Journal of Mathematical Physics, Analysis, Geometry
2008, vol. 4, No. 4, pp. 457–489

On Contraction Properties for Products of Markov Driven Random Matrices

Y. Guivarc'h

IRMAR CNRS Rennes I, Université de Rennes I, Campus de Beaulieu
35042 Rennes Cedex, France

E-mail:yves.guivarch@univ-rennes1.fr

Received March 28, 2008

We describe contraction properties on projective spaces for products of matrices governed by Markov chains which satisfy strong mixing conditions. Assuming that the subgroup generated by the corresponding matrices is "large" we show in particular that the top Lyapunov exponent of their product has multiplicity one and we give an exposition of the related results.

Key words: Lyapunov exponent, Markov chain, martingale, spectral gap, proximal.

Mathematics Subject Classification 2000: 37XX, 22D40, 60BXX, 82B44.

1. Introduction. Notations

Let V be a d-dimensional Euclidean vector space, $i.e$, $V = \mathbb{R}^d$ with its natural scalar product. Let $G = GL(V)$ be the linear group of V and $g_k (k \in \mathbb{Z})$ a sequence of elements of G. We consider the recurrence relation in V

$$v_{n+1} = g_{n+1} v_n\,, \qquad n \in \mathbb{Z}.$$

Then, given $v_0 \in V$, we can express v_n, $n \in \mathbb{N}$, by $v_n = S_n v_0$, where $S_n = g_n \dots g_1 \in G$ is the product of the elements g_k, $1 \leq k \leq n$. In analogy with the constant case $g_k = g$, A. Lyapunov was able to describe the asymptotic behaviour of $S_n v$, $v \in V$, $n \in \mathbb{N}$, in terms of a finite number of exponents $\lambda_1, \lambda_2, \dots, \lambda_p$, $p \leq d$, under a mild growth condition on the sequence g_k (Lyapunov regularity). The numbers λ_i, $1 \leq i \leq p$, are called the Lyapunov exponents and the set $\{\lambda_1, \dots, \lambda_p\}$ is called the Lyapunov spectrum of the sequence $(g_k)_{k \in \mathbb{Z}}$.

Let (Ω, θ, π) be a measured dynamic system where π is a finite θ-invariant and ergodic probability measure, and $g_k(\omega)$ a θ-stationary sequence, $i.e$, $g_k(\omega) = g_0(\theta^k \omega)$, $k \in \mathbb{Z}$, such that $Log\|g_k(\omega)\|$ and $Log\|g_k^{-1}(\omega)\|$ are π-integrable. Using the methods of ergodic theory, V.I. Oseledets showed ([33]) the Lyapunov regularity of the sequence $(g_k(\omega))_{k \in \mathbb{Z}}$, $\pi - a.e.$ In particular the product $S_n(\omega)$ can be reduced to a block-diagonal form where each block has a definite growth exponent λ_i, $1 \le i \le p$. In this setting $S_n(\omega)$ is a G-valued $\mathbb{Z}$-cocycle of (Ω, θ, π), *i.e*, for $m, n \in \mathbb{Z}$:

$$S_{m+n}(\omega) = S_m\ (\theta^n \omega)\ S_n(\omega),$$

with $S_0(\omega) = Id$.

We denote by $P(V)$ the projective space of V and by $x \to g.x$ the projective action of $g \in G$ on $x \in P(V)$. A basic role in this ergodic context is played by the skew product $(\Omega \times P(V), \widetilde{\theta})$ and its $\widetilde{\theta}$-invariant measures with projection π. Here $\widetilde{\theta}$ is the extension of θ:

$$\widetilde{\theta}(\omega, x) = (\theta\omega, g_1(\omega).x).$$

On the other hand, a special situation, where $\pi = \mu^{\otimes \mathbb{Z}}$ is a product measure and the random variables $g_k(\omega)$ are *i.i.d*, has already been deeply studied by H. Furstenberg and H. Kesten. There, the basic object is the random walk $S_n(\omega)$ on G defined by μ, and in particular the Markov chain on $P(V)$ with transition kernel Q_μ defined by

$$Q_\mu(x, A) = \int 1_A(g.x) d\mu(g).$$

The map $\widetilde{\theta}$ considered above can be identified with the shift transformation on the path space of this Markov chain.

If $supp\mu \subset G$ generates a large subgroup denoted by $< supp\mu >$, it was observed by H. Furstenberg that the above Markov chain has nice properties of contraction analogous to those of the iterates of a single positive matrix. For example, if $supp\mu$ is bounded and $< supp\mu >$ is a dense subgroup of the unimodular group $SL(d, \mathbb{R})$, then $\|S_n(\omega)\|$ has exponential growth. This fact was used as a key tool (for $d = 2$) by I. Goldsheid, S.A. Molcanov, L.A. Pastur in order to prove the pure point spectrum property for the Schrödinger operator with random potential on the line. Motivated by this kind of consequence, and going a step further, the author and A. Raugi, and then I. Goldsheid and G.A Margulis, showed simplicity of the Lyapunov spectrum, $(i.e, p = d)$, for the cocycle $S_n(\omega)$, under mild algebraic conditions on $< supp\mu >$. A basic fact, which can be used in a more flexible way, is that the top Lyapunov exponent has multiplicity one. This is the starting point for various nontrivial properties of the cocycle $S_n(\omega)$. Then it is clear that, under mild conditions on $< supp\mu >$, the asymptotic properties of $S_n(\omega)$ can be developed much further and applied to various probabilistic, analytic or geometrical questions. Furthermore, even in the *i.i.d* case, since "large" subgroups play an important role, this topic cannot be considered as a simple extension of Classical Probability Theory, from $\mathbb{R}^*$ to $GL(d, \mathbb{R})$. Here we sketch these developments and we restrict our survey to the case of Markov dependence of the increments $g_k (k \in \mathbb{Z})$. The emphasis is put more on the basic ideas than on the detailed results. We give a detailed exposition of the ideas in the $i.i.d$ case (Sects. 2–4), and we describe briefly the required modifications for the Markovian case (Sect. 5). We observe that this Markovian setting includes the case where π is a Gibbs measure on $\Omega = A^{\mathbb{Z}}$ and $g_0(\omega)$ depends only on a finite number of coordinates. A few applications are described in Sect. 6 and references for other topics are given. We describe now some notations used below.

For a Polish space E, the space of complex continuous functions on E will be denoted $C(E)$, and the space of continuous maps of E into itself by $C(E, E)$. The action of a map u on E will be denoted $x \to u.x$ ($x \in E$) if E is compact. The space of probability measures on a Polish space F will be denoted $M^1(F)$. If $\mu \in M^1(C(E, E))$ and $\rho \in M^1(E)$, we write $\mu * \rho$ for the measure on E given by $\varphi \to \int \varphi(g.x)d\mu(g)d\rho(x)$. A measure $\nu \in M^1(E)$ is said to be μ-stationary if $\mu * \nu = \nu$. In this context, we will consider the Markov chain on E with transition kernel Q_μ defined by

$$Q_\mu \varphi(x) = \int \varphi(g.x)d\mu(g),$$

where φ is a bounded Borel function on E.

The adjoint of $u \in End\ E$, with respect to the given scalar product, will be denoted u^*. For $g \in GL(V)$, we will also denote by g the corresponding projective map on $P(V)$. The elements of $P(V)$ will be represented by vectors of unit length, taken up to sign. In particular, for $x \in P(V)$ and $g \in GL(V)$, $\|gx\| \in \mathbb{R}_+$ is well defined. The wedge products over V will be denoted by $\wedge^k V$ ($1 \le k \le d$). The Euclidean scalar product extends naturally to $\wedge^k V$. The submanifold of $P(\wedge^2 V)$ corresponding to decomposable 2-vectors will be denoted by $P_2(V)$. For $x \in P(V)$, $x \wedge y \in P_2(V)$, $g \in G$, we will consider the following cocycles:

$$\sigma_1(g, x) = Log\|gx\|, \quad \sigma_2(g, x \wedge y) = Log\|g(x \wedge y)\|.$$

Also we will consider the submanifold $P_{1,2}(V) \subset P(V) \times P_2(V)$ of elements $\xi = (x, x \wedge y)$ and the cocycle α defined by

$$\alpha(g, \xi) = Log\frac{\|gx \wedge gy\|}{\|gx\|^2}.$$

For $x, y \in P(V)$, we set $\delta(x, y) = \|x \wedge y\|$. The unique probability measure on $P(V)$, invariant under orthogonal maps will be denoted m, and the orthogonal group of V by $O(d)$. In addition to projective maps, we need also to consider quasiprojective maps corresponding to nonzero endomorphisms of V. If $u \in EndV$ and $x \in P(V)$, then $u.x$ is well defined if x does not belong to the projective subspace defined by $Ker\ u$, again denoted by $Ker\ u$. Then the quasiprojective map u is defined and continuous outside $Ker\ u$. If $\nu \in M^1(P(V))$ satisfies $\nu(Ker\ u) = 0$, then the push forward measure $u.\nu$ is well defined. If $F \subset G$, we will denote by $< F >$ $(resp[F])$ the subgroup (resp subsemigroup) generated by F. Their closures will be written $< F >^-$ and $[F]^-$, respectively. We will say that a measure μ on G has exponential moment and write $\mu \in M^{1.e}(G)$ if there exists $c > 0$ such that

$$\int \|g\|^c d\mu(g) + \int \|g^{-1}\|^c d\mu(g) < +\infty.$$

The unimodular group $SL(V) = SL(d, \mathbb{R}) \subset G$ will be written G_1. Occasionally the projection of $x \in V$ on $P(V)$ will be denoted $\overline{x}$, but in general we will take the same notation for vectors and elements of the projective space. The same abuse of notations will be made for subspaces.

2. Growth of Column Vectors

Let μ be a probability measure on $G_1 = SL(d, \mathbb{R})$ and $\mathbb{L}^2(V)$ the Hilbert space of square integrable functions with respect to Lebesgue measure on V. We say that a subset $S \subset GL(V)$ is strongly irreducible if no nontrivial union of subspaces of V is S-invariant. In particular strong irreducibility implies irreducibility.

Theorem 2.1. *Let $\mu \in M^1(G_1)$ and assume the closed subgroup $< supp\mu >^-$ is strongly irreducible and unbounded. Let $r(\mu)$ be the spectral radius of the convolution operator on $\mathbb{L}^2(V)$ defined by μ. Then $r(\mu) < 1$.*

Corollary 2.2. *Assume furthermore $\int Log\|g\|d\mu(g) < +\infty$. Then there exists a positive number $\lambda(\mu)$ such that*

$$\lim_{n \to +\infty} \frac{1}{n} \int Log\|g\|d\mu^n(g) = \lambda(\mu) \geq \frac{2}{d} Log \frac{1}{r(\mu)} > 0.$$

Furthermore: $\pi - a.e,\ \lim_{n \to +\infty} \frac{1}{n} Log\|S_n(\omega)\| = \lambda(\mu) > 0,$ *where* $S_n(\omega) = g_n \ldots g_1$.

Theorem 2.3. *Assume that μ satisfies the hypothesis of Th. 1, and furthermore $\int Log\|g\|d\mu(g) < +\infty$. Then for every fixed $v \in V \setminus \{0\}$:*

$$\pi - a.e, \lim_{n \to +\infty} \frac{1}{n} Log\|S_n(\omega)v\| = \lambda(\mu) > 0.$$

Also, $\frac{1}{n} \int Log\|gx\|d\mu^n(g)$ converges to $\lambda(\mu)$ uniformly on $P(V)$.

The proof of Th. 2.1 depends on the following lemmas.

Lemma 2.4. *Assume that the subgroup Γ of G_1 is strongly irreducible and unbounded. Then no probability measure on $P(V)$ is Γ-invariant.*

P r o f. Assume $\nu \in M^1(P(V))$ is Γ-invariant. Let $g_n \in G_1$ with $\lim_{n\to+\infty} \|g_n\| = +\infty$ and write $u_n = \frac{g_n}{\|g_n\|}$. Then $\det u_n = \frac{1}{\|g_n\|^d}$ converges to zero. Since $\|u_n\| = 1$, we can extract from u_n a convergent subsequence and assume $\lim_{n\to+\infty} u_n = u$, $\|u\| = 1$, $\det u = 0$. Let $W \subset P(V)$ (resp W') be the projective subspace associated with $Ker\ u$ (resp $Im\ u$). We denote by ν_1 and ν_2 the restrictions of ν to W and $P(V) \setminus W$ and write $\nu = \nu_1 + \nu_2$. We observe that u defines a quasiprojective *map*, again denoted by u, of $P(V) \setminus W$ into $P(V)$. Then we have $\nu = \lim_{n\to+\infty} g_n.\nu = u.\nu_2 + \lim_{n\to+\infty} g_n.\nu_1$. Since $P(V)$ is compact, we can assume that $g_n.\nu_1$ converges to ν_1' concentrated on the subspace $W_1 = \lim_{n\to+\infty} g_n.W$. It follows that $\nu = \lim_{n\to+\infty} g_n.\nu$ is concentrated on the union of W_1 and W'.

Let Φ be the set of subsets X of $P(V)$ which are finite unions of projective subspaces and which satisfy $\nu(X) = 1$. Since any decreasing sequence of elements of Φ is asymptotically constant, Φ has a least element, which is $X_0 = \bigcap_{X\in\Phi} X$. Since $g.\nu = \nu$, one has $g.X_0 = X_0$. This contradicts strong irreducibility of Γ. ∎

P r o f of Theorem 2.1. We denote by $\rho(\mu)$ the convolution operator on $\mathbb{L}^2(V)$ defined by $\rho(\mu)(f)(v) = \int f(g^{-1}v)d\mu(g)$. Since every $g \in G_1$ preserves Lebesgue measure, $\|\rho(\mu)f\|_2 \le 1$. Assume $r(\mu) = 1$ and let $z \in \mathbb{C}$ be a spectral value of $\rho(\mu)$ with $|z| = 1$. Then, either $\lim_{n\to+\infty} \|\rho(\mu)f_n - f_n\|_2 = 0$ for some sequence $f_n \in \mathbb{L}^2(V)$ with $\|f_n\|_2 = 1$, or $Im\ (\rho(\mu) - zI)$ is not dense in $\mathbb{L}^2(V)$. In the second case, duality gives $Ker(\rho(\mu^*) - \overline{z}I) = \{0\}$. Since μ^* satisfies also the hypothesis we can only consider the first case.

Since $|\rho(\mu)|f_n| - |f_n|| \le |\rho(\mu)f_n - f_n|$, we have also

$$\lim_{n\to+\infty} \|\rho(\mu)|f_n| - |f_n|\|_2 = 0, \lim_{n\to+\infty} \|\rho(\mu)|f_n|\|_2 = 1.$$

Hence $\lim_{n\to+\infty} < \rho(\mu)|f_n|, |f_n| >= 1 = \lim_{n\to+\infty} \int < \rho(g)|f_n|, |f_n| > d\mu(g)$.

It follows that there exists a Borel subset S' of $supp\mu$ with $\mu(S') = 1$, and a subsequence n_k such that $< \rho(g)|f_{n_k}|, |f_{n_k}| >$ converges to 1, for every $g \in S'$. For the sake of brevity we write $n_k = n$. The inequality

$$\|\rho(g)|f_n|^2 - |f_n|^2\|_1 \le \|\rho(g)|f_n| - |f_n|\|_2 \|\rho(g)|f_n| + |f_n|\|_2 \le 2\|\rho(g)|f_n| - |f_n|\|_2$$

gives $\lim_{n\to+\infty} \|\rho(g)|f_n|^2 - |f_n|^2\|_1 = 0$ for every $g \in S'$.

We consider the probability measure $\theta_n = |f_n|^2(v)dv$ on V and its projection $\overline{\theta_n}$ on $P(V)$. Then the above relation says that $\lim_{n\to+\infty} g.\theta_n - \theta_n = 0$ in variation norm, hence $\lim_{n\to+\infty} g.\overline{\theta_n} - \overline{\theta_n} = 0$ also in variation. Since $P(V)$ is compact, we can assume $\lim\limits_{n\to\infty} \overline{\theta_n} = \theta$ in weak topology. In particular, for every $g \in S'$: $g.\theta = \theta$. Since $\mu(S') = 1$, S' generates $< supp\mu >^-$, hence $g.\theta = \theta$ for any $g \in < supp\mu >^-$. Lemma 2.4 says that this is impossible. ∎

P r o f of Corollary 2.2. We denote $u_n = \int Log\|g\| d\mu^n(g)$. Since $\|g\| \geq 1$, $u_n \geq 0$. The subadditivity of $Log\|g\|$ implies $u_{m+n} \leq u_m + u_n$, hence $0 \leq u_n \leq nu_1 < +\infty$.

It follows $\lim\limits_{n\to+\infty} \dfrac{u_n}{n} = \mathop{Inf}\limits_{n} \dfrac{u_n}{n} = \gamma \geq 0$.

We consider the $\mathbb{L}^2$-functions on V, f and 1_C defined by

$$f(v) = Inf(1, \|v\|^{-\delta}), \qquad C = \{v \in V; 1 \leq \|v\| \leq 2\}$$

with $2\delta > d$ and we normalize the Lebesgue measure dv on V such that vol $C = 1$.

The theorem gives $\limsup\limits_{n\to+\infty} | < \rho(\mu^n)f, 1_C > |^{1/n} \leq r(\mu)$. On the other hand:

$< \rho(\mu^n)f, 1_C > \geq \int 1_C(v) \frac{1}{\|g^{-1}v\|^{\delta}} d\mu^n(g)\ dv \geq 2^{-\delta} \int \frac{1}{\|g^{-1}\|^{\delta}} d\mu^n(g)$,

$Log < \rho(\mu^n)f, 1_C > \geq -\delta Log2 - \delta \int Log\|g^{-1}\| d\mu^n(g)$,

$\delta \liminf\limits_{n\to+\infty} \frac{1}{n} \int Log \ \ \|g^{-1}\| d\mu^n(g) \geq -Logr(\mu)$,

$\liminf\limits_{n\to+\infty} \frac{1}{n} \int Log\|g^{-1}\| d\mu^n(g) \geq \frac{1}{\delta} Log \frac{1}{r(\mu)}$.

Since δ is arbitrary with $\delta > \frac{d}{2}$, and $r(\mu) = r(\mu^*)$, we get $\gamma \geq \frac{2}{d} Log \frac{1}{r(\mu)} > 0$.

The subadditivity of $Log\|g\|$ implies that

$$Log\|S_{m+n}(\omega)\| \leq Log\|S_m(\omega)\| + Log\|S_n \circ \theta^m(\omega)\|,$$

hence we can apply the subadditive ergodic theorem to the sequence $Log\|S_n(\omega)\|$:

$\frac{1}{n} Log\|S_n(\omega)\|$ converges $\pi - a.e$ and in $\mathbb{L}^1(\Omega)$ to a constant $\lambda(\mu)$. It follows:

$$\lambda(\mu) = \lim_{n\to+\infty} \frac{1}{n} \int Log\|S_n(\omega)\| d\pi(\omega) = \lim_{n\to+\infty} \frac{1}{n} \int Log\|g\| d\mu^n(g) = \gamma > 0.$$

∎

For the proof of Th. 2.3, we need the following lemmas.

Lemma 2.5. *For any fixed $c \in \mathbb{R}$, the set W of elements v in V such that*

$$\pi - a.e, \qquad \limsup_{n\to+\infty} \frac{1}{n} Log\|S_n(\omega)v\| \leq c$$

is a $supp\mu$-invariant subspace.

P r o f. We observe that if $a, b > 0$, then $Log(a + b) \leq 1 + Sup(Loga, Logb)$. If $v, v' \in V$, it follows

$$Log\|S_n(\omega)(v + v')\| \leq 1 + Sup(Log\|S_n(\omega)v\|, Log\|S_n(\omega)v'\|).$$

Hence, the condition $v, v' \in W$ implies $v + v' \in W$. Also the condition $v \in W$ implies $\lambda v \in W$ for any $\lambda \in \mathbb{R}$. It follows that W is a subspace of V.

We observe that $S_n(\omega) = S_{n-1}(\theta\omega)g_1(\omega)$. Hence, the condition $v \in W$ implies $\pi - a.e,\ \ g_1(\omega)v \in W$.

The $supp\mu$-invariance of W follows ∎

Lemma 2.6. *Let m be the uniform measure on $P(V)$. For any $u \in EndV$ we have*

$$\int Log\|ux\|dm(x) \geq Log\|u\| - Log2.$$

P r o f. We use the polar decomposition of u: $u = kak'$ with $k, k' \in O(d)$, $a = diag(a^1, \dots, a^d)$ with $a^1 \geq a^2 \geq \dots \geq a^d > 0$ and $\|u\| = a^1$. We write dk for the normalized Haar measure on $O(d)$. Then, since m is $O(d)$-invariant:

$$\int Log\|ux\|dm(x) = \int Log\|ake_1\|dk \geq \int Log|a^1 < ke_1, e_1 > |dk.$$

$$\int Log\|ux\|dm(x) \geq Log\ a^1 + \frac{1}{2\pi}\int_0^{2\pi} Log|cos\theta|\ d\theta = Log\|u\| - Log2.$$

∎

Lemma 2.7. *Let $\nu \in M^1(P(V))$ be μ-stationary i.e $\int g.\nu d\mu(g) = \nu$. Then:*

$$\int Log\|gx\|\ d\mu(g)d\nu(x) = \lambda(\mu).$$

P r o f. Let $\gamma_\nu = \int Log\|gx\|d\mu(g)d\nu(x)$. The finiteness of γ_ν follows from μ-integrability of $Log\|g\|$. Since ν is μ-stationary, for any $n \in \mathbb{N}$:

$$n\gamma_\nu = \int Log\|gx\|d\mu^n(g)d\nu(x) = \int Log\|S_n(\omega)x\|d\pi(\omega)d\nu(x).$$

We observe that if $f(\omega, x)$ is given by $f(\omega, x) = Log\|g_1(\omega)x\|$, then

$$Log\|S_n(\omega)x\| = \sum_1^n f \circ \tilde{\theta}^k(\omega, x).$$

Since $|f(\omega, x)| \leq Log\|g_1(\omega)\|$, f is $\pi \otimes \nu$-integrable and we can use the ergodic theorem

$$\pi \otimes \nu - a.e, \lim_{n \to +\infty} \frac{1}{n} Log\|S_n(\omega)x\| = \int f(\omega, x)d\pi(\omega)d\nu(x) = \gamma_\nu.$$

Then Lemma 2.5 and strong irreducibility of $supp\mu$ imply that for every $x \in P(V)$:

$$\pi - a.e, \quad \limsup_{n\to+\infty} \frac{1}{n} Log\|S_n(\omega)x\| \le \gamma_\nu.$$

In particular, the dominated convergence gives, for every $x \in P(V)$:

$$\lim_{n\to+\infty} \operatorname{Sup}_x \frac{1}{n}\int Log\|gx\|d\mu^n(g) \le \gamma_\nu,$$

hence: $\limsup_{n\to+\infty} \frac{1}{n}\int Log\|gx\|dm(x)d\mu^n(g) \le \gamma_\nu$. Using Lemma 2.6, we have

$$\int Log\|g\|d\mu^n(g) \le \int Log\|gx\|dm(x)d\mu^n(g) + Log2,$$

hence: $\gamma_\nu \ge \lim_{n\to+\infty} \frac{1}{n}\int Log\|g\|d\mu^n(g) = \lambda(\mu)$. Since $\gamma_\nu \le \lambda(\mu)$, we conclude $\gamma_\nu = \lambda(\mu)$. ■

The following is a well known fact of Markov chain theory (see [9, 16]).

Lemma 2.8. *Let G be a locally compact group, E be a compact metric G-space, $\mu \in M^1(G)$, $I \subset M^1(E)$ the set of μ-stationary measures on E, f a continuous function on E such that $\nu_1(f) = \nu_2(f)$, for every $\nu_1, \nu_2 \in I$. Then, with $\nu \in I$:*

$$\pi - a.e, \quad \lim_{n\to+\infty} \frac{1}{n}\sum_1^n f(S_k(\omega).x) = \nu(f).$$

The convergence of $\frac{1}{n}\sum_1^n \int f(g.x)d\mu^k(g)$ to $\nu(f)$ is uniform on E.

We will use this lemma if $E = P(V)$ and $f(x) = \int Log\|gx\|d\mu(g)$. In the proof of Th. 3 below we assume $\int Log^2\|g\|d\mu(g) < +\infty$.

P r o f of Theorem 3. We consider the Markov chain on $P(V)$ with transition kernel $Q_\mu(x,.) = \mu * \delta_x$, its space of trajectories $\Omega \times P(V)$, and the random variables $X_k(\omega, x) = Log\|g_k(S_{k-1}.x)\|$, $k \ge 1$. Clearly, $Log\|S_n(\omega)x\| = \sum_1^n X_k(\omega, x)$.

We fix $x \in P(V)$ and we denote by $\mathcal{F}_n$ the σ-field on Ω generated by the random variables $S_k(\omega).x$, $0 \le k \le n$. Then we have $\mathbb{E}(X_k)|\mathcal{F}_{k-1}| = f(S_{k-1}.x)$, hence the sequence $Y_k = X_k - f(S_{k-1}.x)$ is the sequence of increments of the martingale $Z_n = \sum_1^n Y_k$.

If we assume $\int Log^2\|g\|d\mu(g) < +\infty$, then $\underset{k\geq 1}{Sup}\,\mathbb{E}(|Y_k|^2) \leq \displaystyle\int Log^2\|g\|d\mu(g)$ $< +\infty$.

Hence the law of large numbers for martingales gives $\pi - a.e, \lim \frac{1}{n}\sum_{1}^{n} Y_k = 0$.

Using Lemma 2.7, we conclude that for any μ-stationary measure ν, $\nu(f) = \int Log\|gx\|d\mu(g)d\nu(x) = \lambda(\mu)$.

Then Lemma 2.8 implies

$$\pi - a.e, \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{1}^{n} f(S_k.x) = \lambda(\mu).$$

From the convergence of $\frac{1}{n}\sum_{1}^{n} Y_k$ to zero, we get

$$\pi - a.e, \quad \lim_{n\to\infty} \frac{1}{n} Log\|S_n(\omega)x\| = \lambda(\mu).$$

The last assertion is a direct consequence of Lemma 2.8. ■

Remarks.

a) We have used the condition $\int Log^2\|g\|d\mu(g) < +\infty$ instead of $\int Log\|g\|d\mu(g)$ $< +\infty$. A refinement of the above argument gives the complete result (see [9]). It can also be obtained as a consequence of Oseledets' multiplicative ergodic theorem (see [22]).

b) Strong irreducibility of $< supp\mu >$ have been used only in order to get $\lim\limits_{n\to+\infty} \frac{1}{n} Log\|S_n(\omega)v\| > 0$. The proof above shows that under irreducibility of $< supp\mu >$ one gets, for every $v \in V \setminus \{0\}$,

$$\pi - a.e \lim_{n\to+\infty} \frac{1}{n} Log\|S_n(\omega)v\| = \lim_{n\to+\infty} \frac{1}{n} Log\|S_n(\omega)\| = \lambda(\mu) \geq 0.$$

c) A typical example with $< supp\mu >$ irreducible but not strongly irreducible is $G = SL(2,\mathbb{R})$, $\mu = \frac{1}{2}(\delta_a + \delta_b)$ with $a = diag(\lambda, \frac{1}{\lambda})$, $\lambda > 1$, $b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $\lambda(\mu) = 0$.

d) Theorem 2.3 was obtained in [9] by a different argument. Here it is a consequence of Th. 2.1 which can be considered as a special case of the main result of [7].

e) For a corresponding result where independence of increments is replaced by markovian dependence with spectral gap, see [39]. ■

3. Uniqueness of Stationary Measures and Contraction Properties

Here we consider the group $G = GL(V)$, its action on $P(V)$, and a probability measure $\mu \in M^1(G)$. In order to state the results we give some definitions.

Definition 3.1. *An element $g \in GL(V)$ is said to be proximal if one can write*

$$V = \mathbb{R}v_g \oplus V_g^<, \quad gv_g = \sigma(g)v_g, |\sigma(g)| = \lim_{n\to+\infty} \|g^n\|^{1/n}, \quad gV_g^< = V_g^<,$$

and the spectral radius of g on $V_g^<$ is strictly less than $|\sigma(g)|$.

Definition 3.2. *A subsemigroup $S \subset GL(V)$ is said to satisfy condition i.p if S is strongly irreducible and S contains a proximal element.*

Definition 3.3. *A probability measure $\nu \in M^1(P(V))$ is said to be proper if for every proper projective subspace $H \subset P(V)$ one has $\nu(H) = 0$.*

Definition 3.4. *A sequence $g_n \in G$ is said to satisfy the contracting property towards $z \in P(V)$ if one has $\lim_{n\to+\infty} g_n.m = \delta_z$. (Where m is the uniform measure on $P(V)$).*

Theorem 3.5. *Assume that the closed subsemigroup of G generated by suppμ satisfies condition i.p. Then, there exists a measurable map z from Ω to $P(V)$, defined $\pi - a.e$ such that*

$$g_1.(zo\theta) = z.$$

The map z is unique mod π and

$$\pi - a.e, \quad \delta_{z(\omega)} = \lim_{n\to+\infty} g_1 \cdots g_n.m.$$

The Markov operator defined by $x \to \mu * \delta_x$ has a unique stationary measure ν on $P(V)$ and ν is the law of $z(\omega)$. The measure ν is proper.

Corollary 3.6. *Let $z^*(\omega)$ be defined by $\delta_{z^*(\omega)} = \lim_{n\to+\infty} g_1^* \cdots g_n^*.m$ and assume $x \notin Ker z^*(\omega)$. Then, if $S_n(\omega) = g_n \cdots g_1$:*

$$\pi - a.e, \lim_{n\to+\infty} \frac{\|S_n(\omega)x\|}{\|S_n(\omega)\|} = |< z^*(\omega), x >|, \quad \lim_{n\to+\infty} \frac{\|S_n(\omega)x \wedge S_n(\omega)y\|}{\|S_n(\omega)x\|^2} = 0.$$

If furthermore $y \notin Ker z^*(\omega)$:

$$\pi - a.e, \lim_{n\to+\infty} \frac{\delta(S_n(\omega).x, S_n(\omega).y)}{\delta(x,y)} = 0.$$

For fixed ω, the above convergences are uniform if x, y vary in a compact subset of $P(V) \setminus Ker z^*(\omega)$.

The proof of Th. 3.5 depends of the following lemmas.

Lemma 3.7. *Assume $\nu \in M^1(P(V))$ is proper and $g_n \in G$ is a sequence such that $\lim\limits_{n\to+\infty} g_n.\nu = \delta_z$ Then g_n has the contraction property towards z.*

P r o f. We can assume that g_n converges to a quasiprojective map u, $i.e$, for $H \subset P(V)$ a projective subspace and $x \notin H$: $\lim\limits_{n\to+\infty} g_n.x = u.x$.

Using dominated convergence, we get that for any $\varphi \in C(P(V))$:

$$\lim_{n\to+\infty} \int \varphi(g_n.x)d\nu(x) = (u.\nu)(\varphi) = \varphi(z).$$

It follows $u.x = z$ if $x \notin Keru$. Using again dominated convergence, we get: $u.m = \delta_z$, hence $\lim\limits_{n\to+\infty} g_n.m = \delta_z$. ■

Lemma 3.8. *Assume that $[supp\mu]^-$ is strongly irreducible. Then every μ-stationary measure on $P(V)$ is proper.*

P r o f. Let ν be a μ-stationary measure on $P(V)$.

We consider the set $\mathcal{H}$ of projective subspaces $H \subset P(V)$ such that $\nu(H) > 0$ and H has minimal dimension with respect to this condition. We observe that, if $H, H' \in \mathcal{H}$ and $H \neq H'$, then $\nu(H \cap H') = 0$. It follows that for every $\varepsilon > 0$: $\mathcal{H}_\varepsilon = \{H \in \mathcal{H}; \nu(H) \geq \varepsilon\}$ is finite. Hence, there exists $H_0 \in \mathcal{H}$ with $\nu(H_0) = Sup\{\nu(H); H \in \mathcal{H}\}$ and the set $\mathcal{H}_0$ of such subspaces H_0 is finite. On the other hand, the equation $\nu(H) = \int (g.\nu)(H)d\mu(g)$ implies $g^{-1}H_0 \in \mathcal{H}_0$, $\mu - a.e$ for any $H_0 \in \mathcal{H}_0$, hence $(supp\mu)$ $(\mathcal{H}_0) = \mathcal{H}_0$. This contradicts the strong irreducibility assumption. Hence $\mathcal{H} = \phi$, $i.e$, ν is proper. ■

Lemma 3.9. *Let $\varphi \in C(P(V))$ and denote for $(\omega, \eta) \in \Omega \times \Omega$, $\omega = (g_k)_{k\in\mathbb{N}}$, $\eta = (\gamma_k)_{k\in\mathbb{N}}$: $f_n(\omega) = (g_1 \cdots g_n.\nu)(\varphi)$, $f_n^r(\omega, \eta) = (g_1 \cdots g_n.\gamma_0 \cdots \gamma_r.\nu)(\varphi)$.*

Then, if r is fixed, $\pi \otimes \pi - a.e$ $\lim\limits_{n\to+\infty} f_n^r(\omega, \eta) - f_n(\omega) = 0$.

P r o f. We denote by $\mathcal{F}_n$ the σ-field on Ω generated by $g_1(\omega) \cdots g_n(\omega)$.

Since ν is μ-stationary: $\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n$, $i.e$, f_n is a martingale. It follows that f_n and $f_{n+r} - f_n$ are orthogonal, $i.e$, $\mathbb{E}((f_{n+r} - f_n)^2) = \mathbb{E}(f_{n+r}^2) - \mathbb{E}(f_n^2)$.

Then, for any $m > 0$

$$\sum_{n=1}^{m} \mathbb{E}(f_{n+r} - f_n)^2 \leq 2r|\varphi|_\infty^2.$$

The convergence of the series $\sum\limits_{n=1}^{\infty} \mathbb{E}((f_{n+r} - f_n)^2)$ follows. Since

$$\mathbb{E}((f_{n+r} - f_n)^2) = \int |f_n^r(\omega, \eta) - f_n(\omega)|^2 d\pi(\omega)d\pi(\eta),$$

we get the convergence $\pi\otimes\pi - a.e$ of the series $\sum_{n=1}^{\infty} |f_n^r(\omega,\eta) - f_n(\omega)|^2$. In particular, the assertion of the lemma follows. ∎

P r o f of Theorem 3.5. We have observed above that for any $\varphi \in C(P(V)), f_n(\omega)$ is a martingale. Taking φ in a countable dense subset of $C(P(V))$, we get that there exists $\nu_\omega \in M^1(P(V))$ defined $\pi - a.e$ such that

$$\pi - a.e, \quad \lim_{n\to+\infty} g_1 \cdots g_n.\nu = \nu_\omega.$$

In the same way we get, using Lem. 3.5,

$$\pi \otimes \pi - a.e, \lim_{n\to+\infty} g_1 \cdots g_n \gamma_0 \cdots \gamma_r.\nu = \lim_{n\to+\infty} g_1 \cdots g_n.\nu = \nu_\omega.$$

Hence

$$\pi - a.e, \quad \lim_{n\to+\infty} g_1 \cdots g_n \gamma.\nu = \nu_\omega$$

for every $\gamma \in [supp\mu]^-$. Let $n_k(\omega)$ be a subsequence such that $g_1 \cdots g_{n_k}$ converges to a quasiprojective map τ_ω. Since ν and $\gamma.\nu$ are proper

$$\tau_\omega.(\gamma.\nu) = \tau_\omega.\nu = \nu_\omega.$$

Let H_ω be the kernel of τ_ω, γ_1 a proximal element of $[supp\mu]^-$, with attractive fixed point x. Using the strong irreducibility of $[supp\mu]^-$, we can find $\gamma_0 \in [supp\mu]^-$ such that $\gamma_0.x \notin H_\omega$. Then, taking $\gamma = \gamma_0\gamma_1^n (n \in \mathbb{N})$, we get: $\lim_{n\to+\infty} \gamma_0\gamma_1^n.\nu = \delta_{\gamma.x}$. The continuity of τ_ω outside H_ω gives finally

$$\tau_\omega.\nu = \nu_\omega = \tau_\omega.(\gamma.\delta_x) = \delta_{\tau_\omega \gamma.x}.$$

This shows that ν_ω is $\pi - a.e$ a Dirac measure $\delta_{z(\omega)}$, and, furthermore $\tau_\omega(P(V) \setminus H_\omega) = z(\omega)$. In particular,

$$\pi - a.e, \quad \lim_{n\to+\infty} g_1 \cdots g_1.\nu = \tau_\omega.\nu = \delta_{z(\omega)}.$$

This convergence implies

$$\pi - a.e, \quad z(\omega) = g_1.z(\theta\omega),$$

and furthermore ν is the law of $z(\omega)$. Also we have $\mathbb{E}(\delta_{z(\omega)}|\mathcal{F}_n| = g_1 \cdots g_n.\nu$.

Using Lemma 3.8, we know that ν is proper. Then Lemma 3.7 gives that $g_1 \cdots g_n$ has the convergence property towards $z(\omega)$, hence

$$\pi - a.e, \quad \lim_{n\to+\infty} g_1 \cdots g_n.m = \delta_{z(\omega)}.$$

This relation defines $z(\omega)$ independently of ν. Since ν is the law of $z(\omega)$, ν is unique as a μ-stationary measure.

If $z'(\omega)$ is a solution $\pi - a.e$ of the equation $z' = g_1.(z' \circ \theta)$ and ν' is the law of z', we have, using the independence of g_1 and $z' \circ \theta$: $\nu' = \mu * \nu'$.

From above, we have $\nu' = \nu$. Also $\mathbb{E}(\delta_{z'(\omega)}|\mathcal{F}_n) = g_1 \cdots g_n.\nu'$ and from the martingale convergence theorem

$$\pi - a.e, \quad \delta_{z'(\omega)} = \lim_{n\to+\infty} g_1 \cdots g_n.\nu'.$$

Since $\nu' = \nu$, we get $z' = z \quad \pi - a.e.$ ■

For the proof of Cor. 3.6 we need the following.

Lemma 3.10. *Assume $g_n \in G$ is such that g_n^* has the contraction property towards $z^* \in P(V)$. Then, for any $x, y \in P(V)$, with $x \notin Kerz^*$:*

$$\lim_{n\to+\infty} \frac{\|g_n x\|}{\|g_n\|} = |<z^*, x>|, \quad \lim_{n\to+\infty} \frac{\|g_n x \wedge g_n y\|}{\|g_n x\|^2} = 0.$$

Furthermore, the sequence $\frac{\delta(g_n.x, g_n.y)}{\delta(x,y)}$ converges uniformly to zero if x, y vary in a compact subset of $P(V) \setminus Kerz^*$.

P r o f. We use the polar decomposition $G = K\overline{A^+}K : g_n = k_n a_n k'_n$ with $k_n, k'_n \in K = O(d)$, $a_n \in \overline{A^+}$. Then the convergence of $g_n^*.m$ to z^* implies

$$a_n^{(2)} = o(a_n^1), \quad \lim_{n\to+\infty} k_n'^{-1}.\overline{e}_1 = z^*.$$

If $x = \sum_{i=1}^{d} x^i e_i$, we get

$$\|g_n x\|^2 = \sum_{i=1}^{d} |a_n^i < k'_n x, e_i > |^2 \geq |a_n^1 < k'_n x, e_1 > |^2.$$

Since $\|g_n\| = a_n^1$

$$\lim_{n\to+\infty} \frac{\|g_n x\|^2}{\|g_n\|^2} = \lim_{n\to\infty} |< k_n'^{-1} e_1, x > |^2 + \lim_{n\to+\infty} \sum_{i>1} \left(\frac{a_n^i}{a_n^1}\right)^2 < k'_n x, e_i >^2$$

$$= |< z^*, x > |^2.$$

Also $\|g_n x \wedge g_n y\|^2 = \sum_{i<j} (a_n^i a_n^j)^2 | < k'_n (x \wedge y), e_i \wedge e_j > |^2.$

It follows

$$\|g_n x \wedge g_n y\| \leq d a_n^{(1)} a_n^{(2)} \|x \wedge y\|, \ \|g_n x\| \geq a_n^{(1)} | < k'_n x, e_1 > |,$$

$$|g_n y\| \geq a_n^{(1)}| < k'_n y, e_1 > |,$$

$$\frac{\|g_n x \wedge g_n y\|}{\|x \wedge y\| \ \|g_n x\|^2} \leq d \ \frac{a_n^{(2)}}{a_n^{(1)}} \quad \frac{1}{| < k'_n x, e_1 > |^2}.$$

Since $\lim\limits_{n\to+\infty} | < k'_n x, e_1 > | = | < z^*, x > | \neq 0$ and $a_n^{(2)} = 0(a_n^{(1)})$, we get

$$\lim_{n\to+\infty} \ \frac{\|g_n x \wedge g_n y\|}{\|g_n x\|^2} = 0.$$

Also $\frac{\|g_n x \wedge g_n y\|}{\|g_n x\| \ \|g_n y\| \ \|x\wedge y\|} \leq d \ \frac{a_n^{(2)}}{a_n^{(1)}} \ \frac{1}{|<k'_n x, e_1><k'_n y, e_1>|}$.

Since $x, y \notin Kerz^*$:

$$\lim_{n\to+\infty} \frac{\delta(g_n.x, g_n.y)}{\delta(x,y)} = 0.$$

Since $\lim\limits_{n\to+\infty} | < k'_n x, e_1 >< k'_n y, e_1 > | = | < z^*, x > | \ | < z^*, y > |$ is bounded from below on a compact C of $P(V) \setminus Kerz^*$, the convergence to $| < z^*, x >< z^*, y > |$ is uniform on C. ■

P r o f of Corollary 3.6. We observe that if a semigroup S satisfies $i.p$, then the semigroup S^* satisfies also $i.p$. Then the theorem implies the convergence

$$\pi - a.e, \quad \lim_{n\to+\infty} g_1^* \cdots g_n^*.m = \delta_{z^*(\omega)}.$$

If $S_n(\omega = g_n \cdots g_1$, we have $S_n^*(\omega) = g_1^* \cdots g_n^*$. The theorem implies that $S_n^*(\omega)$ has the contracting property towards $z^*(\omega)$, hence the corollary follows from Lemma 3.10. ■

R e m a r k. The weak convergence of measures to a Dirac measure, stated in Th. 2.5, plays an important role in various questions, in particular in the superrigidity of lattices in semisimple groups (see [10, 32]), as well as in compactifications of symmetric spaces (see [24]). The proof given here is borrowed from [21].

4. Angles of Column Vectors: Exponential Decrease

Here we consider the wedge product $\wedge^2 V$ generated by the decomposable 2-vectors $x \wedge y$ $(x, y \in V)$. A natural scalar product on $\wedge^2 V$ is given by $< x \wedge y, x' \wedge y' >= det \begin{pmatrix} < x, x' > & < x, y' > \\ < y', y' > & < y, y' > \end{pmatrix}$. The angle $\theta(x,y)$ between x and y is given by $sin\theta(x,y) = \frac{\|x\wedge y\|}{\|x\| \ \|y\|}$. Here we are interested by the angle $\theta(S_n(\omega)x, S_n(\omega)y)$. We denote by $P_2(V)$ the projection on $P(\wedge^2 V)$ of the cone of decomposable 2-vectors. We note that $\delta(\overline{x},\overline{y}) = sin\theta(x,y)$ defines a distance δ on $P(V)$. We represent an element of $P_2(V)$ by a 2-vector $x \wedge y$ with $\|x \wedge y\| = 1$. Then we will write $\sigma_2(g, x \wedge y) = Log\|gx \wedge gy\|$.

Also we consider the compact space $P_{1,2}(V)$ of contact elements $\xi = (x, x \wedge y)$, where $\|x\| = \|x \wedge y\| = 1$, and the cocycle on $P_{1,2}(V)$ $\alpha(g,\xi) = Log\frac{\|gx \wedge gy\|}{\|gx\|^2}$. This cocycle can be interpreted as an infinitesimal coefficient of expansion of the projective map g, at x in the direction of $(x \wedge y)$.

Here we will assume that $Log\|g\|$ and $Log\|g^{-1}\|$ are μ-integrable. Also we assume that the semigroup $supp\mu^-$ satisfies condition *i.p.*

Theorem 4.1. *Assume $\mu \in M^1(G)$ is such that $Log\|g\|$ and $Log\|g^{-1}\|$ are μ-integrable, and $[supp\mu]^-$ satisfies condition i.p. We denote*

$$\gamma_1 = \lim_{n \to +\infty} \frac{1}{n} \int Log\|g\| d\mu^n(g) \quad , \quad \gamma_2 = \lim_{n \to +\infty} \frac{1}{n} \int Log\|g \wedge g\| d\mu^n(g).$$

Then $\gamma_2 < 2\gamma_1$.

Corollary 4.2. $\lim\limits_{n \to +\infty} \mathop{Sup}\limits_{\|x\|=\|y\|=1} \frac{1}{n} \int Log\frac{\|gx \wedge gy\|}{\|gx\|^2} d\mu^n(g) = \gamma_2 - 2\gamma_1 < 0.$

Corollary 4.3. *Assume $\mu \in M^1(G)$ has an exponential moment, i.e, $\int \|g\|^c d\mu(g) < +\infty$, $\int \|g^{-1}\|^c d\mu(g) < +\infty$ for some $c > 0$. Then, for ε sufficiently small, there exists $\rho(\varepsilon) < 1$ such that*

$$\lim_{n \to +\infty} \mathop{Sup}_{x,y \in P(V)} \left(\int \frac{\delta^\varepsilon(g.x, g.y)}{\delta^\varepsilon(x,y)} d\mu^n(g) \right)^{1/n} = \rho(\varepsilon) < 1.$$

For a continuous function φ on $P(V)$, we write

$$|\varphi| = \mathop{\mathrm{Sup}}_{x \in P(V)} |\varphi(x)|, [\varphi]_\varepsilon = \mathop{\mathrm{Sup}}_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x \wedge y\|^\varepsilon}.$$

We denote by $H_\varepsilon(P(V))$ the space of ε-Hoelder functions on $P(V)$, *i.e*,

$$H_\varepsilon(P(V)) = \{\varphi \in C(P(V)); [\varphi]_\varepsilon < +\infty\} \ ,$$

and we observe that $H_\varepsilon(P(V))$ is a Banach space for the norm

$$\|\varphi\| = |\varphi| + [\varphi]_\varepsilon.$$

If $t \in \mathbb{R}$, we consider the operator P^{it} on $C(P(V))$ defined by $(P^{it}\varphi)(x) = \int \|gx\|^{it} \varphi(g.x) d\mu(g)$. Then P^{it} defines a bounded operator on $H_\varepsilon(P(V))$. Then we have

Corollary 4.4. *With the notations of Cor. 4.3, there exists $C \geq 0$ such that for any $\varphi \in H_\varepsilon(P(V))$ and $t \in \mathbb{R}$:*

$$[P^{it}\varphi]_\varepsilon \leq \rho(\varepsilon)[\varphi]_\varepsilon + |t|C|\varphi|.$$

In particular 1 is an isolated spectral value of P and if $t \neq 0$ the spectral radius of P^{it} is strictly less than one.

For the proof of Th. 4.1 we will need the following lemmas.

Lemma 4.5. *There exists $C > 0$ such that for any $u \in EndV$,*

$$Log\|u \wedge u\| \leq \int Log\|ux \wedge uy\| dm_2(x \wedge y) + C,$$

where m_2 is the uniform measure on $P_2(V)$.

P r o f. We proceed as in Lemma 2.6, *i.e*, we write $u = kak'$ with k, $k' \in O(d)$, $a = diag(a^1, \ldots, a^d)$. Then we get

$$\int Log\|ux \wedge uy\| dm_2(x \wedge y) \geq Log\|u \wedge u\| + \int Log| < x \wedge y, e_1 \wedge e_2 > | dm_2(x \wedge y).$$

Hence it suffices to show that the integral I in the right-hand side is finite. We consider the unit sphere of $\wedge^2 V$, its algebraic submanifold $V_2 = \{(x \wedge y) \in \wedge^2 V; \|x \wedge y\| = 1\}$, and we denote by $\widetilde{m}_2$ its normalized Riemannian measure. Clearly,

$$I = \int Log \ | < x \wedge y, e_1 \wedge e_2 > | d\widetilde{m}_2(x \wedge y).$$

Since the map $x \wedge y \to | < x \wedge y, e_1 \wedge e_2 > |^2$ is a polynomial map, there exists an integer $r > 0$ and $c > 0$ such that

$$\widetilde{m_2}\{x \wedge y \in V_2, \ | < x \wedge y, e_1 \wedge e_2 > |^2 \leq t\} \leq ct^r.$$

Then the push forward of $\widetilde{m_2}$ on $[0, 1]$ by this map has a density f which satisfies $tf(t) \leq ct^{r/2}$. Then

$$\int Log| < x \wedge y, e_1 \wedge e_2 > | dm_2(x \wedge y) = \int_0^1 (Logt) f(t) dt \geq \int_0^1 t^{r/2}(Logt)\frac{dt}{t} > -\infty,$$

since $r > 0$. ∎

Lemma 4.6. *For any μ-stationary measure ρ on $P_2(V)$*

$$\int Log\|gx \wedge gy\| d\mu(g) d\rho(x \wedge y) \leq \gamma_2.$$

The sequence $\frac{1}{n} \int Log\|gx \wedge gy\| d\mu^n(g) dm_2(x \wedge y)$ converges to γ_2. For any cluster value η of the sequence $\left(\frac{1}{n} \sum_0^{n-1} \mu^k\right) * m_2$, one has

$$\gamma_2 = \int Log\|gx \wedge gy\| d\mu(g) d\eta(x \wedge y).$$

P r o f. Let $\mu_n = \frac{1}{n} \sum_0^{n-1} \mu^k$, $\eta \in M^1(P_2(V))$ and

$$I_n(\beta) = \int \sigma_2(g, x \wedge y) d\mu^n(g) d\beta(x \wedge y).$$

Using the cocycle identity for σ_2:

$$I_n(\beta) = I_{n-1}(\beta) + \int f(x \wedge y) d(\mu^{n-1} * \beta)(x \wedge y),$$

with $f(x \wedge y) = \int \sigma_2(g, x \wedge y) d\mu(g)$. Hence, $\frac{1}{n} I_n(\beta) = (\mu_n * \beta)(f)$. If $\beta = \rho$ is μ-stationary,

$$\frac{1}{n} I_n(\rho) = \rho(f) = \int \sigma_2(g, x \wedge y) d\mu(g) d\rho(x \wedge y).$$

Since $I_n(\beta) \leq \int Log\|g \wedge g\| d\mu^n(g)$, the first assertion follows.

If $\beta = m_2$, Lemma 4.5 gives

$$-\frac{C}{n} + \frac{1}{n} \int Log\|g \wedge g\| d\mu^n(g) \leq \frac{I_n(m_2)}{n} \leq \frac{1}{n} \int Log\|g \wedge g\| d\mu^n(g),$$

hence $\lim\limits_{n \to +\infty} \dfrac{I_n(m_2)}{n} = \gamma_2$.

Also, from above $\frac{1}{n}\ I_n(m_2) = (\mu_n * m_2)(f)$. Since f is continuous,

$$\lim_{n \to +\infty} \mu_n * m_2 = \eta(f) = \int Log\|gx \wedge gy\| d\mu(g) d\eta(x \wedge y).$$

Hence $\gamma_2 = \int Log\|gx \wedge gy\| d\mu(g) d\eta(x \wedge y)$. ∎

Lemma 4.7. *Let (X, T, λ) be a measured dynamical system with λ finite T-invariant, f an integrable function. Then, if*

$$\lambda - a.e, \quad \lim_{n \to +\infty} \sum_{0}^{n-1} f \circ T^k = -\infty,$$

then $\int f(x) d\lambda(x) < 0$.

For the proof of this statement see [15].

P r o f of Theorem 4.1. Using Lemma 2.7, we know that for any μ-stationary measure ν on $P(V)$

$$\int Log\|gx\| d\mu(g) d\nu(x) = \gamma_1.$$

On the other hand, Lemma 4.6 gives $\int Log\|gx \wedge gy\| d\mu(g) d\eta(x \wedge y) = \gamma_2$, where η is a cluster value of the sequence $\mu_n * m_2$.

We consider the compact space $P_{1,2}(V)$. Clearly, G acts on $P_{1,2}(V)$ and the maps $\xi \to \overline{x}$ and $\xi \to \overline{x \wedge y}$ and are G-equivariant. It follows from Markov–Kakutani theorem that there exists on $P_{1,2}(V)$ a μ-stationary measure $\widetilde{\eta}$ which has projection η on $P_2(V)$. Its projection ν on $P(V)$ satisfies as above: $\int Log\|gx\| d\mu(g) d\nu(x) = \gamma_1$. Hence

$$\int Log \frac{\|gx \wedge gy\|}{\|gx\|^2} d\mu(g) d\widetilde{\eta}(\xi)$$
$$= \int \sigma_2(g, x \wedge y) d\mu(g) d\eta(\overline{x \wedge y}) - 2 \int \sigma_1(g, \overline{x}) d\mu(g) d\nu(\overline{x}) = \gamma_2 - 2\gamma_1.$$

In particular, there exists a μ-stationary measure ρ on $P_{1,2}(V)$ such that

$$\int \alpha(g, \xi) d\mu(g) d\rho(\xi) = \gamma_2 - 2\gamma_1.$$

On the other hand, every μ-stationary measure ρ' on $P_{1,2}(V)$ satisfies $\int \alpha(g, \xi) d\mu(g) d\rho'(\xi) \le \gamma_2 - 2\gamma_1$. This follows from the fact that the projections ρ'_1, ρ'_2, on $P_1(V)$ and $P_2(V)$ respectively, satisfy

$$\int \sigma(g, x) d\mu(g) d\rho'_1(x) = \gamma_1, \quad \int \sigma_2(g, x \wedge y) d\mu(g) d\rho'_2(x \wedge y) \le \gamma_2$$

in view of Lemmas 2.7 and 4.6. Using this property we see that we can assume ρ to be extremal μ-stationary in the formula $\int \alpha(g, \xi) d\mu(g) d\rho(\xi) = \gamma_2 - 2\gamma_1$.

We consider the transformation $\widehat{\theta}$ on $\Omega \times P_{1,2}(V)$ defined by

$$\widehat{\theta}(\omega, \xi) = (\theta\omega, g_1(\omega).\xi),$$

the function $f(\omega,\xi)=\alpha(g_1(\omega),\xi)$ and the measure $\pi\otimes\rho$ on $\Omega\times P_{1,2}(V)$. We observe that $\Omega\times P_{1,2}(V)$ is the space of trajectories of the Markov chain on $P_{1,2}(V)$ with transition kernel $R_\mu(\xi,.)=\mu*\delta_\xi$. Since ρ is μ-stationary extremal, $\pi\otimes\rho$ is $\widehat{\theta}$-invariant and ergodic. Since $|f(\omega,\xi)|\le 2Log\|g_1\|+2Log\|g_1^{-1}\|$, it follows that f is $\mu\otimes\rho$ integrable.

On the other hand, the cocycle property for $\alpha(g,\xi)$ implies

$$\sum_1^n f\circ\widehat{\theta}^k(\omega,\xi)=\alpha(S_n(\omega),\xi)=Log\frac{\|S_n(\omega)x\wedge S_n(\omega)y\|}{\|S_n(\omega)x\|^2}.$$

We are going to use Cor. 3.6 with $\delta_{z^*}(\omega)=\lim_{n\to+\infty} g_1^*,\dots,g_n^*.m$.

Using Theorem 3.5 and Lemma 3.8, we see that the law of $z^*(\omega)\in P(V)$ gives measure 0 to any projective subspace. In particular, if $x\in P(V)$ is fixed, the condition $<z^*(\omega),x>=0$ is satified $\pi-a.e.$ In other words, using Cor. 3.6

$$\pi\otimes\rho-a.e,\quad \lim_{n\to+\infty}\alpha(S_n(\omega),\xi)=-\infty.$$

From above, this implies

$$\pi\otimes\rho-a.e,\quad \lim_{n\to+\infty}\sum_1^n(f\circ\widehat{\theta}^k)(\omega,\xi)=-\infty.$$

Then, using Lemma 4.7

$$\int f(\omega,\xi)d\pi\otimes\rho(\xi)=\int\alpha(g,\xi)d\mu(g)d\rho(\xi)<0,\quad i.e\ \ \gamma_2<2\gamma_1. \qquad\blacksquare$$

P r o f of Corollary 4.2. We denote $u_n=\operatorname{Sup}_{x\wedge y\in P_2(V)}\int\sigma_2(g,x\wedge y)d\mu^n(g)$ and we observe that using the cocycle identity for σ_2: $u_{m+n}\le u_m+u_n$.

Also, $u_n\le\int Log\|g\wedge g\|d\mu^n(g)$, hence $\limsup_{n\to+\infty}\frac{u_n}{n}\le\gamma_2$. Furthermore, by subadditivity of u_n, the sequence $\frac{u_n}{n}$ converges. It follows

$$\lim_{n\to+\infty}\operatorname{Sup}_{x\wedge y\in P_2(V)}\frac{1}{n}\int\sigma_2(g,x\wedge y)d\mu^n(g)\le\gamma_2.$$

Furthermore Lemma 4.6 implies that there exists $x\wedge y\in P_2(V)$ such that

$$\lim_{n\to+\infty}\frac{1}{n}\int Log\sigma_2(g,x\wedge y)d\mu^n(g)=\gamma_2.$$

Hence $\lim_{n\to+\infty}\frac{1}{n}\operatorname{Sup}_{\|x\wedge y\|=1}\int Log\sigma_2(g,x\wedge y)d\mu^n(g)=\gamma_2.$

Using Th. 2.3 and the uniform convergence of $\frac{1}{n}\int Log\|gx\|d\mu^n(g)$ to γ_1, the statement follows ■

P r o f of Corollary 4.3. We denote

$$u_n(\varepsilon) = \operatorname*{Sup}_{x,y} \int \frac{\delta^\varepsilon(g.x, g.y)}{\delta^\varepsilon(x,y)} d\mu^n(g).$$

Using Schwarz inequality:

$$u_n(\varepsilon) \leq \operatorname*{Sup}_{\|x\|=\|y\|=1} \int \left(\frac{\|gx \wedge gy\|}{\|gx\|^2 \|x \wedge y\|}\right)^\varepsilon d\mu^n(g) = \operatorname*{Sup}_{\|x\|=\|x\wedge y\|=1} \int e^{\varepsilon\alpha(g,\xi)} d\mu^n(g).$$

We observe that

$$e^{\varepsilon\alpha} \leq 1 + \varepsilon\alpha + \varepsilon^2\alpha^2 e^{\varepsilon\alpha}, \quad |\alpha(g,\xi)| \leq 2Log(\|g\|\ \|g^{-1}\|).$$

Using $u^2 e^u \leq e^{3|u|}$, we get for $0 \leq \varepsilon \leq \varepsilon_0$:

$$(\alpha^2 e^{\varepsilon\alpha})(g,\xi) \leq \frac{1}{\varepsilon_0^2}(\|g\|\ \|g^{-1}\|)^{6\varepsilon_0}, \quad u_n(\varepsilon) \leq 1 + \varepsilon \int \alpha(g,\xi) d\mu^n(g) + \varepsilon^2 I_n$$

with $I_n = \frac{1}{\varepsilon_0^2}\int(\|g\|\ \|g^{-1}\|)^{6\varepsilon_0} d\mu^n(g) < +\infty$. Now we observe that $u_{m+n}(\varepsilon) \leq u_m(\varepsilon)u_n(\varepsilon)$ for m, $n \in \mathbb{N}$.

It follows: $\lim\limits_{n\to+\infty}(u_n(\varepsilon))^{1/n} = \operatorname*{Inf}_k (u_k(\varepsilon))^{1/k}$. Hence, in order to show $\lim\limits_{n\to+\infty}(u_n(\varepsilon))^{1/n} \leq \rho(\varepsilon) < 1$, it suffices to find $k \in \mathbb{N}$ with $u_k(\varepsilon) < 1$.

Using Cor. 4.2, we have

$$\lim_{n\to+\infty} \frac{1}{n} \operatorname*{Sup}_\xi \int \alpha(g,\xi) d\mu^n(g) = \gamma_2 - 2\gamma_1 < 0,$$

hence we can fix $k \in \mathbb{N}$ such that $\operatorname*{Sup}_\xi \int \alpha(g,\xi) d\mu^k(g) = c_k < 0$.

Then $u_k(\varepsilon) \leq 1 + c_k\varepsilon + \varepsilon^2 I_k < 0$, if ε is sufficiently small.

The statement follows. ■

P r o f of Corollary 4.4. The inequality follows from Cor. 4.3 and a simple computation. The spectral gap property is a consequence of the spectral theorem of [26]. The fact that $r(P^{it}) < 1$ if $t \neq 0$ follows also from this theorem and Prop. 6.7 (see [20, 25]).

R e m a r k. The fact that, under condition $i.p$, the "ergodic Lemma" 4.7 allows to deduce quantitative information from weak convergence of measures on projective spaces, as in Th. 3.5, was observed in [15]. Under stronger conditions, Th. 4.1 implies simplicity of Lyapunov spectrum for $S_n(\omega)$ in the $i.i.d$ case (see

Sect. 6). This property has played an important role in the study of pure point spectrum for Schrödinger operators on the line if $d = 2$ [13] and more generally in the strip [5], as well as for the study of propagation in inhomogeneous waveguides ([38]). The observation in [15] has been developed in [21] and [12]. In [12], it was observed that condition $i.p$ can be checked from the Zariski density of $[supp\mu]$ in G_1. The relations between condition $i.p$ and Zariski density of $[supp\mu]$ in the context of semi-simple real algebraic groups were studied in [34]. For another approach to proximality properties see [1].

5. Contraction Properties for Transitive Markov Systems

1) Definitions.

Let (X, d) be a compact metric space, $\widehat{X} = C(X, X)$ the semigroup of continuous maps of X into itself. We endow $\widehat{X}$ with the Borel structure defined by uniform convergence, and we write $a.x$ for the action of $a \in \widehat{X}$ on $x \in X$. We consider a class of Markov operators on X defined as follows. Let μ be a positive Radon measure on $\widehat{X}$, $q(x, a)$ a nonnegative continuous function on $X \times \widehat{X}$ such that for every x in X, $\int q(x, a) d\mu(a) = 1$.

For $a \in \widehat{X}$, we denote

$$\overline{q}(a) = \operatorname*{Sup}_{x \in X} q(x, a).$$

Then we consider the Markov transition kernel Q on X:

$$Q\varphi(x) = \int q(x, a)\varphi(a.x) d\mu(a),$$

where $\varphi \in C(X)$. Clearly, Q preserves $C(X)$. We denote $\Omega = \widehat{X}^{\mathbb{N}}$ and for $\omega = (a_k)_{k \in \mathbb{N}}$ and $n \in \mathbb{N}$, we write $q_n(x, \omega) = \prod\limits_{k=1}^{n} q(s_{k-1}(\omega).x, a_k)$, where $s_k(\omega) = a_k \cdots a_1$, $s_0(\omega) = Id$. Then, for $x \in X$, we define a probability measure Q_x on Ω by

$$Q_x(A_1 \times \cdots \times A_n) = \int_{A_1 \times \cdots \times A_n} q_n(x, \omega) d\mu^{\otimes n}(\omega),$$

where A_i, $1 \le i \le n$, is a Borel subset of $\widehat{X}$. Also, if $\sigma \in M^1(X)$, we write $Q_\sigma = \int Q_x d\sigma(x)$. The shift transformation on Ω is denoted θ, $i.e$, $\theta(\omega) = (a_2, a_3, \ldots)$, where $\omega = (a_1, a_2, \ldots) \in \Omega$. If σ is Q-invariant, then Q_σ is θ-invariant. We observe that $X \times \Omega$ can be identified with the space of trajectories of the Markov chain defined by Q. If $(x, \omega) \in X \times \Omega$, a trajectory can be written as the sequence $(s_k.x)_{k \in \mathbb{N}}$, hence the shift $\widetilde{\theta}$ on $X \times \Omega$ is given by $\widetilde{\theta}(x, \omega) = (s_1.x, \theta\omega)$. We will summarize the data (X, q, μ) by $(X, q \otimes \mu)$.

Definition 5.1. *We say that $(X, q \otimes \mu)$ is a transitive Markov system ($t.M.s$) on X if:*

a) For every $a \in supp\mu$, $\underset{x \in X}{Inf}\, q(x,a) > 0$.

b) In the variation norm on $M^1(\Omega)$, Q_x depends continuously on $x \in X$.

c) The equation $Qh = h$, with $h \in C(X)$ implies $h = cte$.

Condition b seems to be very restrictive. However it is satisfied in various situations (see below). If σ is Q-stationary, the above conditions imply that Q_σ is independent of σ and θ-ergodic, hence the main role below will be played by Q_σ, not by σ itself. It is easy to see that, if every $[supp\mu]$-orbit is dense, conditions a,b imply condition c.

2) Some examples.

a) Product measures.

If $q(x,a) = q(a)$, then Q_x is the product measure $Q_x = (q\mu)^{\otimes \mathbb{N}}$, hence condition b is satisfied

b) Doeblin condition.

If $supp\mu$ is equal to the set $\widehat{X}$ of constant maps, then $q(x,a) = q(x,y)$ with $\{y\} = a.X$. If $q(x,y) > 0$, conditions a,b,c are satisfied

c) Quantum measurements (see [29]).

We consider the vector space $W = \mathbb{C}^d$, with the usual scalar product and the vector space $\mathcal{H}$ of Hermitian operators on W, $\mathcal{H}^+ \subset \mathcal{H}$ the cone of nonnegative operators and we denote $q(x,g) = Trg^*xg$ if $x \in \mathcal{H}^+ \setminus \{0\}$ and $g \in G = GL(W)$. Let $X = \{x \in \mathcal{H}^+; Trx = 1\}$ and $\tilde{g}$ be the transformation of X defined by $\tilde{g}.x = \frac{g^*xg}{Tr(g^*xg)}$. If $\Phi = \{a_1, a_2, \cdots, a_p\}$ is a finite subset of G with $\sum_{i=1}^{p} a_i^* a_i = Id$, we have $\sum_{i=1}^{p} q(x, a_i) = 1$, hence we can consider the following Markov operator Q on X:

$$Q\varphi(x) = \sum_{i=1}^{p} q(x, a_i)\varphi(\widetilde{a}_i.x).$$

If $\mu = \sum_{i=1}^{p} \delta_{a_i}$, and $[supp\mu] = [\Phi]$ satisfies the complex version of condition $i.p$ (see [17]), then $(X, q \otimes \mu)$ is a $t.M.s$. This example can also be considered in the framework of [23], and of example d below, with $s = 1$. We can define a norm $\|.\|_1$ on $\mathcal{H}$ as follows. Since $x \in \mathcal{H}$ is conjugate to diag $(\lambda_1, \ldots, \lambda_d)$ with $\lambda_i \in \mathbb{R}$, we can write $\|x\|_1 = \sum_{i=1}^{d} |\lambda_i|$. Then we consider the representation ρ of G in $\mathcal{H}$ defined by $\rho(g)(x) = g^*xg$ and write $q(x,g) = \|\rho(g)x\|_1$. Then X can be considered as a part of the complex projective space $P(\mathcal{H})$ and $\widetilde{g}$ as the restriction to X of the projective map defined by $\rho(g)$.

The corresponding algebraic framework was developed in ([29]).

d) Mellin transforms on $GL(V)$.

Let $G = GL(V)$, $\mu \in M^1(G)$ be as in Sects. 3, 4 and assume $[supp\mu]$ satisfies condition $i.p.$ We fix a norm $v \to \|v\|$ on V, and we consider the function $(g, v) \to \|gv\|^s$ $(s \geq 0)$ on $G \times V$. We assume $\mu \in M^{1,e}(G)$ $i.e$ $\int (\|g\|^c + \|g^{-1}\|^c) d\mu(g) < +\infty$, for some $c > 0$. Then the following function

$$k(s) = \lim_{n \to +\infty} \left(\int \|g\|^s d\mu^n(g) \right)^{1/n}$$

is well defined, strictly convex and analytic on ([0, c[) (see [23]). We consider also the positive operator P^s on $C(P(V))$ defined by

$$P^s \varphi(x) = \int \|gx\|^s \varphi(g.x) d\mu(g).$$

Then, there exists a unique positive continuous function e_s on $P(V)$ such that $P^s e_s = k(s) e_s$. If we define $q_s(x, g) = \frac{\|gx\|^s}{k(s)} \frac{e_{s(g.x)}}{e_s(x)}$ we observe that $\int q_s(x, g) d\mu(g))$ $= 1$. As shown in [23], conditions a, b, c are satisfied by $(X, q_s \otimes \mu)$, $i.e$ $(X, q_s \otimes \mu)$ is a $t.M.s.$ The function $k(s)$ can be considered as a kind of Mellin transform of μ and is useful in the study of various limit theorems of Probability Theory for products of random matrices. This is the case for large deviations (see [31]) and for Cramer estimates of fluctuation theory (see [23, 14]) and below. We observe that the expression of the operator P^s defined above is reminiscent of the transfer operators of thermodynamic formalism (see below). If there exists a closed convex cone sent into its interior by $supp\mu$, then this analogy can be made precisely. However, in general, it is not possible to distinguish a region of attractivity for all the maps $g \in supp\mu$, hence a deeper analysis is needed (see [23]).

e) Gibbs measures (see [37]).

Let A be finite set, $\Omega = A^{\mathbb{N}^*}$, $\Omega_- = A^{-\mathbb{N}}, f(\omega)$ a Holder function on $A^{\mathbb{Z}}$, θ the shift on $A^{\mathbb{Z}}$. If $x \in \Omega_-$, $a \in A$, we define $x.a$ by juxtaposition, and we have an action of A on Ω_- by continuous maps. We can write f uniquely as

$$f = f^- + \varphi o \theta - \varphi + c,$$

where $c \in \mathbb{R}$, φ, f^- are Holder, f^- depends only on the component of ω in Ω_- and $\sum_{a \in A} q(x, a) = 1$, where $q(x, a) = exp\ f^-(x.a)$. The transfer operator Q on Ω_- defined by

$$Q\varphi(x) = \sum_{a \in A} q(x, a) \varphi(x.a)$$

has a unique stationary measure π and the Gibbs measure on $A^{\mathbb{Z}}$, defined by the potential f, is the unique θ-invariant measure on $A^{\mathbb{Z}}$ with projection π on Ω_-. If μ is a counting measure on A and $X = \Omega_-$, $(X, q \otimes \mu)$ is a *t.M.s.* Then the probability Q_x on Ω is the conditional law of $\omega \in \Omega$, given $x \in \Omega_-$.

3) Harmonic kernels and contraction properties ([23]).

Here we consider a *t.M.s* $(X, q \otimes \mu)$, and a Borel map α from $supp\mu \subset \widehat{X}$ to $GL(V)$. If $\omega = (a_k)_{k\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$, we denote $g_k = \alpha(a_k), S_n(\omega) = g_n \cdots g_1$, $s_n(\omega) = a_n \cdots a_1 \in \widehat{X}$. We want to construct an analogue of the martingale of Sect. 3.

Definition 5.2. *Assume $(X, q\otimes\mu)$ is a t.M.s, and $x \to \nu_x$ is a Markov kernel from X to $P(V)$. We say that ν_x is an α-harmonic kernel if:*

a) $x \to \nu_x$ is continuous in variation;

b) $x \to \nu_x$ satisfies the equation $\nu_x = \int q(x, a)\alpha(a).\nu_{a.x} d\mu(a)$.

It follows from this definition that the sequence of measures $\nu_n(\omega, x) = \alpha(a_1) \cdots \alpha(a_n).\nu_{s_{n(\omega)}.x}$ is a Q_x-martingale for any $x \in X$.

Theorem 5.3. *Let $(X, q \otimes \mu)$ be a t.M.s and α a Borel map from $supp\mu$ to $GL(V)$ such that $[\alpha(supp\mu)]$ satisfies condition i.p.*

Then there exists $z(\omega) \in P(V)$ defined $Q_x - a.e$ such that $\lim_{n\to+\infty} g_1 \cdots g_n.m = \delta_{z(\omega)}$.

Furthermore, the Q_x-law of $z(\omega)$ is the unique α-harmonic kernel and $\omega \to z(\omega)$ is the unique Borel map which satisfies

$$Q_x - a.e, \quad g_1(\omega).z(\theta(\omega)) = z(\omega).$$

This theorem can be applied to the ≪dual≫ function of $\alpha(a), i.e(\alpha(a))^*$ since the semigroup $[\alpha^*(supp\mu)]$ satisfies also condition *i.p.* It gives, in turn, information on the product $S_n(\omega)$, using Lem. 3.10.

Corollary 5.4. *For every $x \in P(V)$ and $v, v' \notin Ker\ z^*(\omega)$, one has the $Q_x - a.e$ convergences*

$$\lim_{n\to+\infty} g_1^* \cdots g_n^*.m = \delta_{z*(\omega)},$$

$$\lim_{n\to+\infty} \frac{\|S_n(\omega)v\|}{\|S_n(\omega)\|} = |< z^*(\omega), v >|,$$

$$\lim_{n\to+\infty} \frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v, v')} = 0.$$

For fixed ω, the last convergences are uniform on every compact subset of $P(V) \setminus Ker\ z^*(\omega)$. Furthermore, if $f \in C(P(V))$, the sequence $f_n(x, v) = \int f(S_n(\omega).v)dQ_x(\omega)$ is equicontinuous on $X \times P(V)$.

We consider now the situation of example d. We fix $s \geq 0$, we assume that $\mu \in M^1(G)$ satisfies $\int \|g\|^c d\mu(g) < +\infty$, we denote if $s \in [0, c[$, $k(s) = \lim_{n\to+\infty} (\int \|g\|^s d\mu^n(g))^{1/n}$ and we consider the positive operator P^s on $C(P(V))$ defined by

$$P^s\varphi(x) = \int \|gx\|^s \varphi(g.x) d\mu(g).$$

We assume that $[supp\mu] \subset G = GL(V)$ satisfies the condition $i.p$, hence there exists a unique normalized positive and continuous function e_s on $P(V)$ such that $P^s e_s = k(s)e_s$.

Then we write $q_s(x, g) = \frac{\|gx\|^s}{k(s)} \frac{e_s(g.x)}{e_s(x)}$ and we consider the $t.M.s$ $(P(V), q\otimes\mu)$. We denote by Q_x^s the Markov measure on $\Omega = G^{\mathbb{N}}$ defined by q_s and μ. Here we consider the function $\alpha^*(g) = g^* \in GL(V)$ and apply the above corollary to this situation. In particular, we denote $z_s^*(\omega)$ the point of $P(V)$ defined by

$$Q_x^s - a.e, \quad \delta_{z_s^*(\omega)} = \lim_{n\to+\infty} g_1^* \cdots g_n^*.m.$$

Then we can compare Q_x^s and Q_y^s in terms of $z_s^*(\omega)$, as follows.

Corollary 5.5. *For every $x, y \in P(V)$ the Markov measures Q_x^s and Q_y^s on $G^{\mathbb{N}}$ are equivalent and*

$$\frac{dQ_x^s}{dQ_y^s}(\omega) = \left|\frac{< z_s^*(\omega), x >}{< z_s^*(\omega), y >}\right|^s \quad \frac{e_s(y)}{e_s(x)}.$$

In particular, for the laws ν_x^s and ν_y^s of $z_s^*(\omega)$ we have $\frac{d\nu_x^s}{d\nu_y^s}(z) = \left|\frac{<z,x>}{<z,y>}\right|^s \frac{e_s(y)}{e_s(x)}$.

4) Angles of column vectors: exponential decrease.

Here we give a quantitative version of the contraction property studied in Sect. 4 and in the above paragraph. We consider the $t.M.s$ $(X, q \otimes \mu)$, a Borel map α from $supp\mu \subset \widehat{X}$ into $GL(V)$ and we assume the finiteness of the integrals $\int Log\|\alpha(a)\|\overline{q}(a)d\mu(a)$ and $\int Log\|\alpha(a)^{-1}\|\overline{q}(a)d\mu(a)$.

We denote by π a stationary measure on X and by Q_π the corresponding Markov measure on $G^{\mathbb{N}}$. With the above notations we define

$$\gamma_1^q = \lim_{n\to+\infty} \frac{1}{n} \int Log\|S_n(\omega)\| dQ_\pi(\omega),$$

$$\gamma_2^q = \lim_{n\to+\infty} \frac{1}{n} \int Log\|S_n(\omega) \wedge S_n(\omega)\| dQ_\pi(\omega).$$

Theorem 5.6. *Let $(X, q\otimes\mu)$ be a t.M.s, α a Borel map of $\widehat{X}$ into $GL(V)$ such that $[supp\alpha(\mu)]$ satisfies condition i.p. We assume the finiteness of the integrals $\int Log\|\alpha(a)\|\overline{q}(a)d\mu(a)$ and $\int Log\|\alpha(a)^{-1}\|\overline{q}(a)d\mu(a)$.*

Then the sequence

$$\operatorname*{Sup}_{(x,v,v')\in X\times P(V)\times \overline{P(V)}} \int Log\frac{\delta(S_n(\omega).v, S_n(\omega).v')}{\delta(v,v')}dQ_x(\omega)$$

converges to $\gamma_2^q - 2\gamma_1^q < 0$.

In the special case of a $t.M.s$ associated with a Gibbs measure we have

Corollary 5.7. *Assume A is a finite set, π is a Gibbs measure on $\Omega = A^{\mathbb{N}}$ defined by a Holder potential, α a Borel map from A to $GL(V)$ such that the semigroup $[\alpha(A)]$ satisfies condition i.p. Then one has the inequality $\gamma_2^q < 2\gamma_1^q$, where $\gamma_1^q = \lim_{n\to+\infty}\frac{1}{n}\int Log\|S_n(\omega)\|d\pi(\omega)$, $\gamma_2^q = \lim_{n\to+\infty}\frac{1}{n}\int Log\|S_n(\omega)\wedge S_n(\omega)\|d\pi(\omega)$.*

In the situation of example d above, under exponential moment and $i.p$ conditions (see Subsects. 3 and 4 above), we can develop, following [23], a spectral analysis of the operators $P^s(s\geq 0)$ on the space $H_\varepsilon(P(V))$ of ε-Holder functions. For a subset $S\subset G$ we write $\gamma^\infty(S) = \lim_{n\to+\infty}\frac{1}{n}Log \operatorname{Sup}\{\|g\|, g\in S^n\}$. This gives in particular

Corollary 5.8. *With the above hypothesis and notation above, the operator P^s on $H_\varepsilon(P(V))$, defined by*

$$P^s\varphi(v) = \int \|gv\|^s\varphi(g.v)d\mu(g),$$

has spectral radius $k(s)$. It has the unique normalized eigenfunction e_s and eigenmeasure ν_s:

$$P^s e_s = k(s)e_s,\quad P^s\nu_s = k(s)\nu_s,\quad |e_s|_\infty = 1,\quad \nu_s(e_s) = 1,$$

where $e_s > 0$.

For ε small, one has the direct sum decomposition $P^s = k(s)(\nu_s\otimes e_s + R_s)$, where R_s commutes with P^s and has spectral radius $r_s(\varepsilon) < 1$. The function $k(s)$ is analytic on $[0, c[$ and $k'(0) = \gamma_1$. If $c = \infty$, then $\lim_{s\to+\infty}\frac{Logk(s)}{s} = \gamma^\infty(supp\mu)$.

Let Q^s be the Markov operator defined by $Q^s\varphi = \frac{1}{k(s)e_s}P^s(e_s\varphi)$ and $\rho_{n,s}(\varepsilon) = \operatorname*{Sup}_{v,v'}\int\frac{\delta^\varepsilon(S_n(\omega).v, S_n(\omega).v')}{\delta^\varepsilon(v,v')}dQ_v^s(\omega)$.

Then, for ε small, we have $\lim_{n\to+\infty}(\rho_{n,s}(\varepsilon))^{1/n} = \rho_s(\varepsilon) < 1$.

In particular, if s is fixed, the resolvent $(\lambda I - Q^s)^{-1}$ has a simple pole at $\lambda = 1$ and is holomorphic in the domain $\{\lambda\in\mathbb{C}\ ;\ |\lambda| > r_s(\varepsilon),\ \lambda\neq 1\}$.

6. On Some Consequences

1) Lyapunov spectrum.

Let (Ω, θ, π) be a measured dynamical system where π is a θ-invariant and ergodic probability measure, α a Borel function from Ω to $G = GL(V)$. We assume that the functions $Log\|\alpha(\omega)\|$ and $Log\|\alpha^{-1}(\omega)\|$ are π-integrable. We write for $i \in \mathbb{N}$, $\alpha(\theta^i \omega) = g_i(\omega)$ and we consider the product $S_n(\omega) = g_n(\omega) \cdots g_1(\omega) \in G$.

In general, if $v \in V$, the asymptotic behaviour of $S_n(\omega)v$ is described by the multiplicative ergodic theorem of V.I. Oseledets ([33]). For a recent detailed proof of this result see ([36]). A more elementary approach is to consider the quantities $(1 \leq i \leq d)$:

$$\gamma_1 = \lim_{n \to +\infty} \frac{1}{n} \int Log\|S_n(\omega)\| d\pi(\omega),$$
$$\gamma_2 = \lim_{n \to +\infty} \frac{1}{n} \int Log\|S_n(\omega) \wedge S_n(\omega)\| d\pi(\omega),$$
$$\gamma_i = \lim_{n \to +\infty} \frac{1}{n} \int Log\| \wedge^i S_n(\omega)\| d\pi(\omega),$$

where the limits of the quantities under the integrals exist $a.e$ by the subadditive ergodic theorem. The following result (see [35]) allows to define the so-called Lyapunov spectrum of $S_n(\omega)$.

Theorem 6.1. *Assume* (Ω, θ, π) *and* $\alpha(\omega)$ *are as above. Then we have the convergence*

$$\pi - a.e, \lim_{n \to +\infty} \frac{1}{2n} Log(S_n^* S_n) = \wedge(\omega),$$

where $\wedge(\omega)$ *is a symmetric endomorphism of* V.

The spectrum of $\wedge(\omega)$ is constant $\pi - a.e$, and of the form $(\lambda_1, \lambda_2, \ldots, \lambda_p)$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_p$, and λ_j, $1 \leq j \leq p$, has multiplicity $m_j > 0$. Each λ_j is called a Lyapunov exponent of $S_n(\omega)$, and λ_1 is called the top Lyapunov exponent. Clearly, $\lambda_1 = \gamma_1$. If $m_1 = 1$, then $\lambda_2 = \gamma_1 + \gamma_2$, hence $\lambda_2 - \lambda_1 = \gamma_2 - 2\gamma_1 < 0$ controls the exponential decay of θ $((S_n(\omega)v,\ S_n(\omega)v')$, where v, v' are "typical" vectors. Conversely, if $\gamma_2 - 2\gamma_1 < 0$, then $m_1 = 1$.

These facts allow to translate Th. 5.6 into

Theorem 6.2. [15]. *Assume* $(X, q \otimes \mu)$ *is a t.M.s and the Borel map* α *from* $\widehat{X}$ *to* G *is such that the integrals* $\int Log\|\alpha(a)\| \overline{q}(a) d\mu(a)$ *and* $\int Log\|\alpha^{-1}(a)\| \overline{q}(a) d\mu(a)$ *are finite, and* $[\alpha(supp\mu)]$ *satisfies condition i.p. Then the top Lyapunov exponent of*

$$S_n(\omega) = g_n(\omega) \cdots g_1(\omega)$$

has multiplicity 1.

In order to deal more generally with the irreducibility and proximality questions, it is convenient to recall the

Definition 6.3. *Let U be a subset of G, $I(U)$ the set of real polynomials in the coefficients of g and $(detg)^{-1}$ which vanishes on U. Then*

$$U^{-Z} = \{g \in G; \forall P \in I(U), P(g) = 0\}$$

is called the Zariski closure of U.

The Zariski topology on G is defined by its closed sets, *i.e* sets U with $U = U^{-Z}$. If U is a semigroup, then U^{-Z} is a closed Lie subgroup of G with a finite number of connected components. An important fact observed in [12] and [34] is that, if $S \subset G$ is a subsemigroup, a proximal element exists in S iff such an element exists in S^{-Z}. Taking this into account, Th. 5.2 gives the following extension of an important result of [12].

Corollary 6.4. *Assume $(X, q \otimes \mu)$ is a t.M.s and $[\alpha(supp\mu)]^{-Z}$ contains $SL(V)$. Then the Lyapunov spectrum of $S_n(\omega)$ is simple, i.e each Lyapunov exponent has multiplicity one.*

For some applications of geometrical character (see, for example, [17, 18, 8]) it is convenient to have "intrinsic" forms of the above. Then we consider a semi-simple algebraic group $\mathbf{G}$, defined over $\mathbb{R}$. We denote by $G_{\mathbb{R}}$ the group of its real points and we use as a tool the Zariski topology on $G_{\mathbb{R}}$. We assume $\mathbf{G}$ to the Zariski connected. For a Lie subgroup L of $G_{\mathbb{R}}$ we denote its Lie algebra by the calligraphic letter $\mathcal{L}$. We consider a maximal connected subgroup $A \subset G$ such that Ad A is diagonal, a maximal compact subgroup K and the polar decomposition $G = K\overline{A}^+K$, where A^+ is an open Weyl chamber of A and $\overline{A^+}$ is its closure. If $d \in A$, write $Log\ d$ for the unique element of $\mathcal{A}$ such that $expLogd = d$. We write, if $g \in G, g = kd(g)k'$, with $d(g) \in \overline{A}^+$, $k, k' \in K$, and we fix a norm on $\mathcal{A} \subset \mathcal{G}$.

Let $(X, q\otimes\mu)$ be a $t.M.s$, α a Borel function from $\widehat{X}$ to G such that the integral $\int \|Logd(\alpha(u))\|\overline{q}(u)d\mu(u)$ is finite and write $S_n(\omega)$ as $S_n(\omega) = k_n d_n(\omega)k'_n$ with $k_n, k'_n \in K$, $d_n(\omega) \in \overline{A}^+$. Then, using the subadditive ergodic theorem, we can define the "Lyapunov vector" $L(\alpha) \in \overline{\mathcal{A}}^+$ by

$$\pi - a.e, \lim_{n\to+\infty} \frac{1}{n} Logd_n(\omega) = L(\alpha).$$

In particular, in the $i.i.d$ case we write $L(\mu)$ for $L(\alpha)$.

Theorem 6.5. [12]. *Assume $(X, q \otimes \mu)$ is a t.M.s, α a Borel function from $\widehat{X}$ to $G_{\mathbb{R}}$. Assume $[\alpha(supp\mu)]^{-Z} = G_{\mathbb{R}}$.*

Then $L(\alpha) \in \mathcal{A}^+$.

We observe that the condition $L(\alpha) \in \mathcal{A}^+$ can be satisfied under much weaker conditions than $[\alpha(supp\mu)]^{-Z} = G_{\mathbb{R}}$, in particular, if $\mathbf{G}$ has a complex structure. For a detailed study see [17], and for an extension to local fields see [22].

2) Some limit theorems.

Assume here that $\mu \in M^1(G)$ and $S_n(\omega)$ is the product of random matrices $S_n(\omega) = g_n \cdots g_1$. We are interested by refinements of the results in the above paragraphs, *i.e*, by refinements of the law of large numbers for $S_n(\omega)$. Hence we have to consider a possible degeneracy of limiting laws. It turns out that if dim $V > 1$, such degeneracies can be avoided if we assume geometric conditions like Zariski density of $[supp\mu]$ or condition *i.p* for $[supp\mu]$. We recall that if $d = 1$, these degeneracies depend on arithmetic conditions on $supp\mu$. We begin by developing some results of this type and we formulate them for a general semigroup Γ instead of a semigroup of the form $[supp\mu]$.

Definition 6.6. *For a proximal element $g \in GL(V) = G$ we denote $\lambda(g) = Log r(g)$, where $r(g)$ is its spectral radius. For a semi-group $\Gamma \subset G$, we denote by $\Delta(\Gamma)$ the set of its proximal elements.*

Then we have the

Proposition 6.7 ([25]). *Assume $\Gamma \subset G$ satisfies condition i.p. Then $\lambda(\Delta(\Gamma))$ generates a dense subgroup of $\mathbb{R}$.*

If $G_{\mathbb{R}}$ is as in the above paragraph, and $g \in G_{\mathbb{R}}$, we need to consider other notions of proximality related to the actions on the flag spaces of G.

Definition 6.8. *Assume $g \in G_{\mathbb{R}}$ and write $L(g) = \lim_{n \to +\infty} \frac{1}{n} Log d(g^n) \in \overline{\mathcal{A}}^+$. We say that g is flag proximal if $L(g) \in \mathcal{A}^+$. For a semigroup $\Gamma \subset G_{\mathbb{R}}$, we denote by Γ^{prox} the set of its flag proximal elements.*

Then we have [17].

Theorem 6.9. *Assume Γ is a Zariski-dense subsemigroup of $G_{\mathbb{R}}$. Then $L(\Gamma^{prox})$ generates a dense subgroup of $\mathcal{A}$.*

The following is an analogue of the classical renewal theorem, where $\dot{V}$ is the factor space of V by $\pm Id$.

Theorem 6.10. *Assume* $dim\ V > 1$ *and* $\mu \in M^1(G)$ *is such that* $\int Log\|g\|d\mu(g)$ *and* $\int Log\|g^{-1}\|d\mu(g)$ *are finite,* $[supp\mu]$ *satisfies condition* $i.p$ *and*

$$\lambda(\mu) = \lim_{n\to+\infty} \frac{1}{n} \int Log\|g\|d\mu^n(g) > 0.$$

Then, for every $v \in \dot{V} \setminus \{0\}$, *the potential* $\sum_0^\infty \mu^k * \delta_v$ *is finite and*

$$\lim_{v\to 0} \sum_0^\infty \mu^k * \delta_v = \frac{1}{\lambda(\mu)} \nu \otimes \ell,$$

where ν *is the unique* μ*-stationary measure on* $P(V)$ *and* $\ell = \frac{dr}{r}$ *is the radial Lebesgue measure on* $\mathbb{R}_+^*$.

The multidimensional analogues of this result can be applied to dynamical problems like density of orbits for the groups of automorphism action on tori ([19, 25]).

In the case $\lambda(\mu) < 0$ and $\gamma^\infty(supp\mu) > 0$, there exists $\chi > o$ such that $k(\chi) = 1$, as explained in Sect. 5. In particular, we have the so-called Cramer estimate as the consequence of Cor. 5.8.

Theorem 6.11. *With the above notation, assume* $[supp\mu]$ *satisfies condition* $i.p$, $\gamma_1 < 0$, $\gamma^\infty\ (supp\mu) > 0$ *and* $\int \|g\|^c d\mu(g) + \int \|g^{-1}\|^c d\mu(g) < +\infty$ *for some* $c > 0$. *Let* $\chi \in]0, c[$ *be defined by* $k(\chi) = 1$. *Then, for every* $v \in V \setminus \{0\}$, *the sequence of functions* $t^\chi \pi\{\omega \in \Omega; \sup_{n\in\mathbb{N}} \|S_n(\omega)v\| > t\}$ *converges to a positive function on* $P(V)$ *proportional to* $e_\chi(v)$.

This result allows to study the tail of stationary solutions of affine recursions on $\mathbb{R}^d$ of the form $X_{n+1} = A_{n+1}X_n + B_{n+1}$, where $(A_n, B_n) \in Aff(\mathbb{R}^d)$ are $i.i.d$ ([28, 14]).

Furthermore, the existence of such tails allows to obtain fractional expansions of Lyapunov exponents near critical points for some classes of products of random matrices (see [6] for $d = 2$). Near a point $\mu \in M^1(G)$ such that $[supp\mu]$ satisfies condition $i.p$, the top Lyapunov exponent is in general nondifferentiable, but only Hoelder (see [30]).

For the Gaussian behaviour of $Log\|S_n(\omega)v\|$ we refer to [3, 16, 20, 27, 38]. The convergence to the Gaussian law can also be studied in the context of $i.i.d$ random variables taking values in a semi-simple group of the form $G_\mathbb{R}$, as in Subsect. 1. In the notations of Th. 5 we have

Theorem 6.12. *Assume* $\mu \in M^1(G_\mathbb{R})$ *satisfies* $\int exp\ c\|Logd(g)\|d\mu(g) < +\infty$, *for some* $c > 0$, *and* $[supp\mu]^{-Z} = G_\mathbb{R}$. *Then* $\frac{1}{\sqrt{n}}(Logd(S_n) - nL(\mu))$ *converges in law to a Gaussian law on* $\mathcal{A}$ *with full dimension.*

R e m a r k s. This theorem extends the result of [11], which was stated for $G_{\mathbb{R}} = SL(d, \mathbb{R})$. The proof is based on the spectral properties of flag space analogs of the Fourier operators $P^{it} (t \in \mathbb{R})$ from Sect. 4. The fullness of the Gaussian law is a consequence of Th. 6.9 (see [17]).

A special case of interest for Mathematical Physics is $G_{\mathbb{R}} = Sp(2n, \mathbb{R})$.

We observe that the exponential moment condition is not necessary for the validity of Th. 6.12. One can expect that a 2-moment condition is sufficient.

The method used for the proof of Th. 6.5, i.e the construction of a suitable martingale as in Th. 5.3, remains valid in more general settings. For examples of such results see [4, 2].

References

[1] *H. Abels, G.A. Margulis, and G.A. Soifer*, Semigroups Containing Proximal Linear Maps. — *Israel J. Math.* **91** (1995), 1–30.

[2] *C. Bonatti and M. Viana*, Lyapunov Exponents with Multiplicity 1 for Deterministic Products of Matrices. — *Ergodic Theory Dyn. Syst.* **24** (2004). No. 5, 1295–1330.

[3] *P. Bougerol and J. Lacroix*, Products of Random Matrices with Applications to Schröedinger Operators. Progress in Probability and Statistics. **8**, Birkhäuser, Berlin, 1985.

[4] *A. Broise and Y. Guivarch*, Exposants Caractéristiques de l'Algorithme de Perron–Jacobi et de la Transformation Associée. — *Ann. de l'Institut Fourier* **51(3)** (2001), 565–686.

[5] *R. Carmona and J. Lacroix*, Spectral Theory of Random Schröedinger Operators. Probability and its Applications, Birkhäuser, Berlin, (1990).

[6] *C. De Calan, J.M. Luck, T.M. Nieuwenhuizen, and D. Petritis*, On the Distribution of a Random Variable Occuring in i.i.d Disordered Systems. — *J. Phys. A* **18** (1985), No. 3, 501–523.

[7] *Y. Derrienic and Y. Guivarc'h*, Théorème de Renouvellement pour les Groupes non Moyennables. — *CRAS Paris* **277** Ser. A (1973), 613–615.

[8] *A. Furman*, Random Walks on Groups and Random Transformations. In: Handbook on Dynamical Systems. **IA**. Elsevier, Amsterdam, 2002.

[9] *H. Furstenberg*, Non Commuting Random Products. — *TAMS* **108** (1963), No. 3, 377–428.

[10] *H. Furstenberg*, Boundary Theory and Stochastic Processes on Homogeneous Spaces. — *Proc. Symp. Pure Math.* **36** (1972), 193–229.

[11] *I. Goldsheid and Y. Guivarc'h*, Zariski Closure and the Dimension of the Gaussian Law of the Product of Random Matrices. — *Prob. Theory Related Fields* **105** (1996), 109–142.

[12] *I. Goldsheid and G.A. Margulis*, Lyapunov Indices of a Product of Random Matrices. — *Russian Math. Surveys* **44(5)** (1989), 11–81.

[13] *I. Goldsheid, S.A. Molcanov, and L.A. Pastur*, Pure Point Spectrum of One-Dimensional Schröedinger Operators. — *Funct. Anal. Appl.* **11** (1977), 1–10.

[14] *Y. Guivarc'h*, Heavy Tail Properties of Stationary Solutions of Multidimensional Stochastic Recursions. Dynamics and Stochastics. IMS Lecture Notes-Monogr. Ser. **48**. (2006), 85–89.

[15] *Y. Guivarc'h*, Sur les Exposants de Lyapunov des Marches Aléatoires. — *CRAS Paris* **292** (1981), 327–329.

[16] *Y. Guivarc'h*, Limit Theorems for Random Walks and Products of Random Matrices. — *Proc. CIMPA-TIFR School on Probability Measures on Groups: Recent Directions and Trends.* Narosha Publishing House (India, New Delhi), 2006.

[17] *Y. Guivarc'h*, On the Spectrum of a Large Subgroup of a Semisimple Group. — *J. Modern Dynamics* **2** (2008), No. 1, 15–42.

[18] *Y. Guivarc'h*, Renewal Theorems, Products of Random Matrices and Toral Endomorphisms. — *Adv. Stud. Pure Math.* **44**. Potential Theory in Matsue (2006), 53–66.

[19] *Y. Guivarc'h and A. Starkov*, Orbits of Linear Group Actions, Random Walk on Homogeneous Spaces and Toral Automorphisms. — *Ergodic. Theory Dyn. Syst.* **24** (2004), 767–802.

[20] *Y. Guivarc'h, L. Ji, and J.C. Taylor*, Products of Random Matrices and Convergence Theorems. — *Contemp. Math.* **59** (1986), 31–54.

[21] *Y. Guivarc'h and A. Raugi*, Frontière de Furstenberg, Propriétés de Contraction et Théorèmes de Convergence. — *Z. Wahr. Verw. Geb.* **89** (1985), 185–242.

[22] *Y. Guivarc'h and R. Shah*, Asymptotic Properties of Convolution Operators and Limits of Triangular Arrays on Locally Compact Groups. — *TAMS* **357** (2005), 3683–3723.

[23] *Y. Guivarc'h and E. Le Page*, Simplicité de Spectres de Lyapunov et Propriété d'Isolation Spectrale pour une Famille d'Opérateurs de Transfert sur l'Espace Projectif. In: Random Walks and Geometry. Workshop, Vienna, 2001. (V. Kaimanovich and De Gruyter, Eds. (2004), 181–259.)

[24] *Y. Guivarc'h and J.C. Taylor*, Compactifications of Symmetric Spaces. Progress in Math. **156** Birkhauser, Berlin, (1998).

[25] *Y. Guivarc'h and R. Urban*, Semigroup Actions on Tori and Stationary Measures on Projective Spaces. — *Stud. Math.* **171(1)** (2005), 33–66.

[26] *C.T. Ionescu–Tulcea and G. Marinescu*, Théorie Ergodique pour une Classe d'Opérateurs non Complètement Continus. — *Ann. Math.* **52** (1950), 140–147.

[27] *H. Hennion and L. Hervé*, Limit Theorems for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness. Lecture Notes in Math. **1766** Springer, Berlin, (2001).

[28] *H. Kesten*, Random Difference Equations and Renewal Theory for Products of Random Matrices. — *Acta Math.* **131** (1973), 323–344.

[29] *B. Kümmerer and H. Maassen*, Purification of Quantum Trajectories. — In: Dynamics and Stochastics IMS Lecture Notes. Monogr. Ser. **48** (2006), 252–261.

[30] *E. Le Page*, Régularité du Plus Grand Exposant Caractéristique de Produits de Matrices Aléatoires et Applications. — *Ann. IHP (B)* **25(2)** (1989), 109–142.

[31] *A.V. Letchikov*, Products of Unimodular Independent Random Matrices. — *Russian Math. Surveys* **51(1)** (1996), 49–46.

[32] *G.A. Margulis*, Discrete Subgroups of Semi-Simple Lie Groups. — Ergebnisse der Mathematik. Springer, Berlin, (1991).

[33] *V.I. Oseledets*, A Multiplicative Ergodic Theorem: Lyapunov Characteristic Exponents for Dynamical Systems. — *Trans. Moscow Math. Soc.* (AMS) **19** (1968), 197–231.

[34] *G.R. Prasad*, Regular Elements in Zariski Dense Subgroups. — *Quarterly J. Math.* **45(180)** (1994), 541–545.

[35] *M.S. Raghunathan*, A Proof of Oseledec Multiplicative Ergodic Theorem. — *Israël J. Math.* **32** (1979), 356–362.

[36] *A. Raugi*, Théorème Ergodique Multiplicatif. Produits de Matrices Aléatoires Indépendantes. Publ. Inst. Rech. Math. Rennes 1996/1997. Univ. Rennes 1, Rennes, 1996.

[37] *Ya G. Sinaï*, Gibbs Measures in Ergodic Theory. — *Russian Math. Surveys* **27(4)** (1972), 21–69.

[38] *V.I. Tutubalin*, A Central Limit Theorem for Products of Random Matrices and Some of its Applications. — *Symp. Math.* Acad. Press **21** (1979), 101–116.

[39] *A.D. Virtser*, Matrix and Operator Random Products. — *Theor. Veroyatnosti i Primenen.* **24** (1979), No. 2, 361–370.