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From Laplacian Transport to Dirichlet-to-Neumann (Gibbs) Semigroups

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The paper gives a short account of some basic properties of *Dirichlet-to-Neumann* operators $\Lambda_{\gamma,\partial\Omega}$ including the corresponding semigroups motivated by the Laplacian transport in anisotropic media ($\gamma \neq I$) and by elliptic systems with dynamical boundary conditions. To illustrate these notions and the properties we use the explicitly constructed *Lax semigroups*. We demonstrate that for a general smooth bounded convex domain $\Omega \subset \mathbb{R}^d$ the corresponding Dirichlet-to-Neumann semigroup $\{U(t) := e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ in the Hilbert space $L^2(\partial\Omega)$ belongs to the *trace-norm* von Neumann–Schatten ideal for any $t > 0$. This means that it is in fact an *immediate Gibbs* semigroup. Recently H. Emamirad and I. Laadnani have constructed a *Trotter–Kato–Chernoff* product-type approximating family $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ *strongly* converging to the semigroup $U(t)$ for $n \to \infty$. We conclude the paper by discussion of a conjecture about convergence of the *Emamirad–Laadnani approximantes* in the *trace-norm* topology.

Key words: Laplacian transport, Dirichlet-to-Neumann operators, Lax semigroups, Dirichlet-to-Neumann semigroups, Gibbs semigroups.

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1. Laplacian Transport and Dirichlet-to-Neumann Operators

E x a m p l e 1.1. It is well known (see, e.g., [LeUl]) that the problem of determining a *conductivity matrix* field $\gamma(x) = [\gamma_{i,j}(x)]_{i,j=1}^d$, for x in a bounded open domain $\Omega \subset \mathbb{R}^d$, is related to "measuring" the elliptic *Dirichlet-to-Neumann*

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map for associated conductivity equation. Notice that the solution of this problem has a lot of practical applications in various domains: geophysics, electrochemistry etc. It is also an important diagnostic tool in medicine, e.g., in the *electrical impedance tomography*; the tissue in the human body is an example of highly anisotropic conductor [BaBr].

Under the assumption that there is no sources or sinks of current the potential $v(x)$, $x \in \Omega$, for a given voltage $f(\omega)$, $\omega \in \partial\Omega$, on the (smooth) boundary $\partial\Omega$ of Ω is a solution of the Dirichlet problem:

$$\begin{cases} \mathrm{div}(\gamma\nabla \mathrm{v}) = 0 \quad \text{in} \ \ \Omega, \\ v|_{\partial\Omega} = f \quad \text{on} \ \ \partial\Omega. \end{cases} \tag{\textbf{P1}}$$

Then the corresponding to (**P1**) Dirichlet-to-Neumann map (operator) $\Lambda_{\gamma,\partial\Omega}$ is defined by

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f / \partial \nu_\gamma := \nu \cdot \gamma \ \nabla v_f \ |_{\partial\Omega} \ . \tag{1.1}$$

Here ν is the unit outer-normal vector to the boundary at $\omega \in \partial\Omega$ and the function $u := u_f$ is the solution of the Dirichlet problem (**P1**).

The Dirichlet-to-Neumann operator (1.1) is also called the *voltage-to-current* map, since the function $\Lambda_{\gamma,\partial\Omega} f$ gives the induced current flux trough the boundary $\partial\Omega$. The key (*inverse*) problem is whether one can determine the conductivity matrix γ by knowing electrical boundary measurements, i.e., the corresponding Dirichlet-to-Neumann operator? Unfortunately, this operator does not determine the matrix γ uniquely, see e.g. [GrUl] and references there.

E x a m p l e 1.2. The problem of electrical current flux in the form (**P1**) is an example of the so-called *Laplacian transport*. Besides the voltage-to-current problem the motivation to study this kind of transport comes for instance from the *transfer* across biological membranes, see e.g. [Sap], [GrFiSap].

Let some "species" of concentration $C(x)$, $x \in \mathbb{R}^d$, diffuse in the *isotropic* bulk ($\gamma = I$) from a (distant) source localized on the closed boundary $\partial\Omega_0$ towards a *semipermeable* compact interface $\partial\Omega$ on which they disappear at a given rate W. Then the *steady* concentration field (Laplacian transport with a diffusion coefficient D) obeys the set of equations

$$\begin{cases} \Delta C = 0, \ x \in \Omega_0 \setminus \Omega, \\ C(\omega_0 \in \partial\Omega_0) = C_0, \ \text{at the source}, \\ (-D) \ \partial_\nu C(\omega) = W \ (C(\omega) - 0), \ \text{on the interface} \ \omega \in \partial\Omega. \end{cases} \tag{\textbf{P2}}$$

Let $C = C_0(1-u)$. Then $\Delta u = 0$, $x \in \Omega$. If we put $\mu := D/W$, then the boundary conditions on $\partial\Omega$ take the form: $(I + \mu\partial_\nu)u \ |_{\partial\Omega} (\omega) = 1 \ |_{\partial\Omega} (\omega)$, where $(1 \ |_{\partial\Omega})(\omega) = \chi_{\partial\Omega}(\omega)$ is a characteristic function of the set $\partial\Omega$, and $u(\omega_0) = 0$, $\omega_0 \in \partial\Omega_0$ on the source boundary.

Consider now the following auxiliary Laplace–Dirichlet problem

$$\Delta u = 0,\ x \in \Omega_0 \setminus \Omega,\quad u\mid_{\partial\Omega}(\omega) = f(\omega \in \partial\Omega) \ \text{ and } \ u\mid_{\partial\Omega_0}(\omega) = 0, \tag{1.2}$$

with solution u_f. Then similarly to (1.1), with the problem (1.2) we can associate a Dirihlet-to-Neumann operator

$$\Lambda_{\gamma=I,\partial\Omega} : f \mapsto \partial_\nu u_f \mid_{\partial\Omega} \tag{1.3}$$

with the domain $\mathrm{dom}(\Lambda_{I,\partial\Omega})$, which belongs to a certain *Sobolev* space (Sect. 2).

The advantage of this approach is that as soon as the operator (1.3) is defined it can be applied for studying the *mixed* boundary value problem (**P2**). This gives in particular the value of the particle flux due to Laplacian transport across the membrane $\partial\Omega$. Indeed, one obtains that $(I + \mu\Lambda_{I,\partial\Omega})u\mid_{\partial\Omega} = 1\mid_{\partial\Omega}$, and that the local (diffusive) particle *flux* is defined as:

$$\phi\mid_{\partial\Omega} := D\ C_0(\partial_n u)\mid_{\partial\Omega} = D\ C_0(\Lambda_{I,\partial\Omega}(I + \mu\Lambda_{I,\partial\Omega})^{-1}1)\mid_{\partial\Omega}\ . \tag{1.4}$$

Then the corresponding total flux across the membrane $\partial\Omega$

$$\Phi := (\phi, 1)_{L^2(\partial\Omega)} = D\ C_0(\Lambda(I + \mu\Lambda_{I,\partial\Omega})^{-1}1, 1)_{L^2(\partial\Omega)} \tag{1.5}$$

is experimentally measurable macroscopic response of the system expressed via transport parameters D, C_0, μ and geometry of $\partial\Omega$. Here $(\cdot,\cdot)_{L^2(\partial\Omega)}$ is a scalar product in the Hilbert space $\partial\mathcal{H} := L^2(\partial\Omega)$.

The aim of this paper is twofold:

(i) to give a short account of some standard results about Dirichlet-to-Neumann operators and related *Dirichlet-to-Neumann semigroups* that solve a certain class of elliptic systems with dynamical boundary conditions;

(ii) to present some recent results concerning the *approximation* theory and the *Gibbs* character of the Dirichlet-to-Neumann semigroups for compact sets Ω with smooth boundaries $\partial\Omega$.

To this end in the next Sect. 2 we recall some fundamental properties of the Dirichlet-to-Neumann operators and semigroups, we illustrate them by a few elementary examples, including the *Lax semigroups* [Lax].

In Section 3 we present the strong *Emamirad–Laadnani approximations* of the Dirichlet-to-Neumann semigroups inspired by the *Chernoff* theory and by its generalizations in [NeZag, CaZag2].

We show in Sect. 4 that for compact sets Ω with smooth boundaries $\partial\Omega$ the Dirichlet-to-Neumann semigroups are in fact (immediate) *Gibbs* semigroups [Zag2].

Some recent results and conjectures about approximations of the Dirichlet-to-Neumann (Gibbs) semigroups in operator and *trace-norm* topologies are collected in the last Sect. 5.

2. Dirichlet-to-Neumann Operators and Semigroups

2.1. Dirichlet-to-Neumann Operators

Let Ω be an open bounded domain in $\mathbb{R}^d$ with a smooth boundary $\partial\Omega$. Let γ be a $C^\infty(\overline{\Omega})$ matrix-valued function on $\overline{\Omega}$, which we call the *Laplacian transport matrix* in domain Ω.

We suppose that the matrix-valued function $\gamma(x) := [\gamma_{i,j}(x)]_{i,j=1}^d$ satisfies the following hypotheses:

(H1) The real coefficients are symmetric and $\gamma_{i,j}(x) = \gamma_{j,i}(x) \in \mathcal{C}^\infty(\overline{\Omega})$.

(H2) There exist two constants $0 < c_1 \leq c_2 < \infty$ such that for all $\xi \in \mathbb{R}^d$ we have

$$c_1\|\xi\|^2 \leq \sum_{i,j=1}^{n} \xi_i\xi_j\gamma_{i,j}(x) \leq c_2\|\xi\|^2. \tag{2.1}$$

Then the *Dirichlet-to-Neumann* operator $\Lambda_{\gamma,\partial\Omega}$ associated with the Laplacian transport in Ω is defined as follows.

Let $f \in C(\partial\Omega)$, and denote by v_f the *unique* solution (see, e.g., [GiTr, Th. 6.25]) of the Dirichlet problem

$$\begin{cases} A_{\gamma,\partial\Omega}\, v := \mathrm{div}(\gamma\ \nabla \mathrm{v}) = 0 \quad \text{in } \Omega, \\ v\ |_{\partial\Omega} = f \quad \text{on } \partial\Omega, \end{cases} \tag{\textbf{P1}}$$

in the Banach space $X := C(\overline{\Omega})$. Here the operator $A_{\gamma,\partial\Omega}$ is defined on its maximal domain

$$\mathrm{dom}(A_{\gamma,\partial\Omega}) := \{u \in X : A_{\gamma,\partial\Omega}\ u \in X\}. \tag{2.2}$$

Definition 2.1. *The Dirichlet-to-Neumann operator is the map*

$$\Lambda_{\gamma,\partial\Omega} : f \mapsto \partial v_f/\partial\nu_\gamma = \nu\cdot\gamma\nabla v_f\ |_{\partial\Omega}, \tag{2.3}$$

with the domain

$$\mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) = \{f \in \partial C(\Omega_R) : v_f \in \mathrm{Ker}(A_{\gamma,\partial\Omega}) \text{ and } |(\nu\cdot\gamma\nabla v_f\ |_{\partial\Omega})| < \infty\}. \tag{2.4}$$

Here ν denotes the unit outer-normal vector at $\omega \in \partial\Omega$, and v_f is the solution of Dirichlet problem **(P1)**.

The solution $v_f := L_{\partial\Omega}f$ of the problem **(P1)** is called the *γ-harmonic lifting* of f, where $L_{\partial\Omega} : C(\partial\Omega) \mapsto C^2(\Omega) \cap C(\overline{\Omega})$ is called the *lifting* operator with domain $\mathrm{dom}(L_{\partial\Omega}) = C(\partial\Omega)$. If $T_{\partial\Omega} : C(\overline{\Omega}) \mapsto C(\partial\Omega)$ denotes the *trace* operator on the smooth boundary $\partial\Omega$, i.e., $v\ |_{\partial\Omega} = T_{\partial\Omega}\ v$, then [Eng]:

$$L_{\partial\Omega} = (T_{\partial\Omega}\ |_{\mathrm{Ker}(A_{\gamma,\partial\Omega})})^{-1} \text{ and } \mathrm{dom}(\Lambda_{\gamma,\partial\Omega}) = T_{\partial\Omega}\{\mathrm{Ker}(A_{\gamma,\partial\Omega})\}. \tag{2.5}$$

Remark 2.2. *Let $\partial X := C(\partial\Omega)$. Then (2.5) implies*

$$T_{\partial\Omega} L_{\partial\Omega}\ u = u\ ,\ u \in \partial X \quad \text{and} \quad L_{\partial\Omega} T_{\partial\Omega}\ w = w\ ,\ w \in \operatorname{Ker}(A_{\gamma,\partial\Omega}). \tag{2.6}$$

One also gets that the lifting operator is bounded: $L_{\partial\Omega} \in \mathcal{L}(\partial X, X)$, whereas the Dirichlet-to-Neumann operator (2.3) is obviously not.

Now let $\mathcal{H}$ be Hilbert space $L^2(\Omega)$ and $\partial\mathcal{H} := L^2(\partial\Omega)$ denote the *boundary space*. In order that the problem **(P1)** admits a *unique* solution v_f, one has to assume that $f \in W_2^{1/2}(\partial\Omega)$, and then v_f belongs the *Sobolev* space $W_2^1(\Omega)$, see e.g. [Tay, Ch.7]. So, we can define Dirichlet-to-Neumann operator in the Hilbert space $\partial\mathcal{H}$ by (2.3) with the domain

$$\operatorname{dom}(\Lambda_{\gamma,\partial\Omega}) := \{f \in W_2^{1/2}(\partial\Omega) :\ \Lambda_{\gamma,\partial\Omega} f \in \partial\mathcal{H} = L^2(\partial\Omega)\}. \tag{2.7}$$

Proposition 2.3. *The Dirichlet-to-Neumann operator (2.3) with domain (2.7) in the Hilbert space $\partial\mathcal{H}$ is unbounded, nonnegative, selfadjoint, first-order elliptic pseudodifferential operator with compact resolvent.*

The complete proof can be found, e.g., in [Tay, Ch. 7], [Tay1]. Therefore, we give here only some comments on these properties of the Dirichlet-to-Neumann operator (2.3) in $\partial\mathcal{H} = L^2(\partial\Omega)$.

Remark 2.4. (a) *By virtue of definition (2.3) for any $f \in W_2^{1/2}(\partial\Omega)$ one gets*

$$(f, \Lambda_{\gamma,\partial\Omega} f)_{\partial\mathcal{H}} = \int_{\partial\Omega} d\sigma(\omega)\ \overline{v_f(\omega)}\ \nu \cdot \gamma(\omega)(\nabla v_f)(\omega) \tag{2.8}$$

$$= \int_{\Omega} dx\ \operatorname{div}(\overline{\mathrm{v_f(x)}}\ (\gamma\nabla \mathrm{v_f})(\mathrm{x})) = \int_{\Omega} \mathrm{dx}\ (\nabla\overline{\mathrm{v_f(x)}} \cdot \gamma\ \nabla \mathrm{v_f})(\mathrm{x})) \geq 0,$$

since the matrix γ verifies **(H2)**. *Thus, operator $\Lambda_{\gamma,\partial\Omega}$ is nonnegative.*

(b) *In fact to ensure the existence of the trace $T_{\partial\Omega}(\nu \cdot \gamma\nabla(L_{\partial\Omega} f))$ one has initially to define the operator $\Lambda_{\gamma,\partial\Omega}$ for $f \in W_2^{3/2}(\partial\Omega)$. Then Dirichlet-to-Neumann operator is a selfadjoint extension with domain (2.7) and moreover it is a bounded map $\Lambda_{\gamma,\partial\Omega} : W_2^{1/2}(\partial\Omega) \mapsto W_2^{-1/2}(\partial\Omega)$.*

(c) *By (2.8) and since derivatives of the first-order are involved in (2.3), one can conclude that this operator should be elliptic and pseudodifferential. If $\gamma(x) = I$, then $\Lambda_{I,\partial\Omega}$ is, roughly, the operator $(-\Delta_{\partial\Omega})^{1/2}$, where $\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$ with corresponding induced metric [Tay, Ch.7], [Tay1].*

(d) *Compactness of the imbedding $W_2^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ implies the compactness of the resolvent of $\Lambda_{\gamma,\partial\Omega}$.*

By (a) and (d) the spectrum $\sigma(\Lambda_{\gamma,\partial\Omega})$ of the Dirichlet-to-Neumann operator is a set of nonnegative increasing eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. The rate of increasing is given by the Weyl asymptotic formula, see, e.g., [Hor, Tay]:

Proposition 2.5. *Let $\Lambda_{\gamma,\partial\Omega}(x,\xi)$, for $(x,\xi) \in T^*\partial\Omega$, be the symbol of the first-order elliptic pseudodifferential Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$. Then the asymptotic behaviour of the corresponding eigenvalues as $k \to \infty$ has the form*

$$\lambda_k \sim \left\{ \frac{k}{C(\partial\Omega, \Lambda_\gamma)} \right\}^{1/(d-1)} ,$$

where

$$C(\partial\Omega, \Lambda_\gamma) := \frac{1}{(2\pi)^{d-1}} \int\limits_{\Lambda_{\gamma,\partial\Omega}(x,\xi)\leq 1} dx\, d\xi .$$

Another important result is due to Hislop and Lutzer [HiLu]. It concerns a localization (*rapid decay*) of the γ-harmonic lifting of the corresponding eigenfunctions.

Proposition 2.6. *Let $\{\phi_k\}_{k=1}^{\infty}$ be eigenfunctions of the Dirichlet-to-Neumann operator: $\Lambda_{\gamma,\partial\Omega} = \lambda_k \phi_k$ with $\|\phi_k\|_{L^2(\partial\Omega)} = 1$. Let $v_{\phi_k} := L_{\partial\Omega}\phi_k$ be the γ-harmonic lifting of ϕ_k to Ω corresponding to the problem* **(P1)**. *Then for any compact $\mathcal{C} \subset \Omega$ and $x \in \mathcal{C}$ one gets the representation*

$$|v_{\phi_k}(x)| = \psi(x,p,\mathcal{C})/{\lambda_k}^p \tag{2.9}$$

with arbitrary large $p > 0$. Here $\psi(x,p,\mathcal{C})$ is a decreasing function of the distance $\operatorname{dist}(x,\, \partial\Omega)$.

Since by the Weyl asymptotic formula we have $\lambda_k = O(k^{1/(d-1)})$, the decay implied by the estimate (2.9) is algebraic.

Conjecture 2.7. [HiLu]. *In fact the order of decay instead of $\psi(x,p,\mathcal{C})/{\lambda_k}^p$ is exponential: $O(\exp[-\,k \operatorname{dist}(\mathcal{C},\, \partial\Omega)])$.*

2.2. Example of a Dirichlet-to-Neumann Operator

To illustrate the results mentioned above we consider a simple example which will be useful below for contraction of the *Lax semigroups*.

Consider a homogeneous isotropic case: $\gamma(x) = I$, and let $\Omega = \Omega_R := \{x \in \mathbb{R}^{d=3} : \|x\| < R\}$. Then $A_{\gamma,\partial\Omega_R} = \Delta_{\partial\Omega_R}$ and for the *harmonic lifting* of

$$f(\omega) = \sum_{l,m} f_{l,m}^{(R)}\, Y_{l,m}(\theta,\varphi) \in W_2^{1/2}(\partial\Omega_R)$$

we obtain

$$v_f(r,\theta,\varphi) = \sum_{l,m} \left(\frac{r}{R}\right)^l \; f_{l,m}^{(R)} \; Y_{l,m}(\theta,\varphi), \tag{2.10}$$

since the *spherical* functions $\{Y_{l,m}\}_{l=0,|m|\le l}^{\infty}$ form a complete orthonormal basis in the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R,\ d\theta\, \sin\theta\, d\varphi)$.

Definition (2.3) and (2.10) imply that nonnegative, selfadjoint, first-order elliptic pseudodifferential Dirichlet-to-Neumann operator

$$(\Lambda_{I,\partial\Omega_R} f)(\omega = (R,\theta,\varphi)) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \left(\frac{l}{R}\right) \; f_{l,m}^{(R)} \; Y_{l,m}(\theta,\varphi) \tag{2.11}$$

has discrete spectrum $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \ l/R\}_{l=0,|m|\le l}^{\infty}$ with spherical eigenfunctions

$$(\Lambda_{I,\partial\Omega_R} Y_{l,m})(R,\theta,\varphi) = \left(\frac{l}{R}\right) \; Y_{l,m}(\theta,\varphi) \tag{2.12}$$

and multiplicity m. The operator (2.11) is obviously unbounded and it has a compact resolvent.

Remark 2.8. *Since by virtue of (2.10) the γ-harmonic lifting of the eigenfunction $Y_{l,m}$ to the ball Ω_R is*

$$v_{Y_{l,m}}(r,\theta,\varphi) = \left(\frac{r}{R}\right)^l \; Y_{l,m}(\theta,\varphi),$$

one can check the localization (Prop. 2.6) and Conjecture about the exponential decay explicitly. For distances $0 < \operatorname{dist}(x,\partial\Omega_R) = R - r \ll R$, one obtains $|v_{Y_{l,m}}(r,\theta,\varphi)| = O(e^{-l(R-r)/R})$.

2.3. Dirichlet-to-Neumann Semigroups on ∂X

To define the Dirichlet-to-Neumann semigroups on the *boundary* Banach space $\partial X = C(\partial\Omega)$ we can follow the line of reasoning of [Esc] or [Eng]. To this end consider in $X = C(\Omega)$ the following elliptic system with the *dynamical boundary conditions*

$$\begin{cases} \operatorname{div}(\gamma \nabla \mathrm{u}(\mathrm{t},\cdot)) = 0 & \text{in } (0,\infty)\times\Omega, \\ \partial u(t,\cdot)/\partial t + \partial u(t,\cdot)/\partial\nu_\gamma = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,\cdot) = f & \text{on } \partial\Omega. \end{cases} \tag{\textbf{P2}}$$

Proposition 2.9. *The problem* **(P2)** *has a unique solution $u_f(t,x)$ for any $f \in C(\partial\Omega)$. Its trace on the boundary $\partial\Omega$ has the form*

$$u_f(t,\omega) := (T_{\partial\Omega} u_f(t,\cdot))(\omega) = (U(t)f)(\omega), \tag{2.13}$$

where the family of operators $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\ge 0}$ is a C_0-semigroup generated by the Dirichlet-to-Neumann operator of the problem **(P1)**.

The following key result about the properties of the Dirichlet-to-Neumann semigroups on the boundary Banach space $\partial X = C(\partial\Omega)$ is due to Escher–Engel [Esc, Eng] and Emamirad–Laadnani [EmLa]:

Proposition 2.10. *The semigroup* $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ *is analytic, compact, positive, irreducible and Markov* C_0*-semigroup of contractions on* $C(\partial\Omega)$*.*

Remark 2.11. *The complete proof can be found in the papers quoted above. So, here we make only some comments and hints concerning Prop. 2.10.*

2.4. Dirichlet-to-Neumann Semigroups on $\partial\mathcal{H}$

The Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_{t\geq 0}$ on $\partial\mathcal{H}$ is defined by selfadjoint and nonnegative Dirichlet-to-Neumann generator $\Lambda_{\gamma,\partial\Omega}$ of Prop. 2.3.

Proposition 2.12. *The Dirichlet-to-Neumann semigroup* $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ *on the Hilbert space* $\partial\mathcal{H}$ *is a holomorphic quasisectorial contraction with values in the trace-class* $\mathfrak{C}_1(\partial\mathcal{H})$ *for* $\mathrm{Re}\,(t) > 0$*.*

Remark 2.13. *The first part of the statement follows from Prop. 2.3. Since the generator* $\Lambda_{\gamma,\partial\Omega}$ *is selfadjoint and nonnegative, the semigroup* $\{U(t)\}_t$ *is holomorphic and quasisectorial contraction for* $\mathrm{Re}\,(t) > 0$*, see, e.g., [CaZag1, Zag1]. The compactness of the resolvent of* $\Lambda_{\gamma,\partial\Omega}$ *implies the compactness of* $\{U(t)\}_{t>0}$*, but to prove the last part of the statement we need a supplementary argument about asymptotic behaviour of its eigenvalues given by the Weyl asymptotic formula (Prop. 2.5).*

This behaviour of eigenvalues implies the second part of Prop. 2.12:

Lemma 2.14. *The Dirichlet-to-Neumann semigroup* $U(t)$ *has values in the trace-class* $\mathfrak{C}_1(\partial\mathcal{H})$ *for any* $t > 0$*.*

P r o f. Since the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is selfadjoint, we have to prove that

$$\|U(t)\|_1 = \sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} < \infty \tag{2.14}$$

for $t > 0$. Here $\|\cdot\|_1$ denotes the norm in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. Then the Weyl asymptotic formula implies that there exists a bounded M and a function $r(k)$ such that

$$\begin{aligned}\sum_{k\geq 1} \mathrm{e}^{-t\lambda_k} &\leq \sum_{k\geq 1} \exp\{-t[(k/c)^{\frac{1}{d-1}} + r(k)]\} \\ &\leq \mathrm{e}^{tM} \sum_{k\geq 1} \exp\{-t(k/c)^{\frac{1}{d-1}}\}.\end{aligned}$$

Here $c := C(\partial\Omega, \Lambda_\gamma)$ and the last sum converges for any $t > 0$, which proves the equation (2.14). ∎

2.5. Example: Lax Semigroups

A beautiful example of explicit representation of the Dirichlet-to-Neumann semigroup (2.13) is due to Lax [Lax, Ch. 36].

Let $\gamma(x) = I$, and $\Omega = \Omega_R$ (see Sect. 2.2). Following [Lax] we define the mapping

$$K(t) : v(x) \mapsto v(e^{-t/R}\, x) \ \text{ for any } \ u \in C(\Omega_R), \tag{2.15}$$

which is a semigroup for the parameter $t \geq 0$ in the Banach space $X = C(\Omega_R)$:

$$(K(\tau)K(t)v)(x) = v(e^{-\tau/R}\, e^{-t/R}\, x) = v(e^{-(\tau+t)/R}\, x)\ ,\ \tau, t \geq 0\ ,\ x \in \Omega_R\,. \tag{2.16}$$

Remark 2.15. *It is clear that if $v(x)$ is $(\gamma = I)$-harmonic in $C(\Omega_R)$, then the function: $x \mapsto v(e^{-t/R}\, x)$ is also harmonic. Therefore,*

$$u_f(t, x) := v_f(e^{-t/R}\, x) = (K(t)L_{\partial\Omega_R} f)(x) = (L_{\partial\Omega_R} f_t)(x),\ x \in \Omega_R, \tag{2.17}$$

is the harmonic lifting of the function $f_t(\omega) := v_f(e^{-t/R}\, \omega)$, $\omega \in \partial\Omega_R$, where v_f solves the problem **(P1)** *for $\gamma = I$. Since in the spherical coordinates $x = (r, \theta, \varphi)$ one has*

$$\partial u_f(t, x)/\partial t = -\partial_r v_f(e^{-t/R} r, \theta, \varphi) e^{-t/R}\ (r/R)$$

and

$$\partial u_f(t, R, \theta, \varphi)/\partial \nu_I = \partial_r v_f(e^{-t/R} r, \theta, \varphi) e^{-t/R},$$

we get that $\partial u_f(t, \omega)/\partial t + \partial u_f(t, \omega)/\partial \nu_I = 0$, i.e., the function (2.17) is a solution of the problem **(P2)**.

Hence, according to (2.13) and (2.17) the operator family

$$S(t) := T_{\partial\Omega_R} K(t) L_{\partial\Omega_R},\ t \geq 0, \tag{2.18}$$

defines the Dirichlet-to-Neumann semigroup corresponding to the problem **(P2)** for $\gamma(x) = I$, and $\Omega = \Omega_R$, which is known as the *Lax semigroup*. By virtue of (2.17) and (2.18) the action of this semigroup is known explicitly:

$$(S(t)f)(\omega) = v_f(e^{-t/R}\omega),\ \omega \in \partial\Omega_R. \tag{2.19}$$

Notice that the semigroup relation

$$S(\tau)S(t) = T_{\partial\Omega_R} K(\tau) L_{\partial\Omega_R} T_{\partial\Omega_R} K(t) L_{\partial\Omega_R} = S(\tau + t), \tag{2.20}$$

follows from the properties of lifting and trace operators (see Remark 2.2), from identity (2.16) and definition (2.18). One finds the generator $\Lambda_{\gamma=I,\partial\Omega_R}$ of this semigroup from the limit

$$\begin{aligned} 0 &= \lim_{t\to 0} \sup_{\omega\in\partial\Omega_R} \left|\frac{1}{t}(f - S(t)f)(\omega) - (\Lambda_{\gamma=I,\partial\Omega_R} f)(\omega)\right| \qquad (2.21) \\ &= \lim_{t\to 0} \sup_{\omega\in\partial\Omega_R} \left|\frac{1}{t}(v_f(R, \theta, \varphi) - v_f(e^{-t/R}\, R, \theta, \varphi)) - (\Lambda_{\gamma=I,\partial\Omega_R} f)(R, \theta, \varphi)\right|. \end{aligned}$$

Then the operator

$$(\Lambda_{\gamma=I,\partial\Omega_R} f)(R,\theta,\varphi) = \partial_r v_f(r=R,\theta,\varphi) \tag{2.22}$$

for any function f from the domain

$$\mathrm{dom}(\Lambda_{I,\partial\Omega_R}) = \{f \in \partial C(\Omega_R) : v_f \in \mathrm{Ker}(A_{I,\partial\Omega_R}) \text{ and } |(\partial_r v_f)\,|_{\partial\Omega_R}\,| < \infty\} \tag{2.23}$$

is identical to (2.4) for the case $\gamma = I$ and $\partial\Omega = \partial\Omega_R$. Therefore, the generator (2.22) of the Lax semigroup is the Dirichlet-to-Neumann operator in this particular case of the Banach space $\partial X = C(\partial\Omega_R)$.

Similarly, we can consider the Lax semigroup (2.18) in the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R,\ d\theta\,\sin\theta\,d\varphi)$. Since the generator of this semigroup is a particular case of the Dirichlet-to-Neumann operator (2.11), by (2.12) and (2.10) we again obtain the corresponding action in the explicit form

$$\begin{aligned}
&(S(t)f)(\omega) &(2.24)\\
&= (e^{-t\Lambda_{I,\partial\Omega_R}} f)(\omega)) = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\sum_{s=0}^{\infty}\frac{(-t)^s}{s!}\left(\frac{l}{R}\right)^s f_{l,m}^{(R)}\, Y_{l,m}(\theta,\varphi)\\
&= \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}(e^{-\,t/R})^l\, f_{l,m}^{(R)}\, Y_{l,m}(\theta,\varphi) = v_f(e^{-t/R}\omega),\ \omega\in\partial\Omega_R,
\end{aligned}$$

which coincides with (2.19).

Notice that for $t > 0$ the Lax semigroups have their values in the trace-class $\mathfrak{C}_1(\partial\mathcal{H})$. This explicitly follows from (2.12), i.e., from the fact that the spectrum of the semigroup generator $\sigma(\Lambda_{I,\partial\Omega_R}) := \{\lambda_{l,m} = \ l/R\}_{l=0,|m|\le l}^{\infty}$ is discrete and

$$\mathrm{Tr}\, S(t) = \sum_{l=0}^{\infty}(2l+1)\ e^{-tl/R} < \infty. \tag{2.25}$$

The last is proven in the whole generality in Th. 2.14.

3. Product Approximations of Dirichlet-to-Neumann Semigroups

3.1. Approximating Family

Since in contrast to the Lax semigroup ($\gamma = I$) the action of the general Dirichlet-to-Neumann semigroup for $\gamma \neq I$ is known only implicitly (2.13), it is useful to construct converging approximations, which are simpler for calculations and analysis.

One of them is the Emamirad–Laadnani *approximation* [EmLa], which is motivated by the explicit action (2.19), (2.24) of the Lax semigroup

$$\begin{aligned}&(S(t)f)(\omega) = (T_{\partial\Omega_R} K_R(t) L_{\partial\Omega_R} f)(\omega) = v_f(e^{-t/R}\omega), \ \omega \in \partial\Omega_R, \quad (3.1)\\ &K_R(t) : v(x) \mapsto v(e^{-t/R}\, x) \ \text{ for any } \ v \in C(\Omega_R)\, (\text{or } \mathcal{H}(\Omega_R)).\end{aligned}$$

The suggestion of [EmLa] consists in substituting the family $\{K_R(t)\}_{t\geq 0}$ by the $\gamma-$*deformed* operator family

$$K_{\gamma,R}(t) : v(x) \mapsto v(e^{-(t/R)\ \gamma(x)}\, x) \ \text{ for any } \ v \in C(\Omega_R)\, (\text{or } \mathcal{H}(\Omega_R)). \qquad (3.2)$$

Definition 3.1. *For the ball Ω_R the Emamirad–Laadnani approximating family $\{V_{\gamma,R}(t) := V_{\gamma,\partial\Omega_R}(t)\}_{t\geq 0}$ is defined by*

$$(V_{\gamma,R}(t)f)(\omega) := (T_{\partial\Omega_R} K_{\gamma,R}(t) L_{\partial\Omega_R} f)(\omega) = v_f(e^{-(t/R)\ \gamma(\omega)}\ \omega), \ \omega \in \partial\Omega_R. \ (3.3)$$

Remark 3.2. (a) *Notice that the approximating family (3.3) is not a semigroup*

$$\begin{aligned}&(V_{\gamma,R}(t)V_{\gamma,R}(s)f)(\omega) = (T_{\partial\Omega_R} K_{\gamma,R}(t) L_{\partial\Omega_R} \widetilde{f}(s))(\omega) \qquad (3.4)\\ &= v_{\widetilde{f}(s)}(e^{-(t/R)\ \gamma(\omega)}\ \omega) \neq v_f(e^{-((t+s)/R)\ \gamma(\omega)}\ \omega) = (V_{\gamma,R}(t+s)f)(\omega).\end{aligned}$$

(b) *This family is strongly continuous at $t = 0$:*

$$\lim_{t\searrow 0} V_{\gamma,R}(t) f = f \ \text{ for any } \ f \in \partial X\, (\text{or } \partial\mathcal{H}). \qquad (3.5)$$

(c) *By definition (3.3) this family has the derivative at $t = +0$:*

$$(\partial_t V_{\gamma,R}(t)f)(\omega)\ |_{t=0} = -\nu(\omega)\cdot\gamma(\omega)(\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega_R} f)(\omega), \qquad (3.6)$$

which for any $f \in \mathrm{dom}(\Lambda_{\gamma,\partial\Omega_R})$ coincides with the (minus) Dirichlet-to-Neumann operator (2.3).

3.2. Strong Approximation of the Dirichlet-to-Neumann Semigroups

By virtue of Remark 3.2 the Emamirad–Laadnani approximation family verifies the conditions of the Chernoff *approximation theorem* ([Che, Th. 1.1]):

Proposition 3.3. *Let $\{\Phi(s)\}_{s\geq 0}$ be a family of the linear contractions on a Banach space $\mathfrak{B}$ and let X_0 be the generator of a C_0-contraction semigroup. Define $X(s) := s^{-1}(I - \Phi(s))$, $s > 0$. Then for $s \to +0$ the family $\{X(s)\}_{s>0}$ converges strongly in the resolvent sense to the operator X_0 if and only if the sequence $\{\Phi(t/n)^n\}_{n\geq 1}$, $t > 0$, converges strongly to e^{-tX_0} as $n \to \infty$ uniformly on any compact t-intervals in $\mathbb{R}^1_+$.*

Notice that $\{V_{\gamma,R}(t)\}_{t\geq 0}$ in the Banach space ∂X is the family of contractions because of the *maximum principle* for the γ-harmonic functions v_f. Since the Dirichlet-to-Neumann operator (2.3) is densely defined and closed, Remark 3.2 (c) implies that the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the strong resolvent sense.

Similar arguments are valid for the case of the Hilbert space $\partial\mathcal{H}$. By virtue of Remark 2.4 the Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega}$ is nonnegative and self-adjoint. This implies again that (3.3) is the family of contractions in $\partial\mathcal{H}$ and that by Remark 3.2 (c) the family $X(s) := s^{-1}(I - V_{\gamma,R}(s))$ converges for $s \to +0$ to $X_0 = \Lambda_{\gamma,\partial\Omega_R}$ in the *strong* resolvent sense.

Resuming the above observations we obtain the *strong* approximation of the Dirichlet-to-Neumann semigroup $U(t)$

Corollary 3.4. [EmLa]

$$\lim_{n\to\infty}(V_{\gamma,R}(t/n))^n f = U(t)f, \quad \textit{for every } f \in \partial X \textit{ or } \partial\mathcal{H}, \tag{3.7}$$

uniformly on any compact t*-intervals in* $(0,\infty)$.

The Emamirad–Laadnani approximation theorem (Cor. 3.4) has the following important extension to more *general* geometry than the ball [EmLa].

Definition 3.5. *We say that a bounded smooth domain* Ω *in* $\mathbb{R}^d$ *has the* property of the interior ball *if for any* $\omega \in \partial\Omega$ *there exists a tangent to* $\partial\Omega$ *at* ω *plane* $\mathcal{T}_\omega$*, and such that one can construct a ball tangent to* $\mathcal{T}_\omega$ *at* ω*, which is totally included in* Ω.

If Ω has this property, then with any point $\omega \in \partial\Omega$, one can associate a *unique* point x_ω, which is the center of the *biggest* ball $B(x_\omega, r_\omega)$ of radius r_ω included in Ω. For any $0 < r \leq r_\omega$, we can construct the approximating family $V_r(t)$ related to the ball $B(x_{r,\omega}, r) := \{x \in \Omega : |x - x_{r,\omega}| \leq r\}$ of radius r, which is centered on the line perpendicular to $\mathcal{T}_\omega$ at the point $\omega \in \partial\Omega$, i.e., $x_{r,\omega} = (r/r_\omega)x_\omega + (1 - r/r_\omega)\omega$. Then we define

$$(V_{\gamma,r}(t)f)(\omega) := T_{\partial\Omega}\; v_f\left(x_{r,\omega} + \mathrm{e}^{-(t/r)\gamma(\omega)}(r\;\nu_\omega)\right). \tag{3.8}$$

Here ν_ω is the outer-normal vector at ω, the function $v_f = L_{\partial\Omega}f$ is the γ-harmonic lifting of the boundary condition f on $\partial\Omega$, and $T_{\partial\Omega}$ is the trace operator

$$T_{\partial\Omega} : H^1(\Omega) \ni v \longmapsto v\,|_{\partial\Omega} \in H^{1/2}(\partial\Omega). \tag{3.9}$$

Remark 3.6. *Notice that:*

(a) *since* $\nu_\omega = (\omega - x_{r,\omega})/r$*, one gets* $(V_{\gamma,r}(t = 0)f)(\omega) := (T_{\partial\Omega}\, v_f)(\omega) = f(\omega)$*;*

(b) *by virtue of (3.8) the strong derivative at* $t = 0$ *has the form*

$$(\partial_t V_{\gamma,r}(t=0)f)(\omega) = -\gamma(\omega)\nu_\omega \cdot (\nabla v_f)(\omega) = -(\Lambda_{\gamma,\partial\Omega} f)(\omega),$$

see (3.6).

Proposition 3.7. [EmLa]. *Let* Ω *has the property of interior ball, and let*

$$\inf_{\omega\in\partial\Omega} \{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} > 0,$$
$$\sup_{\omega\in\partial\Omega} \{r > 0 : B(x_\omega, r_\omega) \subset \Omega\} < \infty.$$

For any $0 < s \leq 1$ *we define* V_{γ,sr_ω}, *i.e.,*

$$V_{\gamma,sr_\omega} f(\omega) = v_f \left(x_{s,\omega} + \mathrm{e}^{-(t/(sr_\omega))\gamma(\omega)} (sr_\omega\ \nu_\omega) \right), \tag{3.10}$$

where $x_{s,\omega} = sx_\omega + (1-s)\omega$. *Then for any* $0 < s \leq 1$

$$\lim_{n\to\infty} (V_{\gamma,sr_\omega}(t/n))^n f = U(t)f, \quad \text{for every } f \in \partial X \text{ or } \partial\mathcal{H}, \tag{3.11}$$

uniformly on any compact t*-intervals in* $(0, \infty)$.

Remark 3.8. *By Definition 3.1 for the ball* Ω_R *and the constant matrix-valued function* $\gamma(x) = I$ *one obviously has* $V_{\gamma=I,R}(t) = S(t) = U(t)$. *On the other hand, for a general smooth domain* Ω *with geometry verifying the conditions of Prop. 3.7, one is obliged to consider the family of approximations* V_{γ,sr_ω} *even for the homogeneous case* $\gamma = I$.

4. Dirichlet-to-Neumann Gibbs Semigroups

4.1. Gibbs Semigroups

Since by Lemma 2.14 for any Dirichlet-to-Neumann semigroup we obtain $U(t > 0) \in \mathfrak{C}_1(\partial\mathcal{H})$, then one can check that it is in fact a *Gibbs* semigroup. To this end we recall the main definitions and some results that we need for the proof (see, e.g., [Zag2]).

Let $\mathfrak{H}$ be a separable, infinite-dimensional complex Hilbert space. We denote by $\mathcal{L}(\mathfrak{H})$ the algebra of all bounded operators on $\mathfrak{H}$ and by $\mathfrak{C}_\infty(\mathfrak{H}) \subset \mathcal{L}(\mathfrak{H})$ the subspace of all *compact* operators. The $\mathfrak{C}_\infty(\mathfrak{H})$ is a $*$-*ideal* in $\mathcal{L}(\mathfrak{H})$, that is: if $A \in \mathfrak{C}_\infty(\mathfrak{H})$, then $A^* \in \mathfrak{C}_\infty(\mathfrak{H})$ and if $A \in \mathfrak{C}_\infty(\mathfrak{H})$ and $B \in \mathcal{L}(\mathfrak{H})$, then $AB \in \mathfrak{C}_\infty(\mathfrak{H})$ and $BA \in \mathfrak{C}_\infty(\mathfrak{H})$. We say that a compact operator $A \in \mathfrak{C}_\infty(\mathfrak{H})$ belongs to the *von Neumann–Schatten* $*$-ideal $\mathfrak{C}_p(\mathfrak{H})$ for a certain $1 \leq p < \infty$, if the norm

$$\|A\|_p := \left(\sum_{n\geq 1} s_n(A)^p \right)^{1/p} < \infty, \tag{4.1}$$

where $s_n(A) := \sqrt{\lambda_n(A^*A)}$ are the *singular* values of A defined by the eigenvalues $\{\lambda_n(\cdot)\}_{n\geq 1}$ of nonnegative selfadjoint operator A^*A. Since the norm $\|A\|_p$ is a nonincreasing function of $p > 0$, one gets

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q > \|A\|_\infty (= \|A\|) \tag{4.2}$$

for $1 \leq p \leq q < \infty$. Then for the von Neumann–Schatten ideals this implies the inclusions

$$\mathfrak{C}_1(\mathfrak{H}) \subseteq \mathfrak{C}_p(\mathfrak{H}) \subseteq \mathfrak{C}_q(\mathfrak{H}) \subset \mathfrak{C}_\infty(\mathfrak{H}). \tag{4.3}$$

Let $p^{-1} = q^{-1} + r^{-1}$. Then, by virtue of the *Hölder inequality* applied to (4.1), one gets $\|AB\|_p \leq \|A\|_q \|B\|_r$, if $A \in \mathfrak{C}_q(\mathfrak{H})$ and $B \in \mathfrak{C}_r(\mathfrak{H})$. Consequently, we obtain

Lemma 4.1. *The operator A belongs to the trace-class $\mathfrak{C}_1(\mathfrak{H})$ if and only if there exist two* (Hilbert–Schmidt) *operators K_1, $K_2 \in \mathfrak{C}_2(\mathfrak{H})$, such that $A = K_1 \, K_2$. Similarly, if $K \in \mathfrak{C}_p(\mathfrak{H})$, then $K^p \in \mathfrak{C}_1(\mathfrak{H})$.*

Let K be an integral operator in the Hilbert space $L^2(D, \mu)$. It is a Hilbert–Schmidt operator if and only if its kernel $k(x, y) \in L^2(D \times D, \mu \times \mu)$, and then one gets the estimate $\|K\|_2 \leq \|k\|_{L^2(D\times D, \mu\times\mu)}$.

The proof is quite straightforward and can be found in, e.g., [Kat, Sim].

Definition 4.2. [Zag2]. *Let $\{G(t)\}_{t\geq 0}$ be a C_0-semigroup on $\mathfrak{H}$ with $\{G(t)\}_{t>0} \subset \mathfrak{C}_\infty(\mathfrak{H})$. It is called the immediate Gibbs semigroup if $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t > 0$, and it is called the eventually Gibbs semigroup if there is $t_0 > 0$ such that $G(t) \in \mathfrak{C}_1(\mathfrak{H})$ for any $t \geq t_0$.*

Remark 4.3. (a) *Notice that by Lem. 4.1 any C_0-semigroup such that one has $\{G(t)\}_{t>0} \subset \mathfrak{C}_p(\mathfrak{H})$ for some $p < \infty$ is an immediate Gibbs semigroup.*

(b) *Since compact C_0-semigroups are normcontinuous for any $t > 0$, the immediate Gibbs semigroups are $\|\cdot\|_1$-norm continuous for $t > 0$.*

For more details on the Gibbs semigroups properties we refer to the book [Zag2].

Corollary 4.4. *By virtue of Prop. 2.12, Def. 4.2 and Remark 4.3 the Dirichlet-to-Neumann semigroup $\{U(t) = e^{-t\Lambda_{\gamma,\partial\Omega}}\}_t$ on the Hilbert space $\partial\mathcal{H}$ is a $\|\cdot\|_1$-holomorphic quasisectorial immediate Gibbs for* Re $(t) > 0$.

4.2. Compact and Tr-norm Approximating Family

Proposition 4.5. [EmLa] *For the ball Ω_R the Emamirad–Laadnani approximating family $\{V_{\gamma,R}(t)\}_{t\geq 0}$ consists of compact operators on the Banach space $\partial X = C(\partial\Omega_R)$ for any $t > 0$.*

The proof follows from Def. 3.1 by *Arzela–Ascoli* criterium of compactness, since representation (3.3) and conditions on γ imply the uniform bound and equicontinuity of the sets $\{V_{\gamma,R}(t)(\partial X)\}_t$ for any $t > 0$.

For the case of Hilbert space we recall the following useful condition for characterization of the Tr-class operators [Zag2].

Proposition 4.6. *If $A \in \mathcal{L}(\mathfrak{H})$ and $\sum_{j=1}^{\infty} \|Ae_j\| < \infty$ for an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of $\mathfrak{H}$, then $A \in \mathfrak{C}_1(\mathfrak{H})$.*

Theorem 4.7. *On the Hilbert space $\partial\mathcal{H} = L^2(\partial\Omega_R)$ the approximating family $\{V_{\gamma,R}(t)\}_{t>0} \subset \mathfrak{C}_1(\partial\mathcal{H})$.*

P r o f. Since the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ of the selfadjoint Dirichlet-to-Neumann operator $\Lambda_{\gamma,\partial\Omega_R}$ form an orthonormal basis in $L^2(\partial\Omega_R)$, we apply Prop. 4.6 for this basis.

Let $\partial\Omega_{t,\gamma,R} := \{x_\omega(t) := e^{-(t/R)\ \gamma(\omega)}\ \omega\}_{\omega\in\partial\Omega_R}$. By representation (3.3) and by estimate (2.9) one obtains

$$\begin{aligned} \|V_{\gamma,R}(t)\phi_k\|^2 &= \int_{\partial\Omega_R} d\sigma(\omega)|v_{\phi_k}(x_\omega)|^2 \\ &\le |\partial\Omega_R| \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t,\gamma,R})^2/k^{2p/(d-1)}. \end{aligned} \tag{4.4}$$

Then, by hypothesis **(H2)** on the matrix γ for the norm of the vector x_ω in $\mathbb{R}^d$ one gets the estimate

$$\|x_\omega\| \le \|e^{-(t/R)\ \gamma}\|\ R \le e^{-c_1(t/R)}\ R.$$

Hence, for any $t > 0$ the $\mathrm{dist}(x_\omega, \partial\Omega_R) \ge (1 - e^{-c_1(t/R)})R > 0$, which for the estimates in (2.9) and in (4.4) implies that

$$0 < \inf_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}) \le \sup_{\omega\in\partial\Omega_R} \psi(x_\omega, p, \partial\Omega_{t>0,\gamma,R}).$$

Then, for $2p/(d-1) > 1$ the estimate (4.4) ensures the convergence of the series in the inequality

$$\|V_{\gamma,R}(t)\|_1 \le \sum_{k=1}^{\infty} \|V_{\gamma,R}(t)\phi_k\|,$$

which finishes the proof. ■

5. Concluding Remarks: Trace-Norm Approximations

The strong Emamirad–Laadnani approximation theorem (Cor. 3.4) and the results of Sect. 4.2 proving that Dirichlet-to-Neumann semigroup $U(t)$ and approximants $V_{\gamma,\partial\Omega}(t/n)^n$ belong to $\mathfrak{C}_1(\partial\mathcal{H})$, for all $n \geq 1$ and $t > 0$, motivate the following conjecture:

Conjecture 5.1. [EmZa]. *The Emamirad–Laadnani approximation theorem is valid in the* Tr*-norm topology of* $\mathfrak{C}_1(\partial\mathcal{H})$.

Remark 5.2. *Notice that the strong approximation of the Dirichlet-to-Neumann Gibbs semigroup $U(t)$ by the* Tr*-class family $(V_{\gamma,\partial\Omega}(t/n))^n$ does not lift automatically the topology of convergence to, e.g., operator-norm approximation [Zag2].*

Therefore, to prove Conjecture 5.1 one needs additional arguments similar to those of [CaZag2]. To this end we put the difference in question $\Delta_n(t) := (V_{\gamma,\partial\Omega}(t/n))^n - U(t)$ in the following form:

$$\begin{aligned} \Delta_n(t) \;&=\; \{(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\}(V_{\gamma,R}(t/n))^{m_n} \\ &+\; (U(t/n))^{k_n}\{(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\}. \end{aligned} \tag{5.1}$$

Here for any $n > 1$, we define two variables $k_n = [n/2]$ and $m_n = [(n+1)/2]$, where $[x]$ denotes the *integer part* of $x \geq 0$, i.e., $n = k_n + m_n$. Then, for the estimate of $\Delta_n(t)$ in the $\mathfrak{C}_1(\partial\mathcal{H})$-topology one gets

$$\begin{aligned} \|\Delta_n(t)\|_1 \;&\leq\; \|(V_{\gamma,\partial\Omega}(t/n))^{k_n} - (U(t/n))^{k_n}\|\;\|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \\ &+\; \|(U(t/n))^{k_n}\|_1\;\|(V_{\gamma,\partial\Omega}(t/n))^{m_n} - (U(t/n))^{m_n}\|. \end{aligned} \tag{5.2}$$

In spite of Remark 5.2, the *explicit* representation of approximants $\{(V_{\gamma,\partial\Omega}(t/n))^n\}_{n\geq 1}$ allows to prove the corresponding operator-norm estimate.

Theorem 5.3. [EmZa]. *Let $V_{\gamma,\partial\Omega_R}(t)$ be defined by (3.3). Then one gets the estimate*

$$\|(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t)\| \leq \varepsilon(n), \quad \lim_{n\to\infty} \varepsilon(n) = 0, \tag{5.3}$$

uniformly for any t-compact in $\mathbb{R}^1_+$.

To establish (5.3) we use the "telescopic" representation

$$\begin{aligned} &(V_{\gamma,\partial\Omega_R}(t/n))^n - U(t) \\ &= \sum_{s=0}^{n-1} (V_{\gamma,\partial\Omega_R}(t/n))^{(n-s-1)}\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}(U(t/n))^s, \end{aligned} \tag{5.4}$$

and the operator-norm estimate of $\{V_{\gamma,\partial\Omega_R}(t/n) - U(t/n)\}$ for large n.

The next auxiliary result establishes a relation between the family of operators $V_{\gamma,\partial\Omega_R}(t)$ and the Dirichlet-to-Neumann semigroup $U(t)$.

Lemma 5.4. [EmZa]. *There exists a bounded operator* $W_{\gamma,\partial\Omega_R}(t)$ *on* $\partial\mathcal{H}$ *such that*

$$V_{\gamma,\partial\Omega_R}(t) = W_{\gamma,\partial\Omega_R}(t)U(t) \tag{5.5}$$

for any $t \geq 0$.

Now we return to the main inequality (5.2). To estimate the *first* term in the right-hand side of (5.2) we need Th. 5.3 and the *Ginibre–Gruber* inequality [CaZag2]

$$\|(V_{\gamma,\partial\Omega}(t/n))^{m_n}\|_1 \leq C\ U(m_n t/n).$$

To establish the latter we use representation (5.5) given by Lem. 5.4.

To estimate the *second* term one needs only the result of Th. 5.3. All together this gives a proof of Conjecture 5.1 at least for the ball Ω_R.

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