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A Note on Limit Shapes of Minimal Difference Partitions

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We provide a variational derivation of the limit shape of minimal difference partitions and discuss the link with exclusion statistics.

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This paper is dedicated to Professor Leonid Pastur for his 70th anniversary

A partition of a natural integer E [1] is a decomposition of E as a sum of a nonincreasing sequence of positive integers $\{h_j\}$, i.e., $E = \sum_j h_j$ such that $h_j \geq h_{j+1}$, for $j = 1, 2 \ldots$. For example, 4 can be partitioned in 5 ways: 4, $3+1$, 2+2, 2+1+1, and 1+1+1+1. Partitions can be graphically represented by Young diagrams (also called Ferrers diagrams) where h_j corresponds to the height of the j-th column. The $\{h_j\}$'s are called the parts or the summands of the partition. One can put several constraints on such partitions. For example, one can take the number of columns N to be fixed or put restrictions on the heights. In this paper we focus on a particular constrained partition problem called the minimal difference p partitions (MDP–p). The MDP–p problem is defined by restricting the height difference between two neighboring columns, $h_j - h_{j+1} \geq p$. For instance the only allowed partitions of 4 with $p = 1$ are 4 and $3 + 1$. A typical Young diagram for MDP–p problem is shown in Fig. 1. Consider now the set of all possible partitions of E satisfying $E = \sum_j h_j$ and $h_j - h_{j+1} \geq p$. Since this is a finite set, one can put a uniform probability measure on it, which means that

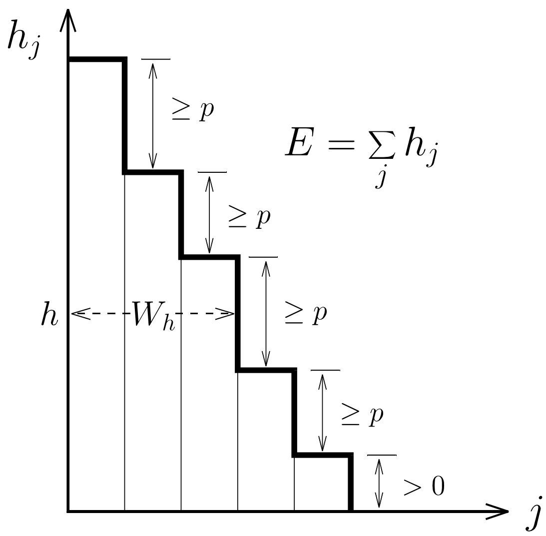

Fig. 1: A typical Young diagram for MDP–p problem. The thick solid border shows the height profile or the outer perimeter. W_h is the width of the Young diagram at a height h, i.e., W_h is the number of columns whose heights $\geq h$.

all partitions are equiprobable. Then, a natural question is: what is the typical shape of a Young diagram when $E \to \infty$?

In the physics literature this problem was first raised by Temperley, who was interested in determining the equilibrium profile of a simple cubic crystal grown from the corner of three walls at right angles. The two dimensional version of the problem — where walls (two) are along the horizontal and the vertical axes and E "bricks" (molecules) are packed into the first quadrant one by one such that each brick, when it is added, makes two contacts along faces — corresponds to the $p = 0$ partition problem. Temperley [2] computed the equilibrium profile of this two dimensional crystal. In the mathematics literature the investigation of the shape of random Young tableaux was started by Vershik and Kerov [3] and independently by Logan and Shepp [4]. The case of uniform random partitions was treated by Vershik and collaborators [5–7] who obtained the limit shapes for the $p = 0$ and $p = 1$ cases and also the average deviations from the limit shapes [8]. Some of these results were extended by Romik [9] to the MDP–p for $p = 2$. These problems belong to the general framework of asymptotic combinatorics, a subject which displays unexpected links with random matrix theory. In this note we compute the limit shapes of MDP–p for all $p \geq 0$ by a variational approach and mention an interesting link with exclusion statistics.

We first recapitulate the arguments used in [10, 11] (see also [12] for a similar approach) to compute the limit shapes of the Young diagrams of unrestricted partitions ($p = 0$). Let $P = (i, h_i)$ and $Q = (j, h_j)$ be two points belonging to the outer perimeter of the Young diagram of a given partition. We evaluate the total number of subdiagrams which connects these two points. These subdiagrams are lattice staircases with the only restriction that each step either goes right or downward. The total number of horizontal steps is $j - i$, the total number of vertical steps is $h_i - h_j$, and the total number of steps is $j - i + h_i - h_j$. Therefore, the total number of configurations is

$$\Omega_0(P, Q) \equiv \Omega_0(i, h_i; j, h_j) = \binom{j - i + h_i - h_j}{j - i}. \tag{1}$$

If P and Q are far apart (i.e., $a = j - i \gg 1, b = h_i - h_j \gg 1$) we may use the Stirling formula which gives

$$\ln \Omega_0(P, Q) = -a \ln \frac{a}{a + b} - b \ln \frac{b}{a + b} = \sqrt{a^2 + b^2}\ \phi(\overrightarrow{n}), \tag{2}$$

where $\overrightarrow{n} \equiv (n_1, n_2) = (b, a)/\sqrt{a^2 + b^2}$ is the unit vector orthogonal to $\overrightarrow{PQ}$ and

$$\phi(\overrightarrow{n}) = -n_1 \ln \frac{n_1}{n_1 + n_2} - n_2 \ln \frac{n_2}{n_1 + n_2}. \tag{3}$$

Heuristically one expects that in the limit $E \to \infty, h \to \infty, W_h \to \infty$, the profile of the Young diagram will be described by a smooth curve $y = y(x)$ where $y = h/\sqrt{E}$ and $x = W_h/\sqrt{E}$ are the scaling variables. The normal vector can be parameterized as

$$\overrightarrow{n} = \left(-\frac{y'(x)}{\sqrt{1 + y'^2(x)}}, \frac{1}{\sqrt{1 + y'^2(x)}} \right).$$

Therefore

$$\phi(\overrightarrow{n}) = \frac{y'(x)}{\sqrt{1 + y'^2(x)}} \ln \left[-\frac{y'(x)}{1 - y'(x)} \right] - \frac{1}{\sqrt{1 + y'^2(x)}} \ln \left[\frac{1}{1 - y'(x)} \right]. \tag{4}$$

In the lattice model, the points P and Q were taken to be far apart. However in the new scale (x, y) one now assumes that they are close enough in order to ensure that the interface is locally flat. The total number of Young diagrams Ω with a given area E is obtained by adding all such local configuration, i.e.

$$\Omega = \exp \left(\sqrt{E} \int_0^\infty \mathrm{d}x \sqrt{1 + y'^2(x)}\, \phi(\overrightarrow{n}) \right), \tag{5}$$

with the area constraint

$$\int_0^\infty y(x)\mathrm{d}x = 1. \tag{6}$$

For large E, the most dominant contribution to Ω arises from the optimal curve $y = y(x)$ which maximizes the integral in (5) with the constraint (6). This optimal curve describe the limit shape of the Young diagrams. Thus we are led to the variational problem of extremizing

$$\mathcal{L}_0 = \int_0^\infty \mathrm{d}x \left[y'(x) \ln \frac{-y'(x)}{1 - y'(x)} - \ln \frac{1}{1 - y'(x)} \right] - \lambda \int_0^\infty y(x)\, \mathrm{d}x, \tag{7}$$

where λ is a Lagrange multiplier. This leads to the Euler-Lagrange equation, which simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{-y'(x)}{1 - y'(x)} = -\lambda. \tag{8}$$

We solve this equation with the boundary conditions $y(\infty) = 0$ and $y(x \to 0) \to \infty$. The later condition follows from the fact that $y \equiv h/\sqrt{E} \sim \ln E$ when $x \equiv W_h/\sqrt{E} \to 0$ for large E [13]. Therefore $y(0)$ diverges in the limit $E \to \infty$. The solution gives the equation of the limiting shape as

$$y(x) = -\frac{1}{\lambda} \ln\bigl(1 - \mathrm{e}^{-\lambda x}\bigr) \quad \text{with} \quad \lambda = \frac{\pi}{\sqrt{6}}, \tag{9}$$

where λ is obtained by using the constraint (6).

The goal of this paper is to extend this derivation to the MDP–p with $p > 0$. This is a priori nontrivial since now one has to take into account the restriction on the steps. In the following we will use an exact correspondence between MDP–p with $p > 0$ and an unrestricted partition ($p = 0$).

Let $\{h_j\}$ denote the set of nonzero heights in a given unrestricted partition ($p = 0$) $E = \sum_{j=1}^N h_j$, where $h_j \geq h_{j+1}$ for all $j = 1, 2, \ldots, N-1$. Let us now define a new set of heights $h'_j = h_j + p(N - j)$ for $j = 1, 2, \ldots, N$. Thus $h'_j - h'_{j+1} = h_j - h_{j+1} + p$ for all $j = 1, 2, \ldots, N-1$ and $h'_N = h_N > 0$. Since $h_j - h_{j+1} \geq 0$, the new heights thus satisfy the constraint $h'_j - h'_{j+1} \geq p$ for all $j = 1, 2, \ldots, N-1$. Since the mapping is one to one, the total number of local MDP–p configuration satisfies

$$\Omega_p(i, h'_i; j, h'_j) = \Omega_0(i, h_i; j, h_j).$$

Moreover, $h_i - h_j = h'_i - h'_j - p(j - i)$. Therefore using (1),

$$\Omega_p = \binom{(j-i)(1-p) + h'_i - h'_j}{j - i}.$$

The fact that the mapping does not preserve the total area does not spoil the argument since here we only deal with local MDP–p configurations. The area constraint is a global one which is implemented at the end of the calculation via a Lagrange multiplier. Following the same steps as before we arrive at the variational problem of extremizing

$$\mathcal{L}_p = \int_0^\infty \mathrm{d}x \left[\big(p + y'(x)\big) \ln \frac{-p - y'(x)}{1 - p - y'(x)} - \ln \frac{1}{1 - p - y'(x)} \right] - \lambda \int_0^\infty y(x)\,\mathrm{d}x. \quad (10)$$

Using the same Euler–Lagrange formalism, finally leads us to the equation of the limit shape for $p > 0$,

$$y = -\frac{1}{\lambda} \ln(1 - \mathrm{e}^{-\lambda x}) - px. \quad (11)$$

The Lagrange multiplier λ in (11) can be determined by using condition $y \geq 0$ and the normalization $\int_0^{x_m} y(x)\,\mathrm{d}x = 1$, where x_m is the solution of the equation $y(x_m) = 0$. Writing $\exp(x_m) = y^*$, it satisfies $y^* - y^{*1-p} = 1$, and in terms of y^* one finds

$$\lambda^2 \equiv \lambda^2(p) = \frac{\pi^2}{6} - \mathrm{Li}_2(1/y^*) - \frac{p}{2}(\ln y^*)^2, \quad (12)$$

where $\mathrm{Li}_2(z) = \sum_{k=1}^\infty z^k k^{-2}$ is the dilogarithm function. $\lambda(p)$ is a constant which depends on the parameter p. For instance for $p = 0, 1$ and 2, one finds $\lambda(0) = \pi/\sqrt{6}$, $\lambda(1) = \pi/\sqrt{12}$ and $\lambda(2) = \pi/\sqrt{15}$ in agreement with the earlier known results [5, 9]. Figure 2 shows the limit shapes for the MDP–p with $p = 0, 1, 2,$ and 3.

Equation (11) implies that the inverse function $x(y) = \lambda^{-1} \ln \phi(y)$ satisfies

$$\phi(y) - \mathrm{e}^{-\lambda y} \phi(y)^{1-p} = 1. \quad (13)$$

Amazingly this equation appears in several apparently unrelated contexts.

1. The generating function $S(t) = 1 + \sum_{k=1}^\infty s_k(q) t^k$ for the number of connected clusters $s_k(q)$ of size k in a q-ary tree satisfies [14]

$$S(t) - tS^q(t) = 1. \quad (14)$$

This establishes a formal link between two different combinatorial objects, on one hand the q-ary trees and on the other hand the MDP–p problem with $p = 1 - q$. In graph theory [15], $s_k(q)$ is known as the generalized Catalan number, which is given by

$$s_k(q) = \frac{1}{k} \binom{qk}{k-1}.$$

2. Consider the generating function of MDP–p problem

$$Z(x, z) = \sum_E \sum_N \rho_p(E, N) x^E z^N,$$

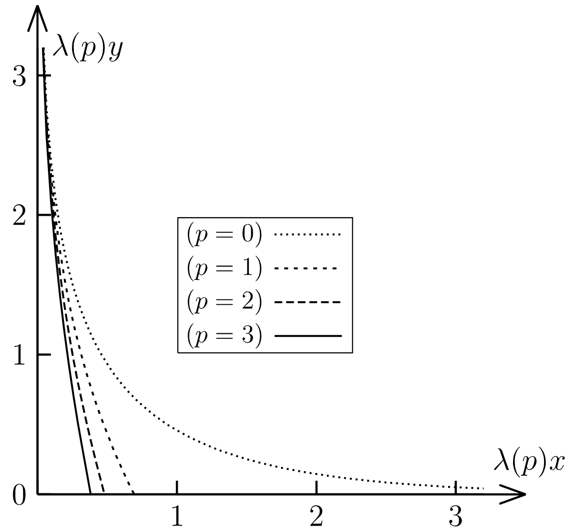

Fig. 2: Limit shapes for the minimal difference p partitions with $p = 0, 1, 2$, and 3, where $\lambda(0) = \pi/\sqrt{6}$, $\lambda(1) = \pi/\sqrt{12}$, $\lambda(2) = \pi/\sqrt{15}$, and $\lambda(3) = 0.752617\ldots$.

where $\rho_p(E, N)$ is the total number of MDP–p of E in N parts. In the limit $E \to \infty$ the number of such partitions will be controlled by the singularities of $Z(x, z)$ near $x = 1$. By setting $x = e^{-\beta}$, one gets for $\beta \to 0$ [16]

$$\ln Z(x, z) \to \int_0^\infty \ln y_p \left(z\mathrm{e}^{-\beta\epsilon} \right) \, \mathrm{d}\epsilon, \tag{15}$$

where the function $y_p(t)$ is given by the solution of the equation

$$y_p(t) - t\, y_p^{1-p}(t) = 1. \tag{16}$$

3. In the physics literature (13) also arises in the context of exclusion statistics. Exclusion statistics [17–21] — a generalization of Bose and Fermi statistics—can be defined in the following thermodynamical sense. Let $Z(\beta, z)$ denote the grand partition function of a quantum gas of particles at inverse temperature β and fugacity z. Such a gas is said to obey exclusion statistics with parameter $0 \leq p \leq 1$, if $Z(\beta, z)$ can be expressed as an integral representation

$$\ln Z(\beta, z) = \int_0^\infty \tilde{\rho}(\epsilon) \ln y_p \left(z\mathrm{e}^{-\beta\epsilon} \right) \, \mathrm{d}\epsilon, \tag{17}$$

where $\tilde{\rho}(\epsilon)$ denotes a single particle density of states and the function $y_p(t)$, which encodes fractional statistics, is given by the solution of (13). The well-known microscopic quantum mechanical realizations of exclusion statistics are the Lowest Landau Level (LLL) anyone model [18] and the Calogero model [19], with $\tilde{\rho}(\epsilon)$ being, respectively, the LLL density of states and the free one dimensional density of states.

The fact that the same equation appears in all three cases is obviously not fortuitous. The link between 2 and 3 follows from the fact that exclusion statistics have a combinatorial interpretation in terms of minimal difference partitions which generalize the usual combinatorial interpretation of the Bose statistic (resp Fermi) in terms of partitions without (with) restrictions. Let us briefly recall this correspondence. Let n_i be the number of columns of height $h = i$ in a Young diagram of a given partition of E, then $E = \sum_i n_i \epsilon_i$ can be interpreted as the total energy of a noninteracting quantum gas of bosons where $\epsilon_i = i$ for $i = 1, 2, \ldots, \infty$ are equidistant single particle energy levels and $n_i = 0, 1, 2, \ldots, \infty$ represents the occupation number of the i-th level. If one now puts the restriction that $h_j > h_{j+1}$ (e.g. allowed partitions of 4 are: 4 and $3+1$), then the restricted partition problem corresponds to a noninteracting quantum gas of fermions, for which $n_i = 0, 1$. If, in addition, one restricts the number of summands to be N, then clearly $N = \sum_i n_i$ represents the total number of particles. For example, if $E = 4$ and $N = 2$, the allowed partitions are $3+1$ and $2+2$ in the unrestricted problem, whereas the only allowed restricted partition is $3+1$. The number $\rho(E, N)$ of ways of partitioning E into N parts is simply the micro-canonical partition function of a gas of quantum particles with total energy E and total number of particles N:

$$\rho(E, N) = \sum_{\{n_i\}} \delta\left(E - \sum_{i=1}^{\infty} n_i \epsilon_i\right) \delta\left(N - \sum_{i=1}^{\infty} n_i\right). \tag{18}$$

For both unrestricted and restricted partitions, one can readily check that the grand partition function $Z(e^{-\beta}, z) = \sum_N \sum_E z^N \mathrm{e}^{-\beta E} \rho(E, N)$, in the limit $\beta \to 0$, is nothing but the one in (15), with $p = 0$ and $p = 1$ respectively.

For a quantum gas obeying exclusion statistics with parameter p it is a priori not obvious how to provide a combinatorial interpretation since the underlying physical models with exclusion statistics describe interacting models. However in some specific cases , such as the Calogero model, one can show that the spectrum can be parameterized as a free spectrum with some restrictions on the quantum numbers which reflect the fact that the Pauli principle is replaced by a stronger exclusion principle [22, 23]. This exclusion is enforced at the level of the Young diagrams by the constraint $h_j - h_{j+1} \geq p$. The link between 1 and 3 expresses this correspondence in terms of counting of states. Exclusion statistics can be implemented by putting n particles in m sites on a one-dimensional lattice, under the restriction that any two particles are at least p sites apart. For a periodic lattice, the number of ways of doing the above is [24]

$$D_{m,n} = \frac{m\Gamma\big(m + (1-p)n\big)}{\Gamma\big(n+1\big)\Gamma\big(m+1-pn\big)}.$$

One can check that $D_{1,n} = s_n(1-p)$ which allows to interpret the generalized Catalan numbers as quantum degeneracy factors.

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