mathbb{C}^p$. Also, for $x = (x_1, x_2, \ldots, x_p)$ put $'x = (x_2, \ldots, x_p)$. Further, by $T_K$ denote a tube set

$$T_K = \{z = x + iy \in \mathbb{C}^p : x \in \mathbb{R}^p, y \in K\},$$

where $K \subset \mathbb{R}^p$ is the base of the tube set.

A vector $\tau \in \mathbb{R}^p$ is called an $\varepsilon$-almost period of the function $f(z)$ on $T_K$ if

$$\sup_{z\in T_K} |f(z+\tau) - f(z)| < \varepsilon.$$

The function $f$ is called *almost periodic on* $T_K$ if for every $\varepsilon > 0$ there exists $L = L(\varepsilon)$ such that every $p$-dimensional cube in $\mathbb{R}^p$ with the side of length $L$ contains at least one $\varepsilon$-almost period of $f$. In particular, when $K = \{0\}$, we get the definition of an almost periodic function on $\mathbb{R}^p$.

A function $f(z)$, $z \in T_\Omega$, where $\Omega$ is a domain in $\mathbb{R}^p$, is called almost periodic if its restriction to $T_K$ is an almost periodic function for every compact set $K \subset \Omega$.

The spectrum sp$f$ of an almost periodic function $f(z)$ on $T_K$ is the set of vectors $\lambda \in \mathbb{R}^p$ such that the Fourier coefficient

$$a_\lambda(y, f) = \lim_{S\to\infty} \frac{1}{(2S)^p} \int\limits_{|x_j|<S, j=1..p} f(x+iy)e^{-i\langle x,\lambda\rangle} dm_p(x) \tag{5}$$

does not vanish on $K$; here $m_p$ is the Lebesgue measure on $\mathbb{R}^p$. The spectrum of every almost periodic function $f$ is at most countable, therefore we have

$$f(x+iy) \sim \sum a_n(y)e^{i\langle x,\lambda^n\rangle},$$

where $\{\lambda^n\}_{n\in\mathbb{N}} = \mathrm{sp}f$ and $a_n(y) = a_{\lambda^n}(y, f)$. Note that for any given countable set $\{\lambda^n\}$ the function $\sum_{n\in\mathbb{N}} n^{-2}e^{i\langle x,\lambda^n\rangle}$ is almost periodic on $\mathbb{R}^p$ with the spectrum $\{\lambda^n\}$.

In [11] L.I. Ronkin introduced the notion of the Jessen function of an almost periodic holomorphic function $f$ on $T_\Omega$ by the formula

$$J_f(y) = \lim_{S\to\infty} \frac{1}{(2S)^p} \int\limits_{[-S,S]^p} \log|f(x+iy)|dm_p(t).$$

Using the methods of the theory of distributions and peculiar properties of zero sets for holomorphic functions, L.I. Ronkin confirmed that the limit exists and defines a convex function in $y \in \Omega$. He also established the multidimensional analogue of equality (3)

$$\lim_{S\to\infty} \frac{m_{2p-2}\{z = x+iy : x \in [-S, S]^p, y \in \omega, f(z) = 0\}}{(2S)^p} = \kappa_p \mu_J(\omega),$$

where $\mu_J$ is the Riesz measure of $J(y)$, $\omega \subset \overline{\omega} \subset \Omega$, $\mu_J(\partial\omega) = 0$, and the area of zero sets is taken counting the multiplicity.

Also, in [13] L.I. Ronkin proved that the products $b_n(y) = a_n(y)e^{\langle y,\lambda^n\rangle}$ do not depend on $y$ for every holomorphic almost periodic function $f(z)$ on $T_\Omega$; in particular, the coefficient $b_0$ corresponding to the exponent $\lambda = 0$ does not depend on $y$. In the case, the Fourier series turns into the Dirichlet series

$$f(z) \sim \sum_{\lambda^n \in\mathbb{R}^p} b_n e^{i\langle z,\lambda^n\rangle}, \quad b_n \in \mathbb{C}. \tag{6}$$

In [12] L.I. Ronkin obtained the following results.

**Theorem R.** *Let $f$ be a holomorphic almost periodic function on $T_\Omega$. Then the function $J_f(y)$ is linear on the domain $\Omega' \subset \Omega$ if and only if the function $f$ has no zeros in $T_{\Omega'}$. Moreover, in this case*

$$f(z) = \exp\{i\langle c_f, z\rangle + g(z)\}, \quad z \in T_{\Omega'}, \tag{7}$$

*where $c_f \in \mathbb{R}^p$ and $g(z)$ is an almost periodic function on $T_{\Omega'}$.*

In conditions of Th. R, we have

$$J_f(y) = -\langle c_f, y\rangle + \operatorname{Re} b_0, \quad y \in \Omega',$$

where $b_0$ is the corresponding coefficient of the Dirichlet-series expansion of the function $g$. Therefore, the following definition seems to be natural.

**Definition.** *The function $-\mathrm{grad} J_f(y)$, $y \in \Omega$, is the secular vector of the almost periodic holomorphic function $f$ on $T_\Omega$.*

In order to formulate our results, we need some definitions and notations.

A cone $\Gamma \subset \mathbb{R}^p$ is the set with the property $y \in \Gamma, t > 0 \Rightarrow ty \in \Gamma$. We will consider the convex cones with nonempty interior and such that $\overline{\Gamma} \bigcap (-\overline{\Gamma}) = \{0\}$. By $\widehat{\Gamma}$ denote the conjugate cone to $\Gamma$, i.e., $\widehat{\Gamma} = \{x \in \mathbb{R}^p : \langle x, y\rangle \geq 0 \quad \forall y \in \Gamma\}$; note that $\widehat{\widehat{\Gamma}} = \overline{\Gamma}$. As usual, $\mathrm{Int} A$ is the interior of the set $A$, and $H_E(x) = \sup_{\lambda \in E}\langle x, \lambda\rangle$ is the *support function* of the set $E \subset \mathbb{R}^p$.

Let $f$ be a holomorphic almost periodic function on a tube $T_\Gamma$ with an open cone $\Gamma$ in the base. By definition, put

$$h_f(y) = \sup_{x \in \mathbb{R}^p} \overline{\lim_{r \to \infty}} \frac{\ln |f(x + iry)|}{r}, \quad y \in \Gamma.$$

The function $h_f$ is called the *$P$-indicator* of $f$ (see [14, p. 245]).

**Theorem A** ([5]). *Let $\Gamma$ be a closed cone in $\mathbb{R}^p$, and $f(x)$ be an almost periodic function on $\mathbb{R}^p$. Then $f$ is extended holomorphically to $T_{\mathrm{Int}\widehat{\Gamma}}$ with the estimates*

$$\exists b < \infty \quad \forall \Gamma' = \overline{\Gamma'} \subset \mathrm{Int}\widehat{\Gamma} \cup \{0\} \quad \exists B(\Gamma') \quad \forall z \in T_{\Gamma'} \quad |f(z)| \leq B(\Gamma')e^{b|y|}, \tag{8}$$

*if and only if $spf \subset \Lambda + \Gamma$ for some $\Lambda \in \mathbb{R}^p$. If this is the case, then $f(z)$ is almost periodic on $T_{\mathrm{Int}\widehat{\Gamma}}$ and for all $y \in \mathrm{Int}\widehat{\Gamma}$*

$$h_f(y) = H_{\mathrm{sp}f}(-y). \tag{9}$$

For almost periodic functions with bounded spectrum, equality (9) was proved in [4].

The following theorem is the main result of our paper.

**Theorem 1.** *Let $\Gamma$ be a closed cone in $\mathbb{R}^p$, and $f(x)$ be an almost periodic function on $\mathbb{R}^p$ such that $f$ is extended holomorphically to $T_{\mathrm{Int}\widehat{\Gamma}}$ with estimates (8). Then for all $y \in \mathrm{Int}\widehat{\Gamma}$*

$$\lim_{R\to\infty} \frac{J_f(Ry)}{R} = h_f(y). \tag{10}$$

*Furthermore, the secular vector $-\mathrm{grad}J_f(Ry)$ tends to $\mathrm{grad}H_{\mathrm{spf}}(-y)$ as $R \to \infty$ in the sense of distributions.*

R e m a r k. Since $J_f(y)$ is a convex function, we see that the secular vector is a locally integrable function on $\mathrm{Int}\widehat{\Gamma}$.

P r o o f. From the beginning assume that $y^0 = (1, 0, 0, \dots, 0) \in \mathrm{Int}\widehat{\Gamma}$, and we will prove (10) for $y = y^0$.

Put $F(z) = f(z)e^{i\langle z, h_f(y^0)y^0\rangle}(\sup_{x\in\mathbb{R}^p}|f(x)|)^{-1}$, $u(z) = \log|F(z)|$. Note that $F(z)$ is an almost periodic holomorphic function on $T_{\mathrm{Int}\widehat{\Gamma}}$ and $|F(x)| \le 1$ on $\mathbb{R}^p$. Applying the Phragmen–Lindelöf principle* on the complex one–dimensional plane $\{x + wy : w \in \mathbb{C}^+\}$, we get

$$u(x + ity) \le h_F(y)\, t, \quad \forall\, z = x + iy \in T_{\mathrm{Int}\widehat{\Gamma}}, \quad t > 0. \tag{11}$$

Then

$$h_F(y) = h_f(y) - \langle y, h_f(y^0)y^0\rangle, \qquad h_F(y^0) = 0. \tag{12}$$

Take $y = y^0$ in (11). We get

$$u(z_1, {}'x) \le 0 \quad \forall \quad (x_1, {}'x) \in \mathbb{R}^p, \quad y_1 \ge 0. \tag{13}$$

Fix $\varepsilon > 0$. Since $\sup_{x\in\mathbb{R}^p} \overline{\lim}_{r\to\infty} r^{-1}u(x + iry^0) = 0$, we see that for some $x^0 = x^0(\varepsilon) \in \mathbb{R}^p$, $r = r(\varepsilon) > 0$,

$$u(x^0 + iry^0) \ge -\varepsilon r. \tag{14}$$

Using the Poisson formula for the disc $D(x_1^0 + iR, R) = \{z_1 : |z_1 - x_1^0 - iR| < R\} \subset \mathbb{C}^+$ with $R > r$, inequality (13), and Maximum principle for the subharmonic function $u(z_1, {}'x^0)$, we obtain

$$u(x_1^0 + ir, {}'x^0)$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} u(x_1^0 + iR + Re^{i\psi}, {}'x^0)\frac{R^2 - (R-r)^2}{R^2 - 2R(R-r)cos(\pi/2+\psi) + (R-r)^2}d\psi$$

*Suppose $g(z)$ is continuous on $\overline{\mathbb{C}^+}$, holomorphic on $\mathbb{C}^+$, and bounded on $\mathbb{R}$ function, which satisfies the condition $\log^+|g(z)| = O(|z|)$ as $|z| \to \infty$; then for $z = x + iy \in \mathbb{C}^+$ we have $|g(z)| \le \sup_{x\in\mathbb{R}}|g(x)|e^{\sigma^+ y}$, where $\sigma^+ = \limsup_{y\to+\infty} y^{-1}\log|g(iy)|$ (see [8, p. 28]).

$$\leq \frac{r}{4\pi R}\int\limits_{\pi/4}^{3\pi/4} u(x_1^0+iR+Re^{i\psi},' x^0)d\psi \leq r(8R)^{-1}\sup_{\psi\in[\pi/4;3\pi/4]} u(x_1^0+iR+Re^{i\psi},' x^0).$$

Hence (14) implies that $u(x_1^0+iR+Re^{i\psi_0},' x^0) \geq -8\varepsilon R$ for some $\psi_0 \in [\pi/4,\ 3\pi/4]$. The function $u(z_1,' x^0)$ is subharmonic in $z_1 \in \mathbb{C}^+$. Taking into account (13) and the embeddings

$$D(x_1^0+2iR,\ R) \subset D(x_1^0+iR+Re^{i\psi_0},\ R+R\ /\!\!\sqrt{2}) \subset \mathbb{C}^+,$$

we get

$$-8\varepsilon R \leq \frac{2}{\pi R^2(3+2\sqrt{2})}\int\limits_{D(x_1^0+iR+Re^{i\psi_0},\ R+R/\sqrt{2})} u(z_1,' x^0)dm_2(z_1)$$

$$< \frac{1}{3\pi R^2}\int\limits_{D(x_1^0+2iR,\ R)} u(z_1,' x^0)dm_2(z_1). \tag{15}$$

Remind that this inequality is valid for all $R > r$.

Put $u_R(z) = R^{-1}u(Rz)$. From (15) it follows that

$$\int\limits_{D(x_1^0/R+2i,\ 1)} u_R(z_1,' x^0/R)dm_2(z_1) > -24\pi\varepsilon. \tag{16}$$

Furthermore, Th. A implies that the function $h_f(y)$ is continuous. Since (12), we get $h_F(y) < \varepsilon$ for $|y-y^0| < p\delta$ with some $\delta = \delta(\varepsilon) \in (0,\ 1/(p+2))$. If we replace in (11) $y$ by $y/|y|$, $x$ by $Rx$, and $t$ by $R|y|$, we obtain

$$u_R(z) = R^{-1}u(Rx+iRy) \leq \varepsilon|y| \tag{17}$$

for all $z$ from the tube domain

$$T^\delta = \{z = x+iy : \ x\in\mathbb{R}^p,\ \big|y/|y|-y^0\big| < p\delta\}.$$

By definition, put

$$A(x^1) = D(x_1^1+2i,\ 1)\times D(x_2^1,\delta)\times D(x_3^1,\delta)\times\cdots\times D(x_p^1,\delta).$$

It can easily be checked that for all $x^1 = (x_1^1,' x^1) \in \mathbb{R}^p$ we have $A(x^1) \subset T^\delta$. Also, we may assume that $\overline{T^\delta} \subset T_{\mathrm{Int}\widehat{\Gamma}} \cup \{0\}$. Then for all $z_1 \in\ D(x_1^0/R+2i,\ 1)$ the function $u_R(z)$ is subharmonic in $z_2 \in D(x_2^1,\delta)$, $z_3 \in D(x_3^1,\delta),\ldots, z_p \in D(x_p^1,\delta)$. Hence (16) implies

$$\int\limits_{A(x^0/R)} u_R(z)dm_{2p}(z) > -24\delta^{2p-2}\pi^p\varepsilon. \tag{18}$$

Suppose that for some $\tau \in \mathbb{R}^p$ we have

$$|F(x^0 + \tau + iry^0) - F(x^0 + iry^0)| \le e^{-\varepsilon r} - e^{-2\varepsilon r}.$$

Then $|F(x^0 + \tau + iry^0)| \ge e^{-2\varepsilon r}$ and $u(x^0 + \tau + iry^0) \ge -2\varepsilon r$. Using the latter inequality instead of (14), we obtain the relation

$$\int\limits_{A(x^0/R+\tau/R)} u_R(z) dm_{2p}(z) > -48\delta^{2p-2}\pi^p\varepsilon. \tag{19}$$

Put $u_R^+(z) = \max\{u_R(z), 0\}$, $u_R^-(z) = \max\{-u_R(z), 0\}$. From (17) it follows that for all $x^1 \in \mathbb{R}^p$ and all $z \in A(x^1)$ we have

$$u_R(z) < \sqrt{10}\varepsilon. \tag{20}$$

Therefore, by (19),

$$\int\limits_{A(x^0/R+\tau/R)} u_R^-(z) dm_{2p}(z) = \int\limits_{A(x^0/R+\tau/R)} u_R^+(z) dm_{2p}(z)$$

$$- \int\limits_{A(x^0/R+\tau/R)} u_R(z) dm_{2p}(z) \le 52\delta^{2p-2}\pi^p\varepsilon. \tag{21}$$

In the sequel we need the following lemma.

**Lemma 1.** *Let $g(x)$ be an almost periodic function in $x \in \mathbb{R}^p$. Then for any $\eta > 0$ there exist a real $L = L(\eta)$ and a set $E = E_1 \times \dots \times E_p$, $E_j \in \mathbb{R}$, such that $E_j \cap [a, a + L] \ne \emptyset$ for every $a \in \mathbb{R}$, $j = 1, \dots, p$, and each $\tau \in E$ is an $\eta$-almost period of $g$.*

P r o o f. By Bochner's criterium*, any sequence $t_n \in \mathbb{R}$ has a subsequence $t_{n'}$ such that the functions $g(x + (t_{n'},' 0))$ converge uniformly in $x \in \mathbb{R}^p$. In other words, the functions $g(x_1 + t_{n'},' x)$ converge uniformly in $x_1 \in \mathbb{R}$ and $'x \in \mathbb{R}^p$. By Bochner's criterium, the function $g(x_1,' x)$ is almost periodic in $x_1 \in \mathbb{R}$ uniformly in $'x \in \mathbb{R}^{p-1}$. Hence there exist $E_1 \in \mathbb{R}$ and $L = L(\eta)$ such that $E_1 \cap [a, a+L] \ne \emptyset$ for all $a \in \mathbb{R}$ and

$$|g(x_1 + t,' x) - g(x,' x)| < \eta/p \quad \forall x_1 \in \mathbb{R}, \quad \forall' x \in \mathbb{R}^{p-1}, \quad \forall t \in E_1,$$

i.e., each $\tau = (t,' 0)$ for $t \in E_1$ is an $\eta/p$-almost period of $g(x)$. In the same way, we find $E_2, \dots, E_p$. It is clear that every point of $E_1 \times \dots \times E_p$ is an $\eta$-almost period of $g$. ∎

*For almost periodic functions on $\mathbb{R}$ see [10, Ch. VI, §1], or [3, p. 14–16]; the proof for the multidimensional case is similar.

Take $S < \infty$, and let $L = L(\varepsilon, r)$ be real from Lem. 1. It is not difficult to prove that if $R > L\sqrt{2}$, then there exist $\tau_1^1, \dots, \tau_1^{N_1} \in E_1$, $N_1 \le 2\sqrt{2}S + 2$, such that

$$\bigcup_{m=1}^{N_1}\left(\frac{x_1^0+\tau_1^m}{R} - \frac{\sqrt{2}}{2}, \frac{x_1^0+\tau_1^m}{R} + \frac{\sqrt{2}}{2}\right) \supset [-S, S], \tag{22}$$

and each point of $[-S, S]$ is contained in at most two intervals. For the same reasons, if $R > L\sqrt{2}/\delta$, then for $j = 2, \dots, p$ there exist $\tau_j^1, \dots, \tau_j^{N_j} \in E_j$, $N_j \le (2\sqrt{2}S + 2)/\delta$, such that

$$\bigcup_{m=1}^{N_j}\left(\frac{x_j^0+\tau_j^m}{R} - \frac{\delta\sqrt{2}}{2}, \frac{x_j^0+\tau_j^m}{R} + \frac{\delta\sqrt{2}}{2}\right) \supset [-S, S]. \tag{23}$$

Let $F = \{\tau = (\tau_1^{m_1}, \dots, \tau_p^{m_p}) : 1 \le m_1 \le N_1, \dots, 1 \le m_p \le N_p\}$. Note that $F$ contains at most $(2\sqrt{2}S+2)^p\delta^{1-p}$ elements. By definition, put

$$\Pi(S, \delta) = \left\{x + iy : x \in [-S, S]^p, \ |y_1 - 2| < \frac{1}{\sqrt{2}}, \ |y_j| < \frac{\delta}{\sqrt{2}}, \ j = 2, \dots, p\right\}$$

Combining (22) and (23), we get

$$\bigcup_{\tau\in F} A\left(\frac{x^0+\tau}{R}\right) \supset \Pi(S, \delta). \tag{24}$$

Applying Lem. 1 to the function $F(x + iry^0)$ with $\eta = e^{-\varepsilon r} - e^{-2\varepsilon r}$ and using (21) for every $\tau \in F$, we obtain

$$\int\limits_{\Pi(S,\delta)} u_R^-(z)dm_{2p}(z) \le \sum_{\tau\in F} \int\limits_{A(\frac{x^0+\tau}{R})} u_R^-(z)dm_{2p}(z) \le 52(2\sqrt{2}S+2)^p\delta^{p-1}\pi^p\varepsilon.$$

Therefore, we have

$$\overline{\lim_{S\to\infty}}\frac{1}{(2S)^p}\int\limits_{\Pi(S,\delta)} u_R(z)dm_{2p}(z) \ge -52(\sqrt{2}\pi)^p\delta^{p-1}\varepsilon. \tag{25}$$

It follows from the definition of the Jessen function that

$$\overline{\lim_{S\to\infty}}\frac{1}{(2S)^p}\int\limits_{[-S,S]} u_R(x+iy)dm_p(x) = \frac{J_F(Ry)}{R}. \tag{26}$$

The functions $u_R(z)$ are uniformly bounded from above for $z \in T_\delta$.

Applying the Fatou lemma to inequality (25), we get

$$\int\limits_{|y_1-2|<\frac{\sqrt{2}}{2},|y_2|<\frac{\delta}{\sqrt{2}},\dots|y_p|<\frac{\delta}{\sqrt{2}}} J_F(Ry)dy \geq -52(\sqrt{2}\pi)^p\delta^{p-1}\varepsilon R, \qquad (27)$$

for all $R > R(L,\, \delta,\, r,\, \varepsilon)$.

To finish the proof, we need the following simple lemma.

**Lemma 2.** *Let $g(t)$ be a convex negative function on $[-\alpha,\, \alpha]$. Then $g(0) \geq \alpha^{-1}\int_{-\alpha}^{\alpha} g(t)dt$.*

P r o o f. The assertion of Lem. 2 follows immediately from the inequality

$$g(t) \leq g(0)\min\{1 - t/\alpha,\, 1 + t/\alpha\}. \qquad \blacksquare$$

Note that (20) and (26) imply

$$J_F(Ry) \leq \sqrt{10}\varepsilon R \qquad (28)$$

for all $y = (y_1, \dots, y_p)$, $|y_1 - 2| < 1$, $|y_j| < \delta$, $j = 2, \dots, p$. Further, the Jessen function $J_F(Ry)$ is convex in $y$ ([11]). Therefore the function

$$g('y) = \int\limits_{|y_1-2|<\frac{1}{\sqrt{2}}} J_F(Ry)dy_1 - 2\sqrt{5}R\varepsilon$$

satisfies the conditions of Lem. 2 in each variable $y_2, \dots, y_p$ with $\alpha = \delta/\sqrt{2}$. Applying the lemma $p - 1$ times and using inequality (27), we obtain

$$\int\limits_{|y_1-2|<\frac{1}{\sqrt{2}}} J_F(Ry_1,'0)dy_1 \geq -40(2\pi)^p R\varepsilon.$$

Since (13), we see that the integrand is negative. Moreover, it is convex, therefore $J_F(Ry_1,'0)$ is a monotonically decreasing function in $y_1$. Then we have

$$J_F((2 - 1/\sqrt{2})Ry^0) \geq -30(2\pi)^p R\varepsilon.$$

The inequality is valid for all $R > R(\varepsilon)$ and $\varepsilon > 0$. Thus we have

$$\lim_{R\to\infty} \frac{J_F(Ry^0)}{R} = 0. \qquad (29)$$

Since $J_F(y) = J_f(y) - \langle y, h_f(y^0)y^0\rangle$, we obtain (10) for $y = y^0$.

For an arbitrary $y' \in \mathrm{Int}\widehat{\Gamma}$ consider an orthogonal operator $A : \mathbb{R}^p \to \mathbb{R}^p$ such that $A(y^0) = y'$. Put $f_1(z) = f(Az)$. Since $h_{f_1}(y^0) = h_f(y')$ and $J_{f_1}(y^0) = J_f(y')$, we obtain (10) for $y = y'$.

Further, from (11) and Th. A it follows that the function $J_f(Ry)/R$ is bounded from above on every compact subset of $\mathrm{Int}\widehat{\Gamma}$. Then fix $y^1 \in \mathrm{Int}\widehat{\Gamma}$ and take $s > 0$ such that $\{y : |y - y^1| \le s\} \subset \mathrm{Int}\widehat{\Gamma}$. Whenever $|y - y^1| < s$, we have

$$2J_f(Ry^1) \le J_f(R(2y^1 - y)) + J_f(Ry)$$

and

$$\frac{J_f(Ry)}{R} \ge 2\inf_{R\ge 1}\left|\frac{J_f(Ry^1)}{R}\right| - \sup_{R\ge 1}\sup_{|y-y^1|\le s}\frac{\max\{J_f(Ry), 0\}}{R}.$$

This means that the functions $J_f(Ry)/R$ are uniformly bounded from below on every compact subset of $\mathrm{Int}\widehat{\Gamma}$. Using (10) and the Lebesgue theorem, we obtain

$$\int \frac{J_f(Ry)}{R}\varphi(y)dm_p(y) \to \int h_f(y)\varphi(y)dm_p(y) \quad \text{as} \quad R \to \infty$$

for every test function $\varphi$ on $\mathrm{Int}\widehat{\Gamma}$, i.e., (10) is valid in the sense of distributions as well. Therefore,

$$\mathrm{grad} J_f(Ry) \to \mathrm{grad} h_f(y) \quad \text{as} \quad R \to \infty$$

in the sense of distributions and Th. A implies the last assertion of Th. 1. ∎

**Corollary 1.** *Suppose that all conditions of Th. 1 are fulfilled. If $H_{\mathrm{sp}f}(y)$ is nonlinear on $(-\widehat{\Gamma})$, then $f(z)$ has zeros on the set $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ for each $q < \infty$.*

P r o o f. Theorem A yields that the function $h_f(y)$ is nonlinear for $y \in \mathrm{Int}\widehat{\Gamma}$. Now Th. 1 implies that $J_f(y)$ is nonlinear on the set $\{Int\widehat{\Gamma} \cap \{|y| > q\}\}$ for each $q < \infty$. Then Th. R implies that $f(z)$ has zeros on $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$. ∎

**Applications to distribution of values**. Here we apply Th. 1 to prove the multidimensional variant of Th. B:

**Theorem 2.** *Let $\Gamma \subset \mathbb{R}^p$ be a closed convex cone and $f(x)$ be an almost periodic function on $\mathbb{R}^p$ that has a holomorphic extension $f(z)$ to $T_{\mathrm{Int}\widehat{\Gamma}}$ with estimates (8). Then:*

1) *if $(\mathrm{sp}f \setminus \{0\}) \subset \Gamma$, then $f(z)$ tends to a finite limit as $y \to \infty$, $y \in \Gamma'$, uniformly in $x \in \mathbb{R}^p$ for all $\Gamma' = \overline{\Gamma}' \subset \mathrm{Int}\widehat{\Gamma} \cup \{0\}$;*

2) *if $(\mathrm{sp}f \setminus \{0\}) \subset \Lambda + \Gamma$ with some $\Lambda \in \mathrm{sp}f \cap (-\Gamma) \setminus \{0\}$, then the function $f(z)$ tends to $\infty$ as $y \to \infty$, $y \in \Gamma'$, uniformly in $x \in \mathbb{R}^p$ for all $\Gamma' = \overline{\Gamma}' \subset \mathrm{Int}\widehat{\Gamma} \cup \{0\}$;*

3) *if* $(\mathrm{sp}f \setminus \{0\}) \subset \Lambda + \Gamma$ *with some* $\Lambda \in (\overline{\mathrm{sp}f} \setminus \mathrm{sp}f) \cap (-\Gamma) \setminus \{0\}$*, then the function* $f(z)$ *takes every complex value on the set* $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ *for each* $q < \infty$*;*

4) *if* $(\mathrm{sp}f \setminus \{0\}) \subset \Lambda + \Gamma$ *with some* $\Lambda \in \overline{\mathrm{sp}f} \setminus ((-\Gamma) \cup \Gamma)$*, then the function* $f(z)$ *takes every complex value, except for at most one, on the set* $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ *for each* $q < \infty$*;*

5) *if* $(\mathrm{sp}f \setminus \{0\}) \not\subset \Lambda + \Gamma$ *for all* $\Lambda \in \overline{\mathrm{sp}f}$ *and* $\mathrm{sp}f \not\subset \Gamma$*, then the function* $f(z)$ *takes every complex value on the set* $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ *for each* $q < \infty$*.*

R e m a r k. It is clear that we can replace $\mathrm{sp}f \setminus \{0\}$ by $\mathrm{sp}f$ in Cases 1–3. Therefore Th. 2 gives, in a sense, a complete description of the value distributions for our class of almost periodic functions.

P r o o f. Case 1 was proved in [4], Case 2 was proved in [5]. Reduce Case 3 to a one-dimensional one. Take $y^0 \in \mathrm{Int}\widehat{\Gamma}$ such that $\langle y^0, \lambda^k\rangle \neq \langle y^0, \lambda^m\rangle$ for all $k \neq m$, and put $\varphi(w) = f(wy^0)$, $w \in \mathbb{C}$.

First, check that $\mathrm{sp}\varphi = \{\langle y^0, \lambda\rangle : \lambda \in \mathrm{sp}f\}$. This is evident for finite exponential sums. In the general case, take a sequence of Bochner–Feyer exponential sums* $P_n(x)$, which approximates $f(x)$ on $R^p$. Since $\mathrm{sp}P_n \subset \mathrm{sp}f$ and $P_n(uy^0) \to \varphi(u)$ uniformly on $\mathbb{R}$, we see that $\mathrm{sp}\varphi \subset \{\langle y^0, \lambda\rangle : \lambda \in \mathrm{sp}f\}$. On the other hand, if $\lambda \in \mathrm{sp}f$, then

$$a_{\langle y^0,\lambda\rangle}(0, P_n(y^0u)) = a_\lambda(0, P_n) \to a_\lambda(0, f) \neq 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $a_{\langle y^0,\lambda\rangle}(0, \varphi) \neq 0$ and $\langle y^0, \lambda\rangle \in \mathrm{sp}\varphi$.

Note that $\langle y^0, \lambda^n\rangle \to \langle y^0, \Lambda\rangle$ as $\lambda^n \to \Lambda$, $\lambda^n \in \mathrm{sp}f$. Also, since $y^0 \in \mathrm{Int}\widehat{\Gamma}$ and $\lambda - \Lambda \in \Gamma$ for all $\lambda \in \mathrm{sp}f$, we get $\langle y^0, \lambda\rangle > \langle y^0, \Lambda\rangle$. Therefore, $\inf \mathrm{sp}\varphi = \langle y^0, \Lambda\rangle$ and $\langle y^0, \Lambda\rangle \notin \mathrm{sp}\varphi$. From Th. B, i. 3 it follows that $f(z)$ takes every complex value on the set $\{z = wy_0 : \mathrm{Im}w > q\}$ for each $q < \infty$.

Let us consider Case 4. Let $b_0$ be the coefficient of series (6) corresponding to the exponent $\lambda = 0$. Then for any $A \in \mathbb{C} \setminus \{b_0\}$ each function $f(z) - A$ has the spectrum $\mathrm{sp}f \cup \{0\}$. Suppose that the support function $H_{\mathrm{sp}f\cup\{0\}}(y)$ is linear on $(-\widehat{\Gamma})$. Then it is not difficult to prove (for example, see [5, Lem. 2]) that $\mathrm{sp}f \cup \{0\} \subset \Lambda' + \Gamma$ with some $\Lambda' \in (-\Gamma) \cap (\overline{\mathrm{sp}f \cup \{0\}})$. But this is impossible in our case. Hence, the function $H_{\mathrm{sp}f\cup\{0\}}(y)$ is nonlinear on $(-\widehat{\Gamma})$. Now Cor. 1 yields that the function $f(z) - A$ has zeros on $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ for each $q < \infty$.

Let us consider Case 5. Let $b_0$ be the same as in Case 4. The function $f(z) - b_0$ has the spectrum $\mathrm{sp}f \setminus \{0\}$. Note that the support function $H_{\mathrm{sp}f\setminus\{0\}}(y)$

*For almost periodic functions on $\mathbb{R}$ see [10, Ch.VI, §1], or [3, p. 38–45]; consideration in the multidimensional case is similar.

is nonlinear on $(-\widehat{\Gamma})$. Hence Cor. 1 implies that the function $f(z) - b_0$ has zeros on $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ for each $q < \infty$. Further, for any $A \in \mathbb{C} \setminus \{b_0\}$ the function $f(z) - A$ has the spectrum $\mathrm{sp}f \cup \{0\}$. If the support function $H_{\mathrm{sp}f\cup\{0\}}(y)$ is linear on $(-\widehat{\Gamma})$, then $\mathrm{sp}f \cup \{0\} \subset \Lambda' + \Gamma$ with some $\Lambda' \in (-\Gamma) \cap (\overline{\mathrm{sp}f \cup \{0\}})$. The both cases $\Lambda' = 0$ and $\Lambda' \neq 0$ contradict to the conditions of Case 5. Therefore the function $f(z) - A$ has zeros on $\mathrm{Int}T_{\widehat{\Gamma}\cap\{|y|>q\}}$ for each $q < \infty$. ■

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